Multiplied Configurations Induced by Quasi Difference Sets

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Abstract

Quasi difference sets are introduced as a tool to produce partial linear spaces. We characterize geometry and automorphisms of configurations decomposable into components induced by quasi difference sets. In particular, we are interested in series of cyclically inscribed copies of a fixed configuration.

Keywords Partial linear space · Difference set · Quasi difference set · Cyclic projective plane

Mathematics Subject Classification 51D20 · 51E30

1 Introduction

There is well-known construction of a point-block geometry induced by some fixed subset D of a group G (cf. [2, 14]). The idea is simple: points are elements of G , and blocks (lines) are the images of D under (left) translations. If every nonzero element of G can be presented in exactly λ ways as a difference of two elements of D , then D is called a *difference set*, cf. [3]. In this case we obtain a λ -design. Difference sets with $\lambda = 1$ are called *Singer* (or *planar*) difference sets and induce a linear spaces, in particular finite Desarguesian projective planes (see [5, 13]). To get weaker geometries we admit sets with $\lambda \in \{0, 1\}$ and call them *quasi difference sets*. This approach was used in [10] to study configurations that can be visualized as series of polygons, inscribed cyclically one into another. Classical Pappus configuration can be presented this way, for instance.

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Some other variation on difference sets can be found in the literature, e.g. *relative difference sets* (cf. [4]), *affine difference sets* (cf. [6]), or *partial difference sets* (cf. [7]). Defining a partial difference set D we require that every nonzero element a of G can be presented as a corresponding difference in λ_1 ways if $a \in D$, and in λ_2 ways if $a \notin D$. Note that our quasi difference sets are not partial in this sense. One of the most important tasks in the theory of difference sets is to determine conditions of the existence, comp. [1,8]. These are not the questions considered in this paper. Instead, we are mainly interested in the geometry (in the rather classical style) of partial linear spaces determined by quasi difference sets.

We consider configurations which can be defined with the help of arbitrary quasi difference set. Elementary properties of these structures are discussed: we verify satisfiability of Veblen, Pappus, and Desargues axioms. A special emphasis is imposed on structures which arise from groups decomposed into a cyclic group C_k and some other group G . These structures can be seen as *multiplied configurations*—series of cyclically inscribed configurations, each one isomorphic to the configuration associated with G . On the other hand, this construction is just a special case of the operation of “joining” two structures, corresponding to the operation of the direct sum of groups. In some cases corresponding decomposition can be defined within the resulting “join”, in terms of the geometry of the considered structures. This definable decomposition enables us to characterize the automorphism group of such a “join”. Some other techniques are used to determine the automorphism group of cyclically inscribed configuration. Roughly speaking, groups in question are semidirect products of some symmetric group and the group of translations of the underlying group.

The technique of quasi difference sets can be used to produce new configurations, so far not considered in the literature. Many of them seem to be of a real geometrical interest for their own. In the last section we apply our apparatus to get some new configurations arising from the well-known: cyclically inscribed Pappus or Fano configurations, multiplied Pappus configurations, a power of cyclic projective planes.

2 Basic Notions and Definitions

Let $\mathfrak{M} = \langle S, \mathcal{L}, \mid \rangle$ be an incidence point-line geometry. If $p \mid k$ for $p \in S, k \in \mathcal{L}$ then we say that “ p is on the line k ” or “ k passes through the point p ”. $\mathfrak{M}$ is a *partial linear space* iff there are at least two points on every line, there is a line through every point, and any two lines that share two or more points coincide. A partial linear space in which the rank of a point and the size of a line are equal is said to be a *symmetric configuration* or in short just a *configuration*. The set of all lines through a point $p \in S$ is denoted by p^* , and dually we write k^* for the set of all points on a line $k \in \mathcal{L}$. The *rank of a point* p is the number $|p^*|$, and the *size of a line* k is the number $|k^*|$. If $p \neq q$ are two collinear points then we write $p \sim q$ and the line which joins these two points is denoted by $\overline{p, q}$. We also write $k \sqcap l$ for the common point of two intersecting lines k and l .

An *automorphism* (or a *collineation*) of $\mathfrak{M}$ is a pair $\varphi = (\varphi', \varphi'')$ of bijections $\varphi': S \rightarrow S, \varphi'': \mathcal{L} \rightarrow \mathcal{L}$ such that for every $a \in S, l \in \mathcal{L}$ the conditions $a \mid l$ and $\varphi'(a) \mid \varphi''(l)$ are equivalent. A pair $\varkappa = (\varkappa', \varkappa'')$ of bijections $\varkappa': M \rightarrow \mathcal{L}$,

$\mathcal{K}'' : \mathcal{L} \longrightarrow M$ satisfying $a \mid l$ iff $\varphi''(l) \mid \varphi'(a)$ for every $a \in S, l \in \mathcal{L}$ is called a *correlation* of $\mathfrak{M}$. A substructure of $\mathfrak{M}$, whose points are all the points of $\mathfrak{M}$ collinear with $a \in S$, and lines are all the lines of $\mathfrak{M}$ which contain at least two points collinear with a is said to be *the neighborhood of a point a* and it is denoted by $\mathfrak{M}_{(a)}$. Clearly, if $\varphi = (\varphi', \varphi'')$ is an automorphism of $\mathfrak{M}$, then φ maps $\mathfrak{M}_{(a)}$ onto $\mathfrak{M}_{(\varphi'(a))}$. $\mathfrak{M}$ is said to be *Veblenian* iff any line that crosses two sides of a triangle meets also the third side of this triangle. We say that $\mathfrak{M}$ is *Desarguesian* iff it satisfies Desargues axiom: if two triangles are perspective from a point, then they are perspective from a line.

In [10] quasi difference sets are defined to study series of cyclically inscribed n -gons. We briefly recall this construction. Let $G = \langle G, \cdot, 1 \rangle$ be an arbitrary group and $D \subset G$. Let us introduce a point-line geometry

$$\mathbf{D}(G, D) = \langle G, G/D \rangle. \quad (2.1)$$

Every translation $\tau_a : G \ni x \mapsto a \cdot x \in G$ is an automorphism of $\mathbf{D}(G, D)$. This yields that without loss of generality we can assume that $1 \in D$. Let G_D be the stabilizer of D in G . Then, the number of points in $\mathbf{D}(G, D)$ is $|G|$, the number of lines is $\frac{|G|}{|G_D|}$, the size of every line is $|D|$, and the rank of every point is $\frac{|D|}{|G_D|}$.

It was proved in [10] that $\mathbf{D}(G, D)$ is a configuration iff for every $c \in G, c \neq 1$ there is at most one pair $(a, b) \in D \times D$ with $ab^{-1} = c$. Any $D \subseteq G$ satisfying this condition is called a *quasi difference set*, for short QDS. In [10] we were mainly interested in the structures of the form $\mathbf{D}(C_k \oplus C_n, D)$, where $D = \{(0, 0), (1, 0), (0, 1)\}$. In this paper we shall generalize this construction. Let us adopt the following convention: (a) means “coordinates” of the point $a \in G$, and $[a]$ denotes “coordinates” of the line $a \cdot D \in G/D$. Using this we get

$$(a) \mid [b] \text{ iff } a \in b \cdot D \text{ iff } b^{-1} \cdot a \in D. \quad (2.2)$$

Note that if D is QDS then $G_D = \{1\}$. Hence, a is uniquely determined by $[a]$, or in other words $a \cdot D = b \cdot D$ holds only for $a = b$.

3 Generalities

Now, we are going to present some general properties of $\mathfrak{D} = \mathbf{D}(G, D)$. Every automorphism $\varphi = (\varphi', \varphi'')$ of $\mathfrak{D}$ uniquely corresponds to a pair $f = (f', f'')$ of bijections of G determined by

$$\varphi'((a)) = (f'(a)), \quad \varphi''([b]) = [f''(b)].$$

We shall frequently refer to the pair f as to an automorphism of $\mathfrak{D}$. On the other hand, some of the automorphisms of $\mathfrak{D}$ are determined by automorphisms of the underlying group G , namely

Fact 3.1 *Let $f \in \text{Aut}(G)$. The map f determines an automorphism $\varphi = (\varphi', \varphi'')$ of $\mathbf{D}(G, D)$ iff $f(D) = q \cdot D$ for some $q \in G$. Then $\varphi'((a)) = (f(a))$ and $\varphi''([a]) = [f(a) \cdot q]$ for every $a \in G$.*

Proposition 3.2 *Let D be QDS in an abelian group G . Then the map $\varkappa$ defined by*

$$\varkappa((a)) = [a^{-1}], \quad \varkappa([a]) = (a^{-1}) \quad (3.1)$$

is an involutive correlation of the structure $\mathfrak{D} = \mathbf{D}(G, D)$. Consequently, $\mathfrak{D}$ is self-dual. A point a of $\mathfrak{D}$ is selfconjugate under $\varkappa$ iff $a^2 \in D$.

