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ORIGINAL PAPER

Strong Edge Geodetic Problem on Complete Multipartite Graphs and Some Extremal Graphs for the Problem

Sandi Klavžar[1,2,3] · **Eva Zmazek**[1]

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Abstract

A set of vertices X of a graph G is a strong edge geodetic set if, to any pair of vertices from X, we can assign one (or zero) shortest path between them, such that every edge of G is contained in at least one on these paths. The cardinality of a smallest strong edge geodetic set of G is the strong edge geodetic number $\mathrm{sg_e}(G)$ of G. In this paper, the strong edge geodetic number of complete multipartite graphs is determined. Graphs G with $\mathrm{sg_e}(G) = n(G)$ are characterized and $\mathrm{sg_e}$ is determined for Cartesian products $P_n \,\square\, K_m$. The latter result in particular corrects an error from the literature.

Keywords Strong edge geodetic problem · Complete multipartite graph · Edge-coloring · Cartesian product of graphs

Mathematics Subject Classification 05C12 · 05C70

1 Introduction

Covering vertices or edges of a graph by the smallest number of paths is a fundamental optimization problem and appears in the literature in several variations depending upon the properties one requires from the paths. In the isometric path cover problem (alias geodetic cover problem), the aim is to cover all the vertices by a minimum number of

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✉ Sandi Klavžar
sandi.klavzar@fmf.uni-lj.si

Eva Zmazek
eva.zmazek@fmf.uni-lj.si

1 Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia

2 Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

3 Faculty of Natural Sciences and Mathematics, University of Maribor, Maribor, Slovenia

shortest paths [4, 5, 10, 11, 15, 21]. In the path cover problem, we want to cover all the vertices by a minimum number of vertex disjoint paths [6, 7, 20]. Dual concepts have also been studied as, for instance, the k-path covers which are sets S of vertices of a graph G, such that every path of order k in G contains at least one vertex from S; see [2, 3, 9]. In the edge version of the isometric path cover problem, we want to cover all the edges by a minimum number of shortest paths [1, 22, 23]. In this paper, we are interested in the strong edge geodetic problem introduced in [17] as follows.

Let $G = (V(G), E(G))$ be a graph. A set of vertices $X \subseteq V(G)$ is a *strong edge geodetic set* if, to any pair of vertices u and v from X, we can assign a shortest u, v-path P_{uv}, such that every edge $xy \in E(G)$ is contained in at least one on the paths P_{uv}. The cardinality of a smallest strong edge geodetic set of G is the *strong edge geodetic number* $\mathrm{sg_e}(G)$ of G. Such a set is briefly called a $\mathrm{sg_e}(G)$-*set*.

In the seminal paper [17], it was proved, among other results, that the strong edge geodetic problem is $\mathcal{NP}$-complete. In [8], it was further proved that there is no approximation of the strong edge geodetic number with an approximation factor better that $781/780$. Several additional results on the strong edge geodetic number were reported in [25, 26]. In the latter paper, the strong edge geodetic number was determined for Cartesian products $P_n \,\square\, P_k$, where $k \in \{2, 3, 4\}$.

The vertex version of the strong edge geodetic problem is known as the *strong geodetic problem* and was studied for the first time in [16, 18]. The strong geodetic problem is also $\mathcal{NP}$-complete and remains such even when restricted to bipartite graphs and multipartite graphs [13]. Moreover, determining whether a given set X is a strong geodetic set is $\mathcal{NP}$-hard [8] as well.

The strong geodetic number of complete bipartite (resp. mutipartite) graphs received a lot of attention. First, in [13], the problem was solved for balanced complete bipartite graphs $K_{n,n}$. Subsequently, using different approaches, a formula for the strong geodetic number of arbitrary complete bipartite graphs was derived in [14] and in [12]. In [14], a lower bound for the strong geodetic number of a complete multipartite graph was given and it was conjectured that the strong geodetic number remains $\mathcal{NP}$-complete on complete multipartite graphs. In [8], this conjecture was disproved by developing a polynomial algorithm for the strong geodetic number of complete mutipartite graphs. In this direction, we emphasize that in [19], an $O(|E(G)| \cdot |V(G)|^2)$ algorithm for computing the strong geodetic number of an outerplanar graph G was developed. Several additional interesting results on the strong geodetic problem were presented in [24]. Among other results, relations between the strong geodetic number and the connectivity and the diameter were established, and graphs with the strong geodetic number equal to 2, $|V(G)| - 1$, and $|V(G)|$ were characterized.

Motivated by the efforts to determine the strong geodetic number of complete bipartite graphs, we determine in Sect. 2 the strong edge geodetic number of complete bipartite graphs, and using this result, we then determine the strong edge geodetic number of arbitrary complete multipartite graphs. In Sect. 3, we characterize graphs G with $\mathrm{sg_e}(G) = n(G)$ and discuss the graphs with $\mathrm{sg_e}(G) = n(G) - 1$. In particular, we observe that Cartesian products $P_2 \,\square\, K_n$ belong to this family of graphs. This corrects [25, Theorem 13] where it is wrongly stated that $\mathrm{sg_e}(P_2 \,\square\, K_n) = 2n - 2$. We then proceed by determining $\mathrm{sg_e}(P_m \,\square\, K_n)$ for all $m, n \geq 2$.

