Semi-analytical solution for some proportional delay differential equations

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Abstract

In this work, we have developed a semi-analytical method, based on the combination of the differential transform method and the Laplace transform, to solve a class of linear and nonlinear differential equations with a proportional delay. In order to provide sufficient precision to meet computational needs, we have resumed some proofs of differential transform properties. Some illustrative and physical examples presented and compared to analytical solutions show promising results regarding the validity and applicability of this approach.

Keywords DDEs · Proportional delay · Differential transformation method · Multiple time delays

1 Introduction

Beyond the theoretical interest of study, the differential equations with proportional delay (DDEs) modeling by delays is necessary for many applied physical applications. An example in economics is the variations in a status at time t as a function of the status of that time with some delay, which is inevitable in decision-making problems [1]. The DDEs have a wide range of applications in biology; the estimation of delay is a human orthostatic tremor [2], chemical kinetics [3] and fracture prediction under severe and multiaxial dynamic loading [4]. More generally, the DDEs is used by the domains under which some processes whose modeling is accompanied by a delayed dynamic part, where the evolution is determined from a value of the status $u(t)$ that depends on past values of the status, . . . In this case, it is necessary to consider a part of the “history” of the system in order to know its realistic evolution. For such needs, many researchers have been attracted by semi-analytical methods because the closed-form solution of DDEs is not always analytically accessible, or, having a method, but it is complicated, even if DDEs is linear. Therefore, some semi-analytical methods have been proposed such as the Adomian decomposition method [5], Elzaki

transform method [6, 7], He’s variational iteration method [8, 9] and the homotopy perturbation method [10].

The differential transformation method (DTM) has been introduced firstly by Zhou [11] to solve differential equations in electrical circuits domains. Since then, the DTM has been used to solve a large variety of equations. Relevant examples include integrodifferential systems, DDEs, differential-difference equation, differential algebraic equation and one-dimensional planar Bratu problem, [11]–[23]. In [18, 21], the authors have treated some problems using the DTM. In this work, we extend the use of this method for more generalized nonlinear differential equations and improve the convergence rate of DTM, and we have combined the Laplace transform (LT) and DTM to solve this type of equation. Using this proposed technique, we will show that there are a number of positive outcomes that can be achieved; among these are the greater flexibility and facility to overcome the problems of nonlinearity and difficulty of integration and the general delays, while respecting the conditions and features of the problem. In summation, we intend to treat some algebraic nonlinear DDEs of different degrees in the numerical examples.

We will consider the nonlinear differential equations type with a more time delays defined by

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$$u^{(p)}(t) = \varphi\left(t, u\left(\frac{t}{\alpha_1}\right), u\left(\frac{t}{\alpha_2}\right), \dots, u\left(\frac{t}{\alpha_n}\right)\right), \quad (1)$$

where $\alpha_i \geq 1$ and $u^{(p)}$ are the p th derivative of u with respect to t , for $p, n \in \mathbb{N}$, and we suppose in this paper that φ is a polynomial function. It will be easy to see that $\frac{t}{\alpha_i} = t - \tau_i$,

with a delay given by $\tau_i = (1 - \frac{1}{\alpha_i})t$. Many traditional stud-

ies using DDEs consider only one time lag in their proposed models, but choosing two time lags or more time lags could really fit some needs of modeling and can be as close as possible to reality. For example, in [22] the authors have treated the problem HIV-1 infection with two time delays, where the same problem contains an intracellular delay and immune delay. In addition, we noted that these equations can also be classified into some types of higher-order neutral functional differential equations with proportional delay (NFDE). But the NFDE is a more general type of equation when the p th derivative $u^{(p)}$ depends not only upon the history of u but also on the history of $u^{(i)}$, where $i \leq p$.

2 Differential transform method

Following Zhou [11], we define the differential transform of a function u by:

$$U_{t_0}(i) = \frac{1}{i!} \left[\frac{d^i u(t)}{dt^i} \right]_{t=t_0}, \quad (2)$$

and the inverse differential transform is defined by:

$$u(t) = \sum_{i=0}^{\infty} U_{t_0}(i)(t - t_0)^i. \quad (3)$$

The choice of this notation with the index t_0 is justified by the fact that the value of U depends on t_0 . The goal is to provide enough precision to meet calculation needs. This is precisely not to fall into the lack of precision committed by some authors (see the example [21]).

If $t_0 = 0$, we use the symbol U that refers to the differential transform of u without an index in the rest of this article (i.e., $U_0 \equiv U$).

We summarize in Table 1 the main formulas of the monodimensional differential transform (see [23] for more details). We denote that δ is the Kronecker symbol and $p \in \mathbb{N}$.

