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ORIGINAL PAPER

Some Conditions Concerning the Shape Operator of a Real Hypersurface in Complex Projective Space

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Abstract
Let M be a real hypersurface of a complex projective space. For any operator B on M and any nonnull real number k, we can define two tensor fields of type (1,2) on M, $B_F^{(k)}$ and $B_T^{(k)}$. We will classify real hypersurfaces in complex projective space for which $B_F^{(k)}$ and $B_T^{(k)}$ either take values in the maximal holomorphic distribution $\mathbb{D}$ or are parallel to the structure vector field ξ, in the particular case of $B = A$, where A denotes the shape operator of M. We also introduce the concept of $A_F^{(k)}$ and $A_T^{(k)}$ being $\mathbb{D}$-recurrent and classify real hypersurfaces such that either $A_F^{(k)}$ or $A_T^{(k)}$ are $\mathbb{D}$-recurrent.

Keywords kth Generalized Tanaka–Webster connection · Complex projective space · Real hypersurface · Shape operator · Lie derivative

Mathematics Subject Classification 53C15 · 53B25

1 Introduction

Consider the complex projective space $\mathbb{C}P^m$, $m \geq 2$, endowed with the Kaehlerian structure (J, g), where J denotes the complex structure and g the Fubini-Study metric

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of constant holomorphic sectional curvature 4 on $\mathbb{C}P^m$. Let M be a connected real hypersurface in $\mathbb{C}P^m$ without boundary. Denote also by g the induced metric on M and by N a local unit normal vector field on M. The Reeb (or structure) vector field on M is defined by $\xi = -JN$. Let ∇ be the Levi-Civita connection on M and A the shape operator associated to N. For any vector field X tangent to M write $JX = \phi X + \eta(X)N$, where ϕX is the tangential component of JX and $\eta(X) = g(X, \xi)$. Then (ϕ, ξ, η, g) defines an almost contact metric structure on M [1].

The existence of such a structure allows us to define, for any nonnull real number k, the so-called kth Generalized Tanaka–Webster connection on M, $\hat{\nabla}^{(k)}$ [2, 3], given by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for any X, Y tangent to M.

This connection on M is a metric one and any element of the almost contact metric structure is parallel for such a connection. If A satisfies $\phi A + A\phi = 2k\phi$, M becomes a contact manifold and this connection coincides with the Tanaka–Webster connection on M, [14–16].

The tensor field of type $(1, 2)$ obtained as the difference of both connections is called the kth Cho tensor on M (see [6, Proposition 7.10]) and it is given by $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M. From this, for any X tangent to M and any nonnull real number k, we define the kth Cho operator corresponding to X, as $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any Y tangent to M.

The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $T^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M, [2]. Thus we define the kth torsion operator associated to X, for any nonnull real number k and any X tangent to M, by $T^{(k)}_X Y = \hat{T}^{(k)}(X, Y)$, for any Y tangent to M.

If $\mathcal{L}$ denotes the Lie derivative on M, we now that it is given by $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$, for any X, Y tangent to M. If we consider the kth Generalized Tanaka–Webster connection we can also define on M a differential operator of first order, that we call the derivative of Lie type associated to such a connection, $\mathcal{L}^{(k)}$, given by

$$\mathcal{L}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X = \mathcal{L}_X Y + T^{(k)}_X Y$$

for any X, Y tangent to M.

Best known real hypersurfaces in $\mathbb{C}P^m$ are called Hopf and satisfy that the Reeb vector field ξ is an eigenvector of the shape operator, that is, $A\xi = \alpha\xi$, for a certain function α on M, called the Reeb curvature of M. The distribution on M given by $\mathbb{D} = Ker(\eta)$ is ϕ-invariant and called the maximal holomorphic distribution on M.

Takagi classified homogeneous real hypersurfaces in complex projective space (see [11–13]). Kimura [4], proved that Takagi's real hypersurfaces are the unique ones that are Hopf and have constant principal curvatures for A in $\mathbb{C}P^m$. Takagi's list contains the following 6 types of real hypersurfaces

- Type (A_1), geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$, with 2 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$ and $\cot(r)$ with eigenspace $\mathbb{D}$.
- Type (A_2), tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$, with 3 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r)$ and $-\tan(r)$. The corresponding eigenspaces of $\cot(r)$ and $-\tan(r)$ are complementary and ϕ-invariant distributions in $\mathbb{D}$.
- Type (B), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric Q^{m-1}, with 3 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r-\frac{\pi}{4})$ and $-\tan(r-\frac{\pi}{4})$ whose corresponding eigenspaces are complementary and equal dimensional distributions in $\mathbb{D}$ such that $\phi V_{\cot(r-\frac{\pi}{4})} = V_{-\tan(r-\frac{\pi}{4})}$.
- Type (C), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n+1 = m$ and $m \geq 5$, with 5 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r-\frac{\pi}{4})$ with multiplicity 2, $\cot(r-\frac{\pi}{2}) = -\tan(r)$ with multiplicity $m-3$, $\cot(r-\frac{3\pi}{4})$, with multiplicity 2 and $\cot(r-\pi) = \cot(r)$ with multiplicity $m-3$. Moreover $\phi V_{\cot(r-\frac{\pi}{4})} = V_{\cot(r-\frac{3\pi}{4})}$ and $V_{-\tan(r)}$ and $V_{\cot(r)}$ are ϕ-invariant.
- Type (D), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Plucker embedding of the complex Grassmannian manifold $G(2,5)$ in $\mathbb{C}P^9$, with the same principal curvatures as type (C), $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, and the other 4 principal curvatures have the same multiplicity 4 and their eigenspaces have the same behaviour with respect to ϕ as in type (C).
- Type (E), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$. They also have the same principal curvatures as type (C), $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r-\frac{\pi}{4})$ and $\cot(r-\frac{3\pi}{4})$ have multiplicities equal to 6 and $-\tan(r)$ and $\cot(r)$ have multiplicities equal to 8. Their corresponding eigenspaces have the same behaviour with respect to ϕ as in type (C).

We will call type (A) real hypersurfaces to both types (A_1) or (A_2).

