A Novel Peridynamic Mindlin Plate Formulation Without Limitation on Material Constants

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Abstract

In this study, a new peridynamic Mindlin plate formulation is introduced by utilising Euler-Lagrange equations. The classical strain energy density of a material point is converted to its corresponding peridynamic form by using Taylor's expansions. The formulation is suitable for thick plates by considering the transverse shear deformation. Material constants do not have any limitation in the current formulation. Different types of boundary conditions are considered in numerical examples including simply supported, clamped and mixed (clamped-supported). To verify the current formulation, peridynamic solutions of the transverse displacements and rotations are compared against solutions obtained from finite element analysis.

Keywords Mindlin plate · Peridynamics · Non-local · Transverse shear deformation

1 Introduction

Solid mechanics is an important research area focusing on how the materials and structures deform if they are subjected to external loading conditions. As a result of the deformation, permanent damages can occur including plasticity and crack occurrence. Predicting crack initiation and propagation is still one of the challenging areas of solid mechanics. Moreover, with the advancement of technology, interest on nano-scale structures is increasing. Analysis of nano-scale structures can require non-classical approaches including length scale parameters. There are various non-classical approaches available in the literature, and amongst these, peridynamics [1] is the focus of this study. There has been a rapid progress in peridynamics research during the recent years [2–24]. An extensive review of peridynamics is given by Javili et al. [25].

Original peridynamic formulation is suitable for analysis of three-dimensional structures. However, this formulation can be computational expensive for structures which have special shapes that can be considered as beams, plates and shells. To model such structures, standard formulation should be modified by taking into account rotational degrees

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of freedom. Hence, three-dimensional geometries can be represented by one-dimensional models as beams or two-dimensional models as plates and shells. Such formulations are currently available for peridynamics. For instance, a peridynamic formulation to analyse thin plates was introduced in [26]. A non-ordinary state-based Euler-Bernoulli beam formulation was developed in [27]. In another study [28], they extended this formulation to represent the Kirchhoff-Love plate theory. State-based peridynamics was used to develop peridynamic Euler beam and Kirchhoff plate formulations in [29] and [30], respectively. To analyse thick plates, it is essential to take into account transverse shear deformations. Transverse shear deformations were taken into account in [31] as a characteristic of peridynamic Mindlin plate formulation. However, this formulation is limited to Poisson's ratio of $1/3$ for isotropic materials.

Hence, a new peridynamic Mindlin plate formulation is introduced in this study which does not have any limitations on material constants. The formulation is developed by using the Euler-Lagrange equation in conjunction with Taylor's expansion. To verify the formulation, several numerical examples are considered for a Mindlin plate under different boundary conditions, and peridynamic and finite element analysis results are compared with each other.

2 Classical Mindlin Plate Formulation

Mindlin developed a plate theory for thick plates by considering transverse shear deformation. According to Mindlin plate theory, a transverse normal to the mid-plane of the plate in the undeformed state remains straight, and there is no change in its length during deformation. The displacement components of any material point can be represented in terms of mid-plane (xy plane) displacements and rotations as

$$u(x, y, z, t) = z\theta_x(x, y, t) \quad (1)$$

$$v(x, y, z, t) = z\theta_y(x, y, t) \quad (2)$$

$$w(x, y, z, t) = w(x, y, t) \quad (3)$$

where θ_x and θ_y denote the mid-plane rotations about positive y -direction and negative x -direction, respectively. Moreover, w denotes the mid-plane transverse displacements. The positive set of the degrees-of-freedom is shown in Fig. 1.

Thus, the strain-displacement relationships can be written as

$$\epsilon_{xx} = z \frac{\partial \theta_x}{\partial x} \quad (4)$$

$$\epsilon_{yy} = z \frac{\partial \theta_y}{\partial y} \quad (5)$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2} z \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \quad (6)$$

Fig. 1 Degrees-of-freedom in peridynamic Mindlin formulation. **a** xz -plane. **b** yz -plane

$$\gamma_{xz} = \kappa_s \left(\theta_x + \frac{\partial w}{\partial x} \right) \quad (7)$$

$$\gamma_{yz} = \kappa_s \left(\theta_y + \frac{\partial w}{\partial y} \right) \quad (8)$$

$$\epsilon_{zz} = 0 \quad (9)$$

where κ_s is introduced as shear coefficient. Equations (4–11) can be also expressed in indicial notation as

$$\epsilon_{IJ} = \frac{1}{2} z \left(\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I} \right) \quad (10)$$

$$\gamma_{I3} = \kappa_s \left(\theta_I + \frac{\partial w}{\partial x_I} \right) \quad (11)$$

Note that the subscript indices, $I, J, \dots = 1 (= x), 2 (= y)$, and this convention will be applied throughout this study.

The stress-strain relationships can be written for isotropic materials as:

$$\sigma_{xx} = \frac{E}{1 - \nu^2} (\epsilon_{xx} + \nu \epsilon_{yy}) \quad (12)$$

$$\sigma_{yy} = \frac{E}{1 - \nu^2} (\epsilon_{yy} + \nu \epsilon_{xx}) \quad (13)$$

$$\sigma_{xy} = \frac{E}{2(1 + \nu)} (\epsilon_{xy} + \epsilon_{yx}) \quad (14)$$

$$\tau_{xz} = G \gamma_{xz} \quad (15)$$

$$\tau_{yz} = G \gamma_{yz} \quad (16)$$

where E , G , and ν represent elastic modulus, shear modulus and Poisson's ratio, respectively. Note that the transverse normal stress, σ_{zz} , is considered to be small compared with in-plane stresses. Thus, it is discarded from the stress components set for simplification. The stresses can also be written in indicial notation as:

$$\sigma_{IJ} = C_{IJKL} \varepsilon_{KL} \quad (17)$$

$$\tau_{I3} = G \gamma_{I3} \quad (18)$$

where

$$C_{IJKL} = \frac{E}{1-\nu^2} \left[\frac{1-\nu}{2} (\delta_{IL} \delta_{JK} + \delta_{IK} \delta_{JL}) + \nu \delta_{IJ} \delta_{KL} \right] \quad (19)$$

