A Numerical Investigation of the c -Numerical Ranges of Differential Operators

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Abstract

In this paper, we consider the problem of computing the c -numerical range numerically for block differential operators, particularly these of Schrödinger type, Hain–Lüst type, and Stokes type.

Keywords Operator matrices · Block operator matrices · c -Numerical range · Finite difference · Schrödinger operator · Hain–Lüst operator · Stokes operator

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1 Introduction

let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator on a complex Hilbert space $\mathcal{H}$, with inner product $\langle \cdot, \cdot \rangle$, and associated norm $\| \cdot \|$. The numerical range of A is the set $W(A)$ defined by the following:

$$W(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}. \quad (1.1)$$

By the famous Toeplitz–Hausdorff Theorem [1,2], $W(A)$ is convex. This set has been studied extensively for many decades. It is useful in the study and to understand the matrices and operators (see [3–8]), and has many applications in numerical analysis, differential equations, system theory, etc (see [7,9–12]). There are many different

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generalizations of numerical range. For $1 \leq k \leq n$, the k -numerical range of linear operator A on $\mathbb{C}^n$, introduced by Halmos [13] is defined as follows:

$$W_k(A) = \left\{ \sum_{j=1}^k \frac{1}{k} x_j^* A x_j : \{x_1, \dots, x_k\} \text{ is an orthonormal set} \right\} \quad (1.2)$$

which is proved to be convex by Berger [14]. For any $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{R}^k$, the c -numerical range of linear operator A on $\mathbb{C}^n$, first introduced by Marcus [15], is defined as follows:

$$W_c(A) = \left\{ \sum_{i=1}^n c_i x_i^* A x_i : \{x_1, \dots, x_n\} \text{ is an orthonormal set} \right\}. \quad (1.3)$$

Extending the definition in (1.3) for a bounded or an unbounded operator A on a complex Hilbert space $\mathcal{H}$, with inner product $\langle \cdot, \cdot \rangle$, and for $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{R}^k$ the c -numerical range $W_c(A)$ of A is defined as follows:

$$W_c(A) = \left\{ \sum_{j=1}^k c_j \langle A x_j, x_j \rangle : (x_1, \dots, x_k) \text{ is an orthonormal subset of } \mathcal{D}(A) \right\}, \quad (1.4)$$

where $\mathcal{D}(A)$ is the domain of A . Furthermore, if $\mathbf{c}$ consists of just one number c_1 , $W_c(A)$, then we obtain the classical numerical range $W(A)$ (see [13]). In addition, for $c_1 = c_2 = \dots = c_k = 1$, the c -numerical range $W_c(A)$ turns into $W_k(A)$. Indeed, $W_k(A) \subseteq W(A)$. Westwick [16] has shown that $W_c(A)$ is convex for any $c \in \mathbb{R}^n$. In addition, he gave an example which shows that, for complex vectors $c \in \mathbb{C}^n$ with $n \geq 3$, the range $W_c(A)$ may fail to be convex.

The c -numerical range is unitary similarity invariant, $W_c(A) = W_c(UAU^*)$, for any unitary matrix U . It is also transpose invariant, $W_c(A) = W_c(A^T)$. Clearly, we have the basic property:

$$W_c(aA + bI_n) = aW_c(A) + b \sum_{j=1}^k c_j, \quad (1.5)$$

for every $a, b \in \mathbb{C}$. A review of the properties of c -numerical ranges of operator matrices may be found in [17].

In this paper, we consider how to compute the c -numerical range by finite difference methods, which reduce the problem to that of computing the c -numerical range of a (finite) matrix and block matrix.

The paper is organized as follows. In Sect. 2.1, some theoretical results are investigated dealing with the c -numerical range of operators using finite difference method. In Sect. 3, we shall applying these results to compute the c -numerical range of differential operators. We shall consider two particular special cases. In the first

case, $\mathcal{A}$ is the Hain–Lüst operator, it is clear the underlying Hilbert space in this case is $H := L^2(0, 1) \times L^2(0, 1)$ and the operator is as follows:

$$\mathcal{A} = \mathcal{L} := \begin{pmatrix} L & w \\ \tilde{w} & z \end{pmatrix}, \quad (1.6)$$

in which where $L = -\frac{d^2}{dx^2} + q$ is a Schrödinger operator (with bounded potential q), while w , $\tilde{w}$ and z are bounded multiplication operators. The domain of $\mathcal{L}$ is given by the following:

$$\begin{aligned} \mathcal{D}(\mathcal{L}) &= H^2(0, 1) \cap H_0^1(0, 1) \times L^2(0, 1) \\ &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u(0) = 0 = u(1), \int_0^1 (|u''|^2 + |u'|^2 + |u|^2 + |v|^2) < \infty \right\}. \end{aligned}$$

This operator has been introduced by Hain and Lüst [18] in application to problems of magnetohydrodynamics and the problem of this type has been studied in [19–23]. We shall be concerned with the effects of discretization, both of the unbounded second-order differential operator L and of the bounded multiplication operators which appear in the other entries.

In the last case, $\mathcal{A}$ is the Stokes-type system of ordinary differential equations. The underlying Hilbert space in this case is also $H := L^2(0, 1) \times L^2(0, 1)$ and the operator is as follows:

$$\mathcal{A} := \begin{pmatrix} -\frac{d^2}{dx^2} & -\frac{d}{dx} \\ \frac{d}{dx} & -\frac{3}{2} \end{pmatrix}; \quad (1.7)$$

the domain of $\mathcal{A}$ is given by the following:

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u(0) = 0 = u(1), u \in H_0^1(0, 1) \cap H^2(0, 1) \text{ and } v \in H^1(0, 1) \right\}.$$

Note that $\mathcal{A}$ is not closed; however, $\mathcal{A}$ is closable and its closure is self-adjoint.