Ruled real hypersurfaces in $\mathbb{C}P^m$ were introduced by Kimura [5]. The maximal holomorphic distribution $\mathbb{D}$ of such real hypersurfaces is integrable with integral manifolds $\mathbb{C}P^{m-1}$. Equivalently, $g(A\mathbb{D}, \mathbb{D}) = 0$. Kimura gave some minimal examples of this kind of real hypersurfaces.

Let B be a symmetric operator on M. Then we can define on M a couple of tensor fields of type (1,2), for any nonnull real number k, $B_F^{(k)}$ and $B_T^{(k)}$, given, respectively, by

$$B_F^{(k)}(X,Y) = ((\hat{\nabla}_X^{(k)} - \nabla_X)B)Y = F_X^{(k)}BY - BF_X^{(k)}Y = [F_X^{(k)}, B]Y \tag{1.1}$$

and

$$B_T^{(k)}(X,Y) = ((\mathcal{L}_X^{(k)} - \mathcal{L}_X)B)Y = T_X^{(k)}BY - BT_X^{(k)}Y = [T_X^{(k)}, B]Y \tag{1.2}$$

for any X, Y tangent to M.

In [10] we considered the case $A = B$ in (1.1), and proved non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $A_F^{(k)} = 0$, for any nonnull real number k. A similar result for $A = B$ in (1.2) was obtained in [9]. Such conditions imply commutativity of A and either $F_X^{(k)}$ or $T_X^{(k)}$, for any X tangent to M, respectively.

In this paper we want to generalize such results. Then, we will consider the conditions $g(A_F^{(k)}(X, Y), \xi) = 0$, (respectively, $g(A_F^{(k)}(X, Y), Z) = 0$) for any X, Y tangent to M (respectively, for any X, Y tangent to M, $Z \in \mathbb{D}$), obtaining the following

Theorem 1.1 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $g(A_F^{(k)}(X, Y), \xi) = 0$ for any X, Y tangent to M if and only if M is locally congruent to a ruled real hypersurface such that $g(A\xi, \xi) = -k$.*

And

Theorem 1.2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull constant. Then $g(A_F^{(k)}(X, Y), Z) = 0$ for any X, Y tangent to M, $Z \in \mathbb{D}$, if and only if M is locally congruent to a real hypersurface of type (A)*

Similar conditions for $A_T^{(k)}$ give us the following results

Theorem 1.3 *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that $g(A_T^{(k)}(X, Y), \xi) = 0$, for any X, Y tangent to M and any nonnull real number k.*

And

Theorem 1.4 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$ and k a nonnull real number. Then $g(A_T^{(k)}(X, Y), Z) = 0$, for any X, Y tangent to M, $Z \in \mathbb{D}$, if and only if M is locally congruent to a real hypersurface of type (A).*

On the other hand, we will say that A is $(\hat{\nabla}^{(k)}, \nabla)$-recurrent if $((\hat{\nabla}_X^{(k)} - \nabla_X)A)Y = \omega(X)AY$, for any X, Y tangent to M, where ω is a nonnull 1-form on M. This is equivalent to have $A_F^{(k)}(X, Y) = \omega(X)AY$.

Similarly, we will say that A is $(\mathcal{L}^{(k)}, \mathcal{L})$-recurrent if $((\mathcal{L}_X^{(k)} - \mathcal{L}_X)A)Y = \delta(X)AY$, for any X, Y tangent to M and a nonnull 1-form δ on M. This is equivalent to have $A_T^{(k)}(X, Y) = \delta(X)AY$.

If we consider $\mathbb{D} - (\hat{\nabla}^{(k)}, \nabla)$-recurrency or $\mathbb{D} - (\mathcal{L}^{(k)}, \mathcal{L})$-recurrency (the same conditions as above for $X, Y \in \mathbb{D}$) we obtain

Theorem 1.5 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $A_F^{(k)}(X, Y) = \omega(X)AY$, for any $X, Y \in \mathbb{D}$ and a nonnull 1-form ω on M if and only if M is locally congruent either to a real hypersurface of type (A) or to a ruled real hypersurface.*

and

Theorem 1.6 *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that $A_T^{(k)}(X, Y) = \delta(X)AY$, for any $X, Y \in \mathbb{D}$, and a nonnull 1-form δ on M, k being a nonnull real number.*

2 Preliminaries

Any mathematical object in the sequel will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface without boundary in $\mathbb{C}P^m$, $m \geq 2$, and N a locally defined normal unit vector field on M. Let∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M, we write $JX = \phi X + \eta(X)N$, where ϕX denotes the tangential component of JX, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M (see [1]). Therefore,

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we get

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi).$$

From the parallelism of J we obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \phi AX$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As $\mathbb{C}P^m$ has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M.

In the sequel the following result owed to Maeda [7], is needed.

Theorem 2.1 *Let M be a Hopf real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then $\alpha = g(A\xi, \xi)$ is constant and if W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then $2\lambda - \alpha \neq 0$ and $A\phi W = \mu\phi W$, where $\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}$.*

We will also need the following theorem proved by Okumura [8]

Theorem 2.2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then $\phi A = A\phi$ if and only if M is locally congruent to a real hypersurface of type (A).*

3 Proofs of Theorems 1.1 and 1.2

Let us suppose that $g(A_F^{(k)}(X,Y),\xi) = 0$ for any X, Y tangent to M. This yields $g(g(\phi AX,\ AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y, \xi) = 0$, for any X, Y tangent to M. Therefore

$$g(A\phi AX, Y) - g(A\xi,\xi)g(\phi AX, Y) + \eta(Y)g(A\phi AX,\xi) + k\eta(X)g(A\phi Y,\xi) = 0 \tag{3.1}$$

for any X, Y tangent to M.

Let us suppose that M is Hopf, that is, $A\xi = \alpha\xi$. Then (3.1) gives $g(A\phi AX, Y) - \alpha g(\phi AX, Y) = 0$, for any X, Y tangent to M. Thus $A\phi AX = \alpha\phi AX$, for any X tangent to M. If we choose $X \in \mathbb{D}$ such that $AX = \lambda X$, from Theorem 2.1 we should have $A\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then $\lambda\mu = \lambda\alpha$ and either $\lambda = 0$ or $\mu = \alpha$.