The linear elastic strain energy density of the Mindlin plate can be expressed as

$$\tilde{W} = \frac{1}{2} (\sigma_{IJ} \varepsilon_{IJ} + \tau_{I3} \gamma_{I3}) \quad (20)$$

Inserting Eqs. (10), (11) and (17) into Eq. (19) and rearranging indices yields

$$\begin{aligned} \tilde{W} = & \frac{E}{1-\nu^2} z^2 \left[\frac{1-\nu}{4} \left(\frac{\partial \theta_I}{\partial x_J} \frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J} \frac{\partial \theta_J}{\partial x_I} + \frac{\partial \theta_I}{\partial x_I} \frac{\partial \theta_J}{\partial x_J} \right) + \frac{3\nu-1}{4} \frac{\partial \theta_I}{\partial x_I} \frac{\partial \theta_J}{\partial x_J} \right] \\ & + \kappa_s^2 \frac{G}{2} \left(\theta_I + \frac{\partial w}{\partial x_I} \right) \left(\theta_I + \frac{\partial w}{\partial x_I} \right) \end{aligned} \quad (21)$$

For a particular material point on the mid-plane, the average strain energy density can be reasonably calculated by integrating the strain energy density function, Eq. (21), through the transverse direction and dividing by the thickness as

$$\begin{aligned} W = & \frac{E}{1-\nu^2} \frac{h^2}{12} \left[\frac{1-\nu}{4} \left(\frac{\partial \theta_I}{\partial x_J} \frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J} \frac{\partial \theta_J}{\partial x_I} + \frac{\partial \theta_I}{\partial x_I} \frac{\partial \theta_J}{\partial x_J} \right) + \frac{3\nu-1}{4} \frac{\partial \theta_I}{\partial x_I} \frac{\partial \theta_J}{\partial x_J} \right] \\ & + \kappa_s^2 \frac{G}{2} \left(\theta_I + \frac{\partial w}{\partial x_I} \right) \left(\theta_I + \frac{\partial w}{\partial x_I} \right) \end{aligned} \quad (22)$$

3 Peridynamic Mindlin Plate Formulation

Peridynamics (PD) is a continuum mechanics formulation in which material points can interact with each other in a non-local manner. The range of non-local interactions is defined as “horizon,” H . The equation of motion peridynamics can be expressed in discrete form as

$$\rho_{(k)} \ddot{\mathbf{u}}_{(k)} = \sum_{j=1}^N \mathbf{f}_{(k)(j)} V_{(j)} + \mathbf{b}_{(k)} \quad (23)$$

for the material point k where N is the number of material points inside the horizon, ρ is density, $\mathbf{u}$ is the displacement, t is time, V is volume and $\mathbf{b}$ is the body load vector. The interaction force vector, $\mathbf{f}_{(k)(j)}$, between material points k and j can be defined as (see Fig. 2)

Fig. 2 PD force density factor and horizon [32]

$$\mathbf{f}_{(k)(j)} = \mathbf{t}_{(k)(j)} - \mathbf{t}_{(j)(k)} \quad (24)$$

The PD equations of motion can be obtained from Euler-Lagrange's equation as:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{u}}_{(k)}} - \frac{\partial L}{\partial \mathbf{u}_{(k)}} = 0 \quad (25)$$

where L is the Lagrangian which is the difference between the total kinetic energy, T , and total potential energy, U , that can be expressed as

$$T = \frac{1}{2} \int_A \int_{-h/2}^{h/2} \rho \dot{\mathbf{u}}(x, z, t) \cdot \dot{\mathbf{u}}(x, z, t) dz dA \quad (26)$$

and

$$U = \sum_k W_{(k)} V_{(k)} - \sum_k \mathbf{b}_{(k)} \cdot \mathbf{u}_{(k)} V_{(k)} \quad (27)$$

where the generalised displacement vector, $\mathbf{u}$, and generalised body force density vector, $\mathbf{b}$, can be defined as

$$\mathbf{u} = (\theta_1 \quad \theta_2 \quad w)^T \quad (28)$$

and

$$\mathbf{b} = (b_{\theta_1} \quad b_{\theta_2} \quad b_z)^T \quad (29)$$

Here, b_{θ} and b_z correspond to moment and transverse force, respectively. Inserting Eqs. (1), (2) and (3) into (26) and discretizing the function gives the kinetic energy of the system as

$$T = \frac{1}{2} \sum_k \rho_{(k)} \left(\frac{h^2}{12} \left(\dot{\theta}_1^{(k)} \right)^2 + \frac{h^2}{12} \left(\dot{\theta}_2^{(k)} \right)^2 + \dot{w}_{(k)}^2 \right) V_{(k)} \quad (30)$$

Assuming the system is holonomic, the first term of Euler-Lagrange equation can be obtained by substituting Eq. (30) into (25):

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{u}}_{(k)}} = \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{u}}_{(k)}} = \rho_{(k)} \left\{ \begin{array}{c} \frac{h^2}{12} \ddot{\theta}_1^{(k)} \\ \frac{h^2}{12} \ddot{\theta}_2^{(k)} \\ \ddot{w}_{(k)} \end{array} \right\} V_{(k)} \quad (31)$$

In PD theory, the strain energy density has a non-local form. Therefore, the strain energy of a certain material point k can be expressed as

$$W_{(k)} = W_{(k)} \left(\mathbf{u}_{(k)}, \mathbf{u}_{(1^k)}, \mathbf{u}_{(2^k)}, \mathbf{u}_{(3^k)}, \dots \right) \quad (32)$$

where $\mathbf{u}_{(k)}$ represents the displacement vector of the material point k , and $\mathbf{u}_{(i^k)}$ ($i = 1, 2, 3, \dots$) represents the displacement vector of the i th material point inside the horizon of the material point k .

