Existence and Multiplicity Results for Steklov Problems with $p(\cdot)$ -Growth Conditions

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Abstract

Using variational methods, we prove in different situations the existence and multiplicity of solutions for the following Steklov problem:

$$\begin{aligned} -\operatorname{div}(a(x, \nabla u)) + |u|^{p(x)-2}u &= 0, \quad \text{in } \Omega, \\ a(x, \nabla u) \cdot \nu &= g(x, u), \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$ and ν is the unit outward normal vector on $\partial\Omega$. $p : \overline{\Omega} \mapsto \mathbb{R}$, $a : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ and $g : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are fulfilling appropriate conditions.

Keywords Critical point theorem · Variable exponents · Elliptic problem · Nonlinear boundary condition · Variational methods

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1 Introduction and main results

In this article, we consider the elliptic problem with nonlinear boundary conditions and variable exponents

$$\begin{aligned} -\operatorname{div}(a(x, \nabla u)) + |u|^{p(x)-2}u &= 0, \quad \text{in } \Omega, \\ a(x, \nabla u) \cdot \nu &= g(x, u), \quad \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$ and the function involved in this problem will be described in what follows.

$p \in C(\overline{\Omega})$ are variable exponents and throughout this paper, we denote

$$p^- = \min_{x \in \overline{\Omega}} p(x); \quad p^+ = \max_{x \in \overline{\Omega}} p(x); \quad (1.2)$$

$$p^\partial(x) = \begin{cases} (N-1)p(x)/[N-p(x)], & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases} \quad (1.3)$$

and

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : 1 < p^- < p^+ < \infty\}. \quad (1.4)$$

Our exponent p fulfills $p \in C_+(\overline{\Omega})$ and for this function p we introduce a characterization of the Carathéodory function $a : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}^N$.

(H₀) $a(x, -s) = -a(x, s)$ for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$.

(H₁) There exists a Carathéodory function $A : \overline{\Omega} \times \mathbb{R}^N \mapsto \mathbb{R}$ continuously differentiable with respect to its second argument, such that $a(x, s) = \nabla_s A(x, s)$ for all $s \in \mathbb{R}^N$ and a.e. $x \in \overline{\Omega}$.

(H₂) $A(x, 0) = 0$ for a.e. $x \in \overline{\Omega}$.

(H₃) There exists $c > 0$ such that a satisfies the growth condition $|a(x, s)| \leq c(1 + |s|^{p(x)-1})$ for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$, where $|\cdot|$ denotes the Euclidean norm.

(H₄) The monotonicity condition $0 \leq [a(x, s_1) - a(x, s_2)](s_1 - s_2)$ holds for a.e. $x \in \overline{\Omega}$ and all $s_1, s_2 \in \mathbb{R}^N$, with equality if and only if $s_1 = s_2$.

(H₅) The inequalities $|s|^{p(x)} \leq a(x, s)s \leq p(x)A(x, s)$ hold for a.e. $x \in \overline{\Omega}$ and all $s \in \mathbb{R}^N$.

A first remark is that hypothesis (H₀) is only needed to obtain the multiplicity of solutions. As in [5], we have decided to use this kind of function a satisfying (H₀)–(H₅) because we want to assure a high degree of generality to our work. Here, we invoke the fact that, with appropriate choices of a , we can obtain many types of operators. We give, in the following, two examples of well-known operators which are presented in lots of papers.

Example 1.1 If $a(x, s) = |s|^{p(x)-2}s$, then we have $A(x, s) = \frac{1}{p(x)}|s|^{p(x)}$.

The assumptions (H₀) – (H₅) are verified, and we arrive at the $p(x)$ -Laplace operator $\operatorname{div}(a(x, \nabla u)) = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \Delta_{p(x)}u$.

Example 1.2 If $a(x, s) = (1 + |s|^2)^{(p(x)-2)/2}s$, then we obtain the operator $A(x, s) = \frac{1}{p(x)}[(1 + |s|^2)^{p(x)/2} - 1]$. The assumptions (H_0) – (H_5) are also verified, and we find a generalized mean curvature operator $\operatorname{div}(a(x, \nabla u)) = \operatorname{div}((1 + |\nabla u|^2)^{(p(x)-2)/2} \nabla u)$.

The above operator appears in [5] and it is used in the study of two antiplane frictional contact problems of elastic cylinders. Functions fulfilling conditions related to (H_0) – (H_5) are used not only in the framework of the spaces with variable exponents [4, 18], but also in the framework of the classical Lebesgue–Sobolev spaces [21] and the anisotropic spaces with variable exponents.

The study of differential and partial differential equation with variable exponent has been received considerable attention in recent years. This importance reflects directly into a various range of applications. There are applications concerning elastic materials [22], image restoration [6], thermorheological and electrorheological fluids [3, 19] and mathematical biology [12].