2.1 Application of DTM on the proportional delay

In this section, we provide some necessary results proved in [18, 21], and these results can be used in the solution of Eq. (1) with proportional delay.

Table 1 Summarized formulas of the monodimensional DT

Functions	Differential transforms
$h(t) = \alpha u(t) + \beta z(t)$	$H_{t_0}(i) = \alpha U_{t_0}(i) + \beta Z_{t_0}(i)$
$h(t) = \frac{d}{dt} u(t)$	$H_{t_0}(i) = (i+1)U_{t_0}(i+1)$
$h(t) = \frac{d^p}{dt^p} u(t)$	$H_{t_0}(i) = \frac{(i+p)!}{i!} U_{t_0}(i+p)$
$h(t) = u(t)z(t)$	$H_{t_0}(i) = \sum_{l=0}^i U_{t_0}(l)Z_{t_0}(i-l)$
$h(t) = t^p$	$H_{t_0}(i) = \delta(i-p)$

Theorem 1 Let U, F and G be the differential transforms of u, f and g , respectively. For all $\alpha \in \mathbb{N}$, then

- (1) If $u(t) = f\left(\frac{t}{\alpha}\right)$, then $U_{t_0}(i) = \frac{1}{\alpha^i} F_{\frac{t_0}{\alpha}}(i)$.
- (2) If $u(t) = \frac{d^p}{dt^p} f\left(\frac{t}{\alpha}\right)$, then $U_{t_0}(i) = \frac{(i+p)!}{i!} \frac{1}{\alpha^{i+p}} F_{\frac{t_0}{\alpha}}(i+p)$.
- (3) If $u(t) = f\left(\frac{t}{\alpha}\right)g\left(\frac{t}{\beta}\right)$, then $U_{t_0}(i) = \sum_{l=0}^i \frac{1}{\alpha^l \beta^{i-l}} F_{\frac{t_0}{\alpha}}(l)G_{\frac{t_0}{\beta}}(i-l)$.
- (4) If $u(t) = \frac{d^p}{dt^p} f\left(\frac{t}{\alpha}\right) \frac{d^q}{dt^q} g\left(\frac{t}{\beta}\right)$, then

$$U_{t_0}(i) = \sum_{l=0}^i \frac{(l+p)!(i-l+q)}{l!(i-l)! \alpha^{l+p} \beta^{i-l+q}} F_{\frac{t_0}{\alpha}}(l+p)G_{\frac{t_0}{\beta}}(i-l+q).$$

Proof See "Appendix".

Remark 1 We note that in the special case when $t_0 = 0$:

We obtain $\frac{t_0}{\alpha} = \frac{t_0}{\beta} = t_0 = 0$. Then, the differential transforms U, F and G are given in Theorem 1 without index.

3 Description of the method

Now consider Eq. (1) in the following short denotation

$$u^{(p)}(t) = \varphi(t, u).$$

The first step is to take LT of both sides of the differential equation. The purpose is to seek some ways to relax the calculus by replacing the differentiation in the problem with an algebraic expression. The calculations cost reduction is clearly explained in the examples treated in this paper.

Taking the LT of both sides of Eq. (1), we obtain

$$s^p \ell u(t) = \ell \varphi(t, u) + \sum_{i=1}^p s^{p-i} u^{i-1}(0),$$

or

$$\ell u(t) = \frac{1}{s^p} \ell \varphi(t, u) + \sum_{i=1}^p \frac{1}{s^i} u^{i-1}(0).$$

Using the LT inverse, we have

$$u(t) = \ell^{-1} \left\{ \frac{1}{s^p} \ell \varphi(t, u) \right\} + \sum_{k=0}^{p-1} \frac{t^k}{k!} u^k(0). \quad (4)$$

In the second step, we apply the DTM which defines the solution $u(t)$ by the series

$$u(t) = \sum_{i=0}^{\infty} U_i. \quad (5)$$

Substituting Eq. (5) into (4), we get the following algorithm

$$U_0 = u(0), \quad (6)$$

$$U_1 = \sum_{k=1}^{p-1} \frac{t^k}{k!} u^k(0), \quad (7)$$

$$U_i = \ell^{-1} \left\{ \frac{1}{s^p} \ell [A_i] \right\}, \quad i \geq 2, \quad (8)$$

where A_i is the differential transform of $\varphi(t, u)$. The DTM is efficient if φ is a polynomial function in u . For more applicability, this method should be developed to cover more complex DDEs.

4 Numerical examples

In order to describe the technique of the DTM combined with the LT, we give some illustrative numerical examples.