The total potential energy given in Eq. (27) can then be written as

$$U = \sum_k W_{(k)} \left(\mathbf{u}_{(k)}, \mathbf{u}_{(1^k)}, \mathbf{u}_{(2^k)}, \mathbf{u}_{(3^k)}, \dots \right) V_{(k)} - \sum_k \mathbf{b}_{(k)} \cdot \mathbf{u}_{(k)} V_{(k)} \quad (33)$$

Thus, the second term in Eq. (25) can be evaluated as

$$\begin{aligned} -\frac{\partial L}{\partial \mathbf{u}_{(k)}} &= \frac{\partial}{\partial \mathbf{u}_{(k)}} \sum_n W_{(n)} \left(\mathbf{u}_{(n)}, \mathbf{u}_{(1^n)}, \mathbf{u}_{(2^n)}, \mathbf{u}_{(3^n)}, \dots \right) V_{(n)} - \frac{\partial}{\partial \mathbf{u}_{(k)}} \sum_n \mathbf{b}_{(n)} \cdot \mathbf{u}_{(n)} V_{(n)} \\ &= \sum_n \frac{\partial W_{(n)}}{\partial \mathbf{u}_{(k)}} (\delta_{nk} + \delta_{n1^k}) V_{(n)} - \frac{\partial}{\partial \mathbf{u}_{(k)}} \sum_n \mathbf{b}_{(n)} \delta_{nk} V_{(n)} = \left(\frac{\partial W_{(k)}}{\partial \mathbf{u}_{(k)}} V_{(k)} + \sum_j \frac{\partial W_{(j)}}{\partial \mathbf{u}_{(k)}} V_{(j)} \right) - \mathbf{b}_{(k)} V_{(k)} \end{aligned} \quad (34)$$

Inserting Eqs. (31) and (34) into Euler-Lagrange equation yields:

$$\rho_{(k)} \left\{ \begin{array}{c} \frac{h^2}{12} \ddot{\theta}_1^{(k)} \\ \frac{h^2}{12} \ddot{\theta}_2^{(k)} \\ \ddot{w}_{(k)} \end{array} \right\} V_{(k)} = - \left\{ \begin{array}{c} \frac{\partial W_{(k)}}{\partial \theta_1^{(k)}} V_{(k)} + \sum_j \frac{\partial W_{(j)}}{\partial \theta_1^{(k)}} V_{(j)} \\ \frac{\partial W_{(k)}}{\partial \theta_2^{(k)}} V_{(k)} + \sum_j \frac{\partial W_{(j)}}{\partial \theta_2^{(k)}} V_{(j)} \\ \frac{\partial W_{(k)}}{\partial w_{(k)}} V_{(k)} + \sum_j \frac{\partial W_{(j)}}{\partial w_{(k)}} V_{(j)} \end{array} \right\} + \left\{ \begin{array}{c} b_{\theta_1}^{(k)} \\ b_{\theta_2}^{(k)} \\ b_z^{(k)} \end{array} \right\} V_{(k)} \quad (35)$$

As explained in Appendix, the strain energy densities of the material point k and j can be expressed in peridynamic form by transforming all differential terms in Eq. (22) as

$$\begin{aligned} W_{(k)} &= \frac{E}{1-\nu^2} \frac{h^2}{12} \left[\frac{1-\nu}{4} \left(\frac{12}{\pi \delta^2 h} \sum_i \frac{(\theta_l^{(i^k)} - \theta_l^{(k)}) (\theta_j^{(i^k)} - \theta_j^{(k)})}{\xi_{(i^k)(k)}} n_l^{(i^k)(k)} n_j^{(i^k)(k)} V_{(i^k)} \right) + \right. \\ &\quad \left. \frac{3\nu-1}{4} \left(\frac{2}{\pi \delta^2 h} \right)^2 \sum_i \frac{(\theta_l^{(i^k)} - \theta_l^{(k)}) (\theta_j^{(i^k)} - \theta_j^{(k)})}{\xi_{(i^k)(k)}} n_l^{(i^k)(k)} V_{(i^k)} \sum_i \frac{(\theta_j^{(i^k)} - \theta_j^{(k)})}{\xi_{(i^k)(k)}} n_j^{(i^k)(k)} V_{(i^k)} \right] \\ &\quad + \kappa_s^2 \frac{G}{2} \frac{3}{\pi \delta^3 h} \times \sum_i \frac{\left(w_{(i^k)} - w_{(k)} + \frac{\theta_l^{(i^k)} - \theta_l^{(k)}}{2} \xi_{(i^k)(k)} n_l^{(i^k)(k)} \right) \left(w_{(i^k)} - w_{(k)} + \frac{\theta_j^{(i^k)} - \theta_j^{(k)}}{2} \xi_{(i^k)(k)} n_j^{(i^k)(k)} \right)}{\xi_{(i^k)(k)}} V_{(i^k)} \end{aligned} \quad (36)$$

