Sharp Bounds of the Hermitian Toeplitz Determinants for Certain Close-to-Star Functions

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Abstract

The sharp lower and upper bounds of the Hermitian Toeplitz determinants of the second and third order for certain close-to-star functions are computed.

Keywords Hermitian Toeplitz determinant · Univalent function · Close-to-star function · Close-to-convex function · Carathéodory class

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1 Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. Given $r > 0$, let $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$ and $\mathbb{T} := \mathbb{T}_1$. Let $\mathcal{H}$ denote the class of all analytic functions in $\mathbb{D}$ and $\mathcal{A}$ be its subclass of all functions f normalized by $f(0) = 0$ and $f'(0) = 1$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \quad z \in \mathbb{D}. \quad (1.1)$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions.

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Given $q, n \in \mathbb{N}$, define the matrix $T_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1.1) by

$$T_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \bar{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n+q-1} & \bar{a}_{n+q-2} & \dots & a_n \end{bmatrix},$$

where $\bar{a}_k := \overline{a_k}$. When a_n is a real number, $T_{q,n}(f)$ is the Hermitian Toeplitz matrix. In particular, any matrix $T_{q,1}(f)$ so is.

In recent years, many authors studied the estimation of determinants whose entries are coefficients of functions in the class $\mathcal{A}$ or its subclasses. Hankel matrices, i.e., square matrices which have constant entries along the reverse diagonal and the generalized Zalcman functional $J_{m,n}(f) := a_{m+n-1} - a_m a_n$, $m, n \in \mathbb{N}$, are of particular interest (see, e.g., [5–7, 12, 14, 16–19, 23]. The determinants of symmetric Toeplitz matrices, the study of which was initiated in [1], are another example of such interest.

In [8, 10, 13], the study to estimate the determinants $T_{q,n}(f)$ whose entries are coefficients of functions in subclasses of $\mathcal{A}$ was initiated. As it is well known, Hermitian Toeplitz matrices play an important role in functional analysis, applied mathematics and in technical sciences. Further results in this direction were obtained in [15].

In this paper, we continue this research by computing the sharp upper and lower bounds of determinants $T_{2,1}(f)$ and $T_{3,1}(f)$ over subclasses of close-to-star functions.

It can be observed that for each $\theta \in \mathbb{R}$, $\det T_{q,1}(f) = \det T_{q,1}(f_\theta)$, where $f_\theta(z) := e^{-i\theta} f(e^{i\theta} z)$, $z \in \mathbb{D}$, i.e., $\det T_{q,1}(f)$ is rotation invariant.

Theorem 1.1 [8] *Let $\mathcal{F}$ be a subclass of $\mathcal{A}$ such that $\{f \in \mathcal{F} : a_2 = 0\} \neq \emptyset$ and $A_2(\mathcal{F}) := \max\{|a_2| : f \in \mathcal{F}\}$ exists. Then,*

$$1 - A_2^2(\mathcal{F}) \leq \det T_{2,1}(f) \leq 1.$$

Both inequalities are sharp.

Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ of starlike functions, i.e., $f \in \mathcal{S}^*$ if $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is called close-to-star if there exist $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$ such that

$$\operatorname{Re} \frac{e^{i\beta} f(z)}{g(z)} > 0, \quad z \in \mathbb{D}. \quad (1.2)$$

Recall that the class $\mathcal{CST}$ of all close-to-star functions introduced by Reade [24] bear the same relation to the class of close-to-convex functions as the class of starlike functions bear to the class of convex functions ([24, p. 61]). This relationship is called

Alexander type. Namely, $f \in CST$ if and only if a function

$$F(z) := \int_0^z \frac{f(t)}{t} dt, \quad z \in \mathbb{D}$$

is close-to-convex ([11], [9, Vol. II, p. 3]). The class of close-to-convex functions was introduced by Kaplan [11], where the following geometrical interpretation was shown: $f \in \mathcal{A}$ is close-to-convex if and only if there are no sections of the curve $f(\mathbb{T}_r)$, for every $r \in (0, 1)$, in which the tangent vector turns backward through an angle not less than π (cf. [9, Vol. II, p. 4]). An analogous geometrical interpretation for close-to-star functions was shown by Reade: $f \in \mathcal{A}$ is close-to-star if and only if there are no sections of the curve $f(\mathbb{T}_r)$, for every $r \in (0, 1)$, in which the radius vector to the curve $f(\mathbb{T}_r)$ turns backward through an angle not less than π ([24, p. 61]). Recall also that Lewandowski [20] proved that the class of close-to-convex functions is identical to the class of linearly accessible functions introduced by Biernacki [2].

The class of close-to-star functions and their subclasses were studied by various authors (e.g., MacGregor [21], Sakaguchi [26], Causey and Merkes [4]; for further references, see [9, Vol. II, pp. 97–104]).

Given $g \in S^*$ and $\beta \in \mathbb{R}$, let $CST_\beta(g)$ are the subclass of CST of all f satisfying (1.2). The four classes $CST_0(g_i)$, $i = 1, \dots, 4$ where

$$g_1(z) := \frac{z}{1-z^2}, \quad g_2(z) := \frac{z}{(1-z)^2}, \quad g_3(z) := \frac{z}{1+z+z^2}$$

and

$$g_4(z) := \frac{z}{1-z}, \quad z \in \mathbb{D}$$

are particularly interesting and were separately studied by various authors. In [10], the sharp bounds of the second- and third-order Hermitian Toeplitz determinants were computed for the classes $CST_0(g_1)$ and $CST_0(g_2)$. In this paper, we will estimate the second- and third-order Hermitian Toeplitz determinants for the other two classes, i.e., for $CST_0(g_3) =: \mathcal{F}_1$ and $CST_0(g_4) =: \mathcal{F}_2$, in which elements f in view of (1.2) satisfy the condition

$$\operatorname{Re} \left\{ (1+z+z^2) \frac{f(z)}{z} \right\} > 0, \quad z \in \mathbb{D}, \quad (1.3)$$

and

$$\operatorname{Re} \left\{ (1-z) \frac{f(z)}{z} \right\} > 0, \quad z \in \mathbb{D}, \quad (1.4)$$

respectively. Let us add that every function f in $\mathcal{F}_1$ analytic in the closed disk $\overline{\mathbb{D}}$ is starlike in one direction as defined by Robertson [25] (see also [9, Vol. I, pp. 207–208])

and [9, Vol. I, p. 210, Theorem 27] with $\mu = 0$ and $\nu = 2\pi/3$), i.e., intersection of $f(\mathbb{D})$ with the real axis is a line segment covered by f univalently. Note that

$$\det T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \bar{a}_2 & 1 & a_2 \\ \bar{a}_3 & \bar{a}_2 & 1 \end{vmatrix} = 2 \operatorname{Re} \left(a_2^2 \bar{a}_3 \right) - 2|a_2|^2 - |a_3|^2 + 1. \quad (1.5)$$

Let $\mathcal{P}$ be the class of all $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \quad (1.6)$$

having a positive real part in $\mathbb{D}$.

