On the Means of Jordan's Totient Function

Mohammadreza Esfandiari¹

Received: 10 July 2019 / Accepted: 16 January 2020 / Published online: 20 February 2020
© The Author(s) 2020

Abstract

In this paper, we study some important means of Jordan's totient function, especially, we obtain asymptotic formula for geometric mean and harmonic mean. We also study alternating sums of Jordan's totient function and Carleman's inequality for this function.

Keywords Arithmetic function · Average order · Geometric mean · Sum of reciprocal · Alternating sum

Mathematics Subject Classification 11A25 · 11N56 · 11N05

1 Introduction and Summary of the Results

Euler's totient function φ is one of the most famous arithmetic functions used in number theory. Recall that $\varphi(n)$ is defined as the number of positive integers less than or equal to n that are coprime to n . Many generalizations and analogs of Euler's function are known. See the special chapter on this topic in [15] and the references given there. Among the generalizations, the most significant is probably Jordan's totient function. For given positive integer r , Jordan's function J_r is defined by the number of r -tuples of integers $(a_1, a_2, \dots, a_r)$ satisfying $1 \leq a_i \leq n$ and $\gcd(a_1, a_2, \dots, a_r, n) = 1$. It can be shown that (by inclusion–exclusion principle)

$$J_r(n) = n^r \prod_{p|n} \left(1 - \frac{1}{p^r}\right). \quad (1.1)$$

Communicated by Massoud Amini.

✉ Mohammadreza Esfandiari
m.esfandiari@znu.ac.ir

¹ Department of Mathematics, University of Zanjan, University Blvd, 45371-38791 Zanjan, Iran

The function J_r is multiplicative and if p is a prime and α is a positive integer, then $J_r(p^\alpha) = p^{r\alpha} - p^{r(\alpha-1)}$. Dirichlet series related to Jordan's function is as follows:

$$\sum_{n=1}^{\infty} \frac{J_r(n)}{n^s} = \frac{\zeta(s-r)}{\zeta(s)} \quad (\Re(s) > r+1),$$

where $\zeta(s)$ is the Riemann zeta-function, defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\Re(s) > 1$.

Gauss type formula for J_r reads as follows:

$$\sum_{d|n} J_r(d) = n^r,$$

from which, the Möbius inversion formula gives

$$J_r(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^r. \quad (1.2)$$

For more results and applications of this function, as well as its history, we refer the reader to [2,3,13,16,19].

In this note, we study the means of Jordan's totient function. The asymptotic expansion of arithmetic mean of J_r is known in the literature (for example, see [13]) and it is shown that for $r > 1$,

$$A_n(J_r) := \frac{1}{n} \sum_{k \leq n} J_r(k) = \frac{n^r}{(r+1)\zeta(r+1)} + O(n^{r-1}). \quad (1.3)$$

In this note, we obtain the asymptotic behaviour of geometric mean of the values of Jordan's function. More precisely, we prove the following.

Theorem 1.1 *Assume that $r \geq 1$ is a fixed integer. Then*

$$\sum_{k \leq n} \log J_r(k) = rn \log n + (\alpha_r - r)n + \frac{r}{2} \log n + R(n), \quad (1.4)$$

where

$$\alpha_r = \sum_p \frac{1}{p} \log \left(1 - \frac{1}{p^r}\right),$$

and

$$R(n) = \begin{cases} O(\log \log n) & \text{if } r = 1, \\ O_r(1) & \text{if } r > 1. \end{cases}$$

Corollary 1.2 *For $r \geq 1$, we have*

$$G_n(J_r) = B_r n^r + \frac{r}{2} B_r n^{r-1} \log n + E(n), \quad (1.5)$$

where

$$B_r = e^{\alpha_r - r} = \frac{1}{e^r} \prod_p \left(1 - \frac{1}{p^r}\right)^{\frac{1}{p}},$$

and

$$E(n) = \begin{cases} O(\log \log n) & \text{if } r = 1, \\ O_r(n^{r-1}), & \text{if } r > 1. \end{cases}$$

We mention that the case $r = 1$ in the above corollary, which corresponds Euler's function, has been obtained already in [10]. Regarding the harmonic mean, we recall that [12],

$$\sum_{n \leq x} \frac{1}{\varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\log x + \gamma - \sum_p \frac{\log p}{p^2 + p - 1} \right) + O\left(\frac{\log x}{x}\right),$$

where γ is Euler's constant.