$$\begin{aligned}
W_{(j)} = & \frac{E}{1-\nu^2} \frac{h^2}{12} \left[\frac{1-\nu}{4} \left(\frac{12}{\pi \delta^2 h} \sum_i \frac{(\theta_l^{(j)} - \theta_l^{(i)}) (\theta_j^{(i)} - \theta_j^{(i)})}{\xi_{(j)(i)}} n_l^{(i)(k)} n_j^{(i)(k)} V_{(i)} \right) + \right. \\
& \left. \frac{3\nu-1}{4} \left(\frac{2}{\pi \delta^2 h} \right)^2 \sum_i \frac{(\theta_j^{(j)} - \theta_j^{(i)})}{\xi_{(j)(i)}} n_l^{(i)(j)} V_{(i)} \sum_i \frac{(\theta_l^{(j)} - \theta_l^{(i)})}{\xi_{(j)(i)}} n_j^{(i)(j)} V_{(i)} \right] \\
& + \kappa_s^2 \frac{G}{2} \frac{3}{\pi \delta^3 h} \times \sum_i \frac{\left(w_{(j)} - w_{(i)} + \frac{\theta_j^{(j)} - \theta_j^{(i)}}{2} \xi_{(j)(i)} n_l^{(i)(j)} \right) \left(w_{(j)} - w_{(i)} + \frac{\theta_j^{(j)} - \theta_j^{(i)}}{2} \xi_{(j)(i)} n_j^{(i)(j)} \right)}{\xi_{(j)(i)}} V_{(i)}
\end{aligned} \quad (37)$$

with $n_1 = \cos \varphi$, $n_2 = \sin \varphi$ and φ is the orientation of peridynamic bond (interaction).

Inserting Eqs. (36) and (37) into Eq. (35) and renaming the summation indices yield the governing equations of PD Mindlin plate formulation as:

$$\begin{aligned}
\rho_{(k)} \frac{h^2}{12} \ddot{\theta}_L^{(k)} = & \frac{E}{1+\nu} \frac{h^2}{12} \frac{12}{\pi \delta^3 h} \sum_j \frac{(\theta_l^{(j)} - \theta_l^{(k)})}{\xi_{(j)(k)}} n_l^{(j)(k)} n_j^{(j)(k)} V_{(j)} \\
& + \left(\frac{E}{1-\nu^2} \frac{3\nu-1}{4} \right) \frac{h^2}{12} \left(\frac{2}{\pi \delta^2 h} \right)^2 \sum_j \frac{n_L^{(j)(k)}}{\xi_{(j)(k)}} \left(\sum_i \frac{(\theta_l^{(i)} - \theta_l^{(k)})}{\xi_{(i)(k)}} n_l^{(i)(k)} V_{(i)} + \right. \\
& \left. \sum_i \frac{(\theta_j^{(i)} - \theta_j^{(k)})}{\xi_{(i)(k)}} n_j^{(i)(j)} V_{(i)} \right) V_{(j)} \\
& - \kappa_s^2 \frac{G}{2} \frac{6}{\pi \delta^3 h} \sum_j \frac{\left(w_{(j)} - w_{(k)} + \frac{\theta_j^{(j)} + \theta_j^{(k)}}{2} \xi_{(j)(k)} n_j^{(j)(k)} \right)}{\xi_{(j)(k)}} n_L^{(j)(k)} V_{(j)} + b_{\theta_L}^{(k)}
\end{aligned} \quad (38)$$

$$\rho_{(k)} \ddot{w}_{(k)} = \kappa_s^2 G \frac{6}{\pi \delta^3 h} \sum_j \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} + \frac{\theta_j^{(k)} + \theta_j^{(j)}}{2} n_j^{(j)(k)} \right) V_{(j)} + b_z^{(k)} \quad (39)$$

In particular, Eqs. (38) and (39) can be simplified for the Poisson's ratio, $\nu = 1/3$, as

$$\begin{aligned}
\rho_{(k)} \frac{h^2}{12} \ddot{\theta}_L^{(k)} = & c_b \sum_j \frac{(\theta_l^{(j)} - \theta_l^{(k)})}{\xi_{(j)(k)}} n_l^{(j)(k)} n_L^{(j)(k)} V_{(j)} \\
& - \frac{c_s}{2} \sum_j \left(w_{(j)} - w_{(k)} + \frac{\theta_j^{(j)} + \theta_j^{(k)}}{2} \xi_{(j)(k)} n_j^{(j)(k)} \right) n_L^{(j)(k)} V_{(j)} + b_{\theta_L}^{(k)}
\end{aligned} \quad (40)$$

$$\rho_{(k)} \ddot{w}_{(k)} = c_s \sum_j \left(\frac{w_{(j)} - w_{(k)}}{\xi_{(j)(k)}} + \frac{\theta_j^{(k)} + \theta_j^{(j)}}{2} n_j^{(j)(k)} \right) V_{(j)} + b_z^{(k)} \quad (41)$$

where c_b and c_s represent PD material parameters related with bending and transverse shear deformations, respectively, which are defined as

$$c_b = \frac{3Eh}{4\pi \delta^3} \quad (42)$$

and

$$c_s = \frac{9\kappa_s^2 E}{4\pi\delta^3 h} \quad (43)$$

4 Boundary Conditions

As shown in Fig. 3, boundary conditions are imposed by creating a fictitious domain, R_c , which is located outside of the real domain, R . The thickness of this fictitious layer can be specified as twice of the horizon size if $\nu \neq 1/3$, or the size of the horizon if $\nu = 1/3$.