If we suppose that in $\mathbb{D}$ there exists a principal curvature $\lambda \neq 0$, $\mu = \alpha$ yields $\alpha\lambda + 2 = 2\alpha\lambda - \alpha^2$. That is, $\alpha\lambda = \alpha^2 + 2$. This implies that $\alpha \neq 0$ and then $\lambda = \frac{\alpha^2+2}{\alpha}$. As $\mu = \alpha$, we also have $\lambda \neq \mu$ and all the principal curvatures are constant. Therefore $A\phi \neq \phi A$ and M cannot be of type (A). If there is not a vector field $Y \in \mathbb{D}$ such that $AY = 0$, the unique principal curvatures on $\mathbb{D}$ are α and $\frac{\alpha^2+2}{\alpha}$. Looking at Takagi's list, this is impossible.

Therefore, the unique principal curvature in $\mathbb{D}$ is $\lambda = 0$. But then, $\mu = -\frac{2}{\alpha}$ must be equal to 0 too, which is also impossible.

Then we must suppose that M is non Hopf. So we can write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$ and β is a function on M that does not vanish at least on a neighborhood of a point $p \in M$. We will make all the calculations on such a neighborhood.

If we take $Y = \xi$ in (3.1) we get $2g(A\phi A\xi, X) = 0$ for any X tangent to M. That is, $\beta g(A\phi U, X) = 0$ for any X tangent to M, which yields

$$A\phi U = 0. \tag{3.2}$$

Taking $X = \xi$ in (3.1) we obtain $g(A\phi A\xi, Y) - \alpha g(\phi A\xi, Y) + kg(A\phi Y, \xi) = 0$, for any Y tangent to M. Then, from (3.2), $-\alpha\beta g(\phi U, Y) - k\beta g(\phi U, Y) = 0$. As $\beta \neq 0$, if we take $Y = \phi U$ we have

$$\alpha = -k. \tag{3.3}$$

If now we take $Y = \phi U$ in (3.1) it follows $-\alpha g(AX, U) - k\eta(X)g(AU, \xi) = 0$, for any X tangent to M. That is, $-\alpha g(AU, X) - k\beta\eta(X) = 0$. From (3.3) we get $kg(AU, X) - k\beta\eta(X) = 0$, for any X tangent to M. Thus

$$AU = \beta\xi. \tag{3.4}$$

From (3.2) and (3.4) we have that $\mathbb{D}_U = \{X \in \mathbb{D} | g(X, U) = g(X, \phi U) = 0\}$ is A-invariant. Take $X, Y \in \mathbb{D}_U$ in (3.1). Then $g(A\phi AX, Y) - \alpha g(\phi AX, Y) = 0$. From (3.3) this yields

$$A\phi AX + k\phi AX = 0 \tag{3.5}$$

for any $X \in \mathbb{D}_U$. We can also write the equation above (3.5) as $-g(A\phi AY, X) + \alpha g(A\phi Y, X) = 0$, for any $X, Y \in \mathbb{D}_U$. From (3.3) we obtain

$$-A\phi AX - kA\phi X = 0 \tag{3.6}$$

for any $X \in \mathbb{D}_U$. Adding (3.5) and (3.6) we have $k(\phi A - A\phi)X = 0$ for any $X \in \mathbb{D}_U$ and, as $k \neq 0$, we get

$$\phi AX = A\phi X$$

for any $X \in \mathbb{D}_U$. Therefore, if $X \in \mathbb{D}_U$ satisfies $AX = \lambda X$, we obtain $A\phi X = \lambda\phi X$. Moreover, from (3.5) it follows $\lambda^2 + k\lambda = 0$ and either $\lambda = 0$ or $\lambda = -k$.

Let us suppose that there exists $Y \in \mathbb{D}_U$ such that $AY = -kY$ and $A\phi Y = -k\phi Y$. The Codazzi equation yields $(\nabla_Y A)\phi Y - (\nabla_{\phi Y} A)Y = -2\xi$. Therefore, $-k\nabla_Y\phi Y - A\nabla_Y\phi Y + k\nabla_{\phi Y}Y + A\nabla_{\phi Y}Y = -2\xi$. Its scalar product with ξ gives $kg(\phi Y, \phi AY) - g(\nabla_Y\phi Y, -k\xi + \beta U) - kg(Y, \phi A\phi Y) + g(\nabla_{\phi Y}Y, -k\xi + \beta U) = -2$. This yields $\beta g([\phi Y, Y], U) - k^2 - kg(\phi Y, \phi AY) + kg(A\phi Y, \phi Y) + kg(Y, \phi A\phi Y) = -2$. Thus

$$g([\phi Y, Y], U) = -\frac{2}{\beta}. \tag{3.7}$$

Its scalar product with U implies $-kg(\nabla_Y\phi Y, U) - g(\nabla_Y\phi Y, \beta\xi) + kg(\nabla_{\phi Y}Y, U) + g(\nabla_{\phi Y}Y, \beta\xi) = 0$. That is, $kg([\phi Y, Y], U) + \beta g(\phi Y, \phi AY) - \beta g(Y, \phi A\phi Y) = 0$. Then

$$g([\phi Y, Y], U) = 2\beta. \tag{3.8}$$

From (3.7) and (3.8) $\beta = -\frac{1}{\beta}$ would give $\beta^2 = -1$, which is impossible.

We conclude that the unique principal curvature in $\mathbb{D}_U$ is 0 and M is ruled. The converse is straightforward and we finish the proof of Theorem 1.1.

In order to prove Theorem 1.2 let us suppose that $g(A_F^{(k)}(X, Y), Z) = 0$ for any X, Y tangent to M, $Z \in \mathbb{D}$. This implies

$$-\eta(AY)g(\phi AX, Z) - k\eta(X)g(\phi AY, Z) - g(\phi AX, Y)\eta(AZ) + \eta(Y)g(A\phi AX, Z) + k\eta(X)g(A\phi Y, Z) = 0 \tag{3.9}$$

for any X, Y tangent to M, $Z \in \mathbb{D}$.

Let us suppose that M is Hopf and $A\xi = \alpha\xi$. Taking $X = \xi$ in (3.9) we get $-kg(\phi AY, Z) + kg(A\phi Y, Z) = 0$ for any Y tangent to M, $Z \in \mathbb{D}$. As $k \neq 0$, this means that $(A\phi - \phi A)X = 0$ for any $X \in \mathbb{D}$. From Theorem 2.2, M must be locally congruent to a real hypersurface of type (A).