2 An Approximating Discrete Operator

We shall replace the Schrödinger operator by a matrix. Suppose $u \in \mathcal{D}(L)$. Pick $N + 1$ points $x_0, x_1, \dots, x_N$ in the interval $[0, 1]$; for the sake of simplicity, we assume equal spacing, so that $x_0 = 0$, $x_j = jh$, $x_N = 1 = Nh$ with $h = \frac{1}{N}$. We would like to form the vectors:

$$\underline{u} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \end{pmatrix} \in \mathbb{C}^{N-1}. \quad (2.1)$$

Moreover, if we are to use the quadrature estimates, we shall need more smoothness in u . We, therefore, observe that the set

$$\mathcal{C}(L) := C^4(0, 1) \cap H_0^1(0, 1) \quad (2.2)$$

is a core of the operator L ; for $u \in \mathcal{C}(L)$, Eq. (2.1) makes sense. Recall the second-order divided difference approximation of the second derivative:

$$\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1}))}{h^2} = u''(x_j) + \frac{1}{4!}h^2(u^4(\xi_j) + u^4(\eta_j)). \quad (2.3)$$

Hence

$$\begin{pmatrix} u''(x_1) \\ u''(x_2) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} \simeq \frac{1}{h^2} T \mathbf{u},$$

where $T = \text{tridiag}(1, -2, 1)$ is the tridiagonal matrix with entries $T_{jj} = -2$, $T_{j,j\pm 1} = 1$. We also define diagonal matrices:

$$Q_N = \text{diag}(q(x_1), q(x_2), \dots, q(x_{N-1})). \quad (2.4)$$

Our matrix replacement of the Schrödinger operator is a matrix of dimension $(N - 1) \times (N - 1)$ given by the following:

$$L_N := -\frac{1}{h^2} T + Q_N. \quad (2.5)$$

We shall use this technique in this paper for computing *c-numerical range*. Muhammad and Marileta [24] replaced the Hain–Lüst operator and Stokes operator by matrices. To discretize the Hain–Lüst operator, suppose that $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A)$. Pick $N + 1$ points $x_0, x_2, \dots, x_N$ in the interval $[0, 1]$; for the sake of simplicity, we assume equal spacing, so that $x_0 = 0$, $x_j = jh$, $x_N = 1 = Nh$ with $h = \frac{1}{N}$. We would like to form the vectors:

$$\underline{u} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} v(x_1) \\ v(x_2) \\ \vdots \\ v(x_{N-1}) \end{pmatrix} \in \mathbb{C}^{N-1}; \quad (2.6)$$

however, since we only have $v \in L^2(0, 1)$, point values of v are meaningless. Moreover, we shall need more smoothness in u and v if we are to use the quadrature estimates we need. We, therefore, observe that the set

$$\mathcal{C}(A) := (C^4(0, 1) \cap H_0^1(0, 1)) \times C^1(0, 1) \quad (2.7)$$

is a core of the operator A ; for $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}(A)$, Eq. (2.6) makes sense. We also define diagonal matrices:

$$\begin{aligned} B_N &= \text{diag}(w(x_1), w(x_2), \dots, w(x_{N-1})), \\ C_N &= \text{diag}(\tilde{w}(x_1), \tilde{w}(x_2), \dots, \tilde{w}(x_{N-1})), \\ D_N &= \text{diag}(z(x_1), z(x_2), \dots, z(x_{N-1})). \end{aligned}$$

Our matrix replacement of the Hain–Lüst operator is a matrix of dimension $2(N-1) \times 2(N-1)$ given by the following:

$$\mathbb{A}_N := \begin{pmatrix} L_N & B_N \\ C_N & D_N \end{pmatrix}, \quad (2.8)$$

where $L_N := -\frac{1}{h^2}T + Q_N$. To discretize the Stokes operator. Observe that

$$\begin{aligned} & -\left(\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1}))}{h^2}\right) - \left(\frac{v\left(x_{j+\frac{1}{2}}\right) - v\left(x_{j-\frac{1}{2}}\right)}{h}\right) \\ & = -\lambda u(x_j) + O(h^2), \end{aligned} \quad (2.9)$$

and

$$-\left(\frac{u(x_j) - u(x_{j-1}))}{h}\right) - \frac{1}{2}v\left(x_{j-\frac{1}{2}}\right) = \lambda v\left(x_{j-\frac{1}{2}}\right) + O(h^2). \quad (2.10)$$

Hence, our matrix replacement of the Stokes operator is a matrix of dimension $(2N-1) \times (2N-1)$ given by the following:

$$\mathbb{M}_N := \begin{pmatrix} E_N & W_N \\ W_N^T & Z_N \end{pmatrix}, \quad (2.11)$$

where $E_N := -\frac{1}{h^2}T$,

$$W_N = -\frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix},$$

and $Z_N = \frac{-3}{2}I_{N \times N}$. We shall use this technique in this paper for computing *c-numerical range*.

2.1 c -Numerical Range Inclusion

The following theorem shows that by finite difference discretization approximation, we derived that the every point of the c -numerical range of the Schrödinger operator $L = -\frac{d^2}{dx^2} + q(x)$ can be approximated to arbitrary accuracy.

Theorem 2.1 Suppose λ is a point of the c -numerical range of the Schrödinger operator $L = -\frac{d^2}{dx^2} + q(x)$ in which q (the potential in L) in $C^2[0, 1]$. Then, for all $\epsilon > 0$, there exists $\tilde{N}$, such that for all $N \geq \tilde{N}$ the c -numerical range of the $(N - 1) \times (N - 1)$ discretization has a point λ_N with $|\lambda - \lambda_N| < \epsilon$.

Proof Fix $\epsilon > 0$ and $\lambda \in W_c(L)$. Then, there exist $u_1, u_2, \dots, u_k \in \mathcal{C}(L)$, such that

$$\lambda = \sum_{i=1}^k c_j \frac{\langle Lu_i, u_i \rangle_{L^2(0,1)}}{\langle u_i, u_i \rangle_{L^2(0,1)}}.$$

Then, it is clear which we need two types of integral which are estimated by quadrature.