We conclude the introduction by giving some definitions needed. The order of a graph G is denoted by $n(G)$. A vertex u of a graph G is *universal* if $\deg_G(u) = n(G) - 1$. The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. As usual, $\chi'(G)$ is the chromatic index of G. For a positive integer n, the set $\{1, \dots, n\}$ will be dented by $[n]$.

If U is a strong edge geodetic set, then we will denote by $\widehat{U}$ the set of associated paths with endpoints from U which cover all the edges of G. Clearly, $\widehat{U}$ is not unique, but unless stated otherwise, we will assume that $\widehat{U}$ has been selected and is fixed.

2 Complete Multipartite Graphs

In this section, we determine the strong edge geodetic number of complete multipartite graphs. To do so, we first prove the corresponding result for complete bipartite graphs which reads as follows.

Theorem 2.1 *If $n \geq m \geq 2$, then the following holds.*

(i) *If n is even, then*

$$\mathrm{sg_e}(K_{n,m}) = \begin{cases} n+1; & n = m, \\ n; & n \geq m+1. \end{cases}$$

(ii) *If n is odd, then*

$$\mathrm{sg_e}(K_{n,m}) = \begin{cases} n+2; & n = m, \\ n+1; & n = m+1, \\ n; & n \geq m+2. \end{cases}$$

In the rest of the section, we assume that $n \geq m \geq 2$ and that the bipartition of $K_{n,m}$ is (X, Y), where $X = \{x_0, \dots, x_{n-1}\}$ and $Y = \{y_0, \dots, y_{m-1}\}$.

Lemma 2.2 *If U is a strong edge geodetic set of $K_{n,m}$, then $X \subseteq U$ or $Y \subseteq U$.*

Proof Let U be a strong edge geodetic set of the graph $K_{n,m}$. Suppose on the contrary that there exist vertices $x_i \in X \setminus U$ and $y_j \in Y \setminus U$. Because $\mathrm{diam}(K_{n,m}) = 2$ and $x_i y_j$ is an edge of $K_{n,m}$, none of the shortest paths with endpoints from U can cover the edge $x_i y_j$, that is, U cannot be a strong edge geodetic set. □

Lemma 2.3 *If U is a strong edge geodetic set of $K_{n,m}$ and $Y \subseteq U$, then $|U| \geq n + 1$.*

Proof Suppose U is a strong edge geodetic set of $K_{n,m}$, where $U = Y \cup X'$ with $X' \subseteq X$ and $|X'| = k$, $0 \leq k \leq n$. Consider an arbitrary vertex $y_j \in Y$. There are exactly $n - k$ edges between y_j and $X \setminus X'$. Because the shortest paths that cover these edges have both of their endpoints in Y, it has to hold $m - 1 \geq n - k$. This in turn implies that $|U| = |Y| + |X'| = m + k \geq n + 1$. □

Corollary 2.4 *If $n \geq m \geq 2$, then $\mathrm{sg_e}(K_{n,m}) \geq n$. Moreover, if $m = n$, then $\mathrm{sg_e}(K_{n,n}) \geq n+1$.*

Proof If $m = n$, then the second assertion of the corollary follows immediately from Lemmas 2.2 and 2.3. Suppose now that $n > m$ and let U be a smallest strong edge geodetic set of $K_{n,m}$, so that $|U| = \mathrm{sg_e}(K_{n,m})$. Then, $X \subseteq U$ or $Y \subseteq U$ by Lemma 2.2. If $X \subseteq U$, then $\mathrm{sg_e}(K_{n,m}) = |U| \geq |X| = n$. And, if $Y \subseteq U$, then $\mathrm{sg_e}(K_{n,m}) \geq n+1$ follows by Lemma 2.3. □

We have thus established the lower bound for the case when n is even. For n odd, we proceed as follows.

Lemma 2.5 *Let U be a strong edge geodetic set of $K_{n,m}$. If n is odd and $X \subseteq U$, then $|U| \geq \frac{2n}{n+1} + m$.*