4.1 Clamped Boundary Condition

The clamped boundary condition can be imposed by specifying zero transverse displacement and zero rotation for the material points at the clamped boundary as

$$w = 0 \quad (44)$$

$$\theta_l = 0 \quad (45)$$

In this study, these conditions can be obtained via mirror image of the transverse displacements of the material points adjacent to the clamped end and anti-symmetric image of rotation fields as shown in Fig. 4 as

$$w_{(n)(i)} = w_{(n)(i^*)} \quad (46)$$

$$w_{(n)(1)} = w_{(n)(1^*)} = 0 \quad (47)$$

$$\theta_L^{(n)(i)} = -\theta_L^{(n)(i^*)} \quad \text{for} \begin{cases} i = 1, 2, \dots, 6 & \text{if } \nu \neq 1/3 \\ i = 1, 2, 3 & \text{if } \nu = 1/3 \end{cases} \quad (48)$$

Fig. 3 Real domain, R and fictitious region, R_c

4.2 Simply Supported Boundary Condition

The simply supported boundary condition can be obtained by imposing zero transverse displacement and curvature for the material point adjacent to the constrained boundary. The constraint condition related with transverse displacement can be achieved by imposing anti-symmetrical transverse displacement fields in the fictitious region as shown in Fig. 5 which can be expressed as

$$w_{(n)(i)} = -w_{(n)(i^*)} \quad \text{for} \begin{cases} i = 1, 2, \dots, 6 & \text{if } \nu \neq 1/3 \\ i = 1, 2, 3 & \text{if } \nu = 1/3 \end{cases} \quad (49)$$

The curvature condition can be defined as

$$\theta_I^{(n)(i)} = \theta_I^{(n)(i^*)} \quad \text{and} \quad \theta_J^{(n)(i)} = -\theta_J^{(n)(i^*)} (I \neq J) \quad \text{for} \begin{cases} i = 1, 2, \dots, 6 & \text{if } \nu \neq 1/3 \\ i = 1, 2, 3 & \text{if } \nu = 1/3 \end{cases} \quad (50)$$

5 Numerical Results

5.1 Mindlin Plate Subjected to Simply Supported Boundary Conditions

As the first numerical example, a simply supported Mindlin plate is considered as shown in Fig. 6. The plate has a length and width of $L = W = 1$ m. The thickness of the plate is specified as $h = 0.1$ m. The Young's modulus is $E = 200$ GPa and Poisson's ratio is $\nu = 0.3$. The shear coefficient is used as $\kappa_s^2 = \frac{\pi^2}{12}$. For the discretisation of the model, 101 points are used both in directions. The discretisation size is $\Delta x = \frac{1}{101}$ m. The horizon size is chosen as $\delta = 3.606\Delta x$. A distributed transverse load of $p = 100$ N/m is applied along a row of material points through the central line as a body load of $b_z = \frac{pW}{101\Delta W} = 1.01 \times 10^5$ N/m³ as shown in Fig. 7.

The FE model is generated by using SHELL181 element of ANSYS with 50×50 elements. As depicted in Figs. 8 and 9, the PD solution of the transverse displacement, w , and rotations, θ_L , along the central x - and y - axes are compared with results from the FE

Fig. 4 Application of clamped boundary condition

Fig. 5 Application of simply supported boundary condition

method. According to this comparison, it can be concluded that the PD and the FE method results agree well with each other.

5.2 Mindlin Plate Subjected to Clamped Boundary Conditions

In the second case, the same problem is considered as in the previous example case except the boundary conditions being clamped instead of simply supported as shown in Fig. 10. As depicted in Figs. 11 and 12, the PD solution of the transverse displacement, w , and rotations, θ_L , along the central x - and y - axes are compared with the FE method results, and there is a very good agreement between the PD and the FE method results.

5.3 Mindlin Plate Subjected to Mixed (Clamped-Simply Supported) Boundary Conditions

In this final numerical case, as opposed to first and second numerical cases, Mindlin plate is subjected to mixed (clamped-simply supported) boundary conditions as shown in Fig. 13, and the PD solution of the transverse displacement w , and rotations, θ_L , along the central x - and y - axes is compared with the FE method results. As depicted in Figs. 14 and 15, the PD and the FE method results agree very well with each other.

Finally, the capability of the current formulation was demonstrated for a different Poisson's ratio of 0.2. As demonstrated in Figs. 16 and 17, the transverse displacement w , and rotations, θ_L , along the central x - and y - axes results obtained from current formulation compare very well with results obtained from FE analysis. This shows that the current formulation does not have any limitation on material constants.

Fig. 6 Simply supported Mindlin plate and loading condition

Fig. 7 Discretisation and application of the body load

(a)

(b)

Fig. 8 Variation of transverse displacements along the central **a** x - axis, **b** y - axis

(a)

(b)

Fig. 9 Variation of rotations along the central **a** x - axis, **b** y - axis

Fig. 10 Clamped Mindlin plate and loading condition

Fig. 11 Variation of transverse displacements along the central **a** x - axis, **b** y - axis

Fig. 12 Variation of rotations along the central **a** x - axis, **b**

Fig. 13 Mindlin plate subjected to mixed (clamped – simply supported) boundary conditions

Fig. 14 Variation of transverse displacements along the central **a** x - axis, **b** y - axis

6 Conclusions

In this study, a new Mindlin plate formulation was developed. In the current formulation, there is no limitation on material constants as in bond-based peridynamics. Euler-Lagrange equation was utilised in conjunction with Taylor's expansion to determine the equations of motion. Three different numerical examples were considered for a Mindlin plate subjected to different boundary conditions. Peridynamic results were compared against results obtained from finite element analysis, and a good agreement was obtained between the two approaches verifying the newly developed formulation.