In the proof of the main result, we will use the following lemma, which contains the well-known formula for c_2 ([3], [22, p. 166]) and further remarks in [7]). In fact, the formulas below follow also from the Carathéodory–Toeplitz theorem.

Lemma 1.2 *If $p \in \mathcal{P}$ is of the form (1.6), then*

$$|c_n| \leq 2, \quad n \in \mathbb{N}. \quad (1.7)$$

Moreover,

$$c_1 = 2\zeta_1 \quad (1.8)$$

and

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \quad (1.9)$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, 2\}$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (1.8), namely,

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (1.8) and (1.9), namely,

$$p(z) = \frac{1 + (\bar{\zeta}_1 \zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\bar{\zeta}_1 \zeta_2 - \zeta_1)z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \quad (1.10)$$

2 The Class $\mathcal{F}_1$

Let $f \in \mathcal{F}_1$ be the form (1.1). Then by (1.3), there exists $p \in \mathcal{P}$ of the form (1.6) such that

$$(1 + z + z^2) \frac{f(z)}{z} = p(z), \quad z \in \mathbb{D}. \quad (2.1)$$

Substituting the series (1.1) and (1.6) into (2.1) by equating the coefficients, we get

$$a_2 = -1 + c_1 \quad \text{and} \quad a_3 = -c_1 + c_2. \quad (2.2)$$

By (1.7), it follows that $A_2(\mathcal{F}_1) = 3$ for the function $f \in \mathcal{A}$ defined by

$$f(z) = \frac{z(1 - z)}{(1 + z + z^2)(1 + z)}, \quad z \in \mathbb{D}, \quad (2.3)$$

which is extremal. Observe also that $a_2 = 0$ for the function $f \in \mathcal{F}_1$ defined by

$$f(z) = \frac{z}{(1 + z + z^2)(1 - z)}, \quad z \in \mathbb{D}.$$

Therefore by Theorem 1.1, we have

Theorem 2.1 *If $f \in \mathcal{F}_1$, then*

$$-8 \leq \det T_{2,1}(f) \leq 1.$$

Both inequalities are sharp.

Now, we will compute the bounds of $\det T_{3,1}(f)$ in the class $\mathcal{F}_1$.

Theorem 2.2 *If $f \in \mathcal{F}_1$, then*

$$\det T_{3,1}(f) \leq 39. \quad (2.4)$$

The inequality is sharp.

Proof By (2.2) and (1.7), we see that $|a_2| \leq 3$ and $|a_3| \leq 4$. Since $\operatorname{Re}(a_2^2 \bar{a}_3) \leq |a_2|^2 |a_3|$ from (1.5) we get

$$\det T_{3,1}(f) \leq F(|a_2|, |a_3|), \quad (2.5)$$

where

$$F(x, y) := 2x^2y - 2x^2 - y^2 + 1, \quad (x, y) \in [0, 3] \times [0, 4].$$

Observe that the point $(1, 1)$ is the unique solution $(0, 3) \times (0, 4)$ of the system of equations

$$\begin{aligned}\frac{\partial F}{\partial x} &= 4x(y - 1) = 0, \\ \frac{\partial F}{\partial y} &= 2(x^2 - y) = 0.\end{aligned}$$

However,

$$\frac{\partial^2 F}{\partial x^2}(1, 1) \frac{\partial^2 F}{\partial y^2}(1, 1) - \left(\frac{\partial F}{\partial x \partial y}(1, 1) \right)^2 = -16 < 0,$$

so $(1, 1)$ is a saddle point of F .

We now consider F on the boundary of $[0, 3] \times [0, 4]$.

(1) On the side $x = 0$,

$$F(0, y) = 1 - y^2 \leq 1, \quad 0 \leq y \leq 4.$$

(2) On the side $x = 3$,

$$F(3, y) = -17 + 18y - y^2 \leq F(3, 4) = 39, \quad 0 \leq y \leq 4.$$

(3) On the side $y = 0$,

$$F(x, 0) = 1 - 2x^2 \leq 1, \quad 0 \leq x \leq 3.$$

(4) On the side $y = 4$

$$F(x, 4) = -15 + 6x^2 \leq 39, \quad 0 \leq x \leq 3.$$

Therefore, the inequality $F(x, y) \leq 39$ holds for all $(x, y) \in [0, 3] \times [0, 4]$, which in view of (2.5) shows (2.4).

For the function (2.3), $a_2 = -3$ and $a_3 = 4$, which makes equality in (2.4). $\square$

Theorem 2.3 *If $f \in \mathcal{F}_1$, then*

$$\begin{aligned}\det T_{3,1}(f) &\geq \frac{1}{8} \left[\left((6\sqrt{2} - 15) \left(4 + 2\sqrt{2} \right)^{\frac{2}{3}} + \left(3\sqrt{2} - 18 \right) \left(4 + 2\sqrt{2} \right)^{\frac{1}{3}} - 36 \right) \right. \\ &\quad \left. = -10.693535 \dots \right] \quad (2.6)\end{aligned}$$

The inequality is sharp.