Type 1: L^2 integrals of smooth functions

Let $u_1, u_2, \dots, u_k \in C^4(0, 1)$ with Dirichlet boundary conditions at 0 and 1. Then

$$\begin{aligned} \sum_{i=1}^k c_i \langle u_i, u_i \rangle_{L^2(0,1)} &= \sum_{i=1}^k c_i \int_0^1 u_i \overline{u_i} dx = \sum_{i=1}^k \sum_{j=1}^N c_i \int_{x_{j-1}}^{x_j} u_i(x) \overline{u_i(x)} dx, \\ &= \sum_{i=1}^k \sum_{j=1}^N c_i \left[\frac{h}{2} \left(u_i(x_{j-1}) \overline{u_i(x_{j-1})} + u_i(x_j) \overline{u_i(x_j)} \right) + O(h^3) \right] \\ &= \sum_{i=1}^k \sum_{j=1}^{N-1} c_i h |u_i(x_j)|^2 + O(h^2) \quad \text{since } u_i(x_0) = 0 = u_i(x_N) \\ &= \sum_{i=1}^k c_i h \langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} + O(h^2) \end{aligned} \quad (2.12)$$

for some vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k \in \mathbb{C}^{N-1}$, where

$$\underline{u}_i = \begin{pmatrix} u_i(x_1) \\ u_i(x_2) \\ \vdots \\ u_i(x_{N-1}) \end{pmatrix} \in \mathbb{C}^{N-1} \quad \text{for } 1 \leq i \leq k. \quad (2.13)$$

By a straightforward computation, the following result can be verified. □

Lemma 2.2 Suppose that $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k \in \mathbb{C}^{N-1}$, and $u_1, u_2, \dots, u_k$ is an eigenfunction of the Schrödinger operator L ; then

$$\sum_{i=1}^k \left(\frac{1}{h \langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} + O(h^2)} \right) = \sum_{i=1}^k \left(\frac{1}{h \langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}} \right) + O(h^2).$$

Type 2: The integrals $\langle Lu, u \rangle_{L^2}$ for smooth u

For $u_1, u_2, \dots, u_k$ as before, we observe that $Lu_1, Lu_2, \dots, Lu_k$ is Lipschitz, because $u''_1, u''_2, \dots, u''_k \in C^2(0, 1)$ and $q \in C^2(0, 1)$:

$$\begin{aligned} \sum_{i=1}^k c_i \langle Lu_i, u_i \rangle_{L^2(0,1)} &= \sum_{i=1}^k c_i \int_0^1 Lu_i \overline{u_i} \, dx = \sum_{i=1}^k \sum_{j=1}^N c_i \int_{x_{j-1}}^{x_j} Lu_i(x) \overline{u_i(x)} \, dx, \\ &= \sum_{i=1}^k \sum_{j=1}^N c_i \left[\frac{h}{2} \left(Lu_i(x_{j-1}) \overline{u_i(x_{j-1})} + Lu_i(x_j) \overline{u_i(x_j)} \right) + O(h^3) \right] \\ &= \sum_{i=1}^k \sum_{j=1}^{N-1} c_i h \left(Lu_i(x_j) \overline{u_i(x)} \right) + O(h) \\ &= \sum_{i=1}^k \sum_{j=1}^{N-1} c_i h (L_N \underline{u}_i)_j \overline{(\underline{u}_i)_j} + O(h) \\ &= \sum_{i=1}^k c_i h \langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} + O(h); \end{aligned} \quad (2.14)$$

also we have

$$\sum_{i=1}^k (L_N u_i - \lambda \underline{u}_i)_j := \sum_{i=1}^k (Lu_i)(x_j) - \lambda \sum_{i=1}^k u_i(x_j) - \frac{1}{4!} h^2 \sum_{i=1}^k (s_i)_j, \quad (2.15)$$

where $(s_i)_j := u_i^4(\xi_j) + u_i^4(\eta_j)$, and $\xi_j, \eta_j \in (x_{j-1}, x_{j+1})$, for $j = 1, \dots, N-1$ and $i = 1, \dots, k$. Then, by taking inner products on both sides of Eq. (2.15), we get the following:

$$\begin{aligned} \sum_{i=1}^k c_i \langle L_N \underline{u}_i - \lambda \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} &:= \sum_{i=1}^k c_i \langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} - \sum_{i=1}^k c_i \lambda \langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} \\ &\quad - \sum_{i=1}^k c_i \frac{h^2}{4!} \langle \underline{s}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}. \end{aligned} \quad (2.16)$$

Multiplying by h and adding $\sum_{i=1}^k \langle Lu_i, u_i \rangle_{L^2} - h \sum_{i=1}^k \langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}$ to both sides of Eq. (2.16), we obtain the following:

$$h \sum_{i=1}^k c_i \langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} + O(h) := \sum_{i=1}^k c_i \langle Lu_i, u_i \rangle_{L^2} - \sum_{i=1}^k \frac{h^3}{4!} \langle \underline{u}_i, \underline{u}_i \rangle, \quad (2.17)$$

and thus, using Lemma 2.2 and dividing both sides of Eq. (2.17) by $\sum_{i=1}^k \langle u_i, u_i \rangle_{L^2}$, we get the following:

$$\sum_{i=1}^k c_i \left(\frac{h \langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} + O(h) + O(h^3)}{\langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}} + O(h^2)} \right) = \sum_{i=1}^k c_i \left(\frac{\langle Lu_i, u_i \rangle_{L^2}}{\langle u_i, u_i \rangle_{L^2}} \right).$$

This implies that

$$\sum_{i=1}^k c_i \left(\frac{\langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}} \right) = \lambda + O(h).$$

It follows that there exists $\lambda_N \in W_c(L_N)$, such that $|\lambda - \lambda_N| \leq \epsilon$.

We introduce Definition 2.3, which is make sense for any bounded or an unbounded linear operator to bound some crucial parts in our main Theorem 2.6. Suppose that A is an unbounded linear operator $A : \mathcal{D}(A) \subset H \rightarrow H$. Consider a sesquilinear form:

$$a(x) = \langle Ax, x \rangle \text{ for } x \in \mathcal{D}(A).$$

We define R -partial c -numerical range as follows:

Definition 2.3 Let A be an unbounded operator with quadratic form $a(\cdot)$. Then, the set

$$W_{cR}(A) = \left\{ \sum_{i=1}^k c_i a(x_i) \mid (x_1, \dots, x_k) \text{ is an orthonormal vectors in } \mathbb{C}^k \right. \\ \left. \text{and } \left| \sum_{i=1}^k c_i a(x_i) \right| \leq R \right\}$$

is called the R -partial c -numerical range of A .

We recall some lemmas useful to prove the main Theorem 2.6.