In the case when $p(x) = p$ is a constant and $\operatorname{div}(a(x, \nabla u)) = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with Dirichlet boundary conditions, many authors consider the type problem

$$\begin{cases} -\Delta_p(u) = g(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

Ambrosetti and Rabinowitz proposed that the mountain pass theorem in 1973 (see, [1]) critical points theory has became one of the main tools for finding solutions to elliptic problems of variational type. One of the very important hypotheses usually imposed on the nonlinearities is the following Ambrosetti–Rabinowitz type condition [(A-R) type condition for short]: There exists $\mu > p$ such that

$$0 < \mu G(x, t) := \mu \int_0^t g(x, s) ds \leq g(x, t)t, \quad \forall (x, t) \in \Omega \times \mathbb{R}^*. \quad (1.6)$$

This condition ensures that the energy functional $\Phi(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} G(x, u) dx$ associated with the problem (1.5) satisfies the Palais–Smale condition [(PS) condition for short], namely any sequence $(u_n) \subset W_0^{1,p}(\Omega)$ such that $\Phi(u_n)$ bounded and $\Phi'(u_n) \rightarrow 0$ contains a converging subsequence. Clearly, if the condition (1.6) is satisfied, then there exist two positive constants a, b such that

$$G(x, t) \geq a|t|^\mu - b, \quad \forall (x, t) \in \Omega \times \mathbb{R}. \quad (1.7)$$

Equation (1.7) means that g is p -superlinear at infinity in the sense that $\lim_{|t| \rightarrow +\infty} \frac{G(x, t)}{|t|^p} = +\infty$.

Our purpose of this work is to extend some of the known results with Neuman or Dirichlet boundary conditions on bounded domain (see, [5, 7]), and generalize some known results in the Steklov problems (see, [2, 20]). We consider the following cases:

- (i) $g(x, u) = \lambda |u|^{r(x)-2} u + f(x, u)$;
- (ii) $g(x, u) = \lambda f(x, u)$,

where $\lambda \in \mathbb{R}^+$, $r : \bar{\Omega} \mapsto \mathbb{R}$ and $f : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are fulfilling appropriate conditions.

We enumerate now the hypotheses concerning the functions f and F , where $F(x, u) = \int_0^u f(x, s)ds$.

(I0) $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

(I1) There exists $k > 0$ such that f satisfies the growth condition

$$|f(x, t)| \leq k|t|^{q(x)-1},$$

for a.e. $x \in \partial\Omega$ and all $t \in \mathbb{R}$, where $q \in C_+(\bar{\Omega})$ with $p^+ < q^- < q^+ < p^\partial(x)$, where $p^\partial(x)$ is defined in (1.3).

(I'1) There exists $k > 0$ such that f satisfies the growth condition

$$|f(x, t)| \leq k|t|^{q(x)-1},$$

for a.e. $x \in \partial\Omega$ and all $t \in \mathbb{R}$, where $q \in C_+(\bar{\Omega})$ with $q^+ < p^-$.

(I2) There exist $\theta > p^+$ and $l > 0$ such that f satisfies the Ambrosetti–Rabinowitz condition

$$0 < \theta F(x, t) \leq f(x, t)t,$$

for all $|t| > l$ and a.e. $x \in \partial\Omega$, where $F(x, t) = \int_0^t f(x, s)ds$.

(I3) The function is odd with respect to its second variable, that is $f(x, -t) = -f(x, t)$ for a.e. $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

(I4) There exists $t_0 > 0$ such that $F(x, t_0) > 0$ a.e. $x \in \partial\Omega$.

(I5) $\lim_{|t| \rightarrow +\infty} \frac{f(x, t)}{|t|^{p(x)-1}} = 0$, for all $x \in \partial\Omega$.

(I6) $\lim_{|t| \rightarrow 0} \frac{f(x, t)}{|t|^{p(x)-1}} = 0$, for all $x \in \partial\Omega$.

Our main results in this paper are the proofs of the following theorems, which are based on the different version of mountain pass theorem. The first three theorems concern the case (i) and the rest of theorems deal the case (ii).

Theorem 1.3 For $f \equiv 0$, assume that (H₀)–(H₅) hold and let $p, r \in C_+(\bar{\Omega})$, such that $p^+ < r^- \leq r^+ < p^\partial(x)$ for all $x \in \bar{\Omega}$, where $p^\partial(x)$ is defined in (1.3). Then for any $\lambda > 0$ the problem (1.1) in the case (i) possesses a nontrivial weak solutions.

Theorem 1.4 Assume that (H₀)–(H₅) hold, (I0)–(I1)–(I2) and let $p, r \in C_+(\bar{\Omega})$ with $1 < r^- \leq r^+ < p^-$. Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ the problem (1.1) in the case (i) possesses a nontrivial weak solutions.

Theorem 1.5 Assume that (H₀)–(H₅) hold, $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying the conditions (I'1), (I2) and (I3). Let $p, r \in C_+(\bar{\Omega})$ with $1 < r^- \leq r^+ < p^-$. Then, there exists $\lambda_1 > 0$, such that for any $\lambda > \lambda_1$ there exists a sequence (u_k) of nontrivial weak solutions for the problem (1.1) in the case (i). Moreover, $u_k \rightarrow 0$, as $k \rightarrow \infty$.

Theorem 1.6 Assume that (H₀)–(H₅) hold, $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying the conditions (I0), (I4) and (I5). Then, there exists a constant $\lambda_0 > 0$ such that

problem (1.1) in the case (ii) has at least one nontrivial weak solution for every $\lambda > \lambda_0$.

Theorem 1.7 Assume that (H_0) – (H_5) hold, $p \in C_+(\bar{\Omega})$ and $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying the conditions (I0), (I4), (I5) and (I6). Then, the problem (1.1) in the case (ii) has at least two nontrivial weak solutions for every $\lambda > \lambda_0$, where λ_0 is the one found in Theorem 1.6.