Proof Clearly, $\varkappa$ is involutory. Let $a, b \in G$. Then $(a) \mid [b]$ means that $b^{-1} \cdot a \in D$. This is equivalent to $(a^{-1})^{-1} \cdot b^{-1} \in D$, i.e. $\varkappa([b]) = (b^{-1}) \mid [a^{-1}] = \varkappa((a))$. Thus $\varkappa$ is a correlation. Finally, assume that $(a) \mid \varkappa((a)) = [a^{-1}]$. From (2.2) we obtain $a^2 \in D$. $\square$

The correlation defined by (3.1) will be referred to as *the standard correlation* of $\mathbf{D}(G, D)$. Immediate from (2.2) is the following

Lemma 3.3 *Let $D \subseteq G$, $a, b \in G$, and $\mathfrak{D} = \mathbf{D}(G, D)$.*

(i) *The set of the lines of $\mathfrak{D}$ through the point (a) can be identified with $a \cdot D^{-1}$, i.e.*

$$(a)^* = \{[a \cdot d^{-1}]: d \in D\}. \quad (3.2)$$

(ii) *The points (a) and (b) are collinear in $\mathfrak{D}$ iff $a^{-1} \cdot b \in D^{-1}D$. If $a^{-1}b = d_1^{-1}d_2$ with $d_1, d_2 \in D$ then*

$$\overline{a, b} = [a \cdot d_1^{-1}] = [b \cdot d_2^{-1}]. \quad (3.3)$$

(iii) *The lines $[a]$ and $[b]$ of $\mathfrak{D}$ have a common point iff $a^{-1} \cdot b \in DD^{-1}$. If $a^{-1} \cdot b = d_1 \cdot d_2^{-1}$ with $d_1, d_2 \in D$ then*

$$a \sqcap b = (a \cdot d_1) = (b \cdot d_2). \quad (3.4)$$

As a straightforward consequence of Lemma 3.3 we get

Proposition 3.4 *Let $G = \langle G, \cdot, 1 \rangle$ be a group, D be QDS in G with $1 \in D$, and $a_1, a_2 \in G$. Points (a_1) and (a_2) can be joined with a polygonal path in $\mathbf{D}(G, D)$ iff there is a finite sequence $q_1, \dots, q_s$ of elements of $D^{-1}D$ such that $a_1 = q_1 \cdots q_s \cdot a_2$. Consequently, the connected component of the point (1) is isomorphic to $\mathbf{D}(\langle D \rangle_G, D)$, where $\langle D \rangle_G$ is the subgroup of G generated by D . Every two connected components of any two points are isomorphic.*

Corollary 3.5 *$\mathbf{D}(G, D)$ is connected iff D generates the whole group G .*

For $D \subseteq G$ we introduce the following condition:

$$\begin{aligned} &\text{if } d_1, d_2, d_3, d_4 \in D \text{ and } d_1 d_2^{-1} d_3 d_4^{-1} \in DD^{-1} \text{ then} \\ &d_1 d_2^{-1} = 1 \text{ or } d_3 d_4^{-1} = 1 \text{ or } d_1 d_4^{-1} = 1, \text{ or } d_3 d_2^{-1} = 1. \end{aligned} \quad (3.5)$$

The next Lemma explains the meaning of the condition (3.5).

Lemma 3.6 Assume that G is an abelian group, and $D \subset G$ is QDS satisfying (3.5). Let $a \in G$ be a point of $\mathbf{D}(G, D)$, $d_1, d_2 \in D$, and $b_1 = [ad_1^{-1}]$, $b_2 = [ad_2^{-1}]$ be two distinct lines through a . Set $i = 1, 2$.

- (i) If $d'_i \in D$ then $p_i = (ad_i^{-1}d'_i)$ is a point on b_i , and $p_1 \sim p_2$ iff $d'_1 = d'_2$.
- (ii) For every point $p_d = (ad_1^{-1}d)$ on b_1 with $d \in D$, $d \neq d_2$, d_1 there is the unique point $q = (ad_2^{-1}d)$ on b_2 which completes points a, p_d to a triangle. The point p_{d_2} cannot be completed to a triangle this way.
- (iii) If c is a line of $\mathbf{D}(G, D)$ which crosses both of b_1, b_2 and misses a then $c = [ad_1^{-1}d_2^{-1}d]$ for some $d \in D$ with $d \neq d_1, d_2$.
- (iv) For any two distinct lines c_1, c_2 crossing both of b_1, b_2 and missing a , i.e. $c_i = [ad_1^{-1}d_2^{-1}d'_i]$ for $d'_i \in D$, we have $c_1 \cap c_2 = (ad_1^{-1}d_2^{-1}d'_1d'_2)$.

Proof Simple analysis based on Lemma 3.3 and (3.5). $\square$

Now we describe automorphisms of $\mathbf{D}(G, D)$ with a quasi difference set D satisfying (3.5). Let us begin with some rigidity properties.

Lemma 3.7 Let $\mathfrak{D} = \mathbf{D}(G, D)$, where D is QDS that satisfies (3.5) and $p \in G$. Let f be a collineation of $\mathfrak{D}$ such that $f(p) = p$. If f satisfies any of the following:

- (i) f fixes all points on a line through p ,
- (ii) f preserves every line through p ,

then f fixes all the points on the lines through p .

Proof Since $\mathfrak{D}$ has a transitive group of automorphisms, we can assume that $p = (1)$. Let $f = (f', f'')$ be a collineation of $\mathfrak{D}$ satisfying (i) for a line $l_1 = [d_1^{-1}]$ with $d_1 \in D$. Take $d_2 \in D$, $d_2 \neq d_1$ and a line $l_2 = [d_2^{-1}]$. Then the points $(d_1^{-1}d_2)$ on l_1 and $(d_2^{-1}d_1)$ on l_2 are the unique points “between” l_1 and l_2 that are not collinear (cf. Lemma 3.6(ii)). We have $f'(d_1^{-1}d_2) = (d_1^{-1}d_2)$ and $f'(d_2^{-1}d_1) \notin [f''(d_2^{-1})]$. The only point in $\mathfrak{D}_{(p)}$ non-collinear with $(d_1^{-1}d_2)$ lies on $[d_2^{-1}]$; therefore, $f''(d_2^{-1}) = d_2^{-1}$. Thus f preserves every line through p .

The case with the assumption (ii) can be proved in a similar way. $\square$

From Lemma 3.7 we get

Corollary 3.8 Let f be a collineation which fixes a line l of $\mathfrak{D}$ point-wise. Under assumption (3.5) f fixes all points on every line which crosses l . Consequently, if $\mathfrak{D}$ is connected then $f = \text{id}$.

Corollary 3.9 Under assumption (3.5) every automorphism of $\mathfrak{D}$ which has a fixed point p is uniquely determined by its action on the lines through p . Consequently, the point stabilizer $\text{Aut}(\mathfrak{D})_{(p)}$ of the automorphism group of $\mathfrak{D}$ is isomorphic to a subgroup of the permutation group $S_{|D|}$.

4 Products of Difference Sets

Let $G_i = \langle G_i, \cdot_i, 1_i \rangle$ be a group for $i \in I$, and $1_i \in D_i \subset G_i$ for every $i \in I$.

Let $G = \prod_{i \in I} G_i$, i.e. let G be the set of all functions $g: I \rightarrow \bigcup \{G_i: i \in I\}$ with $g(i) \in G_i$. Then the product $\prod_{i \in I} G_i$ is the structure $\langle G, \cdot, 1 \rangle$, where $(g_1 \cdot g_2)(i) = g_1(i) \cdot g_2(i)$ for $g_1, g_2 \in G$, and $1(i) = 1_i$. It is just the standard construction of the direct product of groups. The set

$$\sum_{i \in I} G_i = \{g \in G: g(i) \neq 1_i \text{ for a finite number of } i \in I\}$$

is a subgroup of $\prod_{i \in I} G_i$, denoted by $\sum_{i \in I} G_i$. If $I = \{1, \dots, r\}$ is finite, then $G_1 \oplus \dots \oplus G_r := \prod_{i \in I} G_i = \sum_{i \in I} G_i =: \sum_{i=1}^r G_i$.

For every $j \in I$ we define the standard projection $\pi_j: \prod_{i \in I} G_i \rightarrow G_j$ by $\pi_j(g) = g(j)$, and the standard inclusion $\varepsilon_j: G_j \rightarrow \sum_{i \in I} G_i$ by the conditions $(\varepsilon_j(a))(j) = a$ and $(\varepsilon_j(a))(i) = 1_i$ for $i \neq j$ and $a \in G_j$. Recall that π_j and ε_j are group homomorphisms. We set $\sum_{i \in I} D_i = \bigcup \{\varepsilon_i(D_i): i \in I\}$. For a finite set $I = \{1, \dots, r\}$ we write $\sum_{i \in I} D_i = \sum_{i=1}^r D_i = D_1 \uplus \dots \uplus D_r$.

Proposition 4.1 *If D_i is QDS in G_i for every $i \in I$, then $\sum_{i \in I} D_i$ is QDS in $\sum_{i \in I} G_i$.*

Proof We set $D = \sum_{i \in I} D_i$. Let $g_1, g_2, g_3, g_4 \in D$ and assume that $g_1 g_2^{-1} = g_3 g_4^{-1}$. Let $g_i \in \varepsilon_{j_i}(D_{j_i})$. If $j_1 = j_2$, then $\pi_j(g_1 g_2^{-1}) = 1_j$ for every $j \neq j_1$; thus $\pi_j(g_3 g_4^{-1}) = 1_j$, and thus $j_3 = j_4 = j_1$. From assumption we infer that $g_1 = g_3$ and $g_2 = g_4$. If $j_1 \neq j_2$; analogously, we come to $j_1 = j_3$ and $j_2 = j_4$. Then we obtain $g_1 = g_3$ and $g_2^{-1} = g_4^{-1}$, which yields our claim. $\square$

Let $\mathfrak{D}_i = \mathbf{D}(G_i, D_i)$. We write

$$\sum_{i \in I} \mathfrak{D}_i := \mathbf{D}(\sum_{i \in I} G_i, \sum_{i \in I} D_i).$$

For $I = \{1, \dots, r\}$ we write also $\sum_{i=1}^r \mathfrak{D}_i := \mathbf{D}(G_1, D_1) \oplus \dots \oplus \mathbf{D}(G_r, D_r)$. Note that, for $a, b \in \sum_{i \in I} G_i$, $a \in b \cdot \sum_{i \in I} D_i$ iff $a = b \cdot d$ for some $d \in \bigcup \{\varepsilon_i(D_i): i \in I\}$, i.e. iff there is $i \in I$ such that $a_i \in b_i \cdot D_i$ and $a_j = b_j$ for all $j \neq i$. So, we get

$$(a) \mid [b] \text{ iff } (a_i) \mid [b_i] \text{ in } \mathfrak{D}_i \text{ for some } i \in I \text{ and } a_j = b_j \text{ for } i \neq j \in I. \quad (4.1)$$

Let us consider two particular cases. First, let $\mathfrak{D} = \mathbf{D}(C_k, \{0, 1\}) \oplus \mathbf{D}(G', D')$ and $D = \{0, 1\} \uplus D'$. Let $a = (i, a') \in C_k \times G'$. Then the points of $a \cdot D$ are points of the form $(i, a' \cdot p)$ with $p \in D'$, and—one point— $(i + 1, a')$. Somewhat informally we can say that the line with the coordinates $[i, a']$ consists of the points (i, p) , where $(p) \mid [a']$ and one “extra” point $(i + 1, a')$. In other words, we have a function f_i which assigns a point of $\mathfrak{D}' = \mathbf{D}(G', D')$ to every line of $\mathfrak{D}'$ such that the lines of $\mathfrak{D}$ are of the form $i \times l' \cup \{(i + 1, f_i(l'))\}$, where l' is a line of $\mathfrak{D}'$. Thus $\mathfrak{D}$ is a configuration consisting of k copies of $\mathfrak{D}'$ cyclically inscribed one into another. In the above construction, the function f_i is defined by $f_i([a]) = (a)$.