Lemma 2.4 Let A be an operator which is defined in Definition (2.3); then

$$\overline{W_c(A)} = \bigcup_{R>0} \overline{W_{cR}(A)}.$$

Proof Suppose that $\lambda \in W_c(A)$; then, there exist an orthonormal vectors $x_1, \dots, x_k \in \mathcal{D}(A)$, such that $\lambda = \sum_{i=1}^k c_i a(x_i)$. Therefore, by Definition (2.3), $\lambda \in W_{cR}(A)$ for any $\left| \sum_{i=1}^k c_i a(x_i) \right| \leq R$. Hence, $\overline{W_c(A)} \subseteq \bigcup_{R>0} W_{cR}(A)$. To prove opposite inclusion, it is clear $W_{cR}(A) \subseteq W_c(A)$ for all $R > 0$, so this immediately gives $\bigcup_{R>0} W_{cR}(A) \subseteq W_c(A)$.

The next Lemma on the consequences of constructing a piecewise linear function from a vector in $\mathbb{C}^{N-1}$ by interpolation will be used in the proof of Theorem 2.6 $\square$

Lemma 2.5 Let $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k$ in $\mathbb{C}^{N-1}$, $u_1, u_2, \dots, u_k$ in $H_0^1(0, 1)$ be the piecewise linear functions given by $u_r(x) = u_{r(j-1)} \frac{(x_j-x)}{(x_j-x_{j-1})} + u_{rj} \frac{(x-x_{j-1})}{(x_j-x_{j-1})}$ for $1 \leq r \leq k$, where $x_{j-1} \leq x \leq x_j$ for $J = 1, 2, \dots, N$ and $u_{r0} = 0 = u_{rN}$. Then, $(\|u'_r\|_{L_2}/\|u_r\|_{L_2})^2 \leq 6/h^2$.

Proof For $1 \leq r \leq k$ and let $u_1, u_1, \dots, u_k \in H_0^1(0, 1)$, be the piecewise linear functions given by $u_r(x) = u_{r(j-1)} \frac{(x_j-x)}{(x_j-x_{j-1})} + u_{rj} \frac{(x-x_{j-1})}{(x_j-x_{j-1})}$. Then, for $-h/2 \leq x \leq h/2$, it is not difficult to see that

$$\int_{-h/2}^{h/2} (u_r(x))^2 dx = \frac{h}{2} ((u_{rj}^2 + u_{r(j-1)}^2) - \frac{h}{6} (u_{rj} - u_{r(j-1)})^2);$$

thus

$$\int_0^1 |u_r|^2 = \sum_{j=1}^N h |u_{rj}|^2 - \frac{h}{6} \int_0^1 |u'_r|^2 dx.$$

Rearranging gives

$$h \langle \underline{u}_r, \underline{u}_r \rangle_{\mathbb{C}^{N-1}} = \int_0^1 (|u_r|^2 + \frac{h^2}{6} \|u'_r\|_2^2) dx \geq \|u_r\|_2^2 \quad (2.18)$$

$\square$.

The following theorem is our main result.

Theorem 2.6 Suppose that the coefficient q appearing in the Schrödinger operator $L = -\frac{d^2}{dx^2} + q(x)$ is Lipschitz. Consider the $(N-1) \times (N-1)$ matrix L_N given in (2.5). For each $R > 0$, there exists a constant K_R independent of N , such that $W_{cR}(L_N)$ is contained in a $K_R N^{-1/2}$ -neighborhood of $W_{cR}(L)$.

Proof Suppose that $\lambda_N \in W_{cR}(L_N)$; then, there exist an orthonormal vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k$ in $\mathbb{C}^{N-1}$, such that

$$\lambda_N = c_1 \langle L_N \underline{u}_1, \underline{u}_1 \rangle + c_2 \langle L_N \underline{u}_2, \underline{u}_2 \rangle + \dots + c_k \langle L_N \underline{u}_k, \underline{u}_k \rangle.$$

We begin by constructing a piecewise linear functions $u_1, u_2, \dots, u_k$ in $H_0^1(0, 1)$, such that $u_r(x_j) = u_{rj}$, and $u_{r0} = 0 = u_{rN}$ for $1 \leq r \leq k$. It is clear $u_1, u_2, \dots, u_k$

$\notin \mathcal{D}(L)$, because $u_1, u_2, \dots, u_k \notin H^2(0, 1)$. This is not a problem when using the characterization of $W_c(A)$ in Definition (2.3); we simply note that the quadratic form associated with L is the usual Dirichlet form:

$$\ell(u_1, u_1) := \int_0^1 (|u'_1|^2 + q|u_1|^2) dx, \dots, \ell(u_k, u_k) := \int_0^1 (|u'_k|^2 + q|u_k|^2) dx,$$

so, first, we need to estimate $\mathcal{I}_1$ on the right-hand side of Eq. (2.19):

$$\begin{aligned} & \left| \sum_{i=1}^k c_i \left(\frac{\langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}} \right) - \sum_{i=1}^k c_i \left(\frac{\ell(u_i, u_i)}{\langle u_i, u_i \rangle_{L^2}} \right) \right| \\ & \leq c_1 \underbrace{\left| \frac{\langle L_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}} - \frac{\ell(u_1, u_1)}{\langle u_1, u_1 \rangle_{L^2}} \right|}_{\mathcal{I}_1} + \dots \\ & \quad + c_k \underbrace{\left| \frac{\langle L_N \underline{u}_k, \underline{u}_k \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_k, \underline{u}_k \rangle_{\mathbb{C}^{N-1}}} - \frac{\ell(u_k, u_k)}{\langle u_k, u_k \rangle_{L^2}} \right|}_{\mathcal{I}_k} \end{aligned} \quad (2.19)$$