This paper consists of four sections. Sect. 1, contains an introduction and our main results. In sect. 2, which has a preliminary character, we state some elementary properties concerning the generalized Lebesgue–Sobolev spaces and embedding results. Here, we also collect the ingredients of our proofs. The proofs of our main results are given as follows: the case (i) is studied in Sects. 3 and 4 is concerned by case (ii). The mountain pass theorem of Ambrosetti and Rabinowitz (see for example Theorem 2.3) is used to show Theorems 1.3 and 1.4. For the proof of Theorem 1.5, we use the version of the symmetric mountain pass theorem (see, Theorem 2.4). Finally, the proof of Theorem 1.6 relies on a classical Weierstrass type theorem (see, Theorem 2.5) and the proof of the last theorem is based on Theorem 1.6 and the previous mountain pass theorem of Ambrosetti and Rabinowitz.

2 Preliminaries

We first recall some basic facts about the variable exponent Lebesgue–Sobolev.

For $p \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ u; u : \Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\Omega} |u|^{p(x)} dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \alpha > 0; \int_{\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

which is separable and reflexive Banach space (see, [16]).

Let us define the space

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\| = \inf \left\{ \alpha > 0; \int_{\Omega} \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} dx + \int_{\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}; \quad \forall u \in W^{1,p(x)}(\Omega).$$

The properties of $W^{1,p(x)}(\Omega)$ and the properties concerning the embedding results are given in the following proposition.

Proposition 2.1 ([9–11, 13, 16])

1. $W^{1,p(x)}(\Omega)$ is separable reflexive Banach space;
2. If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.

The mapping

$$\rho(u) := \int_{\Omega} \left[|\nabla u|^{p(x)} + |u|^{p(x)} \right] dx, \quad \forall u \in W^{1,p(x)}(\Omega), \quad (2.1)$$

plays an important role in manipulating the generalized Lebesgue–Sobolev spaces.

Proposition 2.2 For $u, u_k \in W^{1,p(x)}(\Omega)$, $k = 1, 2, \dots$, we have

1. $\|u\| \geq 1$ implies $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
2. $\|u\| \leq 1$ implies $\|u\|^{p^-} \geq \rho(u) \geq \|u\|^{p^+}$,

where $\rho(\cdot)$ is defined by (2.1).

For the proofs of Theorems 1.3 and 1.4, we will apply the following mountain pass theorem of Ambrosetti–Rabinowitz.

Theorem 2.3 [14] Let X endowed with the norm $\|\cdot\|_X$, be a Banach space. Assume that $\phi \in C^1(X; \mathbb{R})$ satisfies the Palais–Smale condition. Also, assume that ϕ has a mountain pass geometry, that is,

- (i) there exist two constants $r > 0$ and $\rho \in \mathbb{R}$ such that $\phi(u) \geq \rho$ if $\|u\|_X = r$;
- (ii) $\phi(0) < \rho$ and there exists $e \in X$ such that $\|e\|_X > r$ and $\phi(e) < \rho$.

Then, ϕ has a critical point $u_0 \in X$ such that $u_0 \neq 0$ and $u_0 \neq e$ with critical value

$$\phi(u_0) = \inf_{\gamma \in P} \sup_{u \in \gamma} \phi(u) \geq \rho > 0,$$

where P denotes the class of the paths $\gamma \in C([0, 1]; X)$ joining 0 to e .

The key argument in the proof of Theorem 1.5 is the following version of the symmetric mountain pass theorem.

Theorem 2.4 ([15]) Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the following two assumptions

- (A₁). $I(u)$ is even, bounded below; $I(0) = 0$ and $I(u)$ satisfies the Palais–Smale condition (PS);
- (A₁). For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$, where Γ_k denotes the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$ with

$$\gamma(A) := \inf \{k \in \mathbb{N}; \exists h : A \rightarrow \mathbb{R}^k \setminus \{0\} \text{ such that } h \text{ is continuous and odd}\}$$

is the genus of A .

Then, $I(u)$ admits a sequence of critical points u_k such that $I(u_k) < 0$; $u_k \neq 0$ and $u_k \rightarrow 0$, as $k \rightarrow \infty$.

Finally, we remind the Weierstrass type theorem that will be used in the proof of Theorem 1.6.

Theorem 2.5 ([8]) *Assume that X is a reflexive Banach space and the function $\Phi : X \rightarrow \mathbb{R}$ is coercive and (sequentially) weakly lower semicontinuous on X . Then, Φ is bounded from below on X and attains its infimum on X .*

3 Proofs of Our Main Results in the Case (i)

The energy functional corresponding to problem (1.1) in the case (i) is defined on $W^{1,p(x)}(\Omega)$ as follows

$$\phi_{\lambda,f}(u) = \int_{\Omega} [A(x, \nabla u) + \frac{|u|^{p(x)}}{p(x)}] dx - \lambda \int_{\partial\Omega} \frac{|u|^{r(x)}}{r(x)} d\sigma - \int_{\partial\Omega} F(x, u) d\sigma, \quad (3.1)$$

where $d\sigma$ is the $N - 1$ dimensional Hausdorff measure restricted to the boundary $\partial\Omega$. Let us recall that a weak solution of problem (1.1) in the case (i) is any $u \in W^{1,p(x)}(\Omega)$ such that

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u) \nabla v dx + \int_{\Omega} |u|^{p(x)-2} u v dx \\ &= \lambda \int_{\partial\Omega} |u|^{r(x)-2} u v d\sigma + \int_{\partial\Omega} f(x, u) v d\sigma \quad \text{for all } v \in W^{1,p(x)}(\Omega). \end{aligned} \quad (3.2)$$

It is known that the weak solutions of problem (1.1) given by (3.2) are exactly the critical points of $\phi_{\lambda,f}$ defined by (3.1).

