Cohomogeneity Two Nonsemisimple Isometric Actions

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Abstract

We describe the orbits of a cohomogeneity two Riemannian G -manifold M from topological point of view, under the conditions that G is nonsemisimple and M decomposes as a product of negatively curved Riemannian manifolds.

Keywords Riemannian manifold · Lie group · Isometry

Mathematics Subject Classification 53C30 · 57S25

1 Introduction

A manifold on which a group G acts is called a G -manifold. In this paper, we consider a complete Riemannian manifold M with the action of a closed and connected Lie subgroup G of isometries. Dimension of the orbit space is called the cohomogeneity of the action. Manifolds having actions of cohomogeneity zero are called homogeneous. A classic theorem about Riemannian manifolds of nonpositive curvature [22] states that a homogeneous Riemannian manifold M of nonpositive curvature is simply connected or it is diffeomorphic to a cylinder over a torus (i.e., it is diffeomorphic to $\mathbb{R}^k \times T^s$, $k + s = \dim M$), and a theorem by S. Kobayashi [11] states that a homogeneous Riemannian manifold of negative curvature is simply connected. Therefore, it is diffeomorphic to $\mathbb{R}^n$, $n = \dim M$. There are many interesting theorems about topological properties of cohomogeneity one G -manifolds of nonpositive curvature [2, 17, 18, 21].

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The authors of [21] studied cohomogeneity one G -manifolds of negative curvature. Among other results, they proved that if M is a nonsimply connected negatively curved cohomogeneity one Riemannian G -manifold and $\dim(M) \geq 3$, then either M is diffeomorphic to $\mathbb{R}^k \times T^s$, $k+s = \dim M$ or $\pi_1(M) = \mathbb{Z}$ and the principal orbits are covered by $S^{n-2} \times \mathbb{R}$, $n = \dim M$, and $\frac{M}{G}$ is homeomorphic to one of the spaces R and $[0, \infty)$. Also, topological properties of cohomogeneity one Riemannian manifolds of nonpositive curvature has been studied by many authors. But, classification of orbits and orbit spaces of cohomogeneity two Riemannian G -manifolds of nonpositive curvature is an open problem. This article follows previous papers [13–16], where we proved various results about topological properties of cohomogeneity two Riemannian G -manifolds. One of the main examples of Riemannian manifolds of nonpositive curvature is the product $M = M_1 \times \cdots \times M_k$ of Riemannian manifolds such that M_i , $1 \leq i \leq k$, has negative curvature. In the paper [15], we studied the orbits of cohomogeneity two G -manifolds of this kind, under the condition that $M^G \neq \emptyset$. In the present paper, we replace the condition $M^G \neq \emptyset$ by the condition that G is nonsemisimple and there is no non-principal orbit of positive dimension. Among other results, we show that if M is not simply connected then either it is homeomorphic to the product of a cohomogeneity one G -manifold with $\mathbb{R}$, or $\pi_1(M) = \mathbb{Z}^p$, $p \geq 1$. Our main result is Theorem 3.2.

The paper is organized as follows: We recall some definitions and prove some statements about Riemannian manifolds of nonpositive curvature in Preliminaries. In the section of results, first we mention a remark about the relations of the orbits of a G -manifold and the orbits of its universal covering manifold, which is important in the proof of our theorem. Then, we give our main theorem and its proof.

2 Preliminaries

In what follows, M is a Riemannian manifold, G is a closed and connected subgroup of the isometries of M , and $M^G = \{x \in M : G(x) = x\}$. All geodesics of M are considered to have unit speed.

A complete connected and simply connected Riemannian manifold of nonpositive curvature is called a Hadamard manifold. An isometry ϕ of a Hadamard manifold M is called elliptic if it has fixed point. ϕ is called hyperbolic (parabolic) if the function $d_\phi^2 : M \rightarrow \mathbb{R}$ defined by $d_\phi^2(x) = d^2(x, \phi(x))$ has minimum point (has no minimum).

An isometry ϕ is called axial if there is a geodesic γ translated by ϕ (there exists a positive constant c such that $\phi(\gamma(t)) = \gamma(t+c)$). γ is called the axis of ϕ .

If γ is a geodesic in the Hadamard manifold M , we denote by $[\gamma]$ the collection of all geodesics which are asymptotic to γ . The collection of all asymptotic classes of the geodesics of M is denoted by $M(\infty)$ and is called the ideal boundary of M (see [8] for details). In fact, we can imagine $M \cup M(\infty)$ as a manifold with boundary such that M is its interior and $M(\infty)$ is its boundary.