Now, let $\mathfrak{D} = \mathbf{D}(G_1, D_1) \oplus \mathbf{D}(G_2, D_2)$. The lines of $\mathfrak{D}$ are of the form $(a_1, a_2) + D_1 \uplus D_2$, which, on the other hand, can be written as $[a_1]^* \times \{(a_2)\} \cup \{(a_1)\} \times [a_2]^*$. Recall, that the lines of the Segre product $\mathfrak{D}^* = \mathbf{D}(G_1, D_1) \otimes \mathbf{D}(G_2, D_2)$ (cf. [9]) are the sets of one of two forms: $[a_1]^* \times \{(a_2)\}$ or $\{(a_1)\} \times [a_2]^*$. Therefore, the lines of $\mathfrak{D}$ are unions of some pairs of the lines of $\mathfrak{D}^*$.

From Proposition 3.4 we have the following:

Fact 4.2 Let D_i be QDS in a group G_i such that $\langle D_i \rangle_{G_i} = G_i$ for all $i \in I$. Then $\sum_{i \in I} D_i$ generates $\sum_{i \in I} G_i$. Consequently, if every one of the structures $\mathfrak{D}_i = \mathbf{D}(G_i, D_i)$ is connected then $\sum_{i \in I} \mathfrak{D}_i$ is connected as well.

Let $J \subset I$; we extend the inclusions ε_i to the map $\varepsilon_J: \sum_{j \in J} G_j \longrightarrow \sum_{i \in I} G_i$ by the condition

$$(\varepsilon_J(a))(i) = \begin{cases} 1_i & \text{for } i \in I \setminus J \\ a(i) & \text{for } i \in J \end{cases} \quad \text{for arbitrary } a \in \sum_{j \in J} G_j. \quad (4.2)$$

Proposition 4.3 Let J be a nonempty proper subset of I . Then

$$\sum_{i \in I} \mathfrak{D}_i \cong \sum_{i \in J} \mathfrak{D}_i \oplus \sum_{i \in I \setminus J} \mathfrak{D}_i.$$

Proposition 4.4 Let $\mathfrak{D}_i = \mathbf{D}(G_i, D_i)$, where D_i is QDS in a group G_i for all $i \in I$. Assume that there is a pair of bijections $\varphi'_i, \varphi''_i: G_i \longrightarrow G_i$ such that the pair $\varkappa_i = (\varkappa'_i, \varkappa''_i)$ of maps

$$\varkappa'_i: (a) \mapsto [\varphi'_i(a)], \quad \varkappa''_i: [a] \mapsto (\varphi''_i(a)) \quad \text{for } a \in G_i \quad (4.3)$$

is a correlation of $\mathfrak{D}_i$ for $i \in I$. Set $\varphi' = \varphi'_1 \times \cdots \times \varphi'_r$ and $\varphi'' = \varphi''_1 \times \cdots \times \varphi''_r$. If $\varphi'_i = \varphi''_i$ for every $i \in I$ (i.e. if $\varkappa_i$ are involutory), then the pair $\varkappa = (\varkappa', \varkappa'')$ of maps

$$\varkappa': (a) \mapsto [\varphi'(a)], \quad \varkappa'': [a] \mapsto (\varphi''(a)) \quad \text{for } a \in \sum_{i \in I} G_i \quad (4.4)$$

is an involutory correlation of $\sum_{i \in I} \mathfrak{D}_i$.

Proof By (4.1) we get $\varkappa''([b]) = (\varphi''(b)) \mid [\varphi'(a)] = \varkappa'((a))$ iff the following holds: $\varkappa''_i([b_i]) = (\varphi''_i(b_i)) \mid [\varphi'_i(a_i)] = \varkappa'_i((a_i))$ and $\varphi'_j(b_j) = \varphi'_j(a_j)$ for $j \neq i$. Now the claim is evident. $\square$

If we assume in Proposition 4.4 that every $\varkappa_i$ is the standard correlation ($\varphi'_i(a) = a^{-1}$, cf. Proposition 3.2), then $\varkappa$ is also the standard correlation. Using (4.1) as in Proposition 4.4 one can also prove the following:

Proposition 4.5 Let $\mathfrak{D}_i = \mathbf{D}(G_i, D_i)$ and $f_i = (f'_i, f''_i)$ be bijections of G_i such that $f'_i: (a) \mapsto (f'_i(a))$ and $f''_i: [a] \mapsto [f''_i(a)]$ yields a collineation of $\mathfrak{D}_i$ for $i \in I$, and let $\mathfrak{D} = \sum_{i \in I} \mathfrak{D}_i$. We set $F' = \prod_{i \in I} f'_i$, $F'' = \prod_{i \in I} f''_i$, and $F = (F', F'')$. Then the pair F is a collineation of $\mathfrak{D}$ iff $f'_i = f''_i$ for every $i \in I$.

Obviously, the pair (f'_i, f''_i) , where $f'_i = f''_i = \tau_{a_i}$ and $a_i \in G_i$, is a collineation of $\mathfrak{D}_i$; therefore, the pair $(\prod_{i \in I} f'_i, \prod_{i \in I} f''_i)$ is an automorphism of $\mathfrak{D}$. But this is a rather trivial result, as $\prod_{i \in I} \tau_{a_i} = \tau_a$. We have also some automorphisms of another type.

Proposition 4.6 Let $\beta \in S_n$, $x \in G^n$ and $h: G^n \longrightarrow G^n$ be the map defined by $h((x_1, \dots, x_n)) = (x_{\beta(1)}, \dots, x_{\beta(n)})$. The pair $F = (F', F'')$ with $F'((x)) = (h(x))$,

Fig. 1 Multiplied Pappus configuration

$F''([x]) = [h(x)]$ is a collineation of

$$\underbrace{\mathbf{D}(G, D) \oplus \mathbf{D}(G, D) \oplus \cdots \mathbf{D}(G, D)}_{n \text{ times}} = \mathbf{D}(G^n, D^n).$$

4.1 Cyclic Multiplying

Let C_k be a cyclic group of the rank k . We use additive notation for these groups. In this section we consider configurations $\mathbf{D}(G, \mathcal{D}_r)$, where $r \geq 2$, $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ and $\mathcal{D}_r = \{e_0, e_1, \dots, e_r\}$ is QDS with $e_0 = (0, \dots, 0)$ and $(e_i)_j = 0$ for $i, j = 1, \dots, r$, $i \neq j$, $(e_i)_i = 1$. The set $\mathcal{D}_r$ will be called *canonical* QDS. Note that $\mathbf{D}(G, \mathcal{D}_r) \cong \sum_{i=1}^r \mathbf{D}(C_{n_i}, \{0, 1\})$ and in view of Proposition 4.3

$$\mathbf{D}(G, \mathcal{D}_r) \cong \mathbf{D}(C_{n_1}, \{0, 1\}) \oplus \mathbf{D}(C_{n_2} \oplus \cdots \oplus C_{n_r}, \mathcal{D}_{r-1}).$$

Thus $\mathbf{D}(G, \mathcal{D}_r)$ generalizes the construction of cyclically inscribed polygons considered in [10]. An interesting example of this type is $\mathbf{D}(C_3 \oplus C_3 \oplus C_3, \mathcal{D}_3)$ —three copies of Pappus configuration cyclically inscribed, see Fig. 1 and a more general case considered in Sect. 6.2.

Lemma 4.7 Let $\mathfrak{M} = \mathbf{D}((C_k)^n, \mathcal{D}_n)$, α be a permutation of the set $\{0, \dots, n\}$ and $i = 0, \dots, n$.

- (i) A permutation α induces a collineation $f = (f', f'')$ of the structure $\mathfrak{M}$ such that $f'(e_0) = e_0$ and $f''(-e_i) = -e_{\alpha(i)}$ for all i .
- (ii) If $f = (f', f'')$ is a collineation induced by α such that $f'(e_0) = e_0$, $f''(-e_i) = -e_{\alpha(i)}$ for all i , then $f' \tau_v(f')^{-1} = \tau_{f'(v)}$.

Proof (i) Assume that $\alpha(0) = 0$ and consider a map $f' : G \rightarrow G$ defined by

$$f'(x_1, \dots, x_n) = (x_{\alpha(1)}, \dots, x_{\alpha(n)}). \quad (4.5)$$

Then $f' \in \text{Aut}(\mathcal{G})$ and $f'(\mathcal{D}_n) = \mathcal{D}_n$, and thus f' determines an automorphism of $\mathfrak{M}$. It is seen that $f'(e_0) = e_0$. In view of Fact 3.1 we get $f''(y_1, \dots, y_n) = (y_{\alpha(1)}, \dots, y_{\alpha(n)})$, so $f''(-e_i) = -e_{\alpha(i)}$.

Let $\alpha(0) = s \neq 0$ be a transposition and a map $f' : G \rightarrow G$ be given by:

$$f'(x_1, \dots, x_n) = \left(x_1, \dots, x_{s-1}, -\sum_{i=1}^n x_i, x_{s+1}, \dots, x_n \right). \quad (4.6)$$

Again $f' \in \text{Aut}(\mathcal{G})$, and note that $f'(\mathcal{D}_n) = -e_s + \mathcal{D}_n$, $f'(e_0) = e_0$. From Fact 3.1 f' induces a collineation of $\mathfrak{M}$ with $f''(y_1, \dots, y_n) = (y_1, \dots, y_{s-1}, -\sum_{i=1}^n y_i - 1, y_{s+1}, \dots, y_n)$. It is easy to check that $f''(-e_i) = -e_{\alpha(i)}$. As every permutation is one of two permutations considered above, we get our claim.