If M is non Hopf we will write $A\xi = \alpha\xi + \beta U$ with the same conditions as in the proof of Theorem 1.1. Taking $X = Y = \xi$ in (3.9) we have $-\alpha\beta g(\phi U, Z) -$

$k\beta g(\phi U, Z) + \beta g(A\phi U, Z) = 0$ for any $Z \in \mathbb{D}$. This gives, bearing in mind that $\beta \neq 0$,

$$A\phi U = (\alpha + k)\phi U. \tag{3.10}$$

If in (3.9) we put $Y = \xi$ we get $-\alpha g(\phi AX, Z) - k\beta\eta(X)g(\phi U, Z) + g(A\phi AX, Z) = 0$, for any X tangent to M, $Z \in \mathbb{D}$. If $Z = \phi U$, we obtain $-\alpha g(AU, X) - k\beta\eta(X) + (\alpha + k)g(AU, X) = 0$, for any X tangent to M. This implies $kAU = k\beta\xi$. As $k \neq 0$ we obtain

$$AU = \beta\xi. \tag{3.11}$$

If we take $Y = \xi$, $X = \phi U$ in (3.9) we have $\alpha(\alpha+k)g(U, Z)-(\alpha+k)g(AU, Z) = 0$, for any $Z \in \mathbb{D}$. From (3.11) we get $\alpha(\alpha + k)g(U, Z) = 0$ for any $Z \in \mathbb{D}$. Taking $Z = U$ we obtain $\alpha(\alpha + k) = 0$.

Let us suppose that $\alpha = -k$. Then (3.10) and (3.11) imply $A\xi = -k\xi+\beta U$, $AU = \beta\xi$, $A\phi U = 0$. If we introduce $X, Y \in \mathbb{D}_U$ in (3.9) we have $-g(\phi AX, Y)g(A\xi, Z) = 0$, for any $Z \in \mathbb{D}$. If $Z = U$ we get $g(\phi AX, Y) = 0$ for any $X, Y \in \mathbb{D}_U$. Now, if we take ϕY instead of Y it follows $g(AX, Y) = 0$ for any $X, Y \in \mathbb{D}_U$ and

$$AX = 0 \tag{3.12}$$

for any $X \in \mathbb{D}_U$. From (3.10), (3.11), (3.12) and the fact that $\alpha = -k$, M should be ruled. But taking $X = \xi$, $Y = U$ in (3.10) we have $k\beta g(\phi U, Z) = 0$ for any $Z \in \mathbb{D}$, which is impossible.

Suppose then that $\alpha = 0$. Therefore, $A\xi = \beta U$, $AU = \beta\xi$ and $A\phi U = k\phi U$. Take $X = \xi$, $Y \in \mathbb{D}_U$ in (3.9). Then $-kg(\phi AY, Z) + kg(A\phi Y, Z) = 0$, for any $Y \in \mathbb{D}_U$, $Z \in \mathbb{D}$. This yields $A\phi Y = \phi AY$ for any $Y \in \mathbb{D}_U$. As $\mathbb{D}_U$ is A-invariant, if $Y \in \mathbb{D}_U$ satisfies $AY = \lambda Y$, $A\phi Y = \lambda\phi Y$. If we take $Y = \xi$, $X \in \mathbb{D}_U$ in (3.9) we obtain $g(A\phi AX, Z) = 0$ for any $X \in \mathbb{D}_U$, $Z \in \mathbb{D}$. Therefore, $A\phi AX = 0$ for any $X \in \mathbb{D}_U$. That is, if $Y \in \mathbb{D}_U$ satisfies $AY = \lambda Y$ we obtain $\lambda = 0$. Therefore $AZ = 0$ for any $Z \in \mathbb{D}_U$. For such a Z Codazzi equation gives $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$. Then $\nabla_Z(\beta U)-A\phi AZ+A\nabla_\xi Z = -\phi Z$. This implies $Z(\beta)U+\beta\nabla_Z U+A\nabla_\xi Z = -\phi Z$ and its scalar product with U implies $Z(\beta) - \beta g(Z, \phi A\xi) = 0$. We have proved that

$$Z(\beta) = 0 \tag{3.13}$$

for any $Z \in \mathbb{D}_U$.

On the other hand, $(\nabla_U A)\xi-(\nabla_\xi A)U = -\phi U$ implies $U(\beta)U+\beta\nabla_U U-\xi(\beta)\xi-\beta\phi A\xi + A\nabla_\xi U = -\phi U$. Its scalar product with ξ gives $-\beta g(U, \phi AU) - \xi(\beta) + \beta g(\nabla_\xi U, U) = 0$. That is,

$$\xi(\beta) = 0 \tag{3.14}$$

and its scalar product with U yields $U(\beta) - \beta^2 g(U, \phi U) = 0$. Thus

$$U(\beta) = 0. \tag{3.15}$$

Also $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$ yields $(\phi U)(\beta)U + \beta\nabla_{\phi U}U + kAU - k\nabla_\xi\phi U + A\nabla_\xi\phi U = U$. Its scalar product with ξ implies $3k\beta + \beta g(\nabla_\xi\phi U, U) = 0$. Then

$$g(\nabla_\xi\phi U, U) = -3k \tag{3.16}$$

and its scalar product with U gives $(\phi U)(\beta) - kg(\nabla_\xi\phi U, U) - \beta g(A\xi, U) = 1$. Therefore, from (3.16),

$$(\phi U)(\beta) = -3k^2 + \beta^2 + 1. \tag{3.17}$$

From (3.13), (3.14), (3.15) and (3.17) we obtain $grad(\beta) = \gamma\phi U$, where $\gamma = -3k^2 + \beta^2 + 1$. As $g(\nabla_X grad(\beta), Y) = g(\nabla_Y grad(\beta), X)$, for any X, Y tangent to M, we get $X(\gamma)g(\phi U, Y) + \gamma g(\nabla_X\phi U, Y) = Y(\gamma)g(\phi U, X) + \gamma g(\nabla_Y\phi U, X)$. If $X = \xi$ we obtain $\gamma g(\nabla_\xi\phi U, Y) = \gamma g(\nabla_Y\phi U, \xi) = -\gamma g(U, AY)$ for any Y tangent to M. If now $Y = U$ it follows $\gamma g(\nabla_\xi\phi U, U) = 0$. From (3.16) we get $-3k\gamma = 0$. Thus $\gamma = 0$ and β is constant.