Proof Substituting (1.8) and (1.9) into (2.2), we get

$$a_2 = -1 + 2\zeta_1, \quad a_3 = -2\zeta_1 + 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2$$

with $\zeta_i \in \overline{\mathbb{D}}$, $i = 1, 2$. Therefore from (1.5), we get

$$\det T_{3,1}(f) = \Psi_1 + \Psi_2, \quad (2.7)$$

where

$$\Psi_1 := -1 + 4|\zeta_1|^2 + 12|\zeta_1|^4 - 4(1 - |\zeta_1|^2)^2|\zeta_2|^2 \quad (2.8)$$

and

$$\begin{aligned} \Psi_2 := & 4 \operatorname{Re} \zeta_1 + 4 \operatorname{Re}(\zeta_1^2) - 24|\zeta_1|^2 \operatorname{Re} \zeta_1 - 8(1 - |\zeta_1|^2) \operatorname{Re}(\zeta_1 \bar{\zeta}_2) \\ & + 4(1 - |\zeta_1|^2) \operatorname{Re} \zeta_2 + 8(1 - |\zeta_1|^2) \operatorname{Re}(\zeta_1^2 \bar{\zeta}_2). \end{aligned}$$

A. Let $\zeta_1 \zeta_2 \neq 0$. Thus $\zeta_1 = re^{i\theta}$ and $\zeta_2 = se^{i\psi}$ with $r, s \in (0, 1]$ and $\theta, \psi \in [0, 2\pi)$. Then,

$$\Psi_2 = \Psi_3 + \Psi_4, \quad (2.9)$$

where

$$\begin{aligned} \Psi_3 := & 4r \cos \theta + 4r^2 \cos 2\theta - 24r^3 \cos \theta \\ = & -4r^2 + 4r \cos \theta - 24r^3 \cos \theta + 8r^2 \cos^2 \theta \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} \Psi_4 := & -8rs(1 - r^2) \cos(\theta - \psi) + 4s(1 - r^2) \cos \psi + 8r^2s(1 - r^2) \cos(2\theta - \psi) \\ = & 4s(1 - r^2) \sqrt{\kappa_1^2 + \kappa_2^2} \sin(\psi + \alpha), \end{aligned} \quad (2.11)$$

where $\alpha \in \mathbb{R}$ is the quantity satisfying

$$\cos \alpha = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}}, \quad \sin \alpha = \frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}} \quad (2.12)$$

with

$$\kappa_1 := -2r \sin \theta + 2r^2 \sin 2\theta, \quad \kappa_2 := 1 - 2r \cos \theta + 2r^2 \cos 2\theta. \quad (2.13)$$

Since $\sin(\psi + \alpha) \geq -1$ and $s \leq 1$, we have

$$\begin{aligned} \Psi_4 \geq & -4(1 - r^2) \sqrt{\kappa_1^2 + \kappa_2^2} \\ = & -4(1 - r^2) \sqrt{1 + 4r^4 - 4r \cos \theta - 8r^3 \cos \theta + 8r^2 \cos^2 \theta}. \end{aligned}$$

Therefore, from (2.9) to (2.11), we get

$$\begin{aligned} \Psi_2 = \Psi_3 + \Psi_4 \geq & 4r \cos \theta + 4r^2 \cos 2\theta - 24r^3 \cos \theta \\ & - 4(1 - r^2)\sqrt{1 + 4r^4 - 4r \cos \theta - 8r^3 \cos \theta + 8r^2 \cos^2 \theta}. \end{aligned} \quad (2.14)$$

Taking into account that $|\zeta_2| \leq 1$ from (2.8), we have

$$\Psi_1 \geq -5 + 12r^2 + 8r^4. \quad (2.15)$$

Thus from (2.7), (2.14) and (2.15), it follows that

$$\det T_{3,1}(f) \geq G(r, \cos \theta), \quad r \in (0, 1], \quad \theta \in [0, 2\pi), \quad (2.16)$$

where

$$G(t, x) := g_1(t, x) - 4(1 - t^2)\sqrt{g_2(t, x)}$$

with

$$g_1(t, x) := -5 + 8t^2 + 8t^4 + 4tx - 24t^3x + 8t^2x^2$$

and

$$g_2(t, x) := 1 + 4t^4 - 4tx - 8t^3x + 8t^2x^2 \quad (2.17)$$

for $t \in [0, 1]$ and $x \in [-1, 1]$.

Let $\Omega := [0, 1] \times [-1, 1]$ and

$$\begin{aligned} \Theta &:= \frac{1}{8} \left[(6\sqrt{2} - 15)(4 + 2\sqrt{2})^{\frac{2}{3}} + (3\sqrt{2} - 18)(4 + 2\sqrt{2})^{\frac{1}{3}} - 36 \right] \\ &= -10.693535 \dots \end{aligned}$$

Now, we will show that

$$\min\{G(t, x) : (t, x) \in \Omega\} = \Theta.$$

A1. We now deal with the critical points of G in the interior of Ω , i.e., in $(0, 1) \times (-1, 1)$. Note that $g_2(t, x) \geq 0$. Moreover, $g_2(t, x) = 0$ holds for $t = \sqrt{2}/2$ and $x = \sqrt{2}/2$. **A1.1.** When $t = \sqrt{2}/2$ and $x = \sqrt{2}/2$, from (2.8), (2.10) and (2.11), we have

$$\Psi_1 = 4 - |\zeta_2|^2, \quad \Psi_3 = -4, \quad \Psi_4 = 0,$$

which yields

$$\det T_{3,1}(f) \geq -1.$$

A1.2. Suppose now that $g_2(t, x) > 0$. Differentiating G with respect to x yields

$$\frac{\partial G}{\partial x}(t, x) = \frac{\partial g_1}{\partial x}(t, x) - 2(1 - t^2)(g_2(t, x))^{-1/2} \frac{\partial g_2}{\partial x}(t, x) = 0. \quad (2.18)$$

A1.2.a. Assume first that $\partial g_1/\partial x = 0$. Then from (2.18), it follows that $\partial g_2/\partial x = 0$, which is possible only for $t = \sqrt{2}/2$ and $x = \sqrt{2}/2$. Further argumentation is as in A1.1. **A1.2.b** Assume now that $\partial g_1/\partial x \neq 0$. Then we can write the equation (2.18) as

$$g_2(t, x)^{1/2} = \frac{2(1 - t^2) \frac{\partial g_2}{\partial x}(t, x)}{\frac{\partial g_1}{\partial x}(t, x)} = \frac{2(1 - t^2)(-1 - 2t^2 + 4tx)}{1 - 6t^2 + 4tx}, \quad (2.19)$$

or equivalently by substituting (2.17) as

$$\begin{aligned} \Phi = \Phi(t, x) := & 128t^8 - 416t^7x + 672t^6x^2 - 512t^5x^3 + 128t^4x^4 - 32t^6 \\ & - 112t^5x + 160t^4x^2 + 52t^4 - 8t^3x - 72t^2x^2 \\ & - 20t^2 + 36tx - 3 = 0. \end{aligned} \quad (2.20)$$