Proof Let U be a strong edge geodetic set of $K_{n,m}$, where $U = X \cup Y'$ with $Y' \subseteq Y$ and $|Y'| = k$. For each edge xy, where $x \in X$ and $y \in Y'$, we put the shortest path xy to $\widehat{U}$. The edges between vertices from X and $Y \backslash Y'$ must be covered by the shortest paths of length 2 with both of their endpoints in X. For each pair of vertices from X, we can put only one shortest path to $\widehat{U}$, so we can only put $\binom{n}{2}$ shortest paths to $\widehat{U}$ to cover the $n \cdot (m-k)$ edges between the vertices from X and the vertices from $Y \backslash Y'$. Moreover, because the degree of every vertex from $Y \backslash Y'$ is n, which we have assumed to be odd, each vertex from $Y \backslash Y'$ must be the central vertex of at least $(n+1)/2$ shortest paths from $\widehat{U}$. Since U is a strong edge geodetic set this implies that $\binom{n}{2} \geq (m-k) \cdot \frac{n+1}{2}$. This inequality rewrites to $k \geq m - n(n-1)/(n+1)$ which in turn implies that $|U| = n + k \geq n + m - n(n-1)/(n+1) = \frac{2n}{n+1} + m$. □

Corollary 2.6 *If $n \geq 3$ is odd, then $\mathrm{sg_e}(K_{n,n}) \geq n+2$ and $\mathrm{sg_e}(K_{n,n-1}) \geq n+1$.*

Proof Let U be a smallest strong edge geodetic set of $K_{n,n}$. By Lemma 2.5

$$|U| \geq n + \frac{2n}{n+1}.$$

As $|U|$ is an integer and $2n/(n+1) > 1$ for $n \geq 2$, we get $|U| = \mathrm{sg_e}(K_{n,n}) \geq n+2$.

Let now U be a smallest strong edge geodetic set of $K_{n,n-1}$. By Lemma 2.2, we have $X \subseteq U$ or $Y \subseteq U$. In the latter case, Lemma 2.3 gives $\mathrm{sg_e}(K_{n,n-1}) \geq n+1$. Assume second that $X \subseteq U$. Then, Lemma 2.5 gives

$$|U| \geq \frac{2n}{n+1} + (n-1) = n + \frac{n-1}{n+1}.$$

Since $\frac{n-1}{n+1} > 0$ for $n \geq 2$ and since $|U|$ is an integer, also in this case, we get $\mathrm{sg_e}(K_{n,n-1}) \geq n+1$. □

So far, we have proved the lower bound for all the cases of Theorem 2.1. In the following, we will construct in each case a strong edge geodetic set of the required cardinality.

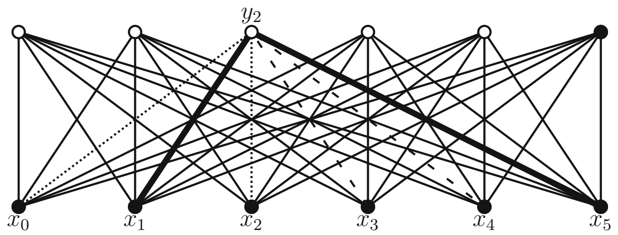

Fig. 1 Shortest paths from $\widehat{U}$ that cover edges incident to y_2

Case 1: n is even.
We first consider $K_{n,n}$ and prove that

$$\mathrm{sg_e}(K_{n,n}) \leq n+1\,. \tag{1}$$

We claim that $U = X \cup \{y_{n-1}\}$ is a strong edge geodetic set of $K_{n,n}$. For every $0 \leq i \leq n-1$, add the shortest path $x_i y_{n-1}$ to $\widehat{U}$ to cover the edge $x_i y_{n-1}$. Then, all the other edges must be covered by shortest paths of the form $x_i y_j x_k$, where $i \neq k$. To do so, we use edge-colorings of K_n. It is well known that $\chi'(K_n) = n-1$ for even n. Let $V(K_n) = \{0, 1, \ldots, n-1\}$. Then, an edge-coloring c of K_n using $n-1$ colors can be defined as follows: if $i, j \in \{0, 1, \ldots, n-2\}$, $i \neq j$, then let $c(ij) = (i+j) \mod (n-1)$, and for for $i \in \{0, 1, \ldots, n-2\}$ let $c(i(n-1)) = 2i \mod (n-1)$.

In the covering of $K_{n,n}$ that we are constructing, we put the shortest path $x_i y_j x_k$ to $\widehat{U}$ if and only if $c(ik) = j$. See Fig. 1, where this construction is illustrated for the case $n = 6$ and the edges incident to y_2. In K_6, we have $c(02) = c(15) = c(34) = 2$; hence, the paths $x_0 y_2 x_2$, $x_1 y_2 x_5$, and $x_3 y_2 x_4$ belong to $\widehat{U}$.

Using this construction, a pair of vertices x_i and x_k is never used twice, and for each vertex $y \in Y \backslash \{y_{n-1}\}$, the shortest paths in $\widehat{U}$ have pairwise different endpoints. Since in c every color is used exactly $n/2$ times, the shortest paths from $\widehat{U}$ passing through y_j cover all the edges incident with y_j. This proves (1).