For more results about reciprocal sums of arithmetic functions, we refer the reader to [4, 7]. For the reciprocal sum of Jordan's function, we prove the following.

Theorem 1.3 Assume that $r > 1$ is a fixed integer. As $x \rightarrow \infty$, we have

$$\sum_{n \leq x} \frac{1}{J_r(n)} = \zeta(r) \beta_r - \frac{\beta'_r}{(r-1)x^{r-1}} + O\left(\frac{1}{x^r}\right),$$

where

$$\beta_r := \prod_p \left(1 + \frac{1}{p^r(p^r - 1)}\right) \quad \text{and} \quad \beta'_r := \prod_p \left(1 + \frac{1}{p(p^r - 1)}\right).$$

Moreover, reciprocal sum of Euler φ function running over squarefree integers has been studied. (For example, see [17].)

$$\sum_{n \leq x} \frac{\mu^2(n)}{\varphi(n)} = \log x + \gamma + \sum_p \frac{\log p}{p(p-1)} + O\left(\frac{1}{\sqrt{x}}\right).$$

We obtain asymptotic formula for reciprocal sum of Jordan's function running over squarefree integers.

Theorem 1.4 For $r > 1$, we have

$$\sum_{n \leq x} \frac{\mu^2(n)}{J_r(n)} = \frac{r\gamma_r}{r-1} - \frac{\gamma'_r x^{1-r}}{r-1} + O\left(x^{1/2-r}\right),$$

where

$$\gamma_r = \frac{1}{\zeta(2)} \prod_p \left(1 + \frac{1}{p^{r-1}(p^r - 1)(p + 1)} \right) \quad \text{and} \quad \gamma'_r = \prod_p \left(1 - \frac{p^r - p}{p^2(p^r - 1)} \right).$$

Recently in [18], L.Tóth studied alternating sums of multiplicative arithmetic functions; similarly, we study alternating sums concerning Jordan's function. We state the following theorems.

Theorem 1.5 Assume that $r \geq 1$ is a fixed integer, we have

$$\sum_{n \leq x} (-1)^{n-1} J_r(n) = \frac{1}{(2^{r+1} - 1)(r + 1)\zeta(r + 1)} x^{r+1} + O(x^r).$$

Theorem 1.6 Assume that $r > 1$ is a fixed integer, we have

$$\sum_{n \leq x} (-1)^{n-1} \frac{1}{J_r(n)} = \zeta(r) \beta_r \beta_r'' + \frac{\beta_r'}{(1 - 2^{r+1})(1 - r)x^{r-1}} + O\left(\frac{1}{x^r}\right),$$

where β_r and β_r' are defined as in Theorem 1.3, and

$$\beta_r'' = \frac{2}{\sum_{v=0}^{\infty} \frac{1}{J_r(2^v)}} - 1 = 2(1 - 2^{-r})^2 - 1.$$

For positive real numbers $a_1, a_2, \dots, a_n$, Carleman's inequality [5] asserts that

$$\sum_{k=1}^n (a_1 \dots a_k)^{\frac{1}{k}} \leq e \sum_{k=1}^n a_k. \quad (1.6)$$