Furthermore, note that by (2.19),

$$0 < \frac{(g_2(t, x))^{1/2}}{2(1 - t^2)} = \frac{-1 - 2t^2 + 4tx}{1 - 6t^2 + 4tx} \quad (2.21)$$

and this inequality holds for

$$0 < t < \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{2t^2 + 1}{4t} < x < \frac{6t^2 - 1}{4t}$$

or

$$\frac{\sqrt{2}}{2} < t < 1 \quad \text{and} \quad \left(x < \frac{2t^2 + 1}{4t} \quad \text{or} \quad x > \frac{6t^2 - 1}{4t} \right).$$

Differentiating G with respect to t and using (2.19) yield

$$\begin{aligned} \frac{\partial G}{\partial t}(t, x) &= \frac{\partial g_1}{\partial t}(t, x) + 8t(g_2(t, x))^{1/2} - 2(1 - t^2)(g_2(t, x))^{-1/2} \frac{\partial g_2}{\partial t}(t, x) \\ &= -\frac{64t^2}{(1 - 6t^2 + 4tx)(-1 - 2t^2 + 4tx)} H(t, x), \end{aligned}$$

where for $(t, x) \in (0, 1) \times (-1, 1)$,

$$H(t, x) := (t - x)(-1 - 8t^2 + 4t^4 + 10tx + 4t^3x - 8t^2x^2).$$

Therefore, each critical point of G satisfies

$$t - x = 0 \quad (2.22)$$

or

$$-1 - 8t^2 + 4t^4 + 10tx + 4t^3x - 8t^2x^2 = 0. \quad (2.23)$$

I. Assume that (2.22) holds. Then, $x = x(t) = t$. Thus by (2.20), we see that

$$\Phi(t, t) = (-3 + 4t^2)(-1 + 2t^2)^2 = 0$$

occurs only when $t = \hat{t}_i$, $i = 1, 2$, where $\hat{t}_1 := \sqrt{3}/2$ and $\hat{t}_2 = \sqrt{2}/2$. Thus,

$$x = x(\hat{t}_1) = \hat{x}_1 = \frac{\sqrt{3}}{2}, \quad x = x(\hat{t}_2) = \hat{x}_2 = \frac{\sqrt{2}}{2}.$$

However, it can be seen that

$$\frac{-1 - 2\hat{t}_1^2 + 4\hat{t}_1\hat{x}_1}{1 - 6\hat{t}_1^2 + 4\hat{t}_1\hat{x}_1} = -1 < 0,$$

which means that the inequality (2.21) is not satisfied for $t = \hat{t}_1$ and $x = \hat{x}_1$.

Note that the case $\hat{t}_2 = \sqrt{2}/2$ and $\hat{x}_2 = \sqrt{2}/2$ was considered in A1.1.

Therefore, G does not have critical point in the interior of Ω in the case of (2.22).

II. Suppose that (2.23) is satisfied. The equation (2.23) as a quadratic one of x with $\Delta := 17 - 44t^2 + 36t^4 > 0$, $t \in (0, 1)$, has two roots, namely,

$$x_i = x_i(t) = \frac{5 + 2t^2 + (-1)^{i+1}\sqrt{17 - 44t^2 + 36t^4}}{8t}, \quad i = 1, 2. \quad (2.24)$$

a. Let $x = x_1$. Then, $\Phi(t, x_1(t)) = 0$ is equivalent to the equation

$$\begin{aligned} & (-39 + 154t^2 - 208t^4 + 96t^6)\sqrt{17 - 44t^2 + 36t^4} \\ & = 161 - 844t^2 + 1716t^4 - 1600t^6 + 576t^8. \end{aligned} \quad (2.25)$$

Squaring both sides of (2.25) leads to

$$64\gamma_1^2(t)\gamma_2^4(t) = 0, \quad (2.26)$$

where for $t \in (0, 1)$,

$$\gamma_1(t) := -1 + t^2, \quad \gamma_2(t) := -1 + 2t^2.$$

We see that there is a unique root $t = \sqrt{2}/2$ of the equation (2.26), which does not satisfy (2.25). **b.** Let $x = x_2$. Then $\Phi(t, x_2(t)) = 0$ is equivalent to the equation

$$\begin{aligned} & -(-39 + 154t^2 - 208t^4 + 96t^6)\sqrt{17 - 44t^2 + 36t^4} \\ & = 161 - 844t^2 + 1716t^4 - 1600t^6 + 576t^8. \end{aligned} \quad (2.27)$$

Squaring the both sides of (2.27) yields again the equation (2.26) having a unique root $t = \sqrt{2}/2$, which satisfies (2.27) also. Since by (2.24), $x_2(t_2) = \sqrt{2}/2$, we see that this case was considered in A1.1.

A2. It remains to consider G in the boundary of Ω .

(1) On the side $t = 0$,

$$G(0, x) \equiv -9 > \Theta, \quad x \in [-1, 1].$$

(2) On the side $t = 1$,

$$G(1, x) = 11 - 20x + 8x^2 \geq G(1, 1) = -1 > \Theta, \quad x \in [-1, 1].$$

(3) On the side $x = -1$,

$$G(t, -1) = -9 - 12t + 12t^2 + 32t^3 + 16t^4 =: \varrho_1(t), \quad t \in [0, 1].$$

Since $\varrho_1'(t) = 0$ occurs only when

$$t = t' := \frac{1}{4}\sqrt[3]{4 + 2\sqrt{2}} + \frac{1}{2\sqrt[3]{4 + 2\sqrt{2}}} - \frac{1}{2} = 0.23784 \dots \in [0, 1] \quad (2.28)$$

and $\varrho_1''(t') > 0$, we have

$$\varrho_1(t) \geq \varrho_1(t') = \Theta, \quad t \in [0, 1].$$

(4) On the side $x = 1$, we have

$$G(t, 1) = -9 + 12t + 12t^2 - 32t^3 + 16t^4 =: \varrho_2(t), \quad t \in [0, 1].$$