Consider now $K_{n,m}$, where $m \leq n-1$ (and n is even). We need to show that $\mathrm{sg_e}(K_{n,m}) \leq n$. For this sake, we claim that X is a strong edge geodetic set. Indeed, use the above edge-coloring c of K_n and for each $y_i \in Y$, $i \in \{0, \ldots, m-1\}$, put all the shortest paths $x_j y_i x_k$ to $\widehat{U}$ for which $c(jk) = i$. By the above argument, X is indeed a strong edge geodetic set, and hence, $\mathrm{sg_e}(K_{n,m}) \leq n$ in this subcase.

Case 2: n is odd.
We first consider $K_{n,n}$ and prove that $\mathrm{sg_e}(K_{n,n}) \leq n+2$. For this purpose, consider the set $U = X \cup \{y_{n-2}, y_{n-1}\}$. The subgraph of $K_{n,n}$ induced by the set of vertices $V(K_{n,n}) \backslash \{x_{n-1}, y_{n-1}\}$ is isomorphic to $K_{n-1,n-1}$. As $n-1$ is even, we can cover its edges by the paths as described in Case 1 to derive (1). Recall that for this covering, the vertices $x_0, \ldots, x_{n-1}$ and y_{n-2} are used. To cover the edges $y_{n-1} x_i$, $0 \leq i \leq n-1$, add the shortest paths $y_{n-1} x_i$ to $\widehat{U}$. Finally, to cover the remaining yet uncovered edges, that is, the edges $x_{n-1} y_i$, where $i \in \{0, \ldots, n-2\}$, put the shortest paths $x_{n-1} y_i x_i$ to $\widehat{U}$.

We next show that $\mathrm{sg_e}(K_{n,n-1}) \leq n+1$. In this subcase, set $U = X \cup \{y_{n-2}\}$. Then, as in the above subcase, cover the edges of the subgraph of $K_{n,n-1}$ induced by the set of vertices $V(K_{n,n-1})\backslash\{x_{n-1}\}$ as described in Case 1 to derive (1). After that, to cover the edges $x_{n-1}y_i$, where $i \in \{0,\ldots,n-2\}$ we add to $\widehat{U}$ the shortest paths $x_{n-1}y_ix_i$.

Consider finally $K_{n,m}$, where $m \leq n-2$. In this case, X is a strong edge geodetic set. For this sake, note that by the second subcase of Case 1, we know that $\{x_0,\ldots,x_{n-2}\}$ is a strong edge geodetic set of the subgraph of $K_{n,m}$ induced by the set $V(K_{n,m})\backslash\{x_{n-1}\}$. To cover the remaining not yet covered edges $x_{n-1}y_i$, where $i \in \{0,\ldots,m-1\}$ we add to $\widehat{U}$ the shortest paths $x_{n-1}y_ix_i$. From here, it is clear that X is a strong edge geodetic set of $K_{n,m}$ and we conclude that in this subcase $\mathrm{sg_e}(K_{n,m}) \leq n$.

We have thus established all the upper bounds which completes the proof of Theorem 2.1. Using it, we can in turn determine the strong edge geodetic number of complete multipartite graphs as follows.

Theorem 2.7 *If $k \geq 2$ and $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, then the following holds.*

(i) *If n_1 is even, then*

$$\mathrm{sg_e}(K_{n_1,\ldots,n_k}) = \begin{cases} \sum\limits_{j=2}^{k} n_j + 1; & n_2 \in \{n_1, n_1+1\}, \\ \sum\limits_{j=2}^{k} n_j; & \textit{otherwise}; \end{cases}$$

(ii) *If n_1 is odd, then*

$$\mathrm{sg_e}(K_{n_1,\ldots,n_k}) = \begin{cases} \sum\limits_{j=2}^{k} n_j + 2; & n_2 = n_1, \\ \sum\limits_{j=2}^{k} n_j; & \textit{otherwise}. \end{cases}$$

Proof Let $k \geq 2$ and $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, and set $G = K_{n_1,\ldots,n_k}$ for the rest of the proof. Let X_i, $i \in [k]$, be the partition sets of G, where $|X_i| = n_i$. Let U be an arbitrary (smallest) strong edge geodetic set of G. If $i \neq j$, then, by Lemma 2.2, we see that $X_i \subseteq U$ or $X_j \subseteq U$. If follows that U contains $k-1$ of the partite sets.

Let $W = \cup_{i=2}^{k} X_i$. Then, it remains to cover the edges between X_1 and each of the X_i, $i \geq 2$. More precisely, we need to cover the edges in induced subgraphs K_{n_1,n_i}, $i \geq 2$, where the partite sets of cardinality n_i are already included. Clearly, no shortest path of G has length greater than 2 and a shortest path of length 2 has both endpoints in the same set of the partition. Hence, an edge of K_{n_1,n_i} can be covered only by vertices in $X_1 \cup X_i$, for every $2 \leq i \leq k$.