3.1 Proof of Theorem 1.3

We prove now that the mountain pass theorem of Ambrosetti–Rabinowitz (Theorem 2.3) can be applied. We organize this proof as follows:

Lemma 3.1 *Assume $p, r \in C_+(\bar{\Omega})$, $a(\cdot, \cdot)$ satisfies (H_0) – (H_5) and $p^+ < r^- \leq r^+ < p^\partial(x)$ for all $x \in \bar{\Omega}$. Then, there exist $\eta, b > 0$ such that $\phi_{\lambda,0}(u) \geq b$ for $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = \eta$.*

Proof According to the fact that

$$|u(x)|^{r^+} + |u(x)|^{r^-} \geq |u(x)|^{r(x)}, \quad \forall x \in \bar{\Omega},$$

we deduce that for all $u \in W^{1,p(x)}(\Omega)$, we have

$$\phi_{\lambda,0}(u) \geq \frac{1}{p^+} \rho(u) - \frac{\lambda}{r^-} \left(\int_{\partial\Omega} |u|^{r^+} d\sigma + \int_{\partial\Omega} |u|^{r^-} d\sigma \right). \quad (3.3)$$

Since $r^- \leq r^+ < p^\partial(x)$ for any $x \in \bar{\Omega}$, then by Proposition 2.1, $W^{1,p(x)}(\Omega)$ is continuously embedded in $L^{r^+}(\partial\Omega)$ and in $L^{r^-}(\partial\Omega)$. It follows that there exist two positive constants C_1 and C_2 such that

$$\int_{\partial\Omega} |u|^{r^+} d\sigma \leq C_1 \|u\|^{r^+}, \int_{\partial\Omega} |u|^{r^-} d\sigma \leq C_2 \|u\|^{r^-}, \forall u \in W^{1,p(x)}(\Omega). \quad (3.4)$$

$$\phi_{\lambda,0}(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{r^-} \left(C_1 \|u\|^{r^+} + C_2 \|u\|^{r^-} \right) \quad \text{if } \|u\| \geq 1$$

and

$$\phi_{\lambda,0}(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{r^-} \left(C_1 \|u\|^{r^+} + C_2 \|u\|^{r^-} \right) \quad \text{if } \|u\| \leq 1.$$

Thus,

$$\phi_{\lambda,0}(u) \geq \|u\|^{p^-} \left(\frac{1}{p^+} - \frac{\lambda}{r^-} \left(C_1 \|u\|^{r^+ - p^-} + C_2 \|u\|^{r^- - p^-} \right) \right) \quad \text{if } \|u\| \geq 1,$$

and

$$\phi_{\lambda,0}(u) \geq \|u\|^{p^+} \left(\frac{1}{p^+} - \frac{\lambda}{r^-} \left(C_1 \|u\|^{r^+ - p^+} + C_2 \|u\|^{r^- - p^+} \right) \right) \quad \text{if } \|u\| \leq 1.$$

As $p^+ < r^- \leq r^+$, the functional $h : [0, 1] \rightarrow \mathbb{R}$ defined by

$$h(t) = \frac{1}{p^+} - \frac{\lambda C_1}{r^-} t^{r^+ - p^+} - \frac{\lambda C_2}{r^-} t^{r^- - p^+}$$

is positive on neighborhood of the origin. So Lemma 3.2 is proved. $\square$

Lemma 3.2 Assume that $p, r \in C_+(\bar{\Omega})$, $a(\cdot, \cdot)$ satisfies (H_0) – (H_5) and $p^+ < r^- \leq r^+ < p^\partial(x)$ for all $x \in \bar{\Omega}$. Then, there exists $e \in W^{1,p(x)}(\Omega)$ with $\|e\| > \eta$ such that $\phi_{\lambda,0}(e) < 0$, where η is given in Lemma 3.1.

Proof Choose $\varphi \in C_0^\infty(\bar{\Omega})$, $\varphi \geq 0$ and $\varphi \not\equiv 0$, on $\partial\Omega$. For $t > 1$, and using (H_2) , (H_3) we have

$$\phi_{\lambda,0}(t\varphi) \leq t\bar{c} \int_{\Omega} |\nabla \varphi| + \frac{\bar{c}_1 t^{p^+}}{p^-} \rho(\varphi) - \frac{\lambda t^{r^-}}{r^+} \int_{\partial\Omega} |\varphi|^{r(x)} d\sigma.$$

Since $p^+ < r^-$, we deduce that $\lim_{t \rightarrow +\infty} \phi_{\lambda,0}(t\varphi) = -\infty$. Therefore, for all $\varepsilon > 0$ there exists $\alpha > 0$ such that $|t| > \alpha$, $\phi_{\lambda,0}(t\varphi) < -\varepsilon < 0$. This completes the proof. $\square$

Lemma 3.3 ([17], Theorem 4.1) *The Carathéodory function $a: \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ described by (H_0) – (H_5) is an operator of type (S_+) that is, if $u_k \rightarrow u$ weakly in $W^{1,p(x)}(\Omega)$ and*

$$\limsup \int_{\Omega} a(x, \nabla u_k)(\nabla u_k - \nabla u) dx \leq 0,$$

then $u_k \rightarrow u$ strongly in $W^{1,p(x)}(\Omega)$.