Example 1 Consider the nonlinear DDEs

$$\frac{du(t)}{dt} = 1 - 2u^2\left(\frac{t}{2}\right), \quad t \in [0, 1] \quad (9a)$$

$$u(0) = 0, \quad (9b)$$

The analytic solution is $u(t) = \sin t$.

Taking the LT of both sides of Eq. (9), to get;

$$s \ell u(t) - u(0) = \frac{1}{s} - 2 \ell \left[u^2\left(\frac{t}{2}\right) \right] \Rightarrow \ell u(t) = \frac{1}{s^2} - \frac{2}{s} \ell \left[u^2\left(\frac{t}{2}\right) \right]. \quad (10)$$

Taking the inverse LT of Eq. (10), we get:

$$u(t) = t - \ell^{-1} \left\{ \frac{2}{s} \ell \left[u^2\left(\frac{t}{2}\right) \right] \right\}. \quad (11)$$

Substituting Eq. (5) into (11), we have the following expression

$$U(i+1) = -\ell^{-1} \left\{ \frac{2}{s} \ell [A_i] \right\} \quad i \geq 0 \quad U(0) = 0 \quad U(1) = t.$$

where A_i is the differential transform of $u^2\left(\frac{t}{2}\right)$ such that

$$A_i = \sum_{l=0}^i \frac{U(l)U(i-l)}{2^l 2^{i-l}}.$$

Then,

$$A_1 = 0 \Rightarrow U(2) = 0;$$

$$A_2 = \frac{t^2}{4} \Rightarrow U(3) = -\ell^{-1} \left[\frac{1}{s^4} \right] = -\frac{t^3}{3!};$$

$$A_3 = 0 \Rightarrow U(4) = 0;$$

$$A_4 = -\frac{t^4}{3!.8} \Rightarrow U(5) = -\ell^{-1} \left[-\frac{4!.2}{3!.8s^6} \right] = \frac{t^5}{5!};$$

and so on.

Then, the series solution according to Eq. (5) is given by

$$u(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

which is formally the same as Maclaurin series of $\sin t$.

Figure 1 shows the comparison between the exact solution and the approximated solution of the system (9). In Fig. 2, we remark that the proposed solution is characterized by two stages of precision: in the first stage, for $t \in [0, 0.6]$ a well precise approximation of solution is obtained; in the second stage, the approximation is just reasonable ($< 2 \times 10^{-4}$).

Fig. 1 The approximate solution is presented by blue stars and the analytical solution red solid line

Fig. 2 Absolute error (Example 1).

Example 2 We consider here an example of DDEs with two-times proportional delay,

$$\frac{du(t)}{dt} = u\left(\frac{t}{2}\right) - 4u\left(\frac{t}{4}\right), \quad t \in [0, 1] \quad (12a)$$

$$u(0) = 1. \quad (12b)$$

The closed-form solution is $u(t) = \frac{3}{4}t^2 - 3t + 1$.

We apply the LT to Eq. (12a), and we get

$$s\ell u(t) - u(0) = \ell\left(u\left(\frac{t}{2}\right) - 4u\left(\frac{t}{4}\right)\right). \quad (13)$$

By the inverse LT of Eq. (13), and using the initial conditions (12b), we obtain

$$u(t) = t + \ell^{-1}\left[\frac{1}{s}\ell(A_i)\right].$$

Putting $A_i = \frac{U(i)}{2^i} - \frac{U(i)}{4^{i-1}}$, this is the differential transform of $u(\frac{t}{2}) - 4u(\frac{t}{4})$.

Typically, we choose $U(0) = 1$ and $U(1) = t$, but with this choice, the proposed technique fails to converge to the solution. The exact solution is given by $u_{\text{exat}}(t) = \frac{3}{4}t^2 - 3t + 1$. The trouble is we do not start by a consistent initial slope because $u'_{\text{exat}}(0) = -3$ and the first-approximation $U(1) = t$. For this reason, we first propose to calculate the derivative $u'(0)$ on the basis of Eq. (12a).

When $t = 0$, we have $u'(0) = u(0) - 4u(0) = -3$ and we propose to put $U(1) = u'(0)t = -3t$. The objective is to ensure the consistency of initial slope.

$$U(i+1) = \ell^{-1}\left\{\frac{1}{s}\ell[A_i]\right\}, \quad i \geq 0;$$

$$U(0) = 1, \quad U(1) = -3t, \quad U(2) = \frac{3}{4}t^2. \quad (14)$$

Then, the exact solution will be given by

$$u(t) = U(0) + U(1) + U(2) = \frac{3}{4}t^2 - 3t + 1.$$

Example 3 We consider the following nonlinear DDEs of the third order

$$\frac{d^3u(t)}{dt^3} = -1 + 2u^2\left(\frac{t}{2}\right), \quad t \in [0, 1] \quad (15a)$$

$$u(0) = 0, \quad u'(0) = 1 \quad \text{and} \quad u''(0) = 0. \quad (15b)$$

The analytical solution is $u(t) = \sin t$.