□

Here, $c_1 \ell(u_1, u_1) = c_1 l_0(u_1, u_1) + c_1 \int_0^1 q|u_1|^2 dx$, where $c_1 l_0(u_1, u_1) = c_1 \int_0^1 |u'_1|^2 dx$. By summation by parts

$$c_1 \int_0^1 |u'_1|^2 dx = \frac{c_1}{h} \sum_{j=1}^N |u_{1j} - u_{1(j-1)}|^2 = c_1 h \left\langle -\frac{1}{h^2} T \underline{u}_1, \underline{u}_1 \right\rangle_{\mathbb{C}^{N-1}}, \quad (2.20)$$

and we are left to deal with

$$\begin{aligned} & \sum_{j=1}^N \int_{x_{j-1}}^{x_j} q(x) |u_1|^2 \\ & = h \sum_{j=1}^{N-1} q(x_j) |u_{1j}|^2 + h \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} (q(x) - q(x_j)) \operatorname{Re}(u_{1j} (u_{1(j-1)} - u_{1j})) dx \\ & \quad + \frac{h}{3} \sum_{j=1}^{N-1} q(x_j) |u_{1j} - u_{1(j-1)}|^2 + \frac{h}{3} \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} (q(x) - q(x_j)) |u_{1j} - u_{1(j-1)}|^2 dx \\ & \quad + h \sum_{j=1}^{N-1} q(x_j) \operatorname{Re}(u_{1j} (u_{1(j-1)} - u_{1j})) + h \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} (q(x) - q(x_j)) |u_{1j}|^2 dx. \end{aligned} \quad (2.21)$$

Assuming that $q(\cdot)$ is Lipschitz, and using the Cauchy–Schwarz inequality, we get the following:

$$\begin{aligned} & \left| h \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} (q(x) - q(x_j)) \operatorname{Re} (u_{1j} (u_{1(j-1)} - u_{1j})) \, dx \right| \\ & \leq \max_{1 \leq k \leq N} \left(\left| \int_{x_{k-1}}^{x_k} (q(x) - q(x_k)) \, dx \right| \right) h \sum_{j=1}^N |u_{1j} (u_{1(j-1)} - u_{1j})| \\ & \leq M_1 h^2 \left(\sum_{j=1}^{N-1} h |u_{1j}|^2 \right)^{1/2} h^{1/2} \left(\sum_{j=1}^{N-1} |u_{1(j-1)} - u_{1j}|^2 \right)^{1/2} \\ & \leq M_1 h^3 \|u'_1\|_2, \end{aligned} \quad (2.22)$$

in which the last inequality follows from (2.22). Here, $\|u'_1\|_2$ can be bounded in a way which depends only on R and not on h . To see this, recall that since $\lambda \in W_R(L_N)$, and by Definition (2.3):

$$\left| h \left\langle -\frac{1}{h^2} T \underline{u}_1, \underline{u}_1 \right\rangle_{\mathbb{C}^{N-1}} + h \langle Q_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}} \right| \leq R h \langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}};$$

this yields, in light of (2.20):

$$\frac{\|u'_1\|_2^2}{h \langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}} \leq \sup(|q|) + R.$$

By Lemma 2.5 for $r = 1$:

$$h \langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}} = \int_0^1 (|u_1|^2 + \frac{h^2}{6} \|u'_1\|_2^2) \, dx \geq \|u_1\|_2^2. \quad (2.23)$$

Thus

$$\frac{\|u'_1\|_2^2}{\|u_1\|_2^2} \leq R + \sup(|q|). \quad (2.24)$$

By the Lipschitz property of q and using (2.22), the fourth term on the right-hand side of Eq. (2.21) satisfies the following:

$$\left| \frac{h}{3} \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} (q(x) - q(x_j)) |u_{1j} - u_{1(j-1)}|^2 \, dx \right| \leq M_1 h^2 \|u'_1\|_2^2, \quad (2.25)$$

while the sixth term admits the bound

$$\left| h \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} (q(x) - q(x_j)) |u_{1j}|^2 dx \right| \leq M_1 h^2. \quad (2.26)$$

The third term of the right-hand side of (2.21) satisfies the following:

$$\begin{aligned} & \left| \frac{h}{3} \sum_{j=1}^{N-1} q(x_j) |u_{1j} - u_{1(j-1)}|^2 \right| \\ & \leq \max_{1 \leq k \leq N} (|q_k|) h \sum_{j=1}^N |u_{1j} - u_{1(j-1)}|^2 \leq M_1 h^2 \|u'_1\|_2^2, \end{aligned} \quad (2.27)$$

and the last inequality again coming from (2.22). The one remaining term of the right-hand side of Eq. (2.21) is estimated by the following:

$$\begin{aligned} & \left| h \sum_{j=1}^{N-1} q(x_j) \operatorname{Re} (u_{1j} (u_{1(j-1)} - u_{1j})) \right| \leq \max_{1 \leq k \leq N} (|q_k|) h \sum_{j=1}^N |(u_{1j} (u_{1(j-1)} - u_{1j}))| \\ & \leq M_1 \left(\sum_{j=1}^{N-1} h |u_{1j}|^2 \right)^{1/2} h^{1/2} \left(\sum_{j=1}^{N-1} |u_{1(j-1)} - u_{1j}|^2 \right)^{1/2} \\ & \leq M_1 h \|u'_1\|_2, \end{aligned} \quad (2.28)$$

with (2.21) again providing the last step. Summing all the contributions, we obtain the following:

$$\left| \int_0^1 q |u_1|^2 - h \langle Q_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}} \right| \leq M_1 h \|u'_1\|_2, \quad (2.29)$$

and so

$$h \langle Q_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}} = \int_0^1 q |u_1|^2 + \zeta_h, \quad (2.30)$$

where $|\zeta_h| \leq M_1 h \|u'_1\|_2$.

We are now in a position to estimate the following:

$$c_1 \left| \frac{\langle L_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}} - \frac{\int_0^1 q |u_1|^2}{\int_0^1 |u_1|^2} \right| =: \varepsilon_q. \quad (2.31)$$

From (2.30) to (2.23):

$$\begin{aligned}\varepsilon_q &= \left| \frac{\int_0^1 q |u_1|^2 + \zeta_h}{(\int_0^1 |u_1|^2) \left(1 + \frac{h^2}{6} \frac{\|u'_1\|_2^2}{\|u\|_2^2}\right)} - \frac{\int_0^1 q |u_1|^2}{\int_0^1 |u_1|^2} \right| \\ &= \left| \frac{\zeta_h - \frac{h^2}{6} \frac{\|u'_1\|_2^2}{\|u_1\|_2^2} \int_0^1 q |u|^2}{(\int_0^1 |u_1|^2) \left(1 + \frac{h^2}{6} \frac{\|u'_1\|_2^2}{\|u\|_2^2}\right)} \right| \\ &\leq \frac{|\zeta_h|}{\|u_1\|_2^2} + \sup(|q|) \frac{h^2}{6} \left(\frac{\|u'_1\|_2}{\|u\|_2} \right)^2.\end{aligned}\quad (2.32)$$