Assume first that $n_2 \geq n_1+2$. Then, by Theorem 2.1 and its proof, we infer that X_2 is a strong edge geodetic set of K_{n_1,n_2} with minimum cardinality. Furthermore, since if $k > 2$, for every $2 < i \leq k$, we have $n_i \geq n_2 \geq n_1+2$, again by Theorem 2.1, we have that X_i is a strong edge geodetic set of K_{n_1,n_i} with minimum cardinality. Therefore, W is a is a strong edge geodetic set of G with minimum cardinality no matter whether

n_1 is even or odd. Moreover, we get the same conclusion if n_1 is odd and $n_2 = n_1 + 1$. Assume next that n_1 is odd and $n_2 = n_1$. Then, Theorem 2.1(ii) implies that the union of X_2 and two vertices of X_1, say u and w is a strong edge geodetic set of K_{n_1,n_2} with minimum cardinality. In this case, we conclude that $W \cup \{u, v\}$ is a strong edge geodetic set of G with minimum cardinality. The cases when n_1 is even and $n_2 \in \{n_1, n_1 + 1\}$ are treated similarly. □

3 Graph with Large Strong Edge Geodetic Sets

In this section, we first characterize graphs G with $\mathrm{sg_e}(G) = n(G)$. After that, we consider graphs G with $\mathrm{sg_e}(G) = n(G) - 1$ and determine $\mathrm{sg_e}(P_n \,\square\, K_m)$. In particular, $\mathrm{sg_e}(P_2 \,\square\, K_m) = 2m - 1$, which corrects a result from [25].

Let G be a graph and $uv \in E(G)$. We say that a vertex v is a *dominant neighbor* of u if $N[u] \subseteq N[v]$, where $N[u] = \{u\} \cup \{x : ux \in E(G)\}$ is the *closed neighborhood* of a vertex u. Vertices u and v of a graph G are *twins* if $N[u] = N[v]$. Note that twins are necessarily adjacent and that if u and v are twins, then u is a dominant neighbor of v and v is a dominant neighbor of u.

The following lemma seems to be of independent interest.

Lemma 3.1 *Let G be a graph and $U \subseteq V(G)$ be a strong edge geodetic set. If v is a dominant neighbor of u, then $u \in U$. In particular, if u and v are twin vertices, then $u \in U$ and $v \in U$.*

Proof Let $uv \in E(G)$ and $N[u] \subseteq N[v]$. If P is a shortest path in G which contains the edge uv, then one of the endpoints of P must be u, for otherwise P would not be shortest. If further u and v are twins, then also $N[v] \subseteq N[u]$ and thus also $v \in U$. □

Proposition 3.2 *Let G be a graph. Then, $\mathrm{sg_e}(G) = n(G)$ if and only if every vertex of G has a dominant neighbor.*

Proof If every vertex of G has a dominant neighbor, then every vertex lies in every strong edge geodetic set by Lemma 3.1. Hence, $\mathrm{sg_e}(G) = n(G)$.

Assume now that a vertex $u \in V(G)$ does not admit a dominant neighbor. We claim that $U = V(G) \setminus \{u\}$ is a strong edge geodetic set of G. Let v be an arbitrary neighbor of u. Since $N[u] \nsubseteq N[v]$, there exists a vertex $w \in N[u] \setminus N[v]$. To cover the edge uv, put the shortest path wuv to $\widehat{U}$. Proceed analogously for every neigbor v' of u, where if the edge $v'u$ has been already covered before, do nothing. In this way, all edges incident with u are covered. Let next xy be an arbitrary edge from $E(G)$ where $\{x, y\} \cap \{u\} = \emptyset$. Then add to $\widehat{U}$ the shortest path xy. Clearly, the paths added so far to $\widehat{U}$ cover all the edges of G and we conclude that $\mathrm{sg_e}(G) < n(G)$. □

Proposition 3.2 implies several results from [25] as for instance [25, Theorem 8] which asserts that if a graph G contains at least two universal vertices, then $\mathrm{sg_e}(G) = n(G)$.

A vertex u of a graph G is *simplicial* if $N(u)$ induces a clique of G. If u is a simplicial vertex and v its arbitrary neighbor, then $N[u] \subseteq N[v]$. Denoting by $s(G)$ the number of simplicial vertices of G Lemma 3.1 thus implies:

Corollary 3.3 *If G is a graph, then* $\mathrm{sg_e}(G) \geq s(G)$.

Since in K_n every vertex is simplicial, Corollary 3.3 implies that $\mathrm{sg_e}(K_n) = n$. We can also deduce this fact from Lemma 3.1 by observing that each pair of vertices of K_n are twins.

Lemma 3.1 implies also the following.

Corollary 3.4 *If a graph G contains a universal vertex, then* $\mathrm{sg_e}(G) \geq n(G) - 1$. *Moreover, if there is only one universal vertex, then* $\mathrm{sg_e}(G) = n(G) - 1$.