Remark 2.1 If M_1 and M_2 are Hadamard manifolds then their product $M = M_1 \times M_2$ is also a Hadamard manifold. Consider unit speed geodesics γ_1 and γ_2 in M_1 and M_2 and put $\gamma(t) = (\gamma_1(\frac{\sqrt{2}}{2}t), \gamma_2(\frac{\sqrt{2}}{2}t))$. Then, γ is a unit speed geodesic in M . Let η_1 and

η_2 be other unit speed geodesics in M_1 and M_2 and η be a geodesic in M , defined in the similar way as γ . If $[\gamma_1] = [\eta_1]$ and $[\gamma_2] = [\eta_2]$, then it is easy to show that $[\gamma] = [\eta]$. Consider the map $\rho : M_1(\infty) \times M_2(\infty) \rightarrow M(\infty)$, defined by $\rho([\gamma_1], [\gamma_2]) = [\gamma]$. Clearly, ρ is a one-to-one and well-defined map. Then, without loss of precision, we can denote $[\gamma]$ by $([\gamma_1], [\gamma_2])$. In this way, $M_1(\infty) \times M_2(\infty)$ can be considered as a subset of $M(\infty)$. We call a point $z \in M(\infty)$ a regular point at infinity, if z belongs to $M_1(\infty) \times M_2(\infty)$ (that is $z = (z_1, z_2)$, $z_1 \in M_1(\infty)$, $z_2 \in M_2(\infty)$). A regular point in the infinity of the product of any finite number of Hadamard manifolds can be defined similarly.

Remark 2.2 A Hadamard manifold M satisfies Axiom 1 (see [8]) if for any distinct points $x, y \in M(\infty)$ there exists a geodesic γ joining x to y .

Lemma 2.3 Let ϕ be a parabolic isometry on a Hadamard manifold M .

- (1) If z is a fixed point at infinity for ϕ , then ϕ leaves each horosphere S centered at z invariant (see [8] for definition of the horosphere).
- (2) If M has strictly negative curvature, then there is a unique fixed point $z \in M(\infty)$ for ϕ .

Proof (1) is a direct consequence of [6, Lemma 3]. If M has strictly negative curvature then by [4, Lemma 9.10], M satisfies Axiom 1. Thus, by [8, Theorem 6.5], we get (2). $\square$

Lemma 2.4 If H is a closed and connected solvable subgroup of the isometries of a Riemannian manifold M of strictly negative curvature, then one of the following statements is true:

- (1) $M^H \neq \emptyset$.
- (2) H translates a unique geodesic.
- (3) There is a unique point in $M(\infty)$ fixed by H .

Proof By [6, Theorem 5], one of the following statements is true:

- (1) $M^H \neq \emptyset$.
- (2) H translates a geodesic.
- (3) There is a point in $M(\infty)$ fixed by H .

Uniqueness of the geodesic in (2) comes from [4, Proposition 4.2]. If (1) and (2) are not true, then all elements of H are parabolic and uniqueness of the fixed point in (3) comes from Lemma 2.3(2). $\square$

Lemma 2.5 Let $M = M_1 \times M_2 \times \cdots \times M_m$ such that for each i , M_i is a Hadamard manifold of negative curvature, and suppose that $\phi \in \text{Iso}(M)$ can be decomposed as the product $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_m$ of non-elliptic isometries $\phi_i \in \text{Iso}(M_i)$. If there is a geodesic $\beta = (\beta_1, \dots, \beta_m)$ such that $\phi(\beta) = \beta$, then β is unique.

Proof Let $\gamma = (\gamma_1, \dots, \gamma_m)$ be another geodesic in M such that $\phi(\gamma) = \gamma$. Then, for each i , $\phi_i(\gamma_i) = \gamma_i$ and $\phi_i(\beta_i) = \beta_i$. We get from the uniqueness of the geodesic left invariant by non-elliptic isometry ϕ_i [4, Proposition 4.2] that $\gamma_i = \beta_i$. Thus, $\beta = \gamma$. $\square$