Lemma 3.4 *Assume that $p, r \in C_+(\bar{\Omega})$, $a(\cdot, \cdot)$ satisfies (H_0) – (H_5) and $p^+ < r^-$. Then, the functional $\phi_{\lambda,0}$ satisfies the Palais–Smale (PS) condition.*

Proof Let $(u_k) \subset W^{1,p(x)}(\Omega)$ be a sequence such that $C = \sup_{k \in \mathbb{N}^*} \phi_{\lambda,0}(u_k)$ and $\phi'_{\lambda,0}(u_k) \rightarrow 0$. Suppose by contradiction that $\|u_k\| \rightarrow \infty$, there exists $k_0 \in \mathbb{N}^*$ such that $\|u_k\| > 1$ for any $k \geq k_0$. Using (H_5) , we have

$$\begin{aligned} C + \|u_k\| &\geq \phi_{\lambda,0}(u_k) - \frac{1}{r^-} \langle \phi'_{\lambda,0}(u_k), u_k \rangle \\ &\geq \int_{\Omega} A(x, \nabla u_k) dx + \int_{\Omega} \frac{1}{p(x)} |u_k|^{p(x)} dx - \lambda \int_{\partial\Omega} \frac{1}{r(x)} |u_k|^{r(x)} d\sigma \\ &\quad - \frac{1}{r^-} \int_{\Omega} [a(x, \nabla u_k) \nabla u_k + |u_k|^{p(x)}] dx + \frac{\lambda}{r^-} \int_{\partial\Omega} |u_k|^{r(x)} d\sigma \\ &\geq \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \rho(u_k) + \lambda \int_{\partial\Omega} \left(\frac{1}{r^-} - \frac{1}{r(x)} \right) |u_k|^{r(x)} d\sigma \\ &\geq \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \rho(u_k) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \|u_k\|^{p^-}. \end{aligned}$$

Since $p^+ < r^-$, this contradicts the fact that $p^- > 1$. So the sequence (u_k) is bounded in $W^{1,p(x)}(\Omega)$. As $W^{1,p(x)}(\Omega)$ is reflexive (Proposition 2.1), for a subsequence still denoted (u_k) , we have $u_k \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$, $u_k \rightarrow u$ in $L^{p(x)}(\Omega)$ and $u_k \rightarrow u$ in $L^{r(x)}(\partial\Omega)$. We have

$$\lim \int_{\Omega} (|u_k|^{p(x)-2} u_k - |u|^{p(x)-2} u)(u_k - u) dx = 0,$$

and

$$\lim \int_{\partial\Omega} (|u_k|^{r(x)-2} u_k - |u|^{r(x)-2} u)(u_k - u) d\sigma = 0.$$

As $\phi'_{\lambda,0}(u_k) \rightarrow 0$, we obtain

$$\limsup \int_{\Omega} [a(x, \nabla u_k)](\nabla u_k - \nabla u) dx \leq 0.$$

Now from Lemma 3.3, we see that $u_k \rightarrow u$ strongly in $W^{1,p(x)}(\Omega)$. $\square$

3.2 Proof of Theorem 1.4

As in the proof of Theorem 1.3, this proof is based on the Theorem 2.3. For deducing that there exists a nontrivial critical point $u_0 \in W^{1,p(x)}(\Omega)$ for the functional $\phi_{\lambda,f}$ with $\phi_{\lambda,f}(u_0) = \inf_{\gamma \in P} \sup_{u \in \gamma} \phi_{\lambda,f}(u) \geq \rho > 0$ where P denotes the class of the paths γ such that $\gamma \in C([0, 1], W^{1,p(x)}(\Omega))$ joining 0 to e , we need to show the following Lemmas.

Lemma 3.5 Assume that $a(\cdot, \cdot)$ and f satisfy (H₀)–(H₅) and (I₀)–(I₂). Let $p, r \in C_+(\overline{\Omega})$ such that $1 < r^- \leq r^+ < p^-$. Then, the functional $\phi_{\lambda,f}$ satisfies the Palais–Smale (PS) condition.

Proof Let $c \geq 0$ and $(u_n) \subset W^{1,p(x)}(\Omega)$ be such that $|\phi_{\lambda,f}(u_n)| < c$ and $\phi'_{\lambda,f}(u_n) \rightarrow 0$ as $n \rightarrow \infty$. We first show that $(u_n)_n$ is bounded. To do so, we argue by contradiction and we assume that, up to a subsequence, $\|u_n\| \rightarrow \infty$.

Then, using (I₂) and (H₅), for sufficiently large n we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq \phi_{\lambda,f}(u_n) - \frac{1}{\theta} \langle \phi'_{\lambda,f}(u_n), u_n \rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \rho(u_n) - \lambda \int_{\partial\Omega} \left(\frac{1}{r(x)} - \frac{1}{\theta} \right) |u|^{r(x)} d\sigma \\ &\quad - \int_{\partial\Omega} \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) d\sigma \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \rho(u_n) - \lambda \left(\frac{1}{r^-} - \frac{1}{\theta} \right) \int_{\partial\Omega} |u_n|^{r(x)} d\sigma \\ &\quad - \int_{\{x \in \partial\Omega; |u_n(x)| > l\}} \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) d\sigma \\ &\quad - |\partial\Omega| \sup\{|F(x, t) - \frac{1}{\theta} f(x, t)t|; x \in \partial\Omega, |t| \leq l\}. \end{aligned}$$