The constant e is the best possible. Recently in [11], Hassani studied Carleman's inequality over the values of arithmetical functions. Hassani studied Carleman's inequality over prime numbers and over reciprocal of the prime numbers, and showed that

$$\begin{aligned} \frac{\sum_{k=1}^n (p_1 \dots p_k)^{\frac{1}{k}}}{\sum_{k=1}^n p_k} &= \frac{1}{e} + O\left(\frac{1}{\log n}\right) \quad \text{and} \\ \times \frac{\sum_{k=1}^n \left(\frac{1}{p_1} \dots \frac{1}{p_k}\right)^{\frac{1}{k}}}{\sum_{k=1}^n \frac{1}{p_k}} &= e + O\left(\frac{1}{\log \log n}\right). \end{aligned}$$

For more results, we refer the reader to [11]. For each positive arithmetical function f , let

$$C_f(n) := \frac{\sum_{k=1}^n (f(1) \dots f(k))^{\frac{1}{k}}}{\sum_{k=1}^n f(k)}.$$

We can write

$$C_f(n) = \frac{\sum_{k=1}^n G_k(f)}{\sum_{k=1}^n f(k)}. \quad (1.7)$$

We prove the following results:

Theorem 1.7 Assume that $r > 1$ is a fixed integer, as $n \rightarrow \infty$ we have

$$C_{J_r}(n) = B_r \zeta(1+r) + \frac{B_r(r+1)}{2} \frac{\log n}{n} + O\left(\frac{1}{n}\right),$$

where B_r is absolute constant defined in Corollary 1.2.

Remark 1.8 The asymptotic behaviour of the ratio of the arithmetic to the geometric means of several numbers of theoretic sequences is studied by M.Hassani (for example, see [8]). As a result, for $r > 1$, as $n \rightarrow \infty$, we have

$$\frac{A_n(J_r)}{G_n(J_r)} = \frac{1}{B_r(r+1)\zeta(r+1)} \left(1 + \frac{r}{2} \frac{\log n}{n} + O\left(\frac{1}{n}\right)\right).$$

2 Preliminaries

To prove our results, we need some results and notations.

Stirling's theorem asserts that (for more details, see [9])

Lemma 2.1 As n tends to infinity, we have

$$\sum_{k=1}^n \log k = n \log n - n + \frac{1}{2} \log n + C + O\left(\frac{1}{n}\right),$$

where

$$C = \log \sqrt{2\pi}.$$

Theorem 3.2 of [1] asserts that

Lemma 2.2 For $\alpha > 0$ and $\alpha \neq 1$, we have

$$\sum_{n \leq x} \frac{1}{n^\alpha} = \frac{x^{1-\alpha}}{1-\alpha} + \zeta(\alpha) + O\left(\frac{1}{x^\alpha}\right).$$

Lemma 1.3 of [8] asserts that

Lemma 2.3 for each real $\alpha > 1$ and each real $z > 1$, we have

$$\sum_{p > z} \frac{1}{p^\alpha} \ll \frac{1}{z^{\alpha-1} \log z}.$$

Chapter 1, Section 1.5 of [14] asserts that

Lemma 2.4 *For $r \geq 2$, there are positive constants c_1 and c_2 such that*

$$c_1 n^r \leq J_r(n) \leq c_2 n^r.$$

In fact, one can show that $c_1 = \frac{1}{\zeta(r)}$ and $c_2 = 1$.

Lemma 2.5 *For $r \geq 1$, we have*

$$\frac{n^r}{J_r(n)} = \sum_{d|n} \frac{\mu^2(d)}{J_r(d)}. \quad (2.1)$$

Lemma 5.2 of [6] asserts that

Lemma 2.6 *For each positive integer k , we have*

$$\sum_{\substack{n \leq x \\ (n,k)=1}} \mu^2(n) = \frac{xk}{\zeta(2)\psi(k)} + O\left(\theta(k)\sqrt{x}\right),$$

where ψ is Dedekind's psi function, defined by $\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$, and $\theta(n) = 2^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime divisors of n .