Remark 2.6 Let G be a connected solvable Lie subgroup of the isometries of $M = M_1 \times M_2 \times \cdots \times M_m$ such that for each i , M_i is simply connected with strictly negative curvature. Since G is connected then each $g \in G$ can be decomposed as $g = g_1 \times \cdots \times g_m$, $g_i \in \text{Iso}(M_i)$ (see [10, Vol. 1, p. 240]). For each $i \in \{1, \dots, m\}$, let $P_i : G \rightarrow \text{Iso}(M_i)$ be the map defined by $P_i(g_1, \dots, g_m) = g_i$. Put $G_i = P_i(G)$. Clearly, G_i is a closed and connected solvable subgroup of the isometries of M_i , and we have:

$$G \subset G_1 \times \cdots \times G_m \quad (*)$$

Lemma 2.7 *Under the assumptions of Remark 2.6, there is a unique subset Ω of M with the property that $G(\Omega) = \Omega$ and Ω is one of the following sets or a product of them.*

- (1) M^G or product of the fixed point set of some elements of $\{G_1, \dots, G_m\}$.
- (2) Image of a geodesic.
- (3) A regular point at infinity of the product of some elements of the set $\{M_1, \dots, M_m\}$.

Proof Since M_i has strictly negative curvature, then by Remark 2.4, one of the following statements is true:

- (I) For all i , $M_i^{G_i} \neq \emptyset$.
- (II) For each i , there is a unique geodesic γ_i in M_i such that $G_i(\gamma_i) = \gamma_i$.
- (III) There is a unique point $\zeta_i \in M_i(\infty)$ for each i , such that $G_i(\zeta_i) = \zeta_i$.
- (IV) $\{G_1, \dots, G_m\}$ is a collection of the groups with properties (I) or (II) or (III).

In the case (I), $(M_1 \times \cdots \times M_m)^{(G_1 \times \cdots \times G_m)} \neq \emptyset$, then $M^G \neq \emptyset$.

In the case (II), $\gamma = (\gamma_1, \dots, \gamma_m)$ is a geodesic such that $(G_1 \times \cdots \times G_m)(\gamma) = \gamma$. Thus, $G(\gamma) = \gamma$ and by Lemma 2.5, γ is unique. In a similar way if the case (III) is true, $\zeta = (\zeta_1, \dots, \zeta_m)$ will be the unique regular point in $M(\infty)$ fixed by G . Now, it is easy to show that if (IV) is true, then there is a unique set Ω , which is a product of the sets similar to (I), (II), (III), such that $G(\Omega) = \Omega$. $\square$

3 Results

Remark 3.1 [5]. If M is a complete and connected Riemannian manifold and G is a connected subgroup of $\text{Iso}(M)$, and if $\tilde{M}$ is the universal Riemannian covering manifold of M with the covering map $\kappa : \tilde{M} \rightarrow M$, then there is a connected covering $\tilde{G}$ of G with the covering map $\pi : \tilde{G} \rightarrow G$, such that $\tilde{G}$ acts isometrically on $\tilde{M}$ and

- (1) Each deck transformation δ of the covering $\kappa : \tilde{M} \rightarrow M$ maps $\tilde{G}$ -orbits onto $\tilde{G}$ -orbits.
- (2) If $x \in M$ and $\tilde{x} \in \tilde{M}$ such that $\kappa(\tilde{x}) = x$ then $\kappa(\tilde{G}(\tilde{x})) = G(x)$.
- (3) $\tilde{M}^{\tilde{G}} = \kappa^{-1}(M^G)$.
- (4) If G is non-semisimple then $\tilde{G}$ is non-semisimple.
- (5) Deck transformation group, which we denote it by Δ , centralizes $\tilde{G}$ (i.e., for each $\delta \in \Delta$ and $\tilde{g} \in \tilde{G}$, $\delta\tilde{g} = \tilde{g}\delta$).

Remark 3.2 Let $\tilde{M}$ be a Hadamard manifold and S be a horosphere in $\tilde{M}$ related to the asymptotic class of geodesics $[\gamma]$ (i.e., all elements of $[\gamma]$ intersect S perpendicularly). The function $f : \tilde{M} \rightarrow \mathbb{R}$, $f(p) = \lim_{t \rightarrow \infty} d(p, \gamma(t)) - t$, is called a Bussmann function. For each point $p \in \tilde{M}$, there is a point $\eta_S(p)$ in S which is the unique point of S nearest p , and the following map is a homeomorphism ([8, pp. 47, 58]):

$$\phi : \tilde{M} \rightarrow S \times \mathbb{R}, \quad \phi(p) = (\eta_S(p), f(p)).$$

Theorem 3.3 (see [15, Theorem 3.5]). *Let M^{n+2} be a nonsimply connected Riemannian manifold which is of cohomogeneity two under the action of a closed and connected subgroup G of isometries, and suppose that M can be decomposed as a product of Riemannian manifolds of negative curvature and $M^G \neq \emptyset$. Then,*