Now, we estimate

$$c_1 \left| \frac{h \langle \frac{-1}{h^2} T_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}}{h \langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}} - \frac{l_0(u, u)}{\langle u_1, u_1 \rangle} \right|. \quad (2.33)$$

From Eqs. (2.20) to (2.23), we have the following:

$$\begin{aligned}c_1 \frac{h \langle \frac{-1}{h^2} T_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}}{h \langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}} &= \frac{c_1 l_0(u, u)}{(\int_0^1 |u_1|^2) \left(1 + \frac{h^2}{6} \frac{\|u'_1\|_2^2}{\|u_1\|_2^2}\right)} \\ &\geq \frac{c_1 l_0(u_1, u_1) \left(1 - \frac{h^2}{6} \frac{\|u'_1\|_2^2}{\|u_1\|_2^2}\right)}{\int_0^1 |u_1|^2}.\end{aligned}\quad (2.34)$$

Thus, we get the following:

$$\begin{aligned}c_1 \left| \frac{h \langle \frac{-1}{h^2} T_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}}{h \langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}} - \frac{l_0(u_1, u_1)}{\langle u_1, u_1 \rangle} c_1 \right| \\ \leq c_1 \frac{l_0(u_1, u_1)}{\int_0^1 |u_1|^2} \frac{h^2}{6} \frac{\|u'_1\|_2^2}{\|u_1\|_2^2} \leq c_1 M_1 \left(\frac{\|u'_1\|_2}{\|u_1\|_2} \right)^4 h^2.\end{aligned}\quad (2.35)$$

Combining Eqs. (2.32), (2.35), and using Eq. (2.24), we get the following:

$$\begin{aligned}c_1 \left| \frac{\langle L_N \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_1, \underline{u}_1 \rangle_{\mathbb{C}^{N-1}}} - \frac{\ell(u_1, u_1)}{\langle u_1, u_1 \rangle_{L^2}} \right| &\leq c_1 \frac{|\zeta_h|}{\|u_1\|_2^2} + c_1 \sup(|q|) \frac{h^2}{6} \left(\frac{\|u'_1\|_2}{\|u_1\|_2} \right)^2 \\ &\quad + c_1 M_1 \left(\frac{\|u'_1\|_2}{\|u_1\|_2} \right)^2 h^2 \leq c_1 M_1 h \sqrt{R} + c_1 M_1 h^2 R \\ &\quad + c_1 M_1 h^2 R^2.\end{aligned}\quad (2.36)$$

By the same argument, the second term and the last term of the right-hand side of Eq. (2.19) satisfies the following:

$$\begin{aligned} c_2 \left| \frac{\langle L_N \underline{u}_2, \underline{u}_2 \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_2, \underline{u}_2 \rangle_{\mathbb{C}^{N-1}}} - \frac{\ell(u_2, u_2)}{\langle u_2, u_2 \rangle_{L^2}} \right| \\ \leq c_2 M_2 h \sqrt{R} + c_k M_2 h^2 R + c_2 M_2 h^2 R^2. \end{aligned} \quad (2.37)$$

and

$$\begin{aligned} c_k \left| \frac{\langle L_N \underline{u}_k, \underline{u}_k \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_k, \underline{u}_k \rangle_{\mathbb{C}^{N-1}}} - \frac{\ell(u_k, u_k)}{\langle u_k, u_k \rangle_{L^2}} \right| \\ \leq c_k M_k h \sqrt{R} + c_k M_2 h^2 R + c_k M_k h^2 R^2. \end{aligned} \quad (2.38)$$

Combining Eqs. (2.36), (2.37), and (2.38), we get the following:

$$\begin{aligned} \left| \sum_{i=1}^k c_i \left(\frac{\langle L_N \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}}{\langle \underline{u}_i, \underline{u}_i \rangle_{\mathbb{C}^{N-1}}} \right) - \sum_{i=1}^k c_i \left(\frac{\ell(u_i, u_i)}{\langle u_i, u_i \rangle_{L^2}} \right) \right| \\ \leq \sum_{i=1}^k \left(c_i M_i h \sqrt{R} + c_i M_i h^2 R + c_i M_i h^2 R^2 \right). \end{aligned}$$

3 Numerical Experiments on Differential Operator

In this section, we study some concrete examples and demonstrate that, in spite of the results obtained in the previous section, practical calculation of the *c-numerical range* is very far from being straightforward. We shall show that the discretization techniques may often yield misleading results, and that a good knowledge of existing analytical estimates of *q-numerical range* is often needed to understand the results. The computations were performed in MATLAB.

3.1 Example 1

Assume that $w : [0, 1] \rightarrow [0, \infty)$, $\tilde{w} : [0, 1] \rightarrow [0, \infty)$, and $z : [0, 1] \rightarrow \mathbb{C}$ are such that $w(x) = 1$, $\tilde{w}(x) = 1$,

$$z(x) = 18e^{2\pi i x} - 20,$$

for each $x \in [0, 1]$.

We introduce the differential expression:

$$\tau_{\tilde{A}} := -\frac{d^2}{dx^2}, \quad \tau_{\tilde{B}} := w(x), \quad \tau_{\tilde{C}} := \tilde{w}(x), \quad \tau_{\tilde{D}} := z(x). \quad (3.1)$$

Fig. 1 The c -numerical range of $W_{\mathbf{c}}(\mathbb{A}_{25})$ and $W_{\tilde{\mathbf{c}}}(\mathbb{A}_{25})$

Let L, B, C, D be the operators in the Hilbert space $L^2(0, 1)$ induced by the differential expressions $\tau_{\tilde{A}}, \tau_{\tilde{B}}, \tau_{\tilde{C}}, \tau_{\tilde{D}}$ with domain:

$$\mathcal{D}(L) := H^2(0, 1) \cap H_0^1(0, 1), \quad \mathcal{D}(B) = \mathcal{D}(C) = \mathcal{D}(D) := L^2(0, 1).$$

In the Hilbert space $L_2^2(0, 1) := L^2(0, 1) \oplus L^2(0, 1)$, we introduce the matrix differential operator:

$$\mathcal{A} = \begin{pmatrix} L & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} w(x) \\ \tilde{w}(x) z(x) \end{pmatrix}, \quad (3.2)$$

on the domain

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1 \in H^2(0, 1) : y_1(0) = 0 = y_1(1) \text{ and } y_2 \in L_2(0, 1) \right\}.$$

The finite difference approximation to the Hain–Lüst operator is the $2(N - 1) \times 2(N - 1)$ matrix $\mathbb{A}_N$ given in Eq. (2.8).