- (ii) Let α be a permutation such that $\alpha(0) = 0$, then f' is given by (4.5) and $(f')^{-1}(x_1, \dots, x_n) = (x_{\alpha^{-1}(1)}, \dots, x_{\alpha^{-1}(n)})$. Therefore $f'\tau_v(f')^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_n) + f'(v) = \tau_{f'(v)}(x_1, \dots, x_n)$. If α is a transposition, then f' is given by (4.6), and thus $f' = (f')^{-1}$. After simple calculation we get the claim. $\square$

Lemma 4.8 *Let $\mathfrak{M} = \mathbf{D}((C_3)^n, \mathcal{D}_n)$. For every point o of $\mathfrak{M}$, any two distinct lines k, l through o and every point p with $o \neq p \nmid k$ there is the unique point q such that $o \neq q \nmid l$ and $p \sim q$.*

Proof As translations are transitive subgroup of $\text{Aut}(\mathfrak{M})$ assume that $o = e_0$. Then, by Lemma 3.3, $k = [-e_i]$, $l = [-e_k]$ and $p = (-e_k + e_j)$ for some $i, j, k = 0, \dots, n$, $i \neq k \neq j$. Assume that q is on $[-e_k]$ and q is collinear with p . Using Lemma 3.3 we get

$$q = \begin{cases} (-e_i + e_j) & \text{if } i \neq j, \\ (-e_i + e_k) & \text{if } i = j. \end{cases} \quad (4.7)$$

$\square$

Note that the condition (3.5) does not hold for all canonical quasi difference sets.

Fact 4.9 *The canonical QDS in $(C_k)^n$ satisfies (3.5) iff $k > 3$.*

Theorem 4.10 *Let $\mathfrak{M} = \mathbf{D}((C_k)^n, \mathcal{D}_n)$ and $k > 3$. The group $\text{Aut}(\mathfrak{M})$ is isomorphic to $S_{n+1} \times (C_k)^n$.*

Proof By Corollary 3.9 every collineation $f = (f', f'')$ fixing e_0 is uniquely determined by a permutation α of lines through e_0 , i.e. $f''(-e_i) = -e_{\alpha(i)}$. From Lemma 4.7(i) every permutation α induces a collineation. So, there is an isomorphism $\xi: f \mapsto \alpha$, and consequently $\text{Aut}(\mathfrak{M})_{(e_0)} \cong S_{n+1}$. Note that every automorphism of $\mathfrak{M}$ is a composition of two maps: a translation and a collineation fixing e_0 . Using Lemma 4.7(ii) we get $(\tau_u f_\alpha)(\tau_v f_\beta) = \tau_u \tau_{f_\alpha(v)} f_{\alpha\beta}$, which closes the proof. $\square$

Next, we consider more general case. Namely, we describe the neighborhood of a point q in a configuration of the following form:

$$\mathfrak{M} = \mathbf{D}(C_k, \{0, 1\}) \oplus \mathbf{D}(G, D) = \mathbf{D}(C_k \oplus G, \{0, 1\} \uplus D),$$

where D is QDS in an abelian group G . Directly from definitions we calculate the following:

Lemma 4.11 *Let $D = \{d_0, \dots, d_n\}$ be QDS in an abelian group $G = \langle G, +, \theta \rangle$, and let $q = (0, \theta) \in C_k \times G$. Set $\mathfrak{M} = \mathbf{D}(C_k, \{0, 1\}) \oplus \mathbf{D}(G, D)$. The lines of $\mathfrak{M}$ through q are the following:*

- (1) $l_i = [0, -d_i]$ for $i = 0, \dots, n$.

Each line $[0, -d_i]$ contains q and the following points:

- (a) $q_{i,j} = (0, -d_i + d_j)$ for $j = 0, \dots, i, i \neq j$;
 (b) $p'_i = (1, -d_i)$.
 (2) $l'' = [-1, \theta] = [k-1, \theta]$.

Its points are q and the following:

- (c) $p''_i = (-1, d_i)$ for $i = 0, \dots, n$.

Then the points $q_{i,j}$ form a substructure isomorphic under the map $(0, a) \mapsto (a)$ to the neighborhood of θ in $\mathbf{D}(G, D)$. Moreover, the following additional lines appear:

- (3) For every $i, j = 0, \dots, n, i \neq j$ the line $l'_{i,j} = [-1, -d_j + d_i]$ joins $p''_i = (-1, d_i) \mid [-1, \theta] = l''$ with $q_{j,i} = (0, -d_j + d_i) \mid [0, -d_j] = l_j$.
 (4) For every i, j as above, the line $l'_{i,j} = l'_{j,i} = [1, -(d_i + d_j)]$ joins $p'_i = (1, -d_i) \mid [0, -d_i] = l_i$ with $p'_j = (1, -d_j) \mid [0, -d_j] = l_j$.

The lines listed above are pairwise distinct.

- (i) If $k > 3$, then no other line appears (Fig. 2).
 (ii) Let $k = 3$. Then $-1 = 2$ holds in C_k , and then another connections are associated with triples $(d_i, d_j, d_r) \in D$ satisfying

$$d_i + d_j + d_r = \theta. \quad (4.8)$$

Namely, let (4.8) be satisfied. Evidently, $-(d_i + d_j) = d_r$.

- (5) The line $l'_{i,j} = [1, -(d_j + d_i)] = [1, d_r]$ passes through $p''_r = (-1, d_r) \mid [-1, \theta] = l''$ and $p'_j = (1, -d_j) \mid [0, -d_j] = l_j$.
 (6) If, moreover, $j \neq i$, then the above line passes through p'_i as well so, it coincides with the line defined in (3.2) (Fig. 3).

Recall from [10] that a collineation of $\mathbf{D}(C_k, \{0, 1\})$ is simply an element of the dihedral group D_k , i.e. it is any map $\alpha_{\varepsilon, q}: i \mapsto \varepsilon i + q$, where $\varepsilon \in \{1, -1\}$. Proposition 4.5 determines all the automorphisms of $\mathbf{D}(C_k \oplus G, \{0, 1\} \uplus D)$ of the form $(i, a) \mapsto (\alpha_{\varepsilon, q}(i), f'(a))$. Still, in this case we should look for automorphisms defined with more complicated formulas.

Fig. 2 The neighborhood of the point $(0, \theta)$ in a configuration $\mathbf{D}(C_k, \{0, 1\}) \oplus \mathbf{D}(G, D)$ for $k > 3$

Fig. 3 The neighborhood of the point $(0, \theta)$ in a configuration $\mathbf{D}(C_3, \{0, 1\}) \oplus \mathbf{D}(G, D)$

Proposition 4.12 Let $\mathfrak{M}_0 = \mathbf{D}(G, D)$, $\mathfrak{M} = \mathbf{D}(C_k \oplus G, \{0, 1\} \uplus D)$ and $f = (f', f'') \in \text{Aut}(\mathfrak{M}_0)$. The following conditions are equivalent:

- (i) There is a collineation $\varphi = (\varphi', \varphi'')$ of $\mathfrak{M}$ such that $\varphi'((0, a)) = (0, f'(a))$ and $\varphi''([0, b]) = [0, f''(b)]$.
- (ii) There is a sequence f_i , $i = 0, \dots, k$, of collineations of $\mathfrak{M}_0$ defined recursively by the formulas: $f_0 = f$ and $f'_{i+1} = f''_i$, where $f_i = (f'_i, f''_i)$.

In the case (ii) we have $\varphi'((i, a)) = (i, f'_i(a))$ and $\varphi''([i, b]) = [i, f''_i(b)]$.

Proof It suffices to note that if (i) holds, then $(1, a) \mid [0, a]$ for every $a \in G$, which gives $\varphi'((1, a)) \mid \varphi''([0, a]) = [0, f''(a)]$, and thus $\varphi'((1, a)) = (1, f''(a))$. Therefore, f'' (as a transformation of points) induces a collineation of $\mathfrak{M}_0$. $\square$

5 Elementary Properties

Now we discuss some elementary axiomatic properties of $\mathbf{D}(G, D)$. Let D be QDS in an abelian group G . Using Lemma 3.6 we prove the following:

Proposition 5.1 *Under assumption (3.5) the structure $\mathbf{D}(G, D)$ is Veblenian.*

Proposition 5.2 *Under assumption (3.5) the structure $\mathbf{D}(G, D)$ is Desarguesian.*

As a consequence of Fact 4.9 and Propositions 5.1, 5.2 we get

Corollary 5.3 *For $k > 3$ the structure $\mathbf{D}((C_k)^n, \mathcal{D}_n)$ is Veblenian and Desarguesian.*

Proposition 5.4 *The structure $\mathfrak{M} = \mathbf{D}((C_3)^n, \mathcal{D}_n)$ is not Veblenian.*

Proof Take $l_i = [-e_i]$ for $i = 1, 2$, and $k_1 = [e_1 + e_2]$, $k_2 = [-e_1 - e_2 + e_3]$. Then $(e_0) \mid l_1, l_2$, and k_1 crosses l_1 in $(-e_1 + e_2)$ and l_2 in $(-e_2 + e_1)$. Furthermore, the lines k_2, l_1 meet in $(-e_1 + e_3)$, and $(-e_2 + e_3)$ is the common point of l_2, k_2 . Suppose that $k_1 \cap k_2 \neq \emptyset$. Then $(e_1 + e_2) + e_t = (-e_1 - e_2 + e_3) + e_s$ for some $s, t = 1, \dots, n$, $s \neq t$, which implies a contradiction: $e_1 + e_2 + e_3 + e_s = e_t$. $\square$

Proposition 5.5 *The structure $\mathfrak{M} = \mathbf{D}((C_3)^n, \mathcal{D}_n)$ is Desarguesian.*

Proof Without loss of generality we can assume that (e_0) is the perspective center of two triangles T_1, T_2 inscribed into three lines l_1, l_2, l_3 of $\mathfrak{M}$ such that the corresponding pairs of their sides intersect each other. We do some calculations based on Lemma 3.3. Note that $l_r = [-e_{i_r}]$ for $r = 1, 2, 3$, $e_{i_r} \in \mathcal{D}_n$. By Lemma 4.8 and (4.7) the sides of our triangles are lines of the form $[-e_i - e_j + e_s]$ or $[e_i + e_j]$ for some $e_i, e_j, e_s \in \mathcal{D}_n$, $s \neq i, j$. The line $[e_{i_1} + e_{i_2}]$ does not meet any other line which crosses both l_1 and l_2 . Thus the sides are lines of the type $[-e_i - e_j + e_s]$. Consequently, the vertices of T_1 are $(-e_{i_1} + e_s)$, $(-e_{i_2} + e_s)$, $(-e_{i_3} + e_s)$, and the vertices of T_2 are $(-e_{i_1} + e_t)$, $(-e_{i_2} + e_t)$, $(-e_{i_3} + e_t)$ for $e_{i_r} \neq e_s, e_t \in \mathcal{D}_n$. Then the points of intersection of the corresponding sides of T_1 and T_2 are $c_1 = (-e_{i_1} - e_{i_2} + e_s + e_t)$, $c_2 = (-e_{i_2} - e_{i_3} + e_s + e_t)$, and $c_3 = (-e_{i_3} - e_{i_1} + e_s + e_t)$. All these points are on the line $[-e_{i_1} - e_{i_2} - e_{i_3} + e_s + e_t]$, which proves the claim. $\square$

The Pappus configuration can be considered as $\mathbf{D}(C_3 \oplus C_3, \mathcal{D}_2)$, cf. [10, 15] (see Fig. 4).