Assume that f is a nonzero complex-valued multiplicative function. Let

$$D(f, s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

denote the Dirichlet series of the function f . Then (see [18, Proposition 1])

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{f(n)}{n^s} = D(f, s) \left(2 \left(\sum_{v=0}^{\infty} \frac{f(2^v)}{2^{vs}} \right)^{-1} - 1 \right). \quad (2.2)$$

Analogous notations and methods in [18], consider the formal power series

$$S_f(x) := \sum_{v=0}^{\infty} a_v x^v,$$

where $a_v = f(2^v)$ ($v \geq 0$), $a_0 = f(1) = 1$. Note that $S_f(x)$ is the Bell series of the function f for the prime $p = 2$. Let

$$\bar{S}_f(n) := \sum_{v=0}^{\infty} b_v x^v$$

be its formal reciprocal power series. Here the coefficients b_v are given by $b_0 = 1$ and $\sum_{j=0}^v a_j b_{v-j} = 0$ ($v \geq 0$).

It follows from (2.2) that the convolution identity

$$(-1)^{n-1} f(n) = \sum_{d|n} h_f(d) f\left(\frac{n}{d}\right) \quad (n \geq 1), \quad (2.3)$$

holds, where the function h_f is multiplicative, $h_f(p^v) = 0$ if $p > 2$, $v \geq 1$ and $h_f(2^v) = 2b_v$ ($v \geq 1$), $h_f(1) = 2b_0 - 1 = 1$. Now, we apply (2.2) and (2.3) to get

$$\sum_{n=1}^{\infty} \frac{h_f(n)}{n^s} = \frac{2}{S_f(1/2^s)} - 1.$$

Thus, for each $r \geq 1$, and for any real number $s > r$, we obtain

$$\sum_{n=1}^{\infty} \frac{h_{j_r}(n)}{n^s} = \frac{2^s - 2^{r+1} + 1}{2^s - 1}.$$

Especially, for $s = r + 1$, we have

$$\sum_{n=1}^{\infty} \frac{h_{j_r}(n)}{n^{r+1}} = \frac{1}{2^{r+1} - 1}. \quad (2.4)$$

Similarly, for reciprocal sum of Jordan's function for $r > 1$, and any real number $s > -r$, we can show that

$$\sum_{n=1}^{\infty} \frac{h_{1/J_r}(n)}{n^s} = \frac{2^{2r+s} - 2^r(2^s + 2) + 1}{2^{2r+s} - 2^{r+s} + 1}.$$

For $s = 1 - r$, we have

$$\sum_{n=1}^{\infty} \frac{h_{1/J_r}(n)}{n^{1-r}} = \frac{1}{1 - 2^{r+1}}. \quad (2.5)$$

3 Proofs

Proof of Theorem 1.1 By definition of $J_r(n)$ in (1.1) and considering Lemma 2.1, we have

$$\begin{aligned} \sum_{k \leq n} \log J_r(k) &= \sum_{k \leq n} \log \left(k^r \prod_{p|k} \left(1 - \frac{1}{p^r} \right) \right) \\ &= r \sum_{k \leq n} \log k + \sum_{k \leq n} \sum_{p|k} \log \left(1 - \frac{1}{p^r} \right) \end{aligned}$$

$$= r \left(n \log n - n + \frac{1}{2} \log n + C + O\left(\frac{1}{n}\right) \right) + \sum_{p \leq n} \left\lfloor \frac{n}{p} \right\rfloor \log \left(1 - \frac{1}{p^r} \right).$$

To approximate the last series, we write

$$\begin{aligned} \sum_{p \leq n} \left\lfloor \frac{n}{p} \right\rfloor \log \left(1 - \frac{1}{p^r} \right) &= n \sum_{p \leq n} \frac{1}{p} \log \left(1 - \frac{1}{p^r} \right) - \sum_{p \leq n} \left\{ \frac{n}{p} \right\} \log \left(1 - \frac{1}{p^r} \right) \\ &= nS_1(n) - S_2(n). \end{aligned} \quad (3.1)$$