Since ϱ_2 is increasing,

$$\varrho_2(t) \geq \varrho_2(0) = -9 > \Theta, \quad t \in [0, 1].$$

B. Suppose that $\zeta_1 = 0$. Then,

$$\Psi_1 = -1 - 4|\zeta_2|^2, \quad \Psi_2 = 4 \operatorname{Re} \zeta_2$$

and, therefore,

$$\det T_{3,1}(f) = -1 + 4 \operatorname{Re} \zeta_2 - 4|\zeta_2|^2 \geq -9.$$

C. Suppose that $\zeta_2 = 0$ and $\zeta_1 = re^{i\theta}$, where $r \in (0, 1]$ and $\theta \in [0, 2\pi)$. Then,

$$\Psi_1 = -1 + 4|\zeta_1|^2 + 12|\zeta_1|^4 = -1 + 4r^2 + 12r^4$$

and

$$\Psi_2 = 4 \operatorname{Re} \zeta_1 + 4 \operatorname{Re}(\zeta_1^2) - 24|\zeta_1|^2 \operatorname{Re} \zeta_1 = 4r \cos \theta + 4r^2 \cos 2\theta - 24r^3 \cos \theta.$$

Thus,

$$\det T_{3,1}(f) = G(r, \cos \theta), \quad r \in (0, 1], \quad \theta \in [0, 2\pi),$$

where

$$G(t, x) := -1 + 12t^4 + 4tx - 24t^3x + 8t^2x^2$$

for $t \in (0, 1]$ and $x \in [-1, 1]$. Set

$$x_w := \frac{-1 + 6t^2}{4t}, \quad t \in (0, 1].$$

Note that $-1 < x_w$ holds for $t \in \left((-2 + \sqrt{10})/6, 1\right)$, and $x_w < 1$ holds for $t \in \left(0, (2 + \sqrt{10})/6\right)$. Hence for $t \in \left((-2 + \sqrt{10})/6, (2 + \sqrt{10})/6\right)$, we have

$$G(t, x) \geq G(t, x_w) = -\frac{3}{2}(2t^2 - 1)^2 \geq -\frac{1}{27}(22 + 4\sqrt{10}) = -1.283300\dots$$

When $t < (-2 + \sqrt{10})/6$, then

$$G(t, x) \geq G(t, -1) = -1 - 4t + 8t^2 + 24t^3 + 12t^4 =: \phi_1(t).$$

Since $\phi_1'(t) = 0$ occurs only when $t = (-3 + \sqrt{15})/6 = 0.145497\dots$ and

$$\phi_1''\left((-3 + \sqrt{15})/6\right) > 0,$$

it follows that

$$\phi_1(t) \geq \phi_1\left(\frac{1}{6}(-3 + \sqrt{15})\right) = -\frac{4}{3}, \quad t \in \left(0, \frac{1}{6}(-2 + \sqrt{10})\right).$$

When $t > (\sqrt{10} + 2)/6$, then

$$G(t, x) \geq G(t, 1) = -1 + 4t + 8t^2 - 24t^3 + 12t^4 =: \phi_2(t).$$

Since ϕ_2 is decreasing, we have

$$\phi_2(t) \geq -1, \quad t \in \left(\frac{1}{6} (2 + \sqrt{10}), 1 \right).$$

Summarizing, from Parts A–C, it follows that the inequality (2.6) holds.

It remains to show that the inequality (2.6) is sharp. It is observed from (2.7), (2.14), (2.15) and (2.16) that $\det T_{3,1}(f) = \Theta$ holds when the following conditions are satisfied:

$$r = t', \quad \cos \theta = -1, \quad s = 1, \quad \sin(\psi + \alpha) = -1, \quad (2.29)$$

where t' is given by (2.28), and where α is determined by the condition (2.12) with κ_1 and κ_2 given by (2.13). Then $\theta = \pi$, $\kappa_1 = 0$ and $\kappa_2 = 1/8((4 + 2\sqrt{2})^{(2/3)} + (2 - \sqrt{2})(4 + 2\sqrt{2})^{(1/3)} + 8) = 1.588825 \dots$. Thus, (2.12) is satisfied if we take $\alpha = \pi/2$. Thus if we put $\psi = \pi$, then ψ satisfies the fourth condition in (2.29). Now, let us consider a function $\tilde{p}$ which has the form (1.10) with $\zeta_1 = -t'$ and $\zeta_2 = -1$, i.e.,

$$\tilde{p}(z) = \frac{1 - z^2}{1 + 2t'z + z^2}, \quad z \in \mathbb{D}.$$

Since $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, from Lemma 1.2 it follows that $\tilde{p}$ belongs to $\mathcal{P}$. Therefore, the extremal function f in the class $\mathcal{F}_1$ for which equality in (2.6) holds satisfies (2.1) with $p = \tilde{p}$. $\square$

3 The Class $\mathcal{F}_2$

Let $f \in \mathcal{F}_2$ be the form (1.1). Then by (1.4), there exists $p \in \mathcal{P}$ of the form (1.6) such that

$$(1 - z) \frac{f(z)}{z} = p(z), \quad z \in \mathbb{D}. \quad (3.1)$$

Putting the series (1.1) and (1.6) into (3.1) by equating the coefficients, we get

$$a_2 = 1 + c_1 \quad \text{and} \quad a_3 = 1 + c_1 + c_2. \quad (3.2)$$

By (1.7), it follows that $A_2(\mathcal{F}_2) = 3$ with the extremal function

$$f(z) = \frac{z(1 + z)}{(1 - z)^2}, \quad z \in \mathbb{D}. \quad (3.3)$$

Note that $a_2 = 0$ for the function $f \in \mathcal{F}_2$ given by

$$f(z) = \frac{z}{1-z^2}, \quad z \in \mathbb{D}.$$

Thus by Theorem 1.1, we have

Theorem 3.1 *If $f \in \mathcal{F}_2$, then*

$$-8 \leq \det T_{2,1}(f) \leq 1.$$

Both inequalities are sharp.

Now, we estimate $\det T_{3,1}(f)$ in the class $\mathcal{F}_2$.

Theorem 3.2 *If $f \in \mathcal{F}_2$, then*

$$\det T_{3,1}(f) \leq 48. \quad (3.4)$$

The inequality is sharp.