Proposition 5.6 *Under assumption (3.5) the structure $\mathbf{D}(G, D)$ does not contain the Pappus configuration.*

Fig. 4 Pappus configuration

Proof Assume that $\mathbf{D}(G, D)$ contains the Pappus configuration. Then $(0, 0), (1, 0), (0, 1) \mid [0, 0]; (0, 0), (1, 2), (0, 2) \mid [0, 2]; (1, 0) \sim (1, 2)$, and $(0, 1) \sim (0, 2)$, which contradicts Lemma 3.6(ii). $\square$

Proposition 5.7 Let D be QDS in an abelian group G such that $1 \in D$. Assume that there are $d_1, d_2, d_3, d_4 \in D \setminus \{1\}$ with $d_1 \neq d_3, d_1^2 = d_2^{-1}$, and $d_3^2 = d_4^{-1}$. Then $\mathfrak{M} = \mathbf{D}(G, D)$ contains Pappus configurations.

Proof From the assumptions we get $d_2 \neq d_4$. Note that incidences indicated in the following table hold in $\mathfrak{M}$:

	(1)	(d_1)	(d_3)	(d_2^{-1})	(d_4^{-1})	($d_2^{-1}d_4^{-1}$)	(d_1d_3)	($d_2^{-1}d_3$)	($d_4^{-1}d_1$)
[1]	×	×	×						
[$d_2^{-1}d_4^{-1}$]				×	×	×			
[d_1d_3]							×	×	×
[d_2^{-1}]	×			×				×	
[d_4^{-1}]	×				×				×
[d_1]		×		×			×		
[$d_1d_4^{-1}$]		×				×			×
[d_3]			×		×		×		
[$d_3d_2^{-1}$]			×			×		×	

Then the map (ψ', ψ'') defined for the points by

(a)	(0, 0)	(1, 0)	(0, 1)	(1, 2)	(2, 1)	(1, 1)	(2, 2)	(0, 2)	(2, 0)
$\psi'((a))$	(1)	(d_1)	(d_3)	(d_2^{-1})	(d_4^{-1})	($d_2^{-1}d_4^{-1}$)	(d_1d_3)	($d_2^{-1}d_3$)	($d_4^{-1}d_1$)

and for the lines by

$[a]$	$[0, 0]$	$[1, 1]$	$[2, 2]$	$[0, 2]$	$[2, 0]$	$[1, 2]$	$[1, 0]$	$[2, 1]$	$[0, 1]$
$\psi''([a])$	$[1]$	$[d_2^{-1}d_4^{-1}]$	$[d_1d_3]$	$[d_2^{-1}]$	$[d_4^{-1}]$	$[d_1]$	$[d_1d_4^{-1}]$	$[d_3]$	$[d_3d_2^{-1}]$

embeds the Pappus configuration into the structure $\mathfrak{M}$. $\square$

6 Examples

The goal of this section was to present some new and (we hope) interesting examples of configurations induced by quasi difference sets. Some of them are copies of one fixed configuration repeatedly inscribed, and the others are a join of a few well-known configurations. We determine automorphisms group of every example. However, in most cases we do not present all details of proofs, as they are very technical. Instead, we show only essential steps in the hope that they suffice to understand the idea and a specificity of a proof.

Let $G = \langle G, +, 0 \rangle$ be an abelian group and $\mathcal{D} \subset G$. Recall, cf. [14], that a *multiplier* α of the set $\mathcal{D}$ is an automorphism of G of the form $x \mapsto \alpha \cdot x$ satisfying $\alpha\mathcal{D} = q + \mathcal{D}$ for some $q \in G$.

6.1 Multi-Fano Configuration

The Fano configuration $\mathfrak{F}$ is a finite projective plane, so it can be obtained as $\mathfrak{F} = \mathbf{D}(C_7, \{0, 1, 3\})$ (cf. [5, 15]). Let us introduce the *multi-Fano* configuration $\mathfrak{F}^+ = \mathbf{D}(C_k \oplus C_7, \{(0, 0), (0, 1), (0, 3), (1, 0)\}) \cong \mathbf{D}(C_k, \{0, 1\}) \oplus \mathfrak{F}$. We refer to the subconfiguration determined by points of the form $\{i\} \times C_7$ with fixed $i = 0, \dots, k-1$ as to a Fano's part of $\mathfrak{F}^+$.

The following two Lemmas can be easily proved by analyzing the neighborhood of a point (cf. Fig. 5).

Lemma 6.1 *If $f \in \text{Aut}(\mathfrak{F}^+)$, and x, y are two points of $\mathfrak{F}^+$, such that $f(x) = y$, then f transforms the Fano's part of $\mathfrak{F}^+_{(x)}$ into the Fano's part of $\mathfrak{F}^+_{(y)}$.*

Lemma 6.2 *If $f \in \text{Aut}(\mathfrak{F}^+)$ and f preserves $\{0\} \times C_7$, then f preserves $\{i\} \times C_7$ for every $i = 0, \dots, k-1$.*

Proposition 6.3 (i) *If $7 \nmid k$, then $\text{Aut}(\mathfrak{F}^+) \cong C_k \oplus C_7$.*
(ii) *If $7 \mid k$, then $\text{Aut}(\mathfrak{F}^+) \cong C_3 \ltimes (C_k \oplus C_7)$.*

Proof Generally, $\text{Tr}(C_k \oplus C_7) \subseteq \text{Aut}(\mathfrak{F}^+)$. Let us take $g = \tau_{-f(0,0)} \circ f$, where $f \in \text{Aut}(\mathfrak{F}^+)$. Then $g((0, 0)) = (0, 0)$. By Lemma 6.1, every collineation $g \in \text{Aut}(\mathfrak{F}^+)_{((0,0))}$ preserves the Fano substructure in the neighborhood of the point $(0, 0)$.

Fig. 5 The neighborhood of a point (i, a) in multi-Fano configuration

According to Lemma 6.2, the Fano substructure is preserved on every of i levels, where $i = 0, 1, \dots, k-1$.

Denote the set $\{0, 1, 3\}$ by $\mathcal{D}$. Note that $\mathcal{D}$ has two multipliers: $2\mathcal{D} = 6 + \mathcal{D}$ and $4\mathcal{D} = 4 + \mathcal{D}$. Thus, in view of Fact 3.1 maps $g_1 = (g'_1, g''_1)$ with $g'_1(x) = 2x$, $g''_1 = \tau_6 g'_1$, and $g_2 = (g'_2, g''_2)$ with $g'_2(x) = 4x$, $g''_2 = \tau_4 g'_2$ are collineations of $\mathfrak{F}$. Moreover, $g_1, g_2 \in \text{Aut}(\mathfrak{F})_{(0)}$. In view of Proposition 4.12, a map $\varphi = (\varphi', \varphi'')$, such that $\varphi'(i, a) = (i, f'_i(a))$ and $\varphi''([i, b]) = [i, f''_i(b)]$, where $(f'_i, f''_i) = f_i \in \text{Aut}(\mathfrak{F})$, $i \in C_k$, is a collineation of $\mathfrak{F}^+$ iff $f'_{i+1} = f''_i$. If we analyze all the elements of $\text{Aut}(\mathfrak{F})_{(0)}$ we note that g_1, g_2 are the unique two maps that can be extended to a collineation of $\mathfrak{F}^+$, since translations are always collineations of $\mathfrak{F}$.

Set $f'_0 = g'_1$, $f''_0 = g''_1 = \tau_6 g'_1$. By induction we get $f'_i = \tau_{6i} g'_1$, $f''_i = \tau_{6(i+1)} g'_1$. In particular, $f''_{k-1} = \tau_{6k} g'_1 = f'_0 = g'_1$. Therefore $\tau_{6k} = \text{id}$, i.e. $6k \equiv 0$ holds in C_7 and thus $7 \mid k$. The same result we obtain for $(f'_0, f''_0) = (g'_2, g''_2)$. To close the proof note that $G = \{g_1, g_2, \text{id}\} \cong C_3$ and $f \tau_{(j,b)} f^{-1}(i, a) = \tau_{f(j,b)}(i, a)$ for $f \in G$. $\square$

6.2 Multi-Pappus Configuration

Since $\mathbf{D}((C_3)^2, \mathcal{D}_2)$ is simply the Pappus configuration, $\mathbf{D}((C_3)^n, \mathcal{D}_n)$ will be called the *multi-Pappus* configuration (there is $\mathbf{D}((C_3)^3, \mathcal{D}_3)$ shown in Fig. 1). Note that, in view of Fact 4.9, Proposition 4.10 cannot be used to characterize the automorphisms group of $\mathbf{D}((C_3)^n, \mathcal{D}_n)$.