To estimate $S_1(n)$, we can write

$$\begin{aligned} S_1(n) &= \sum_{p \leq n} \frac{1}{p} \log \left(1 - \frac{1}{p^r} \right) = \sum_p \frac{1}{p} \log \left(1 - \frac{1}{p^r} \right) - \sum_{p > n} \frac{1}{p} \log \left(1 - \frac{1}{p^r} \right) \\ &= \alpha_r + O\left(\frac{1}{n^r \log n}\right), \end{aligned}$$

where the last approximation is a special case of Lemma 2.3 and given that $\log(1 - \frac{1}{p^r}) \leq \frac{1}{p^r}$, so

$$\sum_{p > n} \frac{1}{p} \log \left(1 - \frac{1}{p^r} \right) \leq \sum_{p > n} \frac{1}{p^{r+1}} \ll \frac{1}{n^r \log n}.$$

To approximate $S_2(n)$, we write

$$S_2(n) = \sum_{p \leq n} \left\{ \frac{n}{p} \right\} \log \left(1 - \frac{1}{p^r} \right) \ll \sum_{p \leq n} \frac{1}{p^r} := R(n) \ll \begin{cases} \log \log n & \text{if } r = 1 \\ 1 & \text{if } r > 1. \end{cases}$$

Combining estimates $S_1(n)$ and $S_2(n)$ in (3.1), we get

$$\sum_{p \leq n} \left\lfloor \frac{n}{p} \right\rfloor \log \left(1 - \frac{1}{p^r} \right) = n\alpha_r + R(n)$$

and, therefore, we have

$$\sum_{k \leq n} \log J_r(k) = rn \log n + n(\alpha_r - r) + \frac{r}{2} \log n + R(n).$$

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2 By definition

$$G_n(J_r) = \left(\prod_{k=1}^n J_r(k) \right)^{\frac{1}{n}};$$

by taking logarithms from both sides, we get

$$n \log G_n(J_r) = \sum_{k \leq n} \log J_r(k).$$

Theorem 1.1 gives

$$\log G_n(J_r) = r \log n + (\alpha_r - r) + \frac{r \log n}{2} + O\left(\frac{R(n)}{n}\right).$$

Now, using this fact $e^x = 1 + x + O(x^2)$ as $x \rightarrow 0$, we have

$$\begin{aligned} G_n(J_r) &= e^{r \log n} e^{(\alpha_r - r)} e^{\frac{r \log n}{2} + O\left(\frac{R(n)}{n}\right)} \\ &= B_r n^r \left(1 + \frac{r \log n}{2} + O_r\left(\frac{R(n)}{n}\right) + O_r\left(\frac{\log^2 n}{n^2}\right)\right) \\ &= B_r n^r + \frac{r}{2} B_r n^{r-1} \log n + E(n), \end{aligned}$$

where the error term $E(n)$ is $O(\log \log n)$ for $r = 1$ and $O(n^{r-1})$ for $r > 1$, which is the desired conclusion. $\square$

Proof of Theorem 1.3 First, by considering Euler's infinite product factorization theorem, it is observed that

$$\beta_r = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{J_r(d) d^r} = \prod_p \left(1 + \frac{1}{p^r(p^r - 1)}\right),$$

and

$$\beta'_r = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{J_r(d) d} = \prod_p \left(1 + \frac{1}{p(p^r - 1)}\right).$$

Now, by considering Lemmas 2.5 and 2.2, we write

$$\begin{aligned} \sum_{n \leq x} \frac{1}{J_r(n)} &= \sum_{n \leq x} \frac{1}{n^r} \sum_{d|n} \frac{\mu^2(d)}{J_r(d)} = \sum_{d \leq x} \frac{\mu^2(d)}{J_r(d)} \sum_{\substack{n \leq x \\ d|n}} \frac{1}{n^r} = \sum_{d \leq x} \frac{\mu^2(d)}{J_r(d) d^r} \sum_{m \leq \frac{x}{d}} \frac{1}{m^r} \\ &= \sum_{d \leq x} \frac{\mu^2(d)}{J_r(d) d^r} \left(\left(\frac{x}{d}\right)^{1-r} + \zeta(r) + O\left(\frac{1}{\left(\frac{x}{d}\right)^r}\right) \right) \\ &= \frac{1}{(1-r)x^{r-1}} \sum_{d \leq x} \frac{\mu^2(d)}{J_r(d) d} + \zeta(r) \sum_{d \leq x} \frac{\mu^2(d)}{J_r(d) d^r} + \sum_{d \leq x} O\left(\frac{\mu^2(d)}{x^r J_r(d)}\right), \end{aligned}$$