Proof By (3.2) and (1.7), we see that $|a_2| \leq 3$ and $|a_3| \leq 5$. As in the proof of Theorem 2.2, the inequality (2.5) holds with the function

$$F(x, y) := 2x^2y - 2x^2 - y^2 + 1, \quad (x, y) \in [0, 3] \times [0, 5].$$

Repeating argumentation in the in the proof of Theorem 2.2, we see that the function F does not have any relative maxima in $(0, 3) \times (0, 5)$.

We consider F on the boundary of $[0, 3] \times [0, 5]$.

(1) On the side $x = 0$,

$$F(0, y) = 1 - y^2 \leq 1, \quad y \in [0, 5].$$

(2) On the side $x = 3$,

$$F(3, y) = -17 + 18y - y^2 \leq F(3, 5) = 48, \quad y \in [0, 5].$$

(3) On the side $y = 0$,

$$F(x, 0) = 1 - 2x^2 \leq 1, \quad x \in [0, 3].$$

(4) On the side $y = 5$,

$$F(x, 5) = -24 + 8x^2 \leq F(3, 5) = 48, \quad x \in [0, 3].$$

Therefore, the inequality $F(x, y) \leq 48$ holds for all $(x, y) \in [0, 3] \times [0, 5]$, which in view of (2.5) shows (3.4).

For the function (3.3), $a_2 = 3$ and $a_3 = 5$, which makes the equality in (3.4). $\square$

Theorem 3.3 *If $f \in \mathcal{F}_2$, then*

$$\det T_{3,1}(f) \geq -\frac{4}{7} \left(5 + 4\sqrt{2} \right) = -6.0896309 \dots \quad (3.5)$$

Proof Substituting (1.8) and (1.9) into (3.2) yields

$$a_2 = 1 + 2\zeta_1, \quad a_3 = 1 + 2\zeta_1 + 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2,$$

for some $\zeta_i \in \overline{\mathbb{D}}$ ($i = 1, 2$). Therefore from (1.5), we get

$$\det T_{3,1}(f) = \Psi_1 + \Psi_2, \quad (3.6)$$

where

$$\Psi_1 := 4|\zeta_1|^2 + 12|\zeta_1|^4 - 4(1 - |\zeta_1|^2)^2|\zeta_2|^2$$

and

$$\begin{aligned} \Psi_2 := & 8 \operatorname{Re}(\zeta_1^2) + 24|\zeta_1|^2 \operatorname{Re} \zeta_1 + 8(1 - |\zeta_1|^2) \operatorname{Re}(\zeta_1 \bar{\zeta}_2) \\ & + 8(1 - |\zeta_1|^2) \operatorname{Re}(\zeta_1^2 \bar{\zeta}_2). \end{aligned}$$

A. Suppose that $\zeta_1 \zeta_2 \neq 0$. Thus, $\zeta_1 = re^{i\theta}$ and $\zeta_2 = se^{i\psi}$ with $r, s \in (0, 1]$ and $\theta, \psi \in [0, 2\pi)$. Then

$$\Psi_1 = 4r^2 + 12r^4 - 4(1 - r^2)^2s^2 \geq -4 + 12r^2 + 8r^4, \quad (3.7)$$

and

$$\Psi_2 = 8r^2 \cos 2\theta + 24r^3 \cos \theta + 8(1 - r^2)rs\sqrt{\kappa_1^2 + \kappa_2^2} \sin(\psi + \alpha), \quad (3.8)$$

where α is the quantity satisfying (2.12) with

$$\kappa_1 := \sin \theta + r \sin 2\theta, \quad \kappa_2 := \cos \theta + r \cos 2\theta. \quad (3.9)$$

Since $\sin(\psi + \alpha) \geq -1$ and $s \leq 1$, from (3.8) we have

$$\Psi_2 \geq 8r^2 \cos 2\theta + 24r^3 \cos \theta - 8(1 - r^2)r\sqrt{1 + 2r \cos \theta + r^2}. \quad (3.10)$$

Thus, from (3.6), (3.7) and (3.10), it follows that

$$\det T_{3,1}(f) \geq -4G(r, \cos \theta), \quad r \in (0, 1], \quad \theta \in [0, 2\pi), \quad (3.11)$$

where, for $t \in [0, 1]$ and $x \in [-1, 1]$,

$$G(t, x) := 1 - t^2 - 2t^4 - 4t^2x^2 - 6t^3x + 2(1 - t^2)t\sqrt{g(t, x)}$$

with

$$g(t, x) := 1 + 2tx + t^2. \quad (3.12)$$

Let $\Omega := [0, 1] \times [-1, 1]$ and

$$\Theta := \frac{1}{7} \left(5 + 4\sqrt{2} \right) = 1.522407 \dots$$

Now, we will show that

$$\max\{G(t, x) : (t, x) \in \Omega\} = \Theta.$$

A1. For this, we first we find the critical points of G in the interior of Ω , i.e., in $(0, 1) \times (-1, 1)$. Note first that $g(t, x) > 0$ in $(0, 1) \times (-1, 1)$. From the equation

$$\frac{\partial G}{\partial x}(t, x) = -8t^2x - 6t^3 + 2(1 - t^2)t^2g(t, x)^{-1/2} = 0,$$

it follows that

$$0 < g(t, x)^{-1/2} = \frac{4x + 3t}{1 - t^2}. \quad (3.13)$$

Hence,

$$4x + 3t > 0. \quad (3.14)$$

Under the condition (3.14), the equation (3.13) by substituting (3.12) can be equivalently written as

$$32tx^3 + 16(4t^2 + 1)x^2 + 2(21t^3 + 12t)x + 8t^4 + 11t^2 - 1 = 0. \quad (3.15)$$

Differentiating G with respect to t leads to the equation

$$\begin{aligned} \frac{\partial G}{\partial t}(t, x) &= -2t - 8t^3 - 8tx^2 - 18t^2x + 2(1 - 3t^2)g(t, x)^{1/2} \\ &\quad + 2(1 - t^2)(x + t)tg(t, x)^{-1/2} = 0, \end{aligned}$$

which by using (3.13) is equivalent to

$$8t^2x^2 + (10t^3 + 4t)x + 7t^2 - 1 = 0. \quad (3.16)$$

The above equation is a quadratic one of x with $\Delta = \Delta(t) := 12 - 36t^2 + 25t^4$, and $\Delta(t) \geq 0$ if and only if $t \in (0, t_1] \cup [t_2, 1)$, where

$$t_1 := \frac{1}{5}\sqrt{2(9 - \sqrt{6})} = 0.723906 \dots, \quad t_2 := \frac{1}{5}\sqrt{2(9 + \sqrt{6})} = 0.957057 \dots$$