Proof Let w be a universal vertex of G. Then, w is a dominant neighbor of every vertex $u \in V(G)\backslash\{w\}$, and hence, Lemma 3.1 implies that $V(G)\backslash\{w\} \subseteq U$ for every strong edge geodetic set U of G. Thus, $\mathrm{sg_e}(G) \geq n(G) - 1$. In the case when w is a unique universal vertex of G, then with the same arguments as we had in the last part of the proof of Proposition 3.2, we infer that $V(G)\backslash\{w\}$ is a strong edge geodetic set. Hence, $\mathrm{sg_e}(G) \leq n(G) - 1$ when G has a unique universal vertex, so that in this case, $\mathrm{sg_e}(G) = n(G) - 1$. □

The second assertion of Corollary 3.4 was earlier presented as [25, Theorem 5]. Moreover, in [25, Theorem 5], it was also claimed that $\mathrm{sg_e}(P_2 \,\square\, K_m) = 2m-2$. It can be checked that the result is not true and that instead the Cartesian products $P_2 \,\square\, K_m$ also belong to the family of graphs G for which $\mathrm{sg_e}(G) \geq n(G) - 1$. More generally, we have the following result.

Theorem 3.5 *If $m \geq 3$ and $n \geq 2$, then*

$$\mathrm{sg_e}(P_n \,\square\, K_m) = \begin{cases} mk; & n = k^2, \\ mk + (m-1); & n = k^2 + h, 1 \leq h \leq k, \\ mk + m; & n = k^2 + h, k+1 \leq h \leq 2k. \end{cases}$$

Proof Set $V(K_m) = [m]$ and $V(P_n) = [n]$ where $i(i+1) \in E(P_n)$ for $i \in [n-1]$. If $y \in V(K_m)$, then we will denote by P_n^y the subgraph of $P_n \,\square\, K_m$ induced by the vertices (i, y), $i \in [n]$. P_n^y is also called a P_n-layer of $P_n \,\square\, K_m$ and is isomorphic to P_n. Throughout the proof, we will use the fact that in a shortest path of $P_n \,\square\, K_m$, there is at most one edge between two distinct P_n-layers.

Consider first the case $n = k^2$, where $k \in \mathbb{N}$. In this case, we claim that $U_1 = \bigcup_{i=1}^{k} \bigcup_{j=1}^{m} \{(i^2, j)\}$ is a strong edge geodetic set of $P_n \,\square\, K_m$. To cover all the edges of $P_n \,\square\, K_m$, we proceed as follows. For every $j \in [m]$, put to $\widehat{U_1}$ the unique shortest path between the vertices $(1, j)$ and (k^2, j). For every pair $y_1, y_2 \in V(K_m)$, $y_1 < y_2$, we put the following shortest paths to $\widehat{U_1}$.

- For every $i \in [k]$, put to $\widehat{U_1}$ the unique shortest path (of length 1) between the vertices (i^2, y_1) and (i^2, y_2).
- For every $2 \leq i \leq k$, and for every $l \in [i-1]$, put to $\widehat{U_1}$ the shortest path between the vertices (l^2, y_1) and (i^2, y_2) that contains the edge $((i-1)^2 + l, y_1)((i-1)^2 + l, y_2)$, and the shortest path between the vertices (l^2, y_2) and (i^2, y_1) that passes through the edge $(i(i-1) + l, y_1)(i(i-1) + l, y_2)$. See Fig. 2 for an example, where the vertices from U_1 are drawn in black.

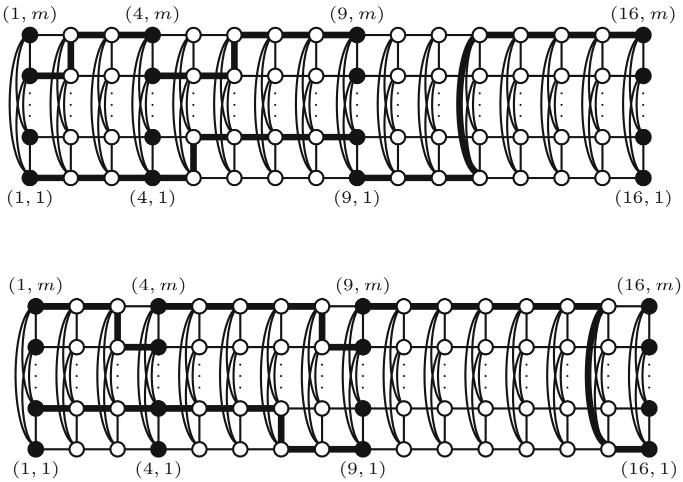

Fig. 2 Shortest paths for $n = 16$ and $(y_1, y_2, i, l) \in \{(m-1, m, 2, 1), (m-1, m, 3, 2), (1, 2, 3, 1), (1, m, 4, 3)\}$

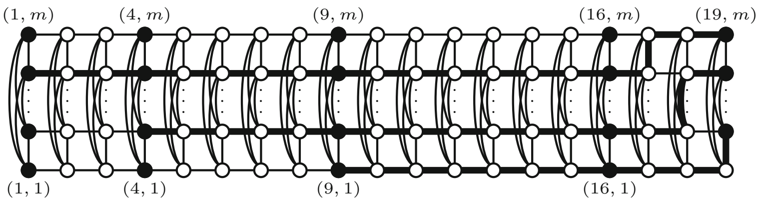

Fig. 3 Shortest paths for $n = 19$ and $(y_1, y_2, i) \in \{(1, 2, 3), (2, m-1, 2), (m-1, m, 1)\}$

The shortest paths from $\widehat{U_1}$ cover all the edges of $P_n \,\square\, K_m$, and hence, we can conclude that $\mathrm{sg_e}(P_n \,\square\, K_m) \leq mk$ when $n = k^2$.