Then $(\nabla_{\phi U} A)U - (\nabla_U A)\phi U = 2\xi$ yields $\beta\phi A\phi U - A\nabla_{\phi U}U - k\nabla_U\phi U + A\nabla_U\phi U = 2\xi$. Its scalar product with ξ gives $kg(U, AU) + \beta g(\nabla_U\phi U, U) = 2$. Therefore,

$$\beta g(\nabla_U\phi U, U) = 2 \tag{3.18}$$

and its scalar product with U implies $-\beta k + \beta g(U, \phi A\phi U) - kg(\nabla_U\phi U, U) = 0$. That is, $-2\beta k = kg(\nabla_U\phi U, U)$. Then

$$g(\nabla_U\phi U, U) = -2\beta. \tag{3.19}$$

From (3.18) and (3.19) we have $-\beta^2 = 1$, which is impossible and this finishes the proof of Theorem 1.2.

4 Proofs of Theorems 1.3 and 1.4

If we suppose that $g(A_T^{(k)}(X, Y), \xi) = 0$ for any X, Y tangent to M we obtain

$$\begin{aligned} &g(\phi AX, AY) - g(\phi A^2Y, X) - g(A\xi, \xi)g(\phi AX, Y) + \eta(Y)g(\phi AX, A\xi) \\ &\quad + k\eta(X)g(\phi Y, A\xi) + g(A\xi, \xi)g(\phi AY, X) - \eta(X)g(\phi AY, A\xi) \\ &\quad - k\eta(Y)g(\phi X, A\xi) = 0 \end{aligned} \tag{4.1}$$

for any X, Y tangent to M.

Let us suppose that M is Hopf, $A\xi = \alpha\xi$, and take $X, Y \in \mathbb{D}$ in (4.1). Then $g(\phi AX, AY) - g(\phi A^2Y, X) - \alpha g(\phi AX, Y) + \alpha g(\phi AY, X) = 0$. Therefore, we obtain $A\phi AX + A^2\phi X - \alpha\phi AX - \alpha A\phi X = 0$ for any $X \in \mathbb{D}$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$, from Theorem 2.1 we know that $A\phi X = \mu\phi X$. Thus $\lambda\mu + \mu^2 - \alpha\lambda - \alpha\mu = 0$. That is, $(\lambda + \mu)\mu - (\lambda + \mu)\alpha = 0$, or $(\lambda + \mu)(\mu - \alpha) = 0$. If $\lambda + \mu = 0$, as $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, we obtain $2\lambda^2 + 2 = 0$, which is impossible. Therefore $\mu = \alpha$ and then, as in the proof of Theorem 1.1, this case is not possible.

Therefore M must be non Hopf. We continue writing $A\xi = \alpha\xi + \beta U$ as in Sect. 3.

Taking $X = \xi$ in (4.1) we obtain $\beta g(\phi U, AY) - \alpha\beta g(\phi U, Y) + kg(\phi Y, A\xi) - g(\phi AY, A\xi) = 0$, for any Y tangent to M. This gives $\beta A\phi U - \alpha\beta\phi U - k\beta\phi U + \beta A\phi U = 0$. Then $2A\phi U - (\alpha + k)\phi U = 0$ and

$$A\phi U = \left(\frac{\alpha + k}{2}\right)\phi U. \tag{4.2}$$

If now we take $Y = \xi$ in (4.1) it follows $2g(\phi AX, A\xi) - g(\phi A^2\xi, X) + \alpha g(\phi A\xi, X) - kg(\phi X, A\xi) = 0$, for any X tangent to M. Then $-2\beta g(A\phi U, X) - g(\phi A(\alpha\xi + \beta U), X) + \alpha\beta g(\phi U, X) + k\beta g(\phi U, X) = 0$, for any X tangent to M. From (4.2) we get $-\alpha g(\phi A\xi, X) - \beta g(\phi AU, X) = 0$, for any X tangent to M. Therefore, $-\alpha\beta\phi U - \beta\phi AU = 0$, or $\phi AU = -\alpha\phi U$. Applying ϕ we obtain

$$AU = \beta\xi - \alpha U. \tag{4.3}$$

Take $X = \phi U$ in (4.1). Then $g(\phi A\phi U, AY) - g(A^2Y, U) - \alpha g(\phi A\phi U, Y) + \eta(Y)g(\phi A\phi U, A\xi) + \alpha g(AY, U) + k\eta(Y)g(U, A\xi) = 0$, for any Y tangent to M. From (4.2) we get $-(\frac{\alpha+k}{2})\, g(AU, Y) - g(A^2U, Y) + \alpha(\frac{\alpha+k}{2})g(U, Y) - \beta(\frac{\alpha+k}{2})\eta(Y) + \alpha g(AU, Y) + k\beta\eta(Y) = 0$, for any Y tangent to M. Therefore, $(\alpha - (\frac{\alpha+k}{2}))AU - A^2U + \alpha(\frac{\alpha+k}{2})U + \beta(k - (\frac{\alpha+k}{2}))\xi = 0$. This and (4.3) yield

$$\alpha k - \alpha^2 - \beta^2 = 0. \tag{4.4}$$

If now we take $Y = \phi U$ in (4.1) we have $g(\phi AX, A\phi U) - g(\phi A^2\phi U, X) - \alpha g(AX, U) - k\eta(X)g(U, A\xi) + \alpha g(\phi A\phi U, X) - \beta g(\phi A\phi U, U)\eta(X) = 0$, for any X tangent to M. This yields $((\frac{\alpha+k}{2}) - \alpha)AU + (\frac{\alpha+k}{2})((\frac{\alpha+k}{2}) - \alpha)U + \beta((\frac{\alpha+k}{2}) - \alpha)\xi = 0$, that is, $(\frac{k-\alpha}{2})AU + (\frac{k+\alpha}{2})(\frac{k-\alpha}{2})U - \beta(\frac{k-\alpha}{2})\xi = 0$. If $\alpha = k$, from (4.4), $\beta = 0$, which is impossible. Therefore, $k \neq \alpha$ and we get

$$AU = \beta\xi - \left(\frac{\alpha + k}{2}\right)U. \tag{4.5}$$

From (4.3) and (4.5), $\alpha = \frac{\alpha+k}{2}$, and then, $\alpha = k$, that we have seen that is impossible, finishing the proof of Theorem 1.3.