Appendix

As explained above, the strain energy density function of Mindlin plate can be expressed as:

Fig. 15 Variation of rotations along the central **a** x - axis, **b** y - axis

Fig. 16 Variation of transverse displacements along the central **a** x - axis, **b** y - axis

Fig. 17 Variation of rotations along the central **a** x - axis, **b** y - axis

Fig. 18 Peridynamic interaction between two material points inside the horizon

$$W = \frac{E}{1-\nu^2} \frac{h^2}{12} \left[\frac{1-\nu}{4} \left(\frac{\partial \theta_I}{\partial x_J} \frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J} \frac{\partial \theta_J}{\partial x_I} + \frac{\partial \theta_I}{\partial x_I} \frac{\partial \theta_J}{\partial x_J} \right) + \frac{3\nu-1}{4} \frac{\partial \theta_I}{\partial x_I} \frac{\partial \theta_J}{\partial x_J} \right] + \kappa_s^2 \frac{G}{2} \left(\theta_I + \frac{\partial w}{\partial x_I} \right) \left(\theta_I + \frac{\partial w}{\partial x_I} \right) \quad (51)$$

In order to obtain the strain energy density function in PD form, it is necessary to transform each differential terms into their equivalent nonlocal form. This can be achieved by utilizing Taylor expansion as:

As shown in Fig. 18, function of rotations, θ , can be Taylor expanded up to 1st-order terms about point $\mathbf{x}$ as:

$$\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x}) = \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \xi n_J \quad (52)$$

$$\theta_K(\mathbf{x} + \xi) - \theta_K(\mathbf{x}) = \frac{\partial \theta_K(\mathbf{x})}{\partial x_L} \xi n_L \quad (53)$$

where $\xi = |\xi|$, and unit direction vector $\mathbf{n}$ is defined as

$$\mathbf{n} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} \quad (54)$$

with φ being the orientation of peridynamic interaction (bond).

Multiplying Eq. (52) with (53) gives

$$\frac{[\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})][\theta_K(\mathbf{x} + \xi) - \theta_K(\mathbf{x})]}{\xi} = \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_K(\mathbf{x})}{\partial x_L} \xi n_J n_L \quad (55)$$

Multiplying both sides of Eq. (55) twice by directional vector yields

$$\frac{[\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})][\theta_K(\mathbf{x} + \xi) - \theta_K(\mathbf{x})]}{\xi} n_R n_S = \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_K(\mathbf{x})}{\partial x_L} \xi n_J n_L n_R n_S \quad (56)$$

Considering $\mathbf{x}$ as a fixed point, integrating both sides of Eq. (56) over a circular domain with centre of $\mathbf{x}$ and radius of δ gives:

$$\begin{aligned} & \int_0^{2\pi} \int_0^\delta \frac{[\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})][\theta_K(\mathbf{x} + \xi) - \theta_K(\mathbf{x})]}{\xi} n_R n_S \xi d\xi d\varphi \\ &= \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_K(\mathbf{x})}{\partial x_L} \int_0^{2\pi} \int_0^\delta \xi n_J n_L n_R n_S \xi d\xi d\varphi \\ &= \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_K(\mathbf{x})}{\partial x_L} (\delta_{JL} \delta_{RS} + \delta_{JR} \delta_{LS} + \delta_{JS} \delta_{LR}) \\ &= \left(\frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_K(\mathbf{x})}{\partial x_J} \delta_{RS} + \frac{\partial \theta_I(\mathbf{x})}{\partial x_R} \frac{\partial \theta_K(\mathbf{x})}{\partial x_S} + \frac{\partial \theta_I(\mathbf{x})}{\partial x_S} \frac{\partial \theta_K(\mathbf{x})}{\partial x_R} \right) \end{aligned} \quad (57)$$

Multiplying both sides of Eq. (57) by $\delta_{RI} \delta_{SK}$ results in:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^\delta \frac{[\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})][\theta_K(\mathbf{x} + \xi) - \theta_K(\mathbf{x})]}{\xi} n_R n_S \delta_{RI} \delta_{SK} \xi d\xi d\varphi \\
&= \frac{\pi \delta^3}{12} \left(\frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_K(\mathbf{x})}{\partial x_J} \delta_{RS} + \frac{\partial \theta_I(\mathbf{x})}{\partial x_R} \frac{\partial \theta_K(\mathbf{x})}{\partial x_S} + \frac{\partial \theta_I(\mathbf{x})}{\partial x_S} \frac{\partial \theta_K(\mathbf{x})}{\partial x_R} \right) \delta_{RI} \delta_{SK}
\end{aligned} \quad (58)$$

Rearranging the dummy indices gives:

$$\begin{aligned}
& \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} + \frac{\partial \theta_I(\mathbf{x})}{\partial x_I} \frac{\partial \theta_J(\mathbf{x})}{\partial x_J} + \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\partial \theta_J(\mathbf{x})}{\partial x_I} \\
&= \frac{12}{\pi \delta^3} \int_0^{2\pi} \int_0^\delta \frac{[\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})][\theta_J(\mathbf{x} + \xi) - \theta_J(\mathbf{x})]}{\xi} n_I n_J \xi d\xi d\varphi
\end{aligned} \quad (59)$$

which can be written in the discretised form as

$$\frac{\partial \theta_I^{(k)}}{\partial x_J} \frac{\partial \theta_I^{(k)}}{\partial x_J} + \frac{\partial \theta_I^{(k)}}{\partial x_I} \frac{\partial \theta_J^{(k)}}{\partial x_J} + \frac{\partial \theta_I^{(k)}}{\partial x_J} \frac{\partial \theta_J^{(k)}}{\partial x_I} = \frac{12}{\pi \delta^3 h} \sum_i \frac{[\theta_I^{(\hat{k})} - \theta_I^{(k)}][\theta_J^{(\hat{k})} - \theta_J^{(k)}]}{\xi_{(\hat{k})}^{(k)}} n_I^{(\hat{k})} n_J^{(\hat{k})} V_{(\hat{k})} \quad (60)$$

where h represents the thickness of the beam in this study.