Assume next that $n = k^2 + h$, where $h \in [k]$. Then, we claim that the set $U_2 = U_1 \cup \bigcup_{j=2}^{m} \{(n, j)\}$ is a strong edge geodetic set of $P_n \,\square\, K_m$.

- First, put all the shortest paths from $\widehat{U_1}$ to $\widehat{U_2}$.
- For every pair $y_1, y_2 \in V(K_m)$, where $y_1 < y_2$, and for every $i \in [h]$, put to $\widehat{U_2}$ the shortest path between the vertices (i^2, y_1) and (n, y_2) that contains the edge $(k^2+i, y_1)(k^2+i, y_2)$. See Fig. 3 for an example, where the vertices from U_2 are again drawn in black.
- For every $y \in \{2, \ldots, m\}$, put to $\widehat{U_2}$ the unique shortest path between the vertices $(1, y)$ and (n, y). Note that all the edges from P_n^1 are already covered by the shortest path from $\widehat{U_2}$ between vertices $(h^2, 1)$ and $(n, 2)$.

Since the shortest paths from $\widehat{U_2}$ cover all the edges of $P_n \,\square\, K_m$, we can conclude that $\mathrm{sg_e}(P_n \,\square\, K_m) \leq mk + (m-1)$, when $n = k^2 + h$ and $h \in [k]$.

Assume finally that $n = k^2 + h$, where $k + 1 \leq h \leq 2k$. In this case, we claim that $U_3 = U_1 \cup \bigcup_{j=1}^{m} \{(n, j)\}$ is a strong edge geodetic set of $P_n \,\square\, K_m$ and proceed as follows.

- Put all the shortest paths from $\widehat{U_1}$ to $\widehat{U_3}$.

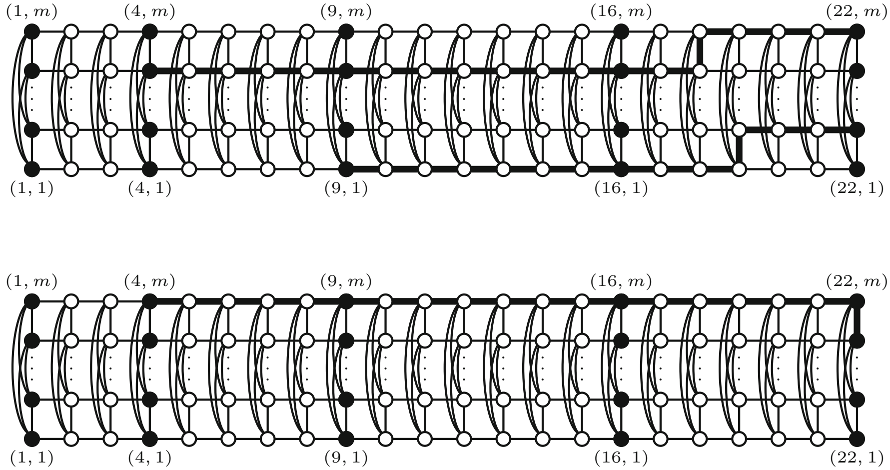

Fig. 4 Shortest paths for $n = 22$ and $(y_1, y_2, i) \in \{(1, 2, 3), (m-1, m, 2)\}$

- For every pair $y_1, y_2 \in V(K_m)$, where $y_1 < y_2$, do the following. For every $i \in [k]$, put to $\widehat{U_3}$ the shortest path between vertices (i^2, y_1) and (n, y_2) that contains the edge $(k^2 + i, y_1)(k^2 + i, y_2)$. Moreover, for every $i \in [h - k]$, also add the shortest path between the vertices (n, y_1) and (i^2, y_2) that contains the edge $(k(k+1) + i, y_1)(k(k+1) + i, y_2)$. In Fig. 4, examples are drawn with the vertices from U_3 again in black.
- For every $j \in [m]$, put to $\widehat{U_3}$ the unique shortest path between the vertices $(1, j)$ and (n, j).

Since the shortest paths from $\widehat{U_3}$ cover all the edges of $P_n \,\square\, K_m$, we get the upper bound $\mathrm{sg_e}(P_n \,\square\, K_m) \leq mk + m$ when $n = k^2 + h$ with $k + 1 \leq h \leq 2k$.