by Lemma 2.4 since $\frac{1}{\zeta(r)}n^r \leq J_r(n) \leq n^r$ for $r > 1$, we see that $\sum_{d=1}^{\infty} \frac{\mu^2(d)}{J_r(d)d}$ converges. So,

$$\begin{aligned} \sum_{n \leq x} \frac{1}{J_r(n)} &= \frac{1}{(1-r)x^{r-1}} \left(\sum_{d=1}^{\infty} \frac{\mu^2(d)}{J_r(d)d} - \sum_{d > x} \frac{\mu^2(d)}{J_r(d)d} \right) \\ &\quad + \zeta(r) \left(\sum_{d=1}^{\infty} \frac{\mu^2(d)}{J_r(d)d^r} - \sum_{d > x} \frac{\mu^2(d)}{J_r(d)d^r} \right) + \sum_{d \leq x} O\left(\frac{\mu^2(d)}{x^r J_r(d)}\right) \\ &= \zeta(r)\beta_r - \frac{\beta'_r}{(r-1)x^{r-1}} + O\left(\frac{1}{x^r}\right). \end{aligned}$$

This completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.4 By considering Lemma 2.6 and using the Abel's summation formula, we get

$$\sum_{\substack{k \leq y \\ (k,d)=1}} \frac{\mu^2(k)}{k^r} = \frac{r}{r-1} \frac{d}{\zeta(2)\psi(d)} - \frac{y^{r-1}d}{(r-1)\zeta(2)\psi(d)} + O\left(y^{1/2-r}2^{\omega(d)}\right).$$

Now, by considering Lemma 2.5, we have

$$\begin{aligned} \sum_{n \leq x} \frac{\mu^2(n)}{J_r(n)} &= \sum_{n \leq x} \frac{\mu^2(n)}{n^r} \sum_{d|n} \frac{\mu^2(d)}{J_r(d)} \\ &= \sum_{d \leq x} \frac{\mu^2(d)}{d^r J_r(d)} \sum_{\substack{k \leq x/d \\ (k,d)=1}} \frac{\mu^2(k)}{k^r} \\ &= \frac{r}{\zeta(2)(r-1)} \sum_{d \leq x} \frac{\mu^2(d)}{d^{r-1} J_r(d)\psi(d)} - \frac{x^{1-r}}{\zeta(2)(r-1)} \sum_{d \leq x} \frac{\mu^2(d)}{J_r(d)\psi(d)} \\ &\quad + O\left(x^{1/2-r} \sum_{d \leq x} \frac{\mu^2(d)2^{\omega(d)}}{d^{1/2} J_r(d)}\right) \\ &= \frac{r\gamma_r}{r-1} - \frac{\gamma'_r x^{1-r}}{r-1} + O\left(x^{1/2-r}\right), \end{aligned}$$

where

$$\gamma_r = \frac{1}{\zeta(2)} \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^{r-1} J_r(d)\psi(d)} = \frac{1}{\zeta(2)} \prod_p \left(1 + \frac{1}{p^{r-1}(p^r-1)(p+1)}\right),$$

and

$$\gamma'_r = \frac{1}{\zeta(2)} \sum_{d=1}^{\infty} \frac{\mu^2(d)}{J_r(d)\psi(d)} = \prod_p \left(1 - \frac{p^r - p}{p^2(p^r - 1)}\right).$$