Using Proposition 2.1 and (I₂), we deduce that, for sufficiently large n ,

$$\begin{aligned} c + 1 + \|u_n\| &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-} - \lambda c_1 \|u_n\|^{r^+} \\ &\quad - |\partial\Omega| \sup\{|F(x, t) - \frac{1}{\theta} f(x, t)t|; x \in \partial\Omega, |t| \leq l\}. \end{aligned}$$

Letting n go to infinity and dividing by $\|u_n\|^{p^-}$ in the above inequality, since $r^+ < p^-$, then we obtain a contradiction. Therefore, $(u_n)_n$ is bounded in $W^{1,p(x)}(\Omega)$. For a subsequence of $(u_n)_n$, $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$, strongly in $L^{r(x)}(\partial\Omega)$ and in $L^{p(x)}(\Omega)$. Therefore, $\langle \phi'_{\lambda,f}(u_n), u_n - u \rangle \rightarrow 0$,

$\int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u) dx \rightarrow 0$, $\int_{\partial\Omega} |u_n|^{r(x)-2} u_n (u_n - u) d\sigma \rightarrow 0$ and by (I1), we have $\int_{\partial\Omega} f(x, u_n)(u_n - u) d\sigma \rightarrow 0$. Thus,

$$\limsup \int_{\Omega} [a(x, \nabla u_n)](\nabla u_n - \nabla u) dx \leq 0.$$

Consequently by Theorem 3.3, $u_n \rightarrow u$ strongly in $W^{1,p(x)}(\Omega)$. $\square$

Lemma 3.6 *Under the same hypotheses of Lemma 3.5, we have*

1. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, we can choose $R > 0$ and $\rho > 0$ so that $\phi_{\lambda,f}(u) \geq R > 0$, for all $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = \rho$.*
2. *There exists $\varphi \in W^{1,p(x)}(\Omega)$; $\varphi > 0$ such that $\phi_{\lambda,f}(t\varphi) \rightarrow -\infty$ as $t \rightarrow +\infty$.*

Proof 1. We have $\phi_{\lambda,f}(u) = \int_{\Omega} [A(x, \nabla u) + \frac{|u|^{p(x)}}{p(x)}] dx - \lambda \int_{\partial\Omega} \frac{|u|^{r(x)}}{r(x)} d\sigma - \int_{\partial\Omega} F(x, u) d\sigma$. Using (H5) and (I1), for all $\|u\| < 1$, we obtain

$$\phi_{\lambda,f}(u) \geq \frac{\|u\|^{p^+}}{p^+} - \frac{\lambda c_1}{r^-} \|u\|^{r^-} - k_2 \|u\|^{q^-},$$

where $c_1, k_2 > 0$. Consequently, we have

$$\phi_{\lambda,f}(u) \geq \|u\|^{p^+} \left(\frac{1}{p^+} - \frac{\lambda c_1}{r^-} \|u\|^{r^- - p^+} - k_2 \|u\|^{q^- - p^+} \right). \quad (3.5)$$

For each $\lambda > 0$, we consider the function $\gamma_{\lambda} : (0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\gamma_{\lambda}(t) = \frac{\lambda c_1}{r^-} t^{r^- - p^+} + k_2 t^{q^- - p^+}. \quad (3.6)$$

It is clear that $\gamma_{\lambda}(t)$ is a continuous function on $(0, +\infty)$. Since $q^- > p^+ \geq p^- > r^+ \geq r^- > 1$, it follows that

$$\lim_{t \rightarrow 0^+} \gamma_{\lambda}(t) = \lim_{t \rightarrow +\infty} \gamma_{\lambda}(t) = 0.$$

Hence, we can find $t_* > 0$ such that $0 < \gamma_{\lambda}(t_*) = \min_{t \in (0, +\infty)} \gamma_{\lambda}(t)$, in which t_* is defined by the equation

$$0 = \gamma'_{\lambda}(t_*) = \frac{\lambda c_1}{r^-} (r^- - p^+) t_*^{r^- - p^+ - 1} + k_2 (q^- - p^+) t_*^{q^- - p^+ - 1}.$$

Thus, $t_* = \left(\frac{\lambda c_1 (p^+ - r^-)}{r^- k_2 (q^- - p^+)} \right)^{\frac{1}{q^- - r^-}}$. Some simple computations imply that

$$\gamma_{\lambda}(t_*) = K \cdot \lambda^{\frac{q^- - p^+}{q^- - r^-}} \rightarrow 0 \text{ as } \lambda \rightarrow 0, \text{ where } K > 0. \quad (3.7)$$

From (3.5), (3.6) and (3.7), there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, we can choose $R > 0$ and $\rho > 0$ so that $\phi_{\lambda,1}(u) \geq R > 0$ for all $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = \rho$.