Suppose now that $g(A_T^{(k)}(X, Y), Z) = 0$ for any X, Y tangent to M, $Z \in \mathbb{D}$. This implies

$$-\eta(AY)g(\phi AX, Z) - k\eta(X)g(\phi AY, Z) + \eta(X)g(\phi A^2Y, Z) + k\eta(AY)g(\phi X, Z)$$
$$-g(\phi AX, Y)g(A\xi, Z) + \eta(Y)g(\phi AX, AZ) + k\eta(X)g(\phi Y, AZ) + g(\phi AY, X)g(A\xi, Z)$$
$$-\eta(X)g(\phi AY, AZ) - k\eta(Y)g(\phi X, AZ) = 0 \tag{4.6}$$

for any X, Y tangent to M, $Z \in \mathbb{D}$.

Let us suppose that M is Hopf with $A\xi = \alpha\xi$. Take $X = \xi$, $Y \in \mathbb{D}$ in (4.6). Then we get $-kg(\phi AY, Z) + g(\phi A^2Y, Z) - kg(\phi Y, AZ) - g(\phi AY, AZ) = 0$, for any $Y, Z \in \mathbb{D}$. Therefore,

$$-k\phi AY + \phi A^2Y + kA\phi Y - A\phi AY = 0 \tag{4.7}$$

for any $Y \in \mathbb{D}$. If we interchange Y and Z we also obtain

$$kA\phi Y - A^2\phi Y - k\phi AY + A\phi AY = 0 \tag{4.8}$$

for any $Y \in \mathbb{D}$. If such a Y satisfies $AY = \lambda Y$, from (4.7) and Theorem 2.1 we obtain

$$(\lambda - \mu)(\lambda - k) = 0 \tag{4.9}$$

where $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. From (4.8) we also get

$$(k - \mu)(\mu - \lambda) = 0. \tag{4.10}$$

From (4.9) and (4.10) either $\lambda = \mu$ for any principal curvature in $\mathbb{D}$, and in this case, from Theorem 2.2, M is locally congruent to a real hypersurface of type (A) or there exists λ such that $\mu \neq \lambda$. Then $\lambda = \mu = k$, which is impossible.

Suppose now that M is non Hopf and write $A\xi$ as before. Take $X = Y = \xi$ in (4.6). Then $-\alpha g(\phi A\xi, Z) - kg(\phi A\xi, Z) + g(\phi A^2\xi, Z) + g(\phi A\xi, AZ) - g(\phi A\xi, AZ) = 0$. Therefore, $-(\alpha + k)\beta g(\phi U, Z) + g(\phi A(\alpha\xi + \beta U), Z) = 0$, for any $Z \in \mathbb{D}$. This yields $-k\beta g(\phi U, Z) + \beta g(\phi AU, Z) = 0$, for any $Z \in \mathbb{D}$. Then $\phi AU = k\phi U$, and applying ϕ we get

$$AU = \beta\xi + kU. \tag{4.11}$$

Take now $X = \xi$, $Y = \phi U$ in (4.6). We obtain $-kg(\phi A\phi U, Z) + g(\phi A^2\phi U, Z) - g(A\xi, U)g(A\xi, Z) - kg(AU, Z) - g(\phi A\phi U, AZ) = 0$, for any $Z \in \mathbb{D}$. If we take $Z = U$ we get $kg(A\phi U, \phi U) - g(A^2\phi U, \phi U) - \beta^2 - k^2 + g(A\phi U, \phi AU) = 0$. That is,

$$2kg(A\phi U, \phi U) = g(A\phi U, A\phi U) + \beta^2 + k^2. \tag{4.12}$$

If we take $X = U$, $Y = \phi U$ in (4.6) we have $-g(\phi AU, \phi U)g(A\xi, Z) + g(\phi A\phi U, U)g(A\xi, Z) = 0$, for any $Z \in \mathbb{D}$. From (4.11) it follows $-kg(A\xi, Z) - g(A\phi U, \phi U)g(A\xi, Z) = 0$, for any $Z \in \mathbb{D}$. If $Z = U$ we get $g(A\phi U, \phi U) = -k$, and from (4.12) $g(A\phi U, A\phi U) + \beta^2 + 3k^2 = 0$, which is impossible, finishing the proof of Theorem 1.4.

5 Proofs of Theorems 1.5 and 1.6

If we suppose that $A_F^{(k)}(X, Y) = \omega(X)AY$ for any $X, Y \in \mathbb{D}$ we get

$$g(\phi AX, AY)\xi - \eta(AY)\phi AX - g(\phi AX, Y)A\xi = \omega(X)AY \tag{5.1}$$

for any $X, Y \in \mathbb{D}$.

Let us suppose that M is Hopf and that $A\xi = \alpha\xi$. Then (5.1) becomes

$$g(\phi AX, AY)\xi - \alpha g(\phi AX, Y)\xi = \omega(X)AY \tag{5.2}$$

for any $X, Y \in \mathbb{D}$. The scalar product of (5.2) and ξ gives $g(\phi AX, AY) - \alpha g(\phi AX, Y) = 0$, for any $X, Y \in \mathbb{D}$. Therefore, we have

$$A\phi AX - \alpha\phi AX = 0 \tag{5.3}$$

for any $X \in \mathbb{D}$, and interchanging X and Y we also get

$$-A\phi AX + \alpha A\phi X = 0 \tag{5.4}$$

for any $X \in \mathbb{D}$. From (5.3) and (5.4) it follows $\alpha(\phi A - A\phi)X = 0$ for any $X \in \mathbb{D}$. Let us suppose that $\alpha = 0$. Then, from (5.3) we obtain $A\phi AX = 0$ for any $X \in \mathbb{D}$ and if we suppose that $AX = \lambda X$, from Theorem 2.1, $\lambda(\frac{2}{2\lambda}) = 0$, which is impossible. Therefore, $\phi A - A\phi = 0$, and from Theorem 2.2, M must be locally congruent to a real hypersurface of type (A). In this case (5.3) gives $A^2\phi X - \alpha A\phi X = 0$ for any $X \in \mathbb{D}$ and also $\phi A^2 X - \alpha\phi AX = 0$. Thus $\mu(\mu - \alpha) = \lambda(\lambda - \alpha) = 0$. We have now that either $\mu = 0$ or $\mu = \alpha$ and, at the same time, either $\lambda = 0$ or $\lambda = \alpha$. These four possibilities give contradictions and M must be non Hopf.