- (a) *M is diffeomorphic to $S^1 \times \mathbb{R}^{n+1}$ or $B \times \mathbb{R}^n$, where B is the Moebius band.*
- (b) *Each principal orbit is diffeomorphic to S^n .*

As we mentioned in Introduction, classification of orbits of cohomogeneity two Riemannian manifolds of nonpositive curvature is an open problem and seems to be a difficult problem in general case. In the following theorem, we consider an important category of Riemannian manifolds of nonpositive curvature, containing products of negatively curved Riemannian manifolds. In direction of [15], we give a description of the manifold and its orbits under the condition that the acting group is non-semisimple.

Theorem 3.4 *Let M^{n+2} , be a nonsimply connected Riemannian manifold such that it can be decomposed as a product of Riemannian manifolds of strictly negative curvature of dimension bigger than two, and let G be a non-semisimple closed and connected subgroup of the isometries of M . If M is of cohomogeneity two under the action of G , without non-principal orbits of positive dimension, then one of the following statements is true:*

- (a) *M is a parabolic manifold homeomorphic to $\frac{S}{\pi_1(M)} \times \mathbb{R}$. Where, S is a horosphere in the universal Riemannian covering of M , and $\frac{S}{\pi_1(M)}$ is a cohomogeneity one G -manifold.*
- (b) *$\pi_1(M) = \mathbb{Z}^p$ for some positive integer p , and all orbits are diffeomorphic to $\mathbb{R}^{n-p} \times T^p$.*
- (c) *M is diffeomorphic to $S^1 \times \mathbb{R}^{n+1}$ or $B \times \mathbb{R}^n$, where B is the Moebius band, and each principal orbit is diffeomorphic to S^n .*
- (d) *$\dim M = 3$ and M has negative curvature. $\pi_1(M) = \mathbb{Z}$ and each orbit is diffeomorphic to S^1 .*

Proof Following Remark 3.1, let $\tilde{M}$ be the universal Riemannian covering manifold of M with the deck transformation group Δ and let $\tilde{G}$ be the corresponding connected covering of G which acts isometrically and by cohomogeneity two on $\tilde{M}$. Since by assumptions of the theorem, M can be decomposed as the product of Riemannian manifolds of strictly negative curvature, then $\tilde{M}$ can be decomposed as $\tilde{M} = M_1 \times M_2 \times \cdots \times M_m$ such that for all i , M_i has strictly negative curvature, and each $\delta \in \Delta$ decomposes as $\delta = \delta_1 \times \delta_2 \times \cdots \times \delta_m$, $\delta_i \in Iso(M_i)$. If $M^G \neq \emptyset$, then we get

part (c) of the theorem from Theorem 3.3. Thus, in the rest of the proof we suppose that $M^G = \emptyset$ which implies $\tilde{M}^{\tilde{G}} = \emptyset$. Then, according to the assumptions of the theorem, all orbits must be regular and of dimension n . Since G is non-semisimple, $\tilde{G}$ is non-semisimple. Let H be a connected solvable normal subgroup of $\tilde{G}$. By Lemma 2.7, there exists a unique subset Ω of $\tilde{M}$ such that $H(\Omega) = \Omega$ and Ω is one of the following sets or product of them.

A : a totally geodesic submanifold of $\tilde{M}$ or a totally geodesic submanifold of the product of some elements of the set $\{M_1, \dots, M_m\}$ (because, fixed point sets of connected subgroups of isometries are totally geodesic submanifolds).

γ : image of a geodesic of $\tilde{M}$ or a geodesic in the product of some elements of the set $\{M_1, \dots, M_m\}$.

ζ : a regular point at infinity of $\tilde{M}$ or a regular point at infinity for product of some elements of the set $\{M_1, \dots, M_m\}$.

That is

$$\Omega \in \{\gamma, \zeta, A, A \times \gamma, A \times \zeta, \zeta \times \gamma, A \times \zeta \times \gamma\} \quad (*)$$

Since H is normal in $\tilde{G}$, for each $g \in \tilde{G}$ we have:

$$g^{-1}Hg = H \Rightarrow g^{-1}Hg(\Omega) = H(\Omega) \Rightarrow g^{-1}Hg(\Omega) = \Omega \Rightarrow H(g\Omega) = g\Omega.$$

Now, from the uniqueness of Ω with the property $H(\Omega) = \Omega$, we get that $g\Omega = \Omega$, then

$$\tilde{G}(\Omega) = \Omega \quad (**)$$

Since the elements of Δ commute with the elements of $\tilde{G}$, for each $\delta \in \Delta$ we have $\tilde{G}(\delta\Omega) = \delta\tilde{G}(\Omega) = \delta\Omega$. Uniqueness of Ω implies that $\delta(\Omega) = \Omega$. Thus,

$$\Delta(\Omega) = \Omega \quad (***)$$

We consider now the seven cases of Ω in (*).