Recalling Eq. (52):

$$\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x}) = \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \xi n_J \quad (61)$$

and multiplying Eq. (61) by a directional vector gives:

$$\frac{\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})}{\xi} n_K = \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} n_J n_K \quad (62)$$

Considering $\mathbf{x}$ as a fixed point, integrating both sides of Eq. (62) over a circular domain with centre of $\mathbf{x}$ and radius of δ yields:

$$\int_0^{2\pi} \int_0^\delta \frac{\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})}{\xi} n_K \xi d\xi d\varphi = \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \int_0^{2\pi} \int_0^\delta n_J n_K \xi d\xi d\varphi = \frac{\partial \theta_I(\mathbf{x})}{\partial x_J} \frac{\pi \delta^2}{2} \delta_{JK} = \frac{\partial \theta_I(\mathbf{x})}{\partial x_K} \frac{\pi \xi^2}{2} \quad (63)$$

which can be rewritten as

$$\frac{\partial \theta_I(\mathbf{x})}{\partial x_K} = \frac{2}{\pi \delta^2} \int_0^{2\pi} \int_0^\delta \frac{\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})}{\xi} n_K \xi d\xi d\varphi \quad (64)$$

Multiplying both sides of Eq. (64) by δ_{IK} gives

$$\frac{\partial \theta_I(\mathbf{x})}{\partial x_I} = \frac{2}{\pi \delta^2} \int_0^{2\pi} \int_0^\delta \frac{\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})}{\xi} n_I \xi d\xi d\varphi \quad (65)$$

Rewriting Eq. (65) with a different index gives:

$$\frac{\partial \theta_J(\mathbf{x})}{\partial x_J} = \frac{2}{\pi \delta^2} \int_0^{2\pi} \int_0^\delta \frac{\theta_J(\mathbf{x} + \xi) - \theta_J(\mathbf{x})}{\xi} n_J \xi d\xi d\varphi \quad (66)$$

Multiplying Eq. (65) with (66) yields:

$$\frac{\partial \theta_I(\mathbf{x})}{\partial x_I} \frac{\partial \theta_J(\mathbf{x})}{\partial x_J} = \left(\frac{2}{\pi \delta^2} \right)^2 \int_0^{2\pi} \int_0^\delta \frac{\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})}{\xi} n_I \xi d\xi d\varphi \int_0^{2\pi} \int_0^\delta \frac{\theta_J(\mathbf{x} + \xi) - \theta_J(\mathbf{x})}{\xi} n_J \xi d\xi d\varphi \quad (67)$$

which can be written in discretised form as

$$\frac{\partial \theta_i^{(k)}}{\partial x_I} \frac{\partial \theta_j^{(k)}}{\partial x_J} = \left(\frac{2}{\pi \delta^2} \right)^2 \sum_i \frac{\theta_I^{(i^k)} - \theta_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{\theta_J^{(i^k)} - \theta_J^{(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} \quad (68)$$

Similar to Eq. (52), the following relationship can be established as:

$$w(\mathbf{x} + \xi) - w(\mathbf{x}) = \frac{\partial w(\mathbf{x})}{\partial x_J} \xi n_J \quad (69)$$

The rotations of material point $\mathbf{x}$ can be estimated as taking the average rotation of point $\mathbf{x}$ and its family member point $\mathbf{x} + \xi$ as:

$$\frac{\theta_I(\mathbf{x} + \xi) + \theta_I(\mathbf{x})}{2} = \theta_I(\mathbf{x}) \quad (70)$$

If Eq. (70) is multiplied by ξn_I :

$$\frac{\theta_I(\mathbf{x} + \xi) - \theta_I(\mathbf{x})}{2} \xi n_I = \theta_I(\mathbf{x}) \xi n_I \quad (71)$$

and added with (69) yields:

$$w(\mathbf{x} + \xi) - w(\mathbf{x}) + \frac{\theta_I(\mathbf{x} + \xi) + \theta_I(\mathbf{x})}{2} \xi n_I = \frac{\partial w(\mathbf{x})}{\partial x_J} \xi n_J + \theta_I(\mathbf{x}) \xi n_I \quad (72)$$

Rewriting Eq. (72) with a different index results in:

$$w(\mathbf{x} + \xi) - w(\mathbf{x}) + \frac{\theta_J(\mathbf{x} + \xi) + \theta_J(\mathbf{x})}{2} \xi n_J = \frac{\partial w(\mathbf{x})}{\partial x_J} \xi n_J + \theta_J(\mathbf{x}) \xi n_J \quad (73)$$

Multiplying Eq. (72) with (73) and then dividing each term by ξ yields:

$$\frac{\left[w(\mathbf{x} + \xi) - w(\mathbf{x}) + \frac{\theta_I(\mathbf{x} + \xi) + \theta_I(\mathbf{x})}{2} \xi n_I \right] \left[w(\mathbf{x} + \xi) - w(\mathbf{x}) + \frac{\theta_J(\mathbf{x} + \xi) + \theta_J(\mathbf{x})}{2} \xi n_J \right]}{\xi} = \left(\frac{\partial w(\mathbf{x})}{\partial x_I} + \theta_I(\mathbf{x}) \right) \left(\frac{\partial w(\mathbf{x})}{\partial x_J} + \theta_J(\mathbf{x}) \right) n_I n_J \quad (74)$$

Considering $\mathbf{x}$ as a fixed point, integrating both sides of Eq. (74) over a circular domain with centre of $\mathbf{x}$ and radius of δ results in

$$\begin{aligned}
& \int_0^{2\pi} \int_0^\delta \frac{\left[w(\mathbf{x} + \boldsymbol{\xi}) - w(\mathbf{x}) + \frac{\theta_I(\mathbf{x} + \boldsymbol{\xi}) + \theta_J(\mathbf{x})}{2} \xi n_J \right] \left[w(\mathbf{x} + \boldsymbol{\xi}) - w(\mathbf{x}) + \frac{\theta_I(\mathbf{x} + \boldsymbol{\xi}) + \theta_I(\mathbf{x})}{2} \xi n_I \right]}{\xi} d\xi d\varphi \\
&= \left(\frac{\partial w(\mathbf{x})}{\partial x_I} + \theta_I(\mathbf{x}) \right) \left(\frac{\partial w(\mathbf{x})}{\partial x_J} + \theta_J(\mathbf{x}) \right) \int_0^{2\pi} \int_0^\delta n_I n_J \xi d\xi d\varphi \\
&= \left(\frac{\partial w(\mathbf{x})}{\partial x_I} + \theta_I(\mathbf{x}) \right) \left(\frac{\partial w(\mathbf{x})}{\partial x_J} + \theta_J(\mathbf{x}) \right) \frac{\pi \delta^3}{3} \delta_{IJ} = \frac{\pi \delta^2}{3} \left(\frac{\partial w(\mathbf{x})}{\partial x_I} + \theta_I(\mathbf{x}) \right)^2
\end{aligned} \tag{75}$$