Using the same methods as in Examples 1 and 2, we get the following relation

$$U(i+2) = \ell^{-1}\left\{\frac{2}{s^3}\ell[A_i]\right\}, \quad i \geq 0, \quad U(0) = 0, \\ U(1) = t, \quad U(2) = -\frac{t^3}{3!},$$

where A_i is the differential transform of $u^2\left(\frac{t}{2}\right)$ with

$$A_i = \sum_{l=0}^i \frac{U(l)U(i-l)}{2^l 2^{i-l}}.$$

Then, we obtain

$$A_1 = 0 \Rightarrow U(3) = 0;$$

$$A_2 = \frac{t^2}{4} \Rightarrow U(4) = \frac{t^5}{5!};$$

$$A_3 = 0 \Rightarrow U(5) = 0;$$

and so on.

Hence, the series solution according to Eq. (5) will be given as

$$u(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots = \sin t.$$

This example shows the performance of this method compared with the heavy calculations of Adomian's polynomials treated in [5] to solve the same example using Adomian decomposition method. Note further that by the

inserting LT we overtake a few numbers of iterations in the calculation of solution compared with the DTM used in [21].

Example 4 (Hutchinson's equation) DDEs have been presented in several biological models. For example, we propose to study the logistic equation [24] or the Hutchinson's equation with a proportional delay for a single-species dynamics. This model is given by:

$$\frac{du(t)}{dt} = Ru(t) - \frac{R}{K}u(t)u\left(\frac{t}{2}\right), \quad t \in [0, 1] \quad (16a)$$

$$u(0) = 1. \quad (16b)$$

Consider $R = 1$ and $K = 1$. Applying the LT to both sides of Eq. (16a), we get

$$\begin{aligned} s\ell u(t) - u(0) &= \ell[u(t)] - \ell\left[u(t)u\left(\frac{t}{2}\right)\right] \\ \Rightarrow \ell u(t) &= \frac{1}{s-1} - \frac{1}{s-1}\ell\left[u(t)u\left(\frac{t}{2}\right)\right]. \end{aligned}$$

Taking the inverse LT of Eq. (17), we get:

$$u(t) = \exp(t) - \ell^{-1}\left\{\frac{1}{s-1}\ell\left[u(t)u\left(\frac{t}{2}\right)\right]\right\}. \quad (18)$$

Substituting Eq. (5) into (18), we have the following expression

$$U(i+1) = -\ell^{-1}\left\{\frac{1}{s-1}\ell[A_i]\right\}, \quad i \geq 0,$$

$$U(0) = 0, \quad U(1) = \exp(t),$$

where A_i is the differential transform of $u(t)u\left(\frac{t}{2}\right)$ such that

$$A_i = \sum_{l=0}^i \frac{U(l)U(i-l)}{2^{i-l}}.$$

Then,

$$A_1 = 0 \Rightarrow U(2) = 0;$$

$$\begin{aligned} A_2 &= \frac{1}{2}\exp(2t) \Rightarrow U(3) = -\frac{1}{2}\ell^{-1}\left[\frac{1}{(s-1)(s-2)}\right] \\ &= \frac{1}{2}\exp(t) - \frac{1}{2}\exp(2t); \end{aligned}$$

$$A_3 = 0 \Rightarrow U(4) = 0;$$

$$\begin{aligned} A_4 &= \frac{5}{16}(\exp(2t) - \exp(3t)) \Rightarrow U(5) \\ &= -\frac{5}{16}\ell^{-1}\left[\frac{1}{(s-1)(s-2)} - \frac{1}{(s-1)(s-3)}\right] \\ &= \frac{5}{16}\ell^{-1}\left[\frac{1}{2(s-1)} - \frac{1}{(s-2)} + \frac{1}{2(s-3)}\right] \\ &= +\frac{5}{16}\left(\frac{1}{2}\exp(t) - \exp(2t) + \frac{1}{2}\exp(3t)\right). \end{aligned}$$

Then, the approximate solution after five iterations is given by

$$u(t) = \frac{63}{32}\exp(t) - \frac{23}{16}\exp(2t) + \frac{15}{32}\exp(3t) \dots$$