Case 1. $\Omega = \gamma$.

$\tilde{G}(\gamma) = \gamma$ and By (**), $\Delta(\gamma) = \gamma$. Since $\tilde{M}^{\tilde{G}} = \emptyset$, then γ is a $\tilde{G}$ -orbit. Since the cohomogeneity is two and all orbits are of dimension n , then $n = 1$. Since by assumptions of the theorem, for all i , $\dim M_i \geq 3$, then $\tilde{M} = M_1$. Thus, M has strictly negative curvature. All $\tilde{G}$ -orbits in $\tilde{M}$ are diffeomorphic to γ , then they are diffeomorphic to R . By Lemma 3.5 in chapter 12 of [7], $\Delta = Z$. Thus, each G -orbit in M will be diffeomorphic to $\frac{R}{\Delta} = \frac{R}{Z} = S^1$. This is part (d) of the theorem.

Case 2. $\Omega = \zeta$.

Let $[\gamma]$ be the asymptotic class of the geodesics related to ζ . We have $\tilde{G}(\zeta) = \zeta$ and by (**), $\Delta(\zeta) = \zeta$ (i.e., $\tilde{G}([\gamma]) = [\gamma]$ and $\Delta([\gamma]) = [\gamma]$). First, suppose that there is an axial element $\delta \in \Delta$ and let λ be the unique geodesic such that $\delta\lambda = \lambda$ (uniqueness of λ comes from Lemma 2.5). If $g \in \tilde{G}$ then $\delta(g\lambda) = g\delta\lambda = g\lambda$. Thus, we get from

the uniqueness of λ that $g\lambda = \lambda$. Then, λ is a $\tilde{G}$ -orbit, and we get part (d) of the theorem as like as Case 1.

Now, suppose that all elements of Δ are non-axial. Non-identity elements of Δ are without fixed points, then they must be parabolic and M will be a parabolic manifold. By Lemma 2.3, for each $\delta \in \Delta$ and each horosphere S related to the asymptotic class $[\gamma]$, $\delta S = S$. Fix a horosphere S related to $[\gamma]$. Put $M_1 = \frac{S}{\Delta}$ and let η_s and f be the maps defined in Remark 3.2. The homeomorphism $\phi : \tilde{M} \rightarrow S \times \mathbb{R}$ mentioned in Remark 3.2, induces a homeomorphism $\phi_1 : \frac{\tilde{M}}{\Delta} = M \rightarrow \frac{S}{\Delta} \times \mathbb{R}$, such that $\phi_1(x) = (\kappa\eta_s(\tilde{x}), f(\tilde{x}))$, $\tilde{x} \in \kappa^{-1}(x)$.

If there is a $g \in \tilde{G}$ which is axial and λ is its unique axis, then we get from the fact that the elements of Δ and g commute that for all $\delta \in \Delta$, $\delta(\lambda)$ is also an axis for g . Since the axis is unique then $\delta(\lambda) = \lambda$. Now, if $g' \in \tilde{G}$, then again we get from the uniqueness of the axis λ for δ that $g'\lambda = \lambda$, thus $G(\lambda) = \lambda$, and we get part (d) of the theorem as like as case 1.

Now, Suppose that all elements of $\tilde{G}$ are non-axial. If for some $g \in \tilde{G}$ and $x \in \tilde{M}$, $gx = x$, then for the geodesic λ in $[\gamma]$ which passes from x , we have $g\lambda = \lambda$, and g must be axial which is contradiction. Thus, we can assume that the elements of $\tilde{G}$ are parabolic and by Lemma 2.3, $\tilde{G}(S) = S$. Thus, S is a cohomogeneity one $\tilde{G}$ -manifold and $\frac{S}{\Delta}$ is a cohomogeneity one G -manifold. This is part (a) of the theorem.

Case 3. $\Omega = A$.