Figure 1 show the attempts to compute approximations to the c -numerical range of the Hain–Lüst operator using matrices (2.8) for different values of $\mathbf{c} = (1, 0, \dots, 0)$ and $\tilde{\mathbf{c}} := \mathbf{c} = (1.1, 0, \dots, -0.1)$ with $N = 25$.

3.2 Analytical Estimates for Example 1

To understand the results in Fig. 1, it is useful to find an analytical estimate for $W_c(A)$. Let $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_k$ be an orthonormal vectors in $\mathcal{D}(A)$, where

$$\underline{y}_1 = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix}, \underline{y}_2 = \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}, \dots, \underline{y}_k = \begin{pmatrix} y_{k1} \\ y_{k2} \end{pmatrix},$$

and let

$$\begin{aligned} \lambda &= \sum_{j=1}^k c_j \langle A \underline{y}_j, \underline{y}_j \rangle \\ &= \sum_{j=1}^k c_j \langle -y''_{j1}, y_{j1} \rangle_{L^2(0,1)} + 2 \sum_{j=1}^k c_j \Re \langle y_{j1}, y_{j2} \rangle_{L^2(0,1)} \\ &\quad + \sum_{j=1}^k c_j \langle y_{j2}, y_{j1} \rangle_{L^2(0,1)}. \end{aligned} \quad (3.3)$$

The first term of Eq. (3.3) gives, as an estimate:

$$\sum_{j=1}^k c_j \langle -y''_{j1}, y_{j1} \rangle_{L^2(0,1)} \geq \pi^2 \sum_{j=1}^k c_j \langle y_{j1}, y_{j1} \rangle_{L^2(0,1)}. \quad (3.4)$$

For the second term on the right-hand side of Eq. (3.3), the Cauchy–Schwarz inequality and Youngs inequality yield the following:

$$2 \sum_{j=1}^k c_j \Re \langle y_{j1}, y_{j2} \rangle_{L^2(0,1)} \geq - \sum_{j=1}^k c_j. \quad (3.5)$$

The third term of the right-hand side of Eq. (3.3) satisfies the following:

$$\sum_{j=1}^k c_j \langle u y_{j2}, y_{j1} \rangle_{L^2(0,1)} \geq \inf_{x \in [0,1]} \Re(u) \sum_{j=1}^k c_j (1 - \langle y_{j1}, y_{j1} \rangle_{L^2(0,1)}). \quad (3.6)$$

Hence, from Eqs. (3.4), (3.5), and (3.6), we get that

$$\begin{aligned} \Re(\lambda) &\geq \left(\pi^2 - \inf_{x \in [0,1]} \Re(u) \right) \sum_{j=1}^k c_j \langle y_{j1}, y_{j1} \rangle_{L^2(0,1)} \\ &\quad + \left(\inf_{x \in [0,1]} \Re(u) - 1 \right) \sum_{j=1}^k c_j. \end{aligned} \quad (3.7)$$

This yields the following:

$$\Re(\lambda) \geq \begin{cases} \left(\inf_{x \in [0,1]} \Re(u) - 1 \right) \sum_{j=1}^k c_j, & \text{if } \pi^2 - \inf_{x \in [0,1]} \Re(u) \geq 0; \\ (\pi^2 - 1) \sum_{j=1}^k c_j, & \text{if } \pi^2 - \inf_{x \in [0,1]} \Re(u) < 0. \end{cases}$$

For our example, these yield $\Re(\lambda) \geq -39 \sum_{j=1}^k c_j$

To estimate $\text{Im}(\lambda)$, observe that

$$\begin{aligned} \text{Im}(\lambda) &= \sum_{j=1}^k c_j \langle u y_{j2}, y_{j1} \rangle_{L^2(0,1)} \\ &\leq \sup_{x \in [0,1]} \Im(u) \sum_{j=1}^k c_j \langle y_{j2}, y_{j2} \rangle_{L^2(0,1)} \leq 18 \sum_{j=1}^k c_j \langle y_{j2}, y_{j2} \rangle_{L^2(0,1)} \quad (3.8) \end{aligned}$$

and

$$\begin{aligned} \text{Im}(\lambda) &= \sum_{j=1}^k c_j \langle u y_{j2}, y_{j1} \rangle_{L^2(0,1)} \\ &\geq \inf_{x \in [0,1]} \Im(u) \sum_{j=1}^k c_j \langle y_{j2}, y_{j2} \rangle_{L^2(0,1)} \geq -18 \sum_{j=1}^k c_j \langle y_{j2}, y_{j2} \rangle_{L^2(0,1)}; \quad (3.9) \end{aligned}$$

hence

$$-18 \sum_{j=1}^k c_j \leq \text{Im}(\lambda) \leq 18 \sum_{j=1}^k c_j.$$

This completes the estimates on $W_c(A)$.

The following result shows that the c -numerical range of Example 1 is unbounded.

Remark 3.1 Let $\eta_1, \eta_2, \dots, \eta_k$ be an orthonormal vectors in $\mathcal{D}(A)$, where

$$\eta_1 = \begin{pmatrix} \sqrt{2} \sin(l\pi x) \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} \sqrt{2} \sin(k\pi x) \\ 0 \end{pmatrix}, \dots, \eta_k = \begin{pmatrix} \sqrt{2} \sin(k\pi x) \\ 0 \end{pmatrix};$$

then

$$\sum_{j=1}^k c_j \langle A \eta_j, \eta_j \rangle_{L^2(0,1)} = l^2 \pi^2 \sum_{j=1}^k c_j.$$

However, $l \in \mathbb{Z}$ can be arbitrary large. This means that $\Re(\lambda)$ is unbounded above.