2. The condition (I2) implies the existence of a positive $k(x)$ such that $F(x, \tau) \geq k(x)|\tau|^\theta$ for all $|\tau| > l$ and a.e. $x \in \partial\Omega$. Thus for $t > 1$, we obtain

$$\begin{aligned}\phi_{\lambda,f}(tu) &\leq \bar{c}t \int_{\Omega} |\nabla u| dx + \frac{\bar{c}}{p^-} t^{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx + \frac{t^{p^+}}{p^-} \int_{\Omega} |u|^{p(x)} dx \\ &\quad - \frac{\lambda}{r^+} t^{r^-} \int_{\partial\Omega} |u|^{r(x)} d\sigma - t^\theta \int_{\{x \in \partial\Omega; |u(x)| > l\}} k(x) |u|^\theta d\sigma \\ &\quad - |\partial\Omega| \inf\{F(x, t) : x \in \partial\Omega, |t| \leq l\}.\end{aligned}$$

Due the fact that $\theta > p^+ > 1$ we arrive at $\phi_{\lambda,f}(\varphi t) \rightarrow -\infty$ as $t \rightarrow +\infty$ where $\varphi(x) = |u(x)| + \epsilon + l$, on $\bar{\Omega}$, where $\epsilon > 0$. Therefore for a fixed $u \neq 0$, we choose $e = t\varphi$ such that $\|e\| > \rho$ and $\phi_{\lambda,f}(e) < 0$. $\square$

3.3 Proof of Theorem 1.5

We show now that the symmetric mountain pass theorem (see Theorem 2.4) can be applied.

Lemma 3.7 *Under the hypotheses of Theorem 1.5, the functional $\phi_{\lambda,f}$ is even, bounded from below, satisfies the Palais–Smale (PS) condition and $\phi_{\lambda,f}(0) = 0$.*

Proof It is clear, by the properties of $a(\cdot, \cdot)$ and f , that $\phi_{\lambda,f} \in C^1$, $\phi_{\lambda,f}$ is even and $\phi_{\lambda,f}(0) = 0$. Using (H₅) and (I'1), we have

$$\phi_{\lambda,f}(u) \geq \int_{\Omega} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \lambda \int_{\partial\Omega} \frac{|u|^{r(x)}}{r(x)} d\sigma - k \int_{\partial\Omega} \frac{|u|^{q(x)}}{q(x)} d\sigma.$$

As $p(x) \leq p^+$, $r^- \leq r(x)$ and $q^- \leq q(x)$ for any $x \in \bar{\Omega}$, we obtain

$$\phi_{\lambda,f}(u) \geq \frac{1}{p^+} \rho(u) - \frac{\lambda}{r^-} \int_{\partial\Omega} |u|^{r(x)} d\sigma - \frac{k}{q^-} \int_{\partial\Omega} |u|^{q(x)} d\sigma.$$

By Proposition 2.1, we deduce that there exist two positive constants C and C' such that

$$\phi_{\lambda,f}(u) \geq \frac{1}{p^+} \rho(u) - \frac{C\lambda}{r^-} \|u\|^{r^+} - \frac{C'k}{q^-} \|u\|^{q^+}.$$

Now using Proposition 2.2, we have

$$\begin{aligned}\phi_{\lambda,f}(u) &\geq \frac{\|u\|^{p^-}}{p^+} - \frac{C\lambda}{r^-} \|u\|^{r^+} - \frac{C'k}{q^-} \|u\|^{q^+}, \quad \text{if } \|u\| > 1, \\ \phi_{\lambda,f}(u) &\geq \frac{\|u\|^{p^+}}{p^+} - \frac{C\lambda}{r^-} \|u\|^{r^-} - \frac{C'k}{q^-} \|u\|^{q^-}, \quad \text{if } \|u\| \leq 1.\end{aligned}$$

Since $p^- > r^+$ and $p^- > q^+$, $\phi_{\lambda,f}$ is bounded from below. $\square$

Lemma 3.8 Assume that $p, r \in C_+(\bar{\Omega})$, $a(\cdot, \cdot)$ satisfies (H_0) – (H_5) and $r^+ < p^-$. Then for each $k \in \mathbb{N}^*$, there exists an $H_k \in \Gamma_k$ such that $\sup_{u \in H_k} \phi_{\lambda,f}(u) < 0$.

Proof Let k -functions $v_1, v_2, \dots, v_k \in C^\infty(\mathbb{R}^N)$ such that

$$\overline{\{x \in \partial\Omega; v_i(x) \neq 0\}} \cap \overline{\{x \in \partial\Omega; v_j(x) \neq 0\}} = \emptyset$$

if $i \neq j$ and

$$|\{x \in \partial\Omega; v_i(x) \neq 0\}| > 0 \quad \forall i, j \in \{1, 2, \dots, k\}.$$

Take $F_k = \text{span}\{v_1, v_2, \dots, v_k\}$; we have $\dim F_k = k$. Denote $S = \{v \in W^{1,p(x)}(\Omega); \|v\| = 1\}$ and for $0 < t \leq 1$, $H_k(t) = t(F_k \cap S)$. For all $t \in]0, 1]$, $\gamma(H_k(t)) = k$. We show now that for any $k \in \mathbb{N}^*$, there exists $t_k \in]0, 1]$ such that $\sup_{u \in H_k(t_k)} \phi_{\lambda,f}(u) < 0$. Indeed, for $0 < t \leq 1$, and using (H_2) , (H_3) and $(I'1)$ we have