Similar to the previous cases, we have $\tilde{G}(A) = A$ and $\Delta(A) = A$. Put $A_1 = \kappa(A)$. A is a nontrivial totally geodesic submanifold of $\tilde{M}$, thus A_1 is a totally geodesic submanifold of M . Since $\Delta(A) = A$, then Δ is equal to deck transformation group of A_1 , and $\pi_1(A_1) = \Delta = \pi_1(M)$. Since all orbits are of dimension n , we have for all $x \in A$:

$$n = \dim \tilde{G}(x) \leq \dim A < \dim \tilde{M} = n + 2 \Rightarrow \dim A = n \quad \text{or} \quad n + 1$$

Now, we consider $\dim A = n$, $\dim A = n + 1$, separately.

I) $\dim A = n$.

In this case, A is a $\tilde{G}$ -orbit and A_1 must be a G -orbit of nonpositive curvature. Thus, we get part (b) of the theorem (Because, a homogeneous Riemannian manifold of nonpositive curvature is diffeomorphic to product of a torus and a euclidean space [22]).

II) $\dim A = n + 1$.

A_1 is a cohomogeneity one G -manifold of non-positive curvature, without singular orbits. Consider the following two cases separately:

(II-1) For all $\delta \in \Delta$, d_δ^2 has no minimum point.

(II-2) There is a $\delta \in \Delta$ such that d_δ^2 has minimum point.

(II-1): In this case by [17, Lemma 3.2], Δ maps each orbit of A onto itself. Thus by a similar way in the proof of [17, Lemma 3.6], all orbits of A_1 are diffeomorphic to $T^p \times \mathbb{R}^{n-p}$ for some nonnegative integer p , thus we get part (b) of the theorem.

(II-2): By [4, Proposition 4.2], the minimum point set of d_δ^2 is equal to the image of all geodesics translated by δ . But by Lemma 2.5, there is at most one geodesic translated by δ . Let γ be the unique geodesic such that $\delta(\gamma) = \gamma$. Since the elements of $\tilde{G}$ and

Δ commute, then we get from the uniqueness of γ that $\tilde{G}(\gamma) = \gamma$, and we get part (d) of the theorem in the similar way as Case 1.

Case 4. $\Omega \in \{A \times \gamma, A \times \zeta, \zeta \times \gamma, A \times \zeta \times \gamma\}$.

By dimensional reasons, this cases can not occur. We give the proof for $\Omega = A \times \gamma$, other cases are similar. We can assume (after a possible rearrangement of the indices) that

$$\begin{aligned}\tilde{M} &= M_1 \times M_2 \times \dots \times M_k \times M_{k+1} \times M_{k+2} \times \dots \times M_m \\ A &\subset M_1 \times M_2 \times \dots \times M_k, \quad \gamma \subset M_{k+1} \times M_{k+2} \times \dots \times M_m\end{aligned}$$

By (**), $\tilde{G}(A \times \gamma) = A \times \gamma$. Thus, $A \times \gamma$ is a union of $\tilde{G}$ -orbits. Since A and γ are nontrivial then the codimension of A in $M_1 \times M_2 \times \dots \times M_k$ is at least 1, and because for all i , $\dim M_i \geq 3$, then the codimension of γ in $M_{k+1} \times M_{k+2} \times \dots \times M_m$ is at least 2. Thus, the codimension of $A \times \gamma$ in $\tilde{M}$ will be at least 3. This is contradiction (because, $A \times \gamma$ is union of orbits which have codimension two in $\tilde{M}$). $\square$

Remark 3.5 In Theorem 3.4, decomposability of M to the product of negatively curved manifolds can be replaced by the weaker condition of decomposability of the universal covering manifold $\tilde{M}$ to negatively curved manifolds and decomposability of Δ (see [17], definition of UND-manifolds and examples).

Remark 3.6 In case (a) of Theorem 3.4, M is homeomorphic to the product of a cohomogeneity one manifold $\frac{S}{\Delta}$ with $\mathbb{R}$. By proof of the theorem, in this case, there is no singular orbit. Since the orbit space of cohomogeneity one manifolds with no singular orbit are homeomorphic to S^1 or R , then the orbit space of M under the action of G will be homeomorphic to $S^1 \times R$ or R^2 . Study of the orbits in this case reduces to the study of the orbits of cohomogeneity one actions on horospheres. In the special case when M has constant negative curvature (decomposition of M has one factor of constant negative curvature), the horospheres of $\tilde{M}$ are isometric to R^{n+1} , and from the known results about cohomogeneity one actions on flat Riemannian manifolds, orbits of M are diffeomorphic to $R^k \times T^{n-k}$, for some positive integer k .

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