Proposition 6.4 *Let $\mathfrak{M} = \mathbf{D}((C_3)^n, \mathcal{D}_n)$ with $n > 2$. Then $\text{Aut}(\mathfrak{M}) \cong S_{n+1} \ltimes (C_3)^n$.*

To prove Proposition 6.4 we use Lemmas 4.7, 4.8 and the following lemma:

Lemma 6.5 *Let $f = (f', f'') \in \text{Aut}(\mathfrak{M})$, p be a point of $\mathfrak{M}$ and l be a line through p . If f'' fixes every line through p and f' fixes every point on l then $f = \text{id}$.*

6.3 Join of the Multi-Pappus Configuration and a Cyclic Projective Plane

Now we focus on

$$\mathfrak{M} = \mathbf{D}((C_3)^k, \mathcal{D}_k) \oplus \mathbf{D}(C_n, D), \quad (6.1)$$

where $\mathfrak{P} = \mathbf{D}(C_n, D)$ is the cyclic projective plane $\text{PG}(2, q)$ with a prime power q , given by a difference set $D = \{d_0, d_1, \dots, d_q\}$ in the group C_n , where $n = q^2 + q + 1$ (cf. [2, 13]). The projective cyclic plane $\text{PG}(2, q)$ can be determined by $q + 1$ distinct difference sets $D^1, D^2, \dots, D^{q+1}$. Quite surprisingly, it turns out that $\mathfrak{M}^i = \mathbf{D}((C_3)^k, \mathcal{D}_k) \oplus \mathbf{D}(C_n, D^i)$, $i = 1, 2, \dots, q + 1$ are not always pairwise isomorphic. Indeed, we will show that $\mathbf{D}(C_3^k, \mathcal{D}_k) \oplus \mathbf{D}(C_{13}, \{0, 1, 3, 9\})$ is not isomorphic to $\mathbf{D}(C_3^k, \mathcal{D}_k) \oplus \mathbf{D}(C_{13}, \{0, 2, 8, 12\})$. In view of Proposition 4.3

$$\mathfrak{M} \cong \underbrace{\mathbf{D}(C_3, \{0, 1\}) \oplus \mathbf{D}(C_3, \{0, 1\}) \oplus \dots \oplus \mathbf{D}(C_3, \{0, 1\})}_{k \text{ times}} \oplus \mathbf{D}(C_n, D).$$

Hence, in this case we can use Lemma 4.11. Every point $p \in \mathfrak{M}$ can be written as $p = (x_k, \dots, x_1, y)$, where $x_k, \dots, x_1 \in C_3$, $y \in C_n$. Let $\theta = (0, \dots, 0) \in (C_3)^k \times C_n$. In this notation, the points and the lines of $\mathfrak{M}_{(\theta)}$ are the following:

$$\begin{aligned} q_{i,j} &= (0, \dots, 0, -d_i + d_j), \\ p'_{m,i} &= (0, \dots, 1_m, 0, \dots, -d_i), \quad p''_{m,i} = (0, \dots, 2_m, 0, \dots, d_i), \\ l_i &= [0, \dots, 0, -d_i], \quad l''_m = [0, \dots, 2_m, 0, \dots, 0], \\ \text{for } d_i, d_j &\in D; \ i, j = 0, \dots, q; \ i \neq j; \quad m = 1, \dots, k, \\ p'_{s,q+r} &= (0, \dots, 1_s, 0, \dots, 2_r, 0, \dots, 0), \quad p''_{s,q+r} = (0, \dots, 2_s, 0, \dots, 1_r, 0, \dots, 0), \\ \text{for } s &= 2, \dots, k; \ r = 1, \dots, s - q + 2. \end{aligned}$$

Using Lemma 4.6 and Proposition 4.5 we prove the following Lemmas:

Lemma 6.6 *Let $F \in \text{Aut}(\mathfrak{M})_{(\theta)}$. Then, F leaves invariant the multi-Pappus sub-configuration $(C_3)^k \times \{0\} \cong \mathbf{D}((C_3)^k, \mathcal{D}_k)$ and the cyclic projective subplane $\{(0, \dots, 0)\} \times C_n \cong \mathbf{D}(C_n, D)$. Moreover, F determines permutations $\alpha \in S_q$ and $\beta \in S_k$ such that $F(l''_m) = l''_{\beta(m)}$, $F(p'_{m,0}) = p'_{\beta(m),0}$ and $F(l_i) = l_{\alpha(i)}$ for $m = 1, \dots, k$, $i = 1, \dots, q$.*

Lemma 6.7 *Let $\beta \in S_k$. We define the map G_β on $C_3^k \oplus C_n$ by the formula $G_\beta((x_k, \dots, x_1, y)) = (x_{\beta(k)}, \dots, x_{\beta(1)}, y)$. Then $G_\beta \in \text{Aut}(\mathfrak{M})$, $G_\beta(\theta) = \theta$, and $G_\beta(p'_{m,0}) = p'_{\beta(m),0}$.*

Lemma 6.8 *Let G_β be the map defined in Lemma 6.7 and $\mathcal{G}_0 = \{G_\beta; \beta \in S_k\}$, $\mathcal{G} = \{\tau_a \circ g; g \in \mathcal{G}_0, a \in (C_3)^k \times C_n\}$. Then $\mathcal{G}_0 \cong S_k$ and $\mathcal{G} \cong S_k \ltimes (C_3^k \oplus C_n)$.*

Lemma 6.9 *If, under notation of Lemma 6.6 we assume $\beta = \text{id}$, then every point $p''_{m,0}, p'_{s,q+r}, p''_{s,q+r}$ is fixed by F . Moreover, for $\alpha \in S_q$ we have $F(p'_{m,i}) = p'_{m,\alpha(i)}$ for $i = 1, \dots, q$.*

Lemma 6.10 *Under assumptions of Lemma 6.9, and the condition*

(a) *for every, except at most one, $d_i \in D$ there exist $d_j, d_r \in D$ such that*

$$d_i + d_j + d_r = 0$$

the permutation α given in Lemma 6.6 satisfies the following: $F(p''_{m,i}) = p''_{m,\alpha(i)}$, $F(q_{i,j}) = q_{\alpha(i),\alpha(j)}$, for $m = 1, \dots, k$; $i, j = 1, \dots, q$, $i \neq j$. Consequently, if $\alpha = \text{id}$, then F is the identity on $\mathfrak{M}_{(\theta)}$.

Let q be a point of $\mathfrak{M}$ and F be an automorphism of $\mathfrak{M}$ with $F(q) = q$. Generalizing the notation of Lemma 6.6 we write ${}^\alpha F_{(q)}$ for the permutation α of $\{1, \dots, q\}$ and ${}^\beta F_{(q)}$ for the permutation β of $\{1, \dots, k\}$ determined by $F \upharpoonright \mathfrak{M}_{(q)}$.

Lemma 6.11 *Let the condition (a) of Lemma 6.10 be satisfied. If $F \in \text{Aut}(\mathfrak{M})$ preserves every line passing through q (in particular, $F(q) = q$), then ${}^\beta F_{(q)}$ and ${}^\alpha F_{(q)}$ are identities, and thus F is the identity on $\mathfrak{M}_{(q)}$.*

Lemma 6.12 *Let the condition (a) of Lemma 6.10 be satisfied. Assume that $F \in \text{Aut}(\mathfrak{M})$ fixes all the points of $\mathfrak{M}_{(q)}$. If $q' \in \mathfrak{M}_{(q)}$, then F fixes the points of $\mathfrak{M}_{(q')}$.*

Since $\mathfrak{M}$ is connected, combining Lemmas 6.10 and 6.12 we obtain

Corollary 6.13 *Let the condition (a) of Lemma 6.10 be satisfied. Assume that $F \in \text{Aut}(\mathfrak{M})$ and q is a point of $\mathfrak{M}$. If $F(q) = q$ and F preserves every line through q , then $F = \text{id}$.*

With the help of Lemma 6.6–Corollary 6.13 we determine automorphisms group of $\mathfrak{M} = \mathbf{D}((C_3)^k, \mathcal{D}_k) \oplus \mathfrak{P}$ in two particular cases: for $\mathfrak{P} = \mathfrak{F} = \text{PG}(2, 2)$ and for $\mathfrak{P} = \text{PG}(2, 3)$. Let us start from $\mathfrak{M} = \mathbf{D}((C_3)^k, \mathcal{D}_k) \oplus \mathfrak{F}$. The obtained structure can be considered as a join of the multi-Pappus and the Fano configuration.

Proposition 6.14 *Let $\mathfrak{M} = \mathbf{D}(C_3^k, \mathcal{D}_k) \oplus \mathbf{D}(C_7, \{0, 1, 3\})$ with $k \geq 2$. Then the group $\text{Aut}(\mathfrak{M})$ is isomorphic to $S_k \ltimes (C_3^k \oplus C_7)$.*

Adopt $\mathfrak{P} = \text{PG}(2, 3)$ instead of Fano configuration. As $\text{PG}(2, 3)$ can be induced by four distinct difference sets $D^1 = \{0, 1, 3, 9\}$, $D^2 = \{0, 2, 8, 12\}$, $D^3 = \{0, 6, 10, 11\}$, $D^4 = \{0, 4, 5, 7\}$, we get

$$\mathfrak{M}^i = \mathbf{D}(C_3^k, \mathcal{D}_k) \oplus \mathbf{D}(C_{13}, D^i),$$

where $i = 1, 2, 3, 4$. Note, that the condition (a) from Lemma 6.10 is satisfied for all sets D^i , but in different ways: for every $d_i \in D^1$ there exist $d_j, d_r \in D^1$ such that $d_i + d_j + d_r = 0$, and in D^2, D^3, D^4 there is exactly one d_i for which do not exist such d_j, d_r . So, the configurations $\mathfrak{M}^j$ are pairwise isomorphic for $j = 2, 3, 4$ (for $\phi(x) = 3x$ we get $\phi(D^2) = D^3$, $\phi(D^3) = D^4$, $\phi(D^4) = D^2$), but are not isomorphic to $\mathfrak{M}^1$. Although $\mathfrak{M}^1$ and $\mathfrak{M}^2$ are not isomorphic, they have isomorphic automorphisms groups. To justify this, we show the complete proof of the following Proposition:

Proposition 6.15 Let $\mathfrak{M}^i = \mathbf{D}(C_3^k, \mathcal{D}_k) \oplus \mathbf{D}(C_{13}, D^i)$, where $i = 1, 2$ and $D^1 = \{0, 1, 3, 9\}$, $D^2 = \{0, 2, 8, 12\}$. Then, the group $\text{Aut}(\mathfrak{M}^i)$ is isomorphic to $S_k \ltimes (C_3^k \oplus C_{13})$.