In the second part of the proof, we need to demonstrate that the obtained upper bounds are sharp, that is, there exist no smaller strong edge geodetic sets as the one constructed above. Let U be a arbitrary strong edge geodetic set of $P_n \,\square\, K_m$.

Assume first that $n = k^2$ for some $k \in \mathbb{N}$. Then, we need to show that $|U| \geq mk$. If for every vertex $y \in K_m$, the set U has at least k vertices in the P_n^y-layer, then clearly $|U| \geq mk$. Assume, therefore, that for some $y_i \in V(K_m)$, the $P_n^{y_i}$- layer contains $k - l$, $l \geq 1$, vertices from U. Since $|V(P_n^{y_i}) \cap U| = k - l$, for every vertex $y \in V(K_m)$, $y \neq y_i$, the strong edge geodetic set U has to have at least x vertices from P_n^y, where $(k - l)x \geq k^2$, to cover all the edges between $P_n^{y_i}$ and P_n^y. Because $x \geq k^2/(k - l) = k + kl/(k - l) \geq k + kl/k = k + l$, we get

$$|U| \geq k - l + (m - 1)(k + l) = mk + (m - 2)l \geq mk + 1,$$

where the last assertion follows, since $m \geq 3$ and $l \geq 1$. We conclude that in any case, $|U| \geq mk$.

Assume second that $n = k^2 + h$, where $1 \leq h \leq k$. Now, we need to prove that $\mathrm{sg_e}(P_n \,\square\, K_m) \geq m(k + 1) - 1$. If, for every vertex $y \in K_m$, the set U has at least $k + 1$ vertices in P_n^y, then clearly $|U| \geq m(k + 1)$ and we are done. Assume therefore that for some $y_i \in V(K_m)$, the set U has $(k+1)-l$, $l \geq 1$, vertices from $P_n^{y_i}$. Since $|V(P_n^{y_i}) \cap U| \leq k+1-l$, for every vertex $y \in V(K_m)$, $y \neq y_i$, the set U has to have at least x vertices from P_n^{y}, where $(k+1-l)x \geq k^2+h$ has to hold to cover all the edges between $P_n^{y_i}$ and P_n^{y}. Because x is an integer, we can compute as follows:

$$x \geq \left\lceil \frac{k^2+h}{k+1-l} \right\rceil = \left\lceil \frac{k(k+h/k)}{k+1-l} \right\rceil = \left\lceil \frac{k(k+1-l)+k(h/k-1+l)}{k+1-l} \right\rceil$$
$$= k + \left\lceil \frac{k(h/k-1+l)}{k+1-l} \right\rceil.$$

Because $l \geq 1$ and therefore $1/(k+1-\ell) \geq 1/k$, we also have

$$x \geq k + \left\lceil \frac{k(h/k-1+l)}{k} \right\rceil = k+l-1+\left\lceil \frac{h}{k} \right\rceil.$$

Since $h \in [k]$, we have $\lceil h/k \rceil = 1$ and therefore $x \geq k+l$. Altogether

$$|U| \geq k+1-l+(m-1)(k+l) = mk+(m-2)l+1$$
$$\geq mk+(m-2)+1 = m(k+1)-1$$

which we wanted to show.

The remaining case is when $n = k^2+h$, where $k+1 \leq h \leq 2k$. Now, we need to prove that $\mathrm{sg_e}(P_n \,\square\, K_m) \geq m(k+1)$. If for every vertex $y \in K_m$, the set U has at least $k+1$ vertices from P_n^{y}, then clearly $|U| \geq m(k+1)$. Assume therefore that for some $y_i \in V(K_m)$, the set U has $(k+1)-l$, $l \geq 1$, vertices from $P_n^{y_i}$. Therefore, $|V(P_n^{y_i}) \cap U| \leq k+1-l$, and hence, for every vertex $y \in V(K_m)$, $y \neq y_i$, the set U has to have at least x vertices from P_n^{y}, where $(k+1-l)x \geq k^2+h$ has to hold to cover all the edges between $P_n^{y_i}$ and P_n^{y}. Because x is an integer and $l \geq 1$, we can similarly as in the previous case estimate that

$$x \geq k+l-1+\lceil h/k \rceil.$$

Because h is an integer between $k+1$ and $2k$, we have $\lceil h/k \rceil = 2$, and therefore, $x \geq k+l+1$. Altogether, we see that

$$|U| \geq k+1-l+(m-1)(k+l+1) = m(k+1)+(m-2)l$$
$$\geq m(k+1)+1\,,$$

where the last assertion holds, since $m \geq 3$ and $l \geq 1$. □

The following special case of Theorem 3.5 has been reported earlier in [25, Theorem 14].

Corollary 3.6 *If $k \geq 2$ and $m \geq 3$, then $\mathrm{sg_e}(P_{k^2} \,\square\, K_m) = mk$.*

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Data Availability Our manuscript has no associated data.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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