As in previous sections we write $A\xi = \alpha\xi + \beta U$. Then (5.1) looks like

$$g(\phi AX, AY)\xi - \beta g(U, Y)\phi AX - g(\phi AX, Y)A\xi = \omega(X)AY \tag{5.5}$$

for any $X, Y \in \mathbb{D}$. Taking $Y = U$ in (5.5) we get $g(\phi AX, AU)\xi - \beta\phi AX - g(\phi AX, U)A\xi = \omega(X)AU$. Its scalar product with U yields $-2\beta g(\phi AX, U) = \omega(X)g(AU, U)$ for any $X \in \mathbb{D}$. If, in particular, $X = U$ we obtain

$$\omega(U)g(AU, U) - 2\beta g(AU, \phi U) = 0. \tag{5.6}$$

Taking the scalar product of (5.5) and ϕU we have $-\beta g(U, Y)g(AX, U) = \omega(X)g(AY, \phi U)$, for any $X, Y \in \mathbb{D}$. If $X = Y = U$ it follows

$$\beta g(AU, U) + \omega(U)g(AU, \phi U) = 0. \tag{5.7}$$

The linear system given by (5.6) and (5.7) satisfies $(\omega(U))^2 + 2\beta^2 \neq 0$, and therefore

$$g(AU, U) = g(AU, \phi U) = 0. \tag{5.8}$$

Taking $X \in \mathbb{D}_U$ in (5.5) and its scalar product with ϕU we obtain $-\beta g(U,Y)g(AU,X) = \omega(X)g(AY,\phi U)$ for any $X \in \mathbb{D}_U$, $Y \in \mathbb{D}$. Bearing in mind (5.8), if $Y = U$ and $X \in \mathbb{D}_U$ we get $-\beta g(AU,X) = 0$, for any $X \in \mathbb{D}_U$. As $\beta \neq 0$, it follows

$$g(AU,X) = 0 \tag{5.9}$$

for any $X \in \mathbb{D}_U$. Now (5.8) and (5.9) yield

$$AU = \beta\xi. \tag{5.10}$$

The scalar product of (5.5) and U gives $\beta g(U,Y)g(A\phi U,X) - \beta g(\phi AX,Y) = \omega(X)g(AY,U) = 0$, for any $X, Y \in \mathbb{D}$. Taking $Y = U$ we have $2\beta g(A\phi U,X) = 0$, for any $X \in \mathbb{D}$. Thus

$$A\phi U = 0. \tag{5.11}$$

Take now $X, Y \in \mathbb{D}_U$ in (5.5). Then, $g(\phi AX,AY)\xi - g(\phi AX,Y)A\xi = \omega(X)AY$, and its scalar product with U yields $\beta g(AX,\phi Y) = 0$, for any $X, Y \in \mathbb{D}_U$. Therefore, $AX = 0$, for any $X \in \mathbb{D}_U$. This, (5.10) and (5.11) imply that M is locally congruent to a ruled real hypersurface, finishing the proof of Theorem 1.5.

If now $A_T^{(k)}(X,Y) = \delta(X)AY$, for any $X, Y \in \mathbb{D}$, we obtain

$$\begin{aligned} g(\phi AX,AY)\xi - \eta(AY)\phi AX - g(\phi A^2Y,X)\xi + k\eta(AY)\phi X \\ -g(\phi AX,AY)A\xi + g(\phi AY,AX)A\xi = \delta(X)AY \end{aligned} \tag{5.12}$$

for any $X, Y \in \mathbb{D}$.

If we suppose that M is Hopf, $A\xi = \alpha\xi$, and take the scalar product of (5.12) and ξ, we get

$$g(\phi AX,AY) - g(\phi A^2Y,X) - \alpha g(\phi AX,Y) + \alpha g(\phi AY,X) = 0 \tag{5.13}$$

for any $X, Y \in \mathbb{D}$. Then (5.13) yields

$$A\phi AX + A^2\phi X - \alpha\phi AX - \alpha A\phi X = 0 \tag{5.14}$$

for any $X \in \mathbb{D}$ and, interchanging X and Y,

$$-A\phi AX - \phi A^2X + \alpha A\phi X + \alpha\phi AX = 0 \tag{5.15}$$

for any $X \in \mathbb{D}$. From (5.14) and (5.15) we have $A^2\phi X - \phi A^2X = 0$, for any $X \in \mathbb{D}$. If we suppose that $AX = \lambda X$, from Theorem 2.1, $A\phi X = \mu\phi X$ and $\mu^2 = \lambda^2$. If $-\lambda = \mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, it yields $\alpha\lambda - 2\lambda^2 = \alpha\lambda + 2$. Therefore, $\lambda^2 + 1 = 0$, which is impossible. Therefore $\lambda = \mu$ and $\phi A = A\phi$. In this case (5.14) becomes $2A^2\phi X - 2\alpha A\phi X = 0$ and then $\mu(\mu - \alpha) = 0$. In the same way, (5.15) implies $-2\phi A^2X + 2\alpha\phi AX = 0$

and $\lambda(\alpha - \lambda) = 0$. The four possibilities that we obtain imply contradictions and M must be non Hopf. Write as usual $A\xi = \alpha\xi + \beta U$.

The scalar product of (5.12) and ϕU gives

$$- \eta(AY)g(AX, U) + k\eta(AY)g(X, U) = \delta(X)g(AY, \phi U) \tag{5.16}$$

for any $X, Y \in \mathbb{D}$. Taking $X = \phi U$ in (5.16) and $Y = U$ we get $-\beta g(A\phi U, U) = \delta(\phi U)g(AU, \phi U)$. Thus

$$(\delta(\phi U) + \beta)g(AU, \phi U) = 0. \tag{5.17}$$

If we put $Y = \phi U$ in (5.16) we obtain

$$\delta(X)g(A\phi U, \phi U) = 0 \tag{5.18}$$

for any $X \in \mathbb{D}$.