Proof Let F be an automorphism of $\mathfrak{M}^i$ and $g = \tau_{-F(\theta)} \circ F$. Then $g(\theta) = \theta$ and $g \in \text{Aut}(\mathfrak{M}^i)$. In view of Lemma 6.6, g leaves the set $\{p'_{m,0} : m = 1, \dots, k\}$ invariant. We set $h = g \circ G_\beta^{-1}$, where G_β is the map defined in Lemma 6.7 and $\beta = {}^\beta g_{(\theta)} \in S_k$. Then $h \in \text{Aut}(\mathfrak{M}^i)$, $h(\theta) = \theta$, and ${}^\beta h_{(\theta)} = \text{id}$. By Lemma 6.9 h preserves the set $\{l''_m : m = 1, \dots, k\}$.

Let $\alpha = {}^\alpha h_{(\theta)}$ be the permutation determined by h , in accordance with Lemma 6.9. Then, from Lemma 6.10 we get $h(q_{i,j}) = h_\alpha(q_{i,j}) := q_{\alpha(i), \alpha(j)}$ for all i, j . Note that if $\alpha \neq \text{id}$, then h_α does not preserve the collinearity in $\mathfrak{M}^1$. For example: $q_{1,2}$, $q_{0,2}$, $q_{3,1}$, $q_{2,1}$ are collinear, but $q_{\alpha(1), \alpha(2)}$, $q_{0, \alpha(2)}$, $q_{\alpha(3), \alpha(1)}$, $q_{\alpha(2), \alpha(1)}$ are not, unless $\alpha = \text{id}$. In $\mathfrak{M}^2$ for all $m = 1, \dots, k$ we have: $p'_{m,1}$ are points of rank 5 on a line of rank 3, $p'_{m,2}$ are points of rank 5 and there is no line of rank 3 passing through these points, and $p'_{m,3}$ are points of rank 6. So, h fixes these points, and thus $\alpha = \text{id}$.

In both cases, h fixes all the lines through θ , so from Corollary 6.13 we get $h = \text{id}$, and thus $g = G_\beta$. Finally, applying Lemma 6.8 we close the proof. $\square$

6.4 A Power of a Cyclic Projective Plane

Let $\mathfrak{P} = \mathbf{D}(C_k, \mathcal{D})$ be a cyclic projective plane determined by a difference set $\mathcal{D}$ in the group C_k . Then $k = q^2 + q + 1$, and $q + 1$ is both the size of a line and the degree of a point of $\mathfrak{P}$. Let us draw our attention to the following structure:

$$\mathfrak{P}^n := \underbrace{\mathfrak{P} \oplus \mathfrak{P} \oplus \dots \oplus \mathfrak{P}}_{n \text{ times}} \quad (6.2)$$

Note that $\mathfrak{P}^n = \mathbf{D}((C_k)^n, D)$, where $D = \mathcal{D} \uplus \dots \uplus \mathcal{D}$. Let us introduce a few auxiliary sets. Namely:

$$\begin{aligned} \text{supp}(x) &:= \{i \in \{1, \dots, n\} : x_i \neq 0\} \text{ for } x \in (C_k)^n, \\ \Xi_\alpha^\gamma &:= \{x \in (C_k)^n : |\text{supp}(x)| = \gamma \text{ and if } i \in \text{supp}(x) \text{ then } x_i = \alpha\}, \\ \Xi_{\{\alpha\}, \{\beta\}}^1 &:= \Xi_\alpha^1 \cup \Xi_\beta^1, \\ \Xi_{\{\alpha, \beta\}}^2 &:= \{x \in (C_k)^n : |\text{supp}(x)| = 2, \{i, j\} = \text{supp}(x) \implies \{x_i, x_j\} = \{\alpha, \beta\}\}, \\ P_i &:= \{x \in (C_k)^n : \text{supp}(x) = \{i\}, \text{ or } x = \theta\}, \\ S_i &:= \{(x) : x \in P_i\}, \quad \mathcal{J}_i = \{[x] : x \in P_i\}, \quad i = 1, \dots, n. \end{aligned}$$

For $i_1 \neq i_2$ we have $\mathcal{J}_{i_1} \cap \mathcal{J}_{i_2} = \{[\theta]\}$ and $S_{i_1} \cap S_{i_2} = \{(\theta)\}$. The sets S_i and $\mathcal{J}_i$ consist of points and lines, respectively, which form a projective plane embedded in $\mathfrak{P}^n_{(\theta)}$. There are n such planes with the common line $[\theta]$, and the common point (θ) in $\mathfrak{P}^n_{(\theta)}$. Note that, the degree of the point (θ) in $\mathfrak{P}^n_{(\theta)}$ equals $qn + 1$, and it is equal to the size of every line through (θ) .

By Lemma 3.3 and (4.1) we can prove the following lemmas:

Lemma 6.16 *The line $[y]$ passes through (θ) iff $[y] \in \mathcal{J}_i$ for some i and $y_i \in -\mathcal{D}$.*

Lemma 6.17 *Let $x \in (C_k)^n$. The point (x) is a point of $\mathfrak{P}_{(\theta)}^n$ iff $x \in P_i$ for some i (i.e. $|\text{supp}(x)| = 1$) or $x \in \Xi_{\{\alpha, \beta\}}^2$ and $\alpha \in -\mathcal{D} \setminus \{0\}$, $\beta \in \mathcal{D} \setminus \{0\}$.*

Lemma 6.18 *If $x \in P_i$ then either the line $[x]$ passes through (θ) and its size in $\mathfrak{P}_{(\theta)}^n$ is $qn + 1$, or the size of $[x]$ in $\mathfrak{P}_{(\theta)}^n$ is $q + 1$; then, in particular, $[x]$ does not contain any point of P_j with $j \neq i$.*

Lemma 6.19 *Let $x \in (C_7)^n$. If the line $[x]$ contains a point of $\mathfrak{P}_{(\theta)}^n$ (i.e. it intersects a line of the form $[y]$ defined in Lemma 6.16), then $|\text{supp}(x)| \leq 3$. Moreover, if $|\text{supp}(x)| = 3$, then the size of $[x]$ in $\mathfrak{P}_{(\theta)}^n$ is 2.*

There are lines in $\mathfrak{P}_{(\theta)}^n$ joining points in P_i with points in $\Xi_{\{\alpha, \beta\}}^2$, where $\alpha \in -\mathcal{D} \setminus \{0\}$, $\beta \in \mathcal{D} \setminus \{0\}$. Namely:

Lemma 6.20 *Let $(x) \in S_i$, $[y] \in \mathcal{J}_i$, and $(x) \nmid [y]$. For every $j \neq i$ there are q lines of the size 2 in $\mathfrak{P}_{(\theta)}^n$, such that each of them joins (x) with one of the pairwise collinear points $(z^1), \dots, (z^q)$, where $z_i^1 = \dots = z_i^q = x_i - y_i$, $\{z_j^1, \dots, z_j^q\} = -\mathcal{D} \setminus \{0\}$, and $z_s^1 = \dots = z_s^q = 0$ for all $s \neq i, j$; $s = 1, \dots, n$.*

Lemma 6.21 *For any two points a, a' of $\mathfrak{P}^n$ there is a sequence $b_0, \dots, b_m$ of points of $\mathfrak{P}^n$ such that $b_0 = a$, $b_m = a'$, and b_j is a point of a projective subplane in $\mathfrak{P}_{(b_{j-1})}^n$ for $j = 1, \dots, m$.*

On this level of generality not much more could be said. Now we consider a power of PG(2, 2) and PG(2, 3).

6.4.1 A Power of the Fano Plane

Let us put $\mathfrak{P} = \mathfrak{F} = \mathbf{D}(C_7, \{0, 1, 3\})$.

Proposition 6.22 *The group $\text{Aut}(\mathfrak{F}^n)$ is isomorphic to $S_n \ltimes (C_7)^n$.*

Proof Let $\mathcal{J}_0''$ be the family of lines of the size 4 in $\mathfrak{F}_{(\theta)}^n$ and $\mathcal{J}_0'$ be the family of the lines of the size 3 in $\mathfrak{F}_{(\theta)}^n$ that are not in any of the $\mathcal{J}_i$. We need Lemma 4.6 and lemmas from Sect. 6.4.

Step 1 Let $y \in (C_7)^n$. Then $[y] \in \mathcal{J}_0''$ iff $y \in \Xi_{\{1,3\}}^2$. Let $y \in \Xi_{\{1,3\}}^2$ and $\text{supp}(y) = \{i_1, i_2\}$. The line $[y]$ does not intersect $[\theta]$, but $[y]$ intersects every of the remaining two lines in $\mathcal{J}_{i_1}$ and in $\mathcal{J}_{i_2}$. Consequently, for any two $i_1, i_2 \in \{1, \dots, n\}$ there is (exactly one) line in $\mathcal{J}_0''$ that crosses two lines in $\mathcal{J}_{i_1}$ and two lines in $\mathcal{J}_{i_2}$. No two distinct lines in $\mathcal{J}_0''$ intersect.

Step 2 If $y \neq \theta$, then $[y]$ is of the size 3 in $\mathfrak{F}_{(\theta)}^n$ iff either $y \in \Xi_{\{1,3\}}^2$ or $y \in P_i$ for some $i \in \{1, \dots, n\}$. This gives, in particular, that $\mathcal{J}_0' = \{[y]: y \in \Xi_{\{1,3\}}^2\}$. Let $\text{supp}(y) = \{i_1, i_2\}$, $y_{i_1} = 1$, and $y_{i_2} = 3$. Lines $[y]$ and $[\theta]$ do not meet. The line $[y]$ crosses two other lines in $\mathcal{J}_{i_2}$ and it crosses exactly one line in $\mathcal{J}_{i_1}$. Consequently, for every two $i_1, i_2 \in \{1, \dots, n\}$ there is (exactly one) line in $\mathcal{J}_0'$ that crosses two lines in $\mathcal{J}_{i_2}$ and crosses exactly one line in $\mathcal{J}_{i_1}$. No two distinct lines in $\mathcal{J}_0'$ intersect.