For $\Delta(t_1) = 0$, the equation (3.16) has a unique root

$$x'_0 := -\frac{2 + 5t_1^2}{8t_1} = \frac{-14 + \sqrt{6}}{4\sqrt{18 - 2\sqrt{6}}} = -0.797790 \dots$$

Analogously, for $\Delta(t_2) = 0$, the equation (3.16) has a unique root

$$x''_0 := -\frac{2 + 5t_2^2}{8t_2} = -\frac{14 + \sqrt{6}}{4\sqrt{18 + 2\sqrt{6}}} = -0.859366 \dots$$

It can be verified that the equation in (3.15) does not hold for $t = t_1$ and $x = x'_0$, and for $t = t_2$ and $x = x''_0$.

Assume now that $t \in (0, t_1) \cup (t_2, 1)$. Thus, there are two roots x_1 and x_2 of (3.16), namely

$$x_j = x_j(t) := \frac{-2 - 5t^2 + (-1)^j \sqrt{12 - 36t^2 + 25t^4}}{8t}, \quad j = 1, 2.$$

I. Consider the case $x = x_1$. Note that $x_1 > -1$ is equivalent to

$$-2 + 8t - 5t^2 > \sqrt{12 - 36t^2 + 25t^4}. \quad (3.17)$$

Observe that $-2 + 8t - 5t^2 > 0$ if and only if $t \in (t_3, 1)$, where

$$t_3 := \frac{1}{5} (4 - \sqrt{6}) = 0.310102 \dots \quad (3.18)$$

Thus for $t \in (t_3, t_1) \cup (t_2, 1)$ by squaring the both sides of (3.17), we get the inequality

$$80t^3 - 120t^2 + 32t + 8 < 0,$$

which holds if and only if $t \in (t_4, t_1) \cup (t_2, 1) =: J_1$, where $t_4 := (5 + \sqrt{65})/20 = 0.653112 \dots$. Moreover, $x_1 < 1$ is equivalent to the inequality

$$-2 - 8t - 5t^2 < \sqrt{12 - 36t^2 + 25t^4},$$

which is true for $t \in J_1$.

Substituting $x = x_1$ into the equation (3.15), we get

$$Q_1(t)\sqrt{12 - 36t^2 + 25t^4} = Q_2(t), \quad (3.19)$$

where for $t \in J_1$,

$$Q_1(t) := 1 - 4t^2 + 3t^4, \quad Q_2(t) := -2 + 7t^2 - 6t^4 + t^6.$$

Since $Q_1(t) < 0$ for $t \in J_1$, and $Q_2(t) < 0$ for $t \in (t_4, t_5)$, where $t_5 := \sqrt{10 - 2\sqrt{17}}/2 = 0.662153\dots$, by squaring both sides of (3.19) we equivalently get the equation

$$8(1 - 2t^2)(1 - 8t^2 + 14t^4)(1 - t)^3(1 + t)^3 = 0, \quad (3.20)$$

which has no solution for $t \in (t_4, t_5)$.

II. Consider now the case $x = x_2$. Note that $x_2 > -1$ is equivalent to

$$2 - 8t + 5t^2 < \sqrt{12 - 36t^2 + 25t^4}. \quad (3.21)$$

Since $2 - 8t + 5t^2 < 0$ for $t \in (t_3, 1)$, where t_3 is given by (3.18), consider $t \in (0, t_3]$. By squaring both sides of (3.21), we get the inequality

$$80t^3 - 120t^2 + 32t + 8 \geq 0,$$

which is true for $t \in (0, t_3]$. Thus, $x_2 > -1$ holds for all $t \in (0, t_1) \cup (t_2, 1)$. Moreover, $x_2 < 1$ is equivalent to the inequality

$$2 + 8t + 5t^2 > \sqrt{12 - 36t^2 + 25t^4}.$$

By squaring both sides, we equivalently get the inequality

$$80t^3 + 120t^2 + 32t - 8 > 0,$$

which is true for $t \in (t_6, t_1) \cup (t_2, 1) =: J_2$, where $t_6 := (-5 + \sqrt{65})/20 = 0.153113\dots$. Thus, we restrict our consideration for $t \in J_2$.

Substituting $x = x_2$ into the equation (3.15) yields the equation (3.20), which has three roots in J_2 , namely,

$$\begin{aligned} \tilde{t}_1 &= \frac{\sqrt{2}}{2} = 0.707106\dots, & \tilde{t}_2 &= \frac{1}{14}\sqrt{56 + 14\sqrt{2}} = 0.621875\dots, \\ \tilde{t}_3 &= \frac{1}{14}\sqrt{56 - 14\sqrt{2}} = 0.429766\dots \end{aligned} \quad (3.22)$$

Since $x_2(\tilde{t}_1) = \tilde{x}_1 = -\sqrt{2}/2$, it follows that $4\tilde{x}_1 + 3\tilde{t}_1 < 0$ which contradicts (3.14). Therefore, $(\tilde{t}_1, \tilde{x}_1)$ is not a critical point of G .

Since

$$x_2(\tilde{t}_2) = \tilde{x}_2 = \frac{7(\sqrt{2} - 4)}{4\sqrt{56 + 14\sqrt{2}}} = -0.5197553794\dots,$$

it follows that

$$4\tilde{x}_2 + 3\tilde{t}_2 = \frac{-16 + 10\sqrt{2}}{\sqrt{56 + 14\sqrt{2}}} = -0.213394046\dots < 0,$$

which contradicts (3.14). Therefore, $(\tilde{t}_2, \tilde{x}_2)$ is not a critical point of G .