This completes the proof of Theorem 1.4. $\square$

Proof of Theorem 1.5 By considering (1.3) and (2.3), we write

$$\begin{aligned} \sum_{n \leq x} (-1)^{n-1} J_r(n) &= \sum_{d \leq x} h_{J_r}(d) \sum_{j \leq \frac{x}{d}} J_r(j) \\ &= \sum_{d \leq x} h_{J_r}(d) \left(\frac{\left(\frac{x}{d}\right)^{r+1}}{(r+1)\zeta(r+1)} + O\left(\frac{x}{d}\right)^r \right) \\ &= \frac{x^{r+1}}{(r+1)\zeta(r+1)} \sum_{d \leq x} \frac{h_{J_r}(d)}{d^{r+1}} + O\left(x^r \sum_{d \leq x} \frac{h_{J_r}(d)}{d^r}\right) \\ &= \frac{x^{r+1}}{(r+1)\zeta(r+1)} \left(\sum_{d=1}^{\infty} \frac{h_{J_r}(d)}{d^{r+1}} - \sum_{d > x} \frac{h_{J_r}(d)}{d^{r+1}} + O(x^r) \right) \\ &= \frac{x^{r+1}}{(2^{r+1}-1)(r+1)\zeta(r+1)} + O(x^r). \end{aligned}$$

This completes the proof of Theorem 1.5. $\square$

Proof of Theorem 1.6 Proof of Theorem 1.6 using (2.3) and (2.5) and Theorem 1.3 is similar to the proof of Theorem 1.5. $\square$

Proof of Theorem 1.7 For $r > 1$, by considering (1.5), write

$$\sum_{k=1}^n G_k(J_r) = B_r \sum_{k=1}^n k^r + \frac{r}{2} B_r \sum_{k=1}^n k^{r-1} \log k + O_r \left(\sum_{k=1}^n k^{r-1} \right).$$

We know

$$\sum_{k=1}^n k^r = \frac{n^{r+1}}{r+1} + O(n^r).$$

Using Abel summation formula, we can write

$$\begin{aligned} \sum_{k=1}^n k^{r-1} \log k &= (n \log n - n + O(\log n)) n^{r-1} \\ &\quad - (r-1) \int_2^n (t \log t - t + O(\log t)) t^{r-2} dt \\ &= n^r \log n - n^r + O(n^{r-1} \log n) - (r-1) \left(\frac{n^r}{r} \log n - \frac{n^r}{r^2} \right) \\ &\quad + (r-1) \left(\frac{n^r}{r} \right) + O(n^{r-1} \log n) \end{aligned}$$

$$= \frac{1}{r} n^r \log n - \frac{1}{r^2} n^r + O(n^{r-1} \log n).$$

So, we get

$$\sum_{k=1}^n G_k(J_r) = \frac{B_r}{r+1} n^{r+1} + \frac{B_r}{2} n^r \log n + O(n^r). \quad (3.2)$$

Also, we know

$$\sum_{k=1}^n J_r(k) = \frac{n^{r+1}}{(r+1)\zeta(r+1)} + O(n^r). \quad (3.3)$$

Now, by considering (3.2) and (3.3), we have

$$\begin{aligned} C_{J_r}(n) &= \frac{\sum_{k=1}^n G_k(J_r)}{\sum_{k=1}^n J_r(k)} = \frac{\frac{B_r}{r+1} n^{r+1} + \frac{B_r}{2} n^r \log n + O(n^r)}{\frac{n^{r+1}}{(r+1)\zeta(r+1)} + O(n^r)} \\ &= \frac{\frac{B_r}{r+1} n^{r+1} \left(1 + \frac{r+1}{2} \frac{\log n}{n} + O\left(\frac{1}{n}\right)\right)}{\frac{n^{r+1}}{(r+1)\zeta(r+1)} \left(1 + O\left(\frac{1}{n}\right)\right)} \\ &= B_r \zeta(1+r) \left(1 + \frac{r+1}{2} \frac{\log n}{n} + O\left(\frac{1}{n}\right)\right) \left(1 + O\left(\frac{1}{n}\right)\right) \\ &= B_r \zeta(1+r) + \frac{B_r(r+1)}{2} \frac{\log n}{n} + O\left(\frac{1}{n}\right). \end{aligned}$$