5 Conclusion

Even if the DDEs are linear, it is not easy to obtain an explicit analytical solution for these types of equations. For this purpose, we have developed a simple method for solving linear and nonlinear differential equations with proportional delay using a combination of the LT and the DTM. The treated examples emphasize that the proposed method is an effective tool to compute a semi-analytical solution with a reasonable estimate. Besides that, the convergence rate can be improved by this combination of LT and DTM. (17)

Compliance with ethical standards

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this article.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors

Appendix

Proof (Theorem 1)

(1) Calculating

$$\frac{d^i}{dt^i}u(t) = \left[\frac{d^i}{dt^i}f\left(\frac{t}{\alpha}\right)\right] = \frac{1}{\alpha^i}\left[f^{(i)}\left(\frac{t}{\alpha}\right)\right].$$

$$\begin{aligned} \text{Therefore } U_{t_0}(i) &= \frac{1}{i!}\left[\frac{d^i}{dt^i}u(t)\right]_{t=t_0} = \frac{1}{i!\alpha^i}\left[f^{(i)}\left(\frac{t}{\alpha}\right)\right]_{t=t_0} \\ &= \frac{F_{\frac{t_0}{\alpha}}(i)}{\alpha^i}. \end{aligned}$$

Using the definition of DT, one gets $U_{t_0}(i) = \frac{1}{\alpha^i}F_{\frac{t_0}{\alpha}}(i)$.

(2) From (1), we have

$$\begin{aligned} U_{t_0}(i) &= \frac{1}{i!}\left[\frac{d^p}{dt^p}u(t)\right]_{t=t_0} = \frac{1}{i!\alpha^{i+q}}\left[f^{(i+q)}\left(\frac{t}{\alpha}\right)\right]_{t=t_0} \\ &= \frac{(i+q)!}{i!\alpha^{i+q}}F_{\frac{t_0}{\alpha}}(i+q). \end{aligned}$$

$$(3) \quad U_{t_0}(i) = \frac{1}{i!} \left[\frac{d^i}{dt^i} u(t) \right]_{t=t_0} = \frac{1}{i!} \left[\frac{d^i}{dt^i} \left(f\left(\frac{t}{\alpha}\right) g\left(\frac{t}{\beta}\right) \right) \right]_{t=t_0}.$$

Using the Leibnitz's theorem we get

$$\begin{aligned} &= \sum_{l=0}^i \binom{i}{l} \frac{(i-l)!}{i!} \frac{1}{\alpha^l} \frac{1}{l!} \left[f^{(l)}\left(\frac{t}{\alpha}\right) \right]_{t=t_0} \frac{1}{\beta^{i-l}} \frac{1}{(i-l)!} \left[g^{(i-l)}\left(\frac{t}{\beta}\right) \right]_{t=t_0} \\ &= \sum_{l=0}^i \frac{1}{\alpha^l \beta^{i-l}} F_{\frac{t_0}{\alpha}}^{(l)} G_{\frac{t_0}{\beta}}^{(i-l)}. \end{aligned}$$

(4) We have

$$U_{t_0}(i) = \frac{1}{i!} \left[\frac{d^i}{dt^i} u(t) \right]_{t=t_0} = \frac{1}{i!} \left[\frac{d^i}{dt^i} \left(\frac{d^p}{dt^p} f\left(\frac{t}{\alpha}\right) \frac{d^q}{dt^q} g\left(\frac{t}{\beta}\right) \right) \right]_{t=t_0}.$$

From the Leibnitz's theorem of i th-derivative of a product we get

$$\begin{aligned} &= \frac{1}{i!} \sum_{l=0}^i \binom{i}{l} \frac{1}{\alpha^{l+p}} \left[f^{(l+p)}\left(\frac{t}{\alpha}\right) \right]_{t=t_0} \frac{1}{\beta^{i-l+q}} \left[g^{(i-l+q)}\left(\frac{t}{\beta}\right) \right]_{t=t_0} \\ &= \frac{1}{i!} \sum_{l=0}^i \binom{i}{l} \left(\frac{(l+p)! F_{\frac{t_0}{\alpha}}^{(l+p)}}{\alpha^{l+p}} \right) \left(\frac{(i-l+q)! G_{\frac{t_0}{\beta}}^{(i-l+q)}}{\beta^{i-l+q}} \right) \\ &= \sum_{l=0}^i \frac{(l+p)!(i-l+q)!}{l!(i-l)! \alpha^{l+p} \beta^{i-l+q}} F_{\frac{t_0}{\alpha}}^{(l+p)} G_{\frac{t_0}{\beta}}^{(i-l+q)}. \end{aligned}$$

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