Take the scalar product of (5.12) and U. Then it follows

$$\begin{aligned} &-\eta(AY)g(\phi AX, U) + k\eta(AY)g(\phi X, U) - \beta g(\phi AX, Y) + \beta g(\phi AY, X) \\ &\quad = \delta(X)g(AY, U) \end{aligned} \tag{5.19}$$

for any $X, Y \in \mathbb{D}$. Taking $Y = \phi U$ in (5.19) we obtain $-\beta g(AX, U) + \beta g(\phi A\phi U, X) = \delta(X)g(A\phi U, U)$. If $X = \phi U$ it follows $-\beta g(A\phi U, U) + \beta g(A\phi U, U) = \delta(\phi U)g(A\phi U, U)$. That is,

$$\delta(\phi U)g(A\phi U, U) = 0. \tag{5.20}$$

Suppose that $\delta(\phi U) = -\beta$. Then, from (5.18), $g(A\phi U, \phi U) = 0$ and from (5.20), $g(A\phi U, U) = 0$. If $\delta(\phi U) \neq -\beta$, from (5.17), $g(AU, \phi U) = 0$. Thus we have proved that always

$$g(AU, \phi U) = 0. \tag{5.21}$$

If we take $X = Y = U$ in (5.19), bearing in mind (5.21), we obtain

$$\delta(U)g(AU, U) = 0. \tag{5.22}$$

If now we take $Y = U$ in (5.16) we get $-\beta g(AU, X) + k\beta g(U, X) = \delta(X)g(AU, \phi U) = 0$, for any $X \in \mathbb{D}$. This yields

$$AU = \beta\xi + kU, \tag{5.23}$$

and from (5.22) we also have $\delta(U) = 0$.

If we take $X = U$ in (5.19) we obtain $-\beta g(\phi AU, Y) + \beta g(\phi AY, U) = 0$ for any $Y \in \mathbb{D}$. This yields $A\phi U = -\phi AU$, and bearing in mind (5.23), we arrive at

$$A\phi U = -k\phi U. \tag{5.24}$$

From (5.23) and (5.24) we know that $\mathbb{D}_U$ is A-invariant. Taking $Y = U$, $X \in \mathbb{D}_U$ in (5.12) we have $-\beta\phi AX + k\beta\phi X = \delta(X)AU$, for any $X \in \mathbb{D}_U$. If we take its scalar product with ξ we get $\beta\delta(X) = 0$, for any $X \in \mathbb{D}_U$. Thus $\delta(X) = 0$ for such an X, and the above equation implies $\phi AX = k\phi X$ for any $X \in \mathbb{D}_U$. If we apply ϕ we obtain $AX = kX$ for any $X \in \mathbb{D}_U$. For such a vector field $AX = kX$, $A\phi X = k\phi X$. Codazzi equation implies $(\nabla_X A)\phi X - (\nabla_{\phi X} A)X = -2\xi$. Therefore, $k\nabla_X\phi X - A\nabla_X\phi X - k\nabla_{\phi X}X + A\nabla_{\phi X}X = -2\xi$. Its scalar product with U gives $-kg([\phi X, X], U) - g(\nabla_X\phi X, \beta\xi + kU) + g(\nabla_{\phi X}X, \beta\xi + kU) = 0$. This yields $\beta g(\phi X, \phi AX) - \beta g(X, \phi A\phi X) = 0$. Thus $2k\beta = 0$, which is impossible and finishes the proof of Theorem 1.6.

Suppose finally that M satisfies $A_F^{(k)}(X, Y) = \omega(X)AY$ for any X, Y tangent to M, From Theorem 1.5 M must be locally congruent to either a real hypersurface of type (A) or to a ruled real hypersurface. Moreover, M must satisfy

$$g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX \\ + k\eta(X)A\phi Y = \omega(X)AY \tag{5.25}$$

for any X, Y tangent to M. Suppose that M is a real hypersurface of type (A) and take $X = \xi$ in (5.25). We get

$$-k\phi AY + kA\phi Y = \omega(\xi)AY \tag{5.26}$$

for any Y tangent to M. As our real hypersurface satisfies $A\phi = \phi A$, from (5.26) we have $\omega(\xi)AY = 0$ for any Y tangent to M. If $\omega(\xi) \neq 0$ we should have $AY = 0$ for any Y tangent to M. That is, M is totally geodesic, which is impossible. Therefore $\omega(\xi) = 0$.

Take then $Y = \xi$ in (5.25). We obtain

$$-\alpha\phi AX + A\phi AX = \alpha\omega(X)\xi \tag{5.27}$$

for any X tangent to M. Consider $X \in \mathbb{D}$ and take the scalar product of (5.27) and $Z \in \mathbb{D}$. This gives $-\alpha g(\phi AX, Z) + g(A\phi AX, Z) = 0$, for any $X, Z \in \mathbb{D}$. Therefore $-\alpha\phi AX + A\phi AX = 0$. As $A\phi = \phi A$ we have $-\alpha\phi AX + \phi A^2X = 0$ for any $X \in \mathbb{D}$ Suppose that $AX = \lambda X$. Then $-\alpha\lambda + \lambda^2 = 0$, and the unique principal curvatures in $\mathbb{D}$ are α and 0. Thus M has, exactly, two distinct constant principal curvatures and looking at Takagi's list M must be locally congruent to a geodesic hypersphere. But a geodesic hypersphere has not such principal curvatures.

If M is ruled and we take $Y = \phi U$ in (5.25) it follows $-g(\phi AX, \phi U)A\xi - k\eta(X)AU = \omega(X)A\phi U = 0$. Then $-g(AX, U)A\xi - k\eta(X)AU = 0$, for any X tangent to M. Its scalar product with U implies $-\beta g(AX, U) = 0$ for any X tangent to M. If $X = \xi$ we get $\beta^2 = 0$, which is impossible. Thus we have obtained the following

Corollary 5.1 *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that $A_F^{(k)}(X, Y) = \omega(X)AY$, for a certain nonnull 1-form ω on M, any X, Y tangent to M and a nonnull real number k.*

Similarly, we have

Corollary 5.2 *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that $A_T^{(k)}(X, Y) = \delta(X)AY$, for a certain nonnull 1-form δ on M, any X, Y tangent to M and a nonnull real number k.*

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