This completes the proof of Theorem 1.7. $\square$

Proof of Lemma 2.5 Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $\alpha_i > 0$. Then since the right-hand side of (2.1) is multiplicative,

$$\begin{aligned} \sum_{d|n} \frac{\mu^2(d)}{J_r(d)} &= \prod_{i=1}^k \sum_{j=0}^{\alpha_i} \frac{\mu^2(p_i^j)}{J_r(p_i^j)} \\ &= \prod_{i=1}^k \left(1 + \frac{1}{p_i^r - 1}\right) = \prod_{i=1}^k \frac{1}{1 - \frac{1}{p_i^r}} = \frac{n^r}{J_r(n)}. \end{aligned}$$

This completes the proof of Lemma 2.5. $\square$

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Apostol, T.M.: Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics, p. xii+338. Springer, New York (1976)
2. Andrica, D., Piticari, M.: On some extensions of Jordan's arithmetic functions. *Acta Univ. Apulensis Math. Inf. No.* **7**, 13–22 (2004)
3. Adhikari, S.D., Sankaranarayanan, A.: On an error term related to the Jordan totient function $J_k(n)$. *J. Number Theory* **34**(2), 178–188 (1990). 521–555
4. Bordellès, O., Cloitre, B.: An alternating sum involving the reciprocal of certain multiplicative functions. *J. Integer Seq.* **16**(2), 3 (2013)
5. Carleman, T.: Sur les fonctions quasi-analytiques, Conférences faites au cinquième congrès des mathématiciens Scandinaves. Helsinki **181–196** (1923)
6. Cohen, E.: Arithmetical functions associated with the unitary divisors at an integer. *Math Zeit.* **74**, 66–80 (1960)
7. De Koninck, J.-M., Ivić, A.: Topics in Arithmetical Functions. Mathematics Studies, 43. North-Holland, Amsterdam (1980)
8. Hassani, M.: A remark on the means of the number of the divisors. *Bull. Iran. Math. Soc.* **42**(6), 1315–1330 (2016)
9. Hassani, M.: On the arithmetic-geometric means of positive integers and the number e . *Appl. Math. E-Notes* **14**, 250–255 (2014)
10. Hassani, M.: Uniform distribution modulo one of some sequences concerning the Euler function. *Rev. Un. Mat. Argentina* **54**(1), 55–68 (2013)
11. Hassani, M.: Carleman's inequality over prime numbers. *Appl. Math. E-Notes*. to appear (2019)
12. Landau, E.: Üdie Zahlentheoretische Function $\varphi(n)$ und ihre Beziehung zum Goldbachschen Satz, Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, pp. 177–186 (1900)
13. McCarthy, P.J.: Introduction to Arithmetical Functions. Springer, New York (1986)
14. Murty, M.R.: Problems in Analytic Number Theory, 2nd edn. Graduate Texts in Mathematics, **206**. Readings in Mathematics. Springer, New York (2008)
15. Sándor, J., Crstici, B.: Handbook of Number Theory. II. Kluwer Academic Publishers, Dordrecht (2004)
16. Sándor, J., Mitrinović, D.S., Crstici, B.: Handbook of Number Theory. I. Second printing of the 1996 original. Springer, Dordrecht, pp. xxvi+622 (2006)
17. Sitaramachandrarao, R.: On an error term of Landau. II, Number theory (Winnipeg, Man., 1983). *Rock. Mt. J. Math.* **15**(2), 579–588 (1985)
18. Tóth, L.: Alternating sums concerning multiplicative arithmetic functions. *J. Integer Seq.* **20**(2), 41 (2017). Art. 17.2.1
19. Thajoddin, S., Vangipuram, S.: A note on Jordan's totient function. *Indian J. Pure Appl. Math.* **19**(12), 1156–1161 (1988)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.