3.3 Example 2

We consider the differential expression:

$$\tau_{\tilde{A}} := -\frac{d^2}{dx^2}, \tau_B := -\frac{d}{dx}, \quad (3.10)$$

$$\tau_C := \frac{d}{dx}, \tau_{D_0} := -\frac{3}{2}. \quad (3.11)$$

Let A_0, B_0, C_0, D_0 be the operators in the Hilbert space $L^2(0, 1)$ induced by the differential expressions $\tau_{\tilde{A}}, \tau_B, \tau_C, \tau_{D_0}$ with domain:

$$\begin{aligned} \mathcal{D}(A_0) &:= H^2(0, 1) \cap H_0^1(0, 1), \quad \mathcal{D}(B_0) := H^1(0, 1), \\ \mathcal{D}(C_0) &:= H_0^1(0, 1), \quad \mathcal{D}(D_0) := L^2(0, 1). \end{aligned}$$

In the Hilbert space $L_2^2(0, 1) := L^2(0, 1) \oplus L^2(0, 1)$, we introduce the matrix differential operator:

$$\mathcal{A} := \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} & -\frac{d}{dx} \\ \frac{d}{dx} & -\frac{3}{2} \end{pmatrix}, \quad (3.12)$$

on the domain

$$\mathcal{D}(\mathcal{A}) := (H^2(0, 1) \cap H^1(0, 1)) \oplus H^1(0, 1). \quad (3.13)$$

The finite difference approximation to the Stokes operator is the $(2N-1) \times (2N-1)$ matrix:

$$\mathbb{M}_N := \begin{pmatrix} E_N & W_N \\ W_N^T & Z_N \end{pmatrix},$$

given in Eq. (2.11). Figure 2 shows an estimate of the *c-numerical range* of the Stokes operator using matrices (2.6) for different values of $\mathbf{c} = (1, 0, \dots, 0)$ and $\tilde{\mathbf{c}} := \mathbf{c} = (1.1, 0, \dots, -0.1)$ with $N = 25$.

3.4 Analytical Estimates for Example 2

To understand the results in Fig. 2, it is useful to find an analytical estimate for $W_c(\mathcal{A}_0)$. Let $\underline{\zeta}_1, \underline{\zeta}_2, \dots, \underline{\zeta}_k$ be an orthonormal vectors in $\mathcal{D}(\mathcal{A}_0)$, where

$$\underline{\zeta}_1 = \begin{pmatrix} \zeta_{11} \\ \zeta_{12} \end{pmatrix}, \quad \underline{\zeta}_2 = \begin{pmatrix} \zeta_{21} \\ \zeta_{22} \end{pmatrix}, \dots, \underline{\zeta}_k = \begin{pmatrix} \zeta_{k1} \\ \zeta_{k2} \end{pmatrix},$$

Fig. 2 The c -numerical range of $W_{\mathbf{c}}(\mathbb{A}_{25})$ and $W_{\bar{\mathbf{c}}}(\mathbb{A}_{25})$

and let

$$\begin{aligned} \lambda &= \sum_{j=1}^k c_j \langle \mathcal{A}_0 \underline{\zeta}_j, \underline{\zeta}_j \rangle \\ &= \sum_{j=1}^k c_j \langle -\zeta''_{j1}, \zeta_{j1} \rangle_{L^2(0,1)} + 2 \sum_{j=1}^k c_j \langle \zeta'_{j1}, \zeta_{j2} \rangle_{L^2(0,1)} \\ &\quad + \sum_{j=1}^k c_j \langle \alpha \zeta_{j2}, \zeta_{j1} \rangle_{L^2(0,1)}. \end{aligned} \quad (3.14)$$

The first term of Eq. (3.14) gives, as an estimate, the following:

$$\sum_{j=1}^k c_j \langle -\zeta''_{j1}, \zeta_{j1} \rangle_{L^2(0,1)} \geq \pi^2 \sum_{j=1}^k c_j \langle \zeta_{j1}, \zeta_{j1} \rangle_{L^2(0,1)}. \quad (3.15)$$

For the second term on the right-hand side of Eq. (3.14), the Cauchy–Schwarz inequality and Youngs inequality yield the following:

$$2 \sum_{j=1}^k c_j \langle \zeta'_{j1}, \zeta_{j2} \rangle_{L^2(0,1)} \geq - \sum_{j=1}^k c_j. \quad (3.16)$$

The third term of the right-hand side of Eq. (3.14) satisfies

$$\sum_{j=1}^k c_j \langle \alpha \zeta_{j2}, \zeta_{j1} \rangle_{L^2(0,1)} = \alpha \sum_{j=1}^k c_j (1 - \langle \zeta_{j1}, \zeta_{j1} \rangle_{L^2(0,1)}). \quad (3.17)$$

Hence, from Eqs. (3.15), (3.16), and (3.17), we get that

$$\Re(\lambda) \geq (\pi^2 - \alpha) \sum_{j=1}^k c_j \langle y_{j1}, y_{j1} \rangle_{L^2(0,1)} + (\alpha - 1) \sum_{j=1}^k c_j. \quad (3.18)$$

This yields the following:

$$\Re(\lambda) \geq \begin{cases} (\alpha - 1) \sum_{j=1}^k c_j, & \text{if } \pi^2 - \alpha \geq 0; \\ (\pi^2 - 1) \sum_{j=1}^k c_j, & \text{if } \pi^2 - \alpha < 0. \end{cases}$$

Because in our case $\alpha = -\frac{3}{2}$, then this implies that $\Re(\lambda) \geq -\frac{5}{2} \sum_{j=1}^k c_j$. On the other hand, it is not difficult to see that $\Im(\lambda) = 0$.

This completes the estimates on $W_c(\mathcal{A}_0)$.

4 Conclusions

The computation of *c-numerical ranges* of operator matrices of differential operators and block operator matrices of differential operators by finite difference techniques is easier to implement than Galerkin discretization methods, but the theoretical analysis is more involved and the results obtained are less esthetically satisfactory. In the case of Galerkin discretizations, the *c-numerical ranges* are nested:

$$W_c(A_{N+1}) \supseteq W_c(A_N) \text{ for all } N \in \mathbb{N}.$$

This is not guaranteed for finite differences, where only have Theorem 2.6. However, in all cases, whether the discretization be by finite difference or variational methods, the numerical results can only be properly understood in conjunction with an understanding of the spectral theory of the original operator.

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