Fig. 6 The neighborhood of the point $(0, 0)$ in $\mathfrak{F} \oplus \mathfrak{F}$

Directly from Step 2 we get:

Step 3 A line l in $\mathfrak{F}^n_{(\theta)}$ belongs to $\mathcal{J}'_0$ iff the size of l equals 3 and no other line of the size 3 in $\mathfrak{F}^n_{(\theta)}$ crosses l .

The next two Steps are immediate from Lemma 6.17 and Steps 1, 2:

Step 4 Let (x) be a point of $\mathfrak{F}^n_{(\theta)}$. Then $x \in P_i$ iff there are two distinct lines of the size 3 that pass through it. The set of points on the lines in $\mathcal{J}'_0 \cup \mathcal{J}''_0$ is the set of points (x) with $x \notin \bigcup_{i=1}^n P_i$.

Step 5 From the above it follows that $\mathfrak{F}^n_{(\theta)}$ contains n subconfigurations isomorphic to a Fano plane (cf. Fig. 6 with points marked by circles and squares). These are precisely substructures of the form $\langle S_i, \mathcal{J}_i, \mathcal{I} \rangle$. Intuitively, we can read Step 1 as “any two Fano subplanes of $\mathfrak{F}^n_{(\theta)}$ are joined by a line of the size 4”. Analogously, Step 2 explains how lines of the size 3 join the Fano subplanes. In view of Steps 3 and 4 an automorphism F preserves the set $\bigcup_{i=1}^n S_i$ and permutes the Fano subplanes. So, F determines a permutation σ such that F maps the set S_i onto $S_{\sigma(i)}$ and it maps the family $\mathcal{J}_i$ onto $\mathcal{J}_{\sigma(i)}$ for every $i = 1, \dots, n$.

Step 6 Obviously, F preserves the set of lines of the size 4 in $\mathfrak{F}^n_{(\theta)}$. Since these lines are of the form $[y]$ with $y \in \Xi_3^2$, we can identify every such a line $[y]$ with the set $\text{supp}(y) \in \mathcal{P}_2(\{1, \dots, n\})$. Every point (x) , where $x \in \Xi_3^3$, is in $\mathfrak{F}^n_{(\theta)}$ the meet of three lines $[y_t]$, $y_t \in \Xi_3^2$, and $t = 1, 2, 3$ iff $\text{supp}(y_t) \subset \text{supp}(x)$. Therefore, lines $\{[y]: y \in \Xi_3^2\}$ together with their intersection points form the structure dual to combinatorial Grassmanian $\mathbf{G}_3(n)$, cf. [12]. The map F determines a permutation F_0 of the lines in $\mathcal{J}''_0$ which, in view of the above, is an automorphism of $\mathbf{G}_3(n)$. The automorphisms group of $\mathbf{G}_3(n)$ is the group S_n (comp. [11]), and thus there is $\sigma' \in S_n$ which determines F_0 . It is seen (cf. Step 1) that $\sigma' = \sigma$. Let $G = G_\sigma$ be the automorphism of $\mathfrak{F}^n_{(\theta)}$ determined by the permutation σ (cf. Lemma 4.6) and let $\varphi = G^{-1} \circ F$. Clearly, φ is an automorphism of $\mathfrak{F}^n_{(\theta)}$, and φ maps every line in $\mathcal{J}''_0$ onto

itself. Consequently, φ maps every family $\mathcal{J}_i \setminus \{[\theta]\}$ onto itself and thus it leaves the line $[\theta]$ invariant.

Step 7 From Step 3, the map φ preserves the family $\mathcal{J}'_0$. Observing intersections of the lines of this family and the lines in the families $\mathcal{J}_i$ (cf. Step 2) we get that every line through θ remains invariant under φ .

Step 8 Let F be an automorphism of $\mathfrak{M}$ such that F leaves every line through a point a invariant. Then $F \upharpoonright \mathfrak{F}^n_{(a)} = \text{id}$.

Step 9 Let a, b be two points of $\mathfrak{F}^n$ such that b is a point of a Fano subplane in $\mathfrak{F}^n_{(a)}$. If F is an automorphism of $\mathfrak{F}^n$ such that $F \upharpoonright \mathfrak{F}^n_{(a)} = \text{id}$ then $F \upharpoonright \mathfrak{F}^n_{(b)} = \text{id}$. $\square$

6.4.2 A Power of the Cyclic Projective Plane $\text{PG}(2, 3)$

Let $\mathfrak{P} = \mathbf{D}(C_{13}, \{0, 1, 3, 9\})$, i.e. let $\mathfrak{P}$ be the cyclic projective plane $\text{PG}(2, 3)$. Note that the set $\{0, 1, 3, 9\}$ is fixed by the multiplier $\alpha = 3$. Thus, this multiplier yields an automorphism of $\mathfrak{P}$ (cf. Fact 3.1). The map $\mu_\alpha: C_{13} \ni x \mapsto \alpha \cdot x$ generates the cyclic group consisting of $\mu_1 = \text{id}$, μ_3 , and μ_9 , which is isomorphic to C_3 . Every element of this group induces an automorphism of $\mathfrak{P}_{(\theta)}$.

Proposition 6.23 *The group $\text{Aut}(\mathfrak{P}^n)$ is isomorphic to $S_n \ltimes ((C_3)^n \ltimes (C_{13})^n)$.*

Proof Let $\mathcal{J}''_0$ be the family of lines of the size 4 in $\mathfrak{P}^n_{(\theta)}$ (Fig. 7) and $\mathcal{J}'_0$ be the family of the lines of the size 3 in $\mathfrak{P}^n_{(\theta)}$ that are not in any of the $\mathcal{J}_i$.

Step 1 Let $y \in (C_{13})^n$. Then $[y] \in \mathcal{J}''_0$ iff $y \in \Xi^2_{\{\alpha, \beta\}}$, where $\alpha, \beta \in \{1, 3, 9\}$. If $[y] \in \mathcal{J}''_0$ and $\text{supp}(y) = \{i_1, i_2\}$ then $[y]$ intersects two out of three lines in $\mathcal{J}_{i_1}$ and in $\mathcal{J}_{i_2}$, but do not intersects $[\theta]$. Consequently, for any two $i_1, i_2 \in \{1, \dots, n\}$ there are exactly nine lines in $\mathcal{J}''_0$ that cross two lines in $\mathcal{J}_{i_1}$ and two lines in $\mathcal{J}_{i_2}$. For every point $(x) \in \mathfrak{P}^n_{(\theta)}$ with $x \notin P_i$ there are two lines $[y] \in \mathcal{J}''_0$ such that $(x) \perp [y]$. Every line in $\mathcal{J}''_0$ intersects four other lines in $\mathcal{J}''_0$ and does not intersect remaining four lines from $\mathcal{J}''_0$.

Step 2 There are no lines of the size 3 in $\mathfrak{P}^n_{(\theta)}$ (i.e. $\mathcal{J}'_0 = \emptyset$).

Step 3 Let us consider the set $\mathcal{J}^\alpha := \{[y] \in \mathcal{J}_i: y_i = \alpha \in -\mathcal{D} \text{ and } i = 1, \dots, n\}$. If F is an automorphism of $\mathfrak{P}^n$ leaving every line in $\mathcal{J}^\alpha$ invariant then $F \upharpoonright \mathfrak{P}^n_{(\theta)} = \text{id}$.

Step 4 Let $\sigma \in S_n$. We define the map h_σ on $(C_{13})^n$ by the formula

$$h_\sigma((x_1, \dots, x_n)) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

By Lemma 4.6 the map h_σ induces the collineation $F_\sigma = (F', F'')$ of $\mathfrak{P}^n$. Moreover, $F'(\theta) = \theta$ and $G'(P_i) = P_{\sigma(i)}$ for all $i = 1, \dots, n$.

Step 5 Let F be an automorphism of $\mathfrak{M}$ such that F leaves every line through a point a invariant. Then $F \upharpoonright \mathfrak{P}^n_{(a)} = \text{id}$.

Step 6 Let a, b be two points of $\mathfrak{P}^n$ such that b is a point of a projective subplane $\text{PG}(2, 3)$ in $\mathfrak{P}^n_{(a)}$. If F is an automorphism of $\mathfrak{P}^n$ such that $F \upharpoonright \mathfrak{P}^n_{(a)} = \text{id}$ then $F \upharpoonright \mathfrak{P}^n_{(b)} = \text{id}$. $\square$

Let us come back the power $\mathfrak{P}^n$ of an arbitrary finite projective plane $\mathfrak{P} = \mathbf{D}(C_k, \mathcal{D})$ induced by a difference set $\mathcal{D}$ in a cyclic group C_k . Observing Propositions 6.22

Fig. 7 The structure of lines of the size 4 in $\mathfrak{P}^2_{(\theta)}$

and 6.23 and their proofs we note that every automorphism F of $\mathfrak{P}^n$ is related to one of the following:

- a multiplier of the set $\mathcal{D}$, and then F leaves $\mathfrak{P}^n_{(\theta)}$ invariant,
- a translation τ of $(C_k)^n$, and then F maps $\mathfrak{P}^n_{(\theta)}$ onto $\mathfrak{P}^n_{(\tau(\theta))}$,
- a permutation $\sigma \in S_n$, and then F permutes the projective subplanes isomorphic to $\mathfrak{P}$.

Moreover, a composition of the two first automorphisms is not commutative, which also does not commute with the third one. We claim that

Conjecture 6.24 *Let $\mathfrak{P} = \mathbf{D}(C_k, \mathcal{D})$ be a finite projective plane induced by a difference set $\mathcal{D}$ in a cyclic group C_k . The group $\text{Aut}(\mathfrak{P}^n)$ is isomorphic to*

$$S_n \ltimes \left(((C_{r_1})^n \oplus \cdots \oplus (C_{r_s})^n) \ltimes (C_k)^n \right),$$

where $C_{r_1}, \dots, C_{r_s}$ are the cyclic groups generated by the multipliers of the set $\mathcal{D}$.

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