which gives

$$\begin{aligned}
& \left(\frac{\partial w(\mathbf{x})}{\partial x_I} + \theta_I(\mathbf{x}) \right)^2 \\
&= \frac{3}{\pi \delta^2} \int_0^{2\pi} \int_0^\delta \frac{\left[w(\mathbf{x} + \boldsymbol{\xi}) - w(\mathbf{x}) + \frac{\theta_I(\mathbf{x} + \boldsymbol{\xi}) + \theta_J(\mathbf{x})}{2} \xi n_J \right] \left[w(\mathbf{x} + \boldsymbol{\xi}) - w(\mathbf{x}) + \frac{\theta_I(\mathbf{x} + \boldsymbol{\xi}) + \theta_I(\mathbf{x})}{2} \xi n_I \right]}{\xi} d\xi d\varphi
\end{aligned} \tag{76}$$

Equation (76) can be written in discretised form as

$$\begin{aligned}
w(\mathbf{x} + \boldsymbol{\xi}) - w(\mathbf{x}) &= \frac{\partial w(\mathbf{x})}{\partial x_J} \xi n_J \left(\frac{\partial w_{(k)}}{\partial x_I} + \theta_I^{(k)} \right)^2 \\
&= \frac{3}{\pi \delta^3 h} \sum_i \frac{\left[w_{(i^k)} - w_{(k)} + \frac{\theta_I^{(k)} + \theta_I^{(i^k)}}{2} \xi_{(i^k)(k)} n_I^{(i^k)(k)} \right] \left[w_{(i^k)} - w_{(k)} + \frac{\theta_J^{(k)} + \theta_J^{(i^k)}}{2} \xi_{(i^k)(k)} n_J^{(i^k)(k)} \right]}{\xi_{(i^k)(k)}}
\end{aligned} \tag{77}$$

Finally, combining Eqs. (60), (68), (77) and (51) gives the strain energy density of the material point k in PD form as

$$\begin{aligned}
W_{(k)} &= \frac{E}{1-\nu^2} \frac{h^2}{12} \left[\frac{1-\nu}{4} \left(\frac{12}{\pi \delta^3 h} \sum_i \frac{\left(\theta_I^{(i^k)} - \theta_I^{(k)} \right) \left(\theta_J^{(i^k)} - \theta_J^{(k)} \right)}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} n_J^{(i^k)(k)} V_{(i^k)} \right) + \right. \\
&\quad \left. \frac{3\nu-1}{4} \left(\frac{2}{\pi \delta^2 h} \right)^2 \sum_i \frac{\left(\theta_I^{(i^k)} - \theta_I^{(k)} \right)}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{\left(\theta_J^{(i^k)} - \theta_J^{(k)} \right)}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} \right] \\
&\quad + \kappa_s^2 \frac{G}{2} \frac{3}{\pi \delta^3 h} \times \sum_i \frac{\left(w_{(i^k)} - w_{(k)} + \frac{\theta_I^{(i^k)} + \theta_I^{(k)}}{2} \xi_{(i^k)(k)} n_I^{(i^k)(k)} \right) \left(w_{(i^k)} - w_{(k)} + \frac{\theta_J^{(i^k)} + \theta_J^{(k)}}{2} \xi_{(i^k)(k)} n_J^{(i^k)(k)} \right)}{\xi_{(i^k)(k)}} V_{(i^k)}
\end{aligned} \tag{78}$$

Regarding the strain energy density for the material point j , a similar form will hold if we replace the index k with j as:

$$\begin{aligned}
W_{(j)} = & \frac{E}{1-\nu^2} \frac{h^2}{12} \left[\frac{1-\nu}{4} \left(\frac{12}{\pi \delta^3 h} \sum_i \frac{(\theta_I^{(j)} - \theta_I^{(i)}) (\theta_J^{(j)} - \theta_J^{(i)})}{\xi_{(j)(i)}} n_I^{(i)(j)} n_J^{(i)(j)} V_{(i)} \right) + \right. \\
& \left. \frac{3\nu-1}{4} \left(\frac{2}{\pi \delta^3 h} \right)^2 \sum_i \frac{(\theta_I^{(j)} - \theta_I^{(i)})}{\xi_{(j)(i)}} n_I^{(i)(j)} V_{(i)} \sum_i \frac{(\theta_J^{(j)} - \theta_J^{(i)})}{\xi_{(j)(i)}} n_J^{(i)(j)} V_{(i)} \right] \\
& + \kappa_s^2 \frac{G}{2} \frac{3}{\pi \delta^3 h} \times \sum_i \frac{\left(w_{(i)} - w_{(j)} + \frac{\theta_I^{(j)} + \theta_I^{(i)}}{2} \xi_{(j)(i)} n_I^{(i)(j)} \right) \left(w_{(i)} - w_{(j)} + \frac{\theta_J^{(j)} + \theta_J^{(i)}}{2} \xi_{(j)(i)} n_J^{(i)(j)} \right)}{\xi_{(j)(i)}} V_{(i)} \quad (79)
\end{aligned}$$

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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