Further,

$$x_2(\tilde{t}_3) = \tilde{x}_3 = \frac{-5 + 3\sqrt{2}}{\sqrt{56 - 14\sqrt{2}}} = -0.1258756 \dots \quad (3.23)$$

Since

$$4\tilde{x}_3 + 3\tilde{t}_3 = \frac{-8 + 9\sqrt{2}}{\sqrt{56 - 14\sqrt{2}}} = 0.7857962 \dots > 0,$$

then $(\tilde{t}_3, \tilde{x}_3)$ satisfies (3.14), and therefore it is a unique critical point of G . Denote

$$\begin{aligned} \lambda_1 &:= \frac{\partial^2 G}{\partial t^2}(\tilde{t}_3, \tilde{x}_3) = \frac{1}{14} (8 - 93\sqrt{2}) = -8.822990 \dots, \\ \lambda_2 &:= \frac{\partial^2 G}{\partial t \partial x}(\tilde{t}_3, \tilde{x}_3) = \frac{1}{7} (-43 + 23\sqrt{2}), \\ \lambda_3 &:= \frac{\partial^2 G}{\partial x^2}(\tilde{t}_3, \tilde{x}_3) = \frac{1}{7} (44 - 39\sqrt{2}). \end{aligned}$$

Since $\lambda_1 < 0$ and

$$\lambda_1 \lambda_3 - \lambda_2^2 = \frac{1}{7} (128 - 32\sqrt{2}) = 11.8207380 \dots > 0,$$

the function G has a local maximum at $(\tilde{t}_3, \tilde{x}_3)$.

A2. It remains to consider G in the boundary of Ω .

(1) On the side $t = 0$,

$$G(0, x) \equiv 1 < \Theta, \quad x \in [-1, 1].$$

(2) On the side $t = 1$,

$$G(1, x) = -2 - 6x - 4x^2 \leq G\left(1, -\frac{3}{4}\right) = \frac{1}{4} < \Theta, \quad x \in [-1, 1].$$

(3) On the side $x = -1$,

$$G(t, -1) = 1 + 2t - 7t^2 + 4t^3 =: \varrho_1(t), \quad t \in [0, 1].$$

Since $\varrho_1'(t) = 0$ if and only if $t = 1/6$ or $t = 1$, we see that

$$\varrho_1(t) \leq \varrho_1\left(\frac{1}{6}\right) = \frac{125}{108} < \Theta, \quad t \in [0, 1].$$

(4) On the side $x = 1$, we have

$$G(t, 1) = 1 + 2t - 3t^2 - 8t^3 - 4t^4 =: \varrho_2(t), \quad t \in [0, 1].$$

Since $\varrho_2'(t) = 0$ occurs only when $t = (\sqrt{3} - 1)/4 \in (0, 1)$ and $\varrho_2''((\sqrt{3} - 1)/4) = -6(1 + \sqrt{3}) < 0$, it follows that

$$\varrho_2(t) \leq \varrho_2\left(\frac{1}{4}(\sqrt{3} - 1)\right) = \frac{3}{16}(3 + 2\sqrt{3}) = 1.2120 \dots < \Theta, \quad t \in [0, 1].$$

B. Suppose that $\zeta_1 = 0$. Then,

$$\det T_{3,1}(f) = -4|\zeta_2|^2 \geq -4.$$

C. Suppose that $\zeta_2 = 0$ and $\zeta_1 = re^{i\theta}$, where $r \in (0, 1]$ and $\theta \in [0, 2\pi)$. Then,

$$\Psi_1 = 4|\zeta_1|^2 + 12|\zeta_1|^4 = 4r^2 + 12r^4,$$

and

$$\Psi_2 = 8 \operatorname{Re}(\zeta_1^2) + 24|\zeta_1|^2 \operatorname{Re} \zeta_1 = 8r^2 \cos 2\theta + 24r^3 \cos \theta.$$

Thus,

$$\det T_{3,1}(f) = G(r, \cos \theta), \quad r \in (0, 1], \quad \theta \in [0, 2\pi),$$

where

$$G(t, x) = -4t^2 + 12t^4 + 24t^3x + 16t^2x^2$$

for $t \in (0, 1]$ and $x \in [-1, 1]$. Set

$$x_w := -\frac{3}{4}t, \quad t \in (0, 1).$$

Since $-1 < x_w < 0$,

$$G(t, x) \geq G(t, x_w) = -4t^2 + 3t^4 := \phi(t).$$

By the fact that $\phi'(t) = 0$ occurs when $t = \sqrt{6}/3 = 0.816496 \dots$ and $\phi''(\sqrt{6}/3) = 16 > 0$, we see that

$$\phi(t) \geq \phi\left(\frac{\sqrt{6}}{3}\right) = -\frac{4}{3}, \quad t \in (0, 1).$$

Summarizing, from Parts A–C, it follows that the inequality (3.5) holds.

Now, we discuss the sharpness of (3.5). It is observed from (3.7), (3.10) and (3.11) that $\det T_{3,1}(f) = -4\Theta$ holds when the following conditions are satisfied:

$$r = \tilde{t}_3, \quad \cos \theta = \tilde{x}_3, \quad s = 1, \quad \sin(\psi + \alpha) = -1, \quad (3.24)$$

where $\tilde{t}_3$ and $\tilde{x}_3$ are given by (3.22) and (3.23), respectively, and α is determined by the condition (2.12) with κ_1 and κ_2 given by (3.9). Set $\theta = \operatorname{Arccos}(\tilde{x}_3)$ so that it satisfies the second condition in (3.24). Then, $\kappa_1 = 0.884712 \dots > 0$ and $\kappa_2 = -0.542022 \dots < 0$. Thus, (2.12) is satisfied if we take

$$\alpha = -\operatorname{Arccos} \left(\frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}} \right) = -0.549672 \dots.$$

Thus if we put

$$\psi = \frac{3\pi}{2} - \alpha = 5.262061 \dots,$$

then ψ satisfies the fourth condition in (3.24). Now, let us consider a function $\tilde{p}$ which has the form (1.10) with $\zeta_1 = \tilde{t}_3 e^{i\theta}$ and $\zeta_2 = e^{i\psi}$. Since $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, in view of Lemma 1.2, we see that $\tilde{p}$ belongs to the class $\mathcal{P}$. Taking now a function

$$\tilde{f}(z) = \frac{z\tilde{p}(z)}{(1-z)}, \quad z \in \mathbb{D},$$

we see that $\tilde{f} \in \mathcal{F}_2$ and $\det T_{3,1}(\tilde{f}) = -4\Theta$, which completes the proof. $\square$

Availability of data and material Not applicable. The manuscript has no associated data.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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