$$\begin{aligned}\sup_{u \in H_k(t)} \phi_{\lambda,f}(u) &\leq \sup_{v \in F_k \cap S} \phi_{\lambda,f}(tv) \\ &= \sup_{v \in F_k \cap S} \left\{ \int_{\Omega} [A(x, t\nabla v) + \frac{t^{p(x)}}{p(x)} |v|^{p(x)}] dx - \lambda \int_{\partial\Omega} \frac{t^{r(x)}}{r(x)} |v|^{r(x)} d\sigma - \int_{\partial\Omega} F(x, tv) d\sigma \right\} \\ &\leq \sup_{v \in F_k \cap S} \left\{ t\bar{c} \int_{\Omega} |\nabla v| dx + \frac{\bar{c}_1 t^{p^-}}{p^-} \rho(v) - \lambda \frac{t^{r^+}}{r^+} \int_{\partial\Omega} |v|^{r(x)} d\sigma \right\} \\ &= \sup_{v \in F_k \cap S} \left\{ \left(t\bar{c} \int_{\Omega} |\nabla v| dx + \frac{\bar{c}_1 t^{p^-}}{p^-} - \lambda \frac{t^{r^+}}{r^+} \int_{\partial\Omega} |v|^{r(x)} d\sigma \right) \right\},\end{aligned}$$

where $\bar{c}, \bar{c}_1 > 0$. Let $c = \min_{v \in F_k \cap S} \int_{\partial\Omega} |v|^{r(x)} d\sigma > 0$, at this point there exist $\lambda_0 > 0$ and for all $\lambda > \lambda_0$, we may choose $t_k \in]0, 1]$ which is small enough such that

$$\sup_{v \in F_k \cap S} \left(t_k \bar{c} \int_{\Omega} |\nabla v| dx + \frac{\bar{c}_1 t_k^{p^-}}{p^-} - \lambda \frac{t_k^{r^+}}{r^+} c \right) < 0, \quad \forall \lambda > \lambda_0 \text{ since } r^+ < p^-.$$

$\square$

Proof of Theorem 1.5 By Lemmas 3.7, 3.8 and Theorem 2.4, the problem (1.1) admits a sequence of non-trivial weak solutions (u_k) , such that $\phi_{\lambda,0}(u_k) < 0$ and $\lim_{k \rightarrow +\infty} u_k = 0$. $\square$

4 Proofs of Our Main Results in Case (ii)

The energy functional corresponding to problem (1.1) the case (ii) is defined on $W^{1,p(x)}(\Omega)$ as follows:

$$\psi_{\lambda,f}(u) = \int_{\Omega} \left[A(x, \nabla u) + \frac{|u|^{p(x)}}{p(x)} \right] dx - \lambda \int_{\partial\Omega} F(x, u) d\sigma. \quad (4.1)$$

Let us recall that a weak solution of problem 1.1; the case (ii) is any $u \in W^{1,p(x)}(\Omega)$ such that

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u) \nabla v dx + \int_{\Omega} |u|^{p(x)-2} u v dx \\ &= \lambda \int_{\partial\Omega} f(x, u) v d\sigma \quad \text{for all } v \in W^{1,p(x)}(\Omega). \end{aligned} \quad (4.2)$$

As in Sect. 3 the weak solutions of problem 1.1 given by (4.2) are exactly the critical points of $\phi_{\lambda,f}$ defined by (4.1).

4.1 Proof of Theorem 1.6

To apply Theorem 2.5, we need the following two Lemmas.

Lemma 4.1 *The functional $\psi_{\lambda,f}$ introduced by (4.1) is (sequentially) weakly lower semicontinuous for every $\lambda > 0$.*

Proof It is known that the functional $u \rightarrow \int_{\Omega} [A(x, \nabla u) + \frac{|u|^{p(x)}}{p(x)}] dx$ defined on $W^{1,p(x)}(\Omega)$ is weakly lower semicontinuous (see, [5, Lemma 5]). At the same time, hypothesis (I5) implies the existence of a positive constant k' such that

$$|f(x, t)| \leq k'(1 + |t|^{p(x)-1}) \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \partial\Omega. \quad (4.3)$$

Hence, since the embedding from $W^{1,p(x)}(\Omega)$ to $L^{p(x)}(\partial\Omega)$ is compact, standard arguments infer that $\psi_{\lambda,f}$ is weakly lower semicontinuous for every $\lambda > 0$ and the proof of this lemma is complete. $\square$

Lemma 4.2 *The functional $\psi_{\lambda,f}$ introduced by (4.1) is coercive and satisfies the Palais–Smale condition, for every $\lambda > 0$.*

Proof We first show that $\psi_{\lambda,f}$ is coercive. Using (I5), for $\epsilon > 0$, we find $\delta > 0$ such that

$$|f(x, t)| \leq \epsilon |t|^{p(x)-1} \quad \text{for all } |t| \geq \delta \text{ and a.e. } x \in \partial\Omega.$$

We integrate the previous relation and we obtain

$$|F(x, t)| \leq \frac{\epsilon}{p^-} |t|^{p(x)} + \max_{|s| \leq \delta} |f(x, s)| |t| \quad \text{for all } t \in \mathbb{R}. \quad (4.4)$$

By (H₅) and Proposition 2.2 we have that, for all $u \in W^{1,p(x)}(\Omega)$ with $\|u\| > 1$,

$$\int_{\Omega} \left[A(x, \nabla u) + \frac{|u|^{p(x)}}{p(x)} \right] dx \geq \frac{1}{p^+} \|u\|^{p^-}. \quad (4.5)$$

Then, by (4.4) and (4.5) we deduce that, $\forall u \in W^{1,p(x)}(\Omega)$ with $\|u\| > 1$,

$$\psi_{\lambda,f}(u) \geq \left(\frac{1}{p^+} - \frac{\lambda c \epsilon}{p^-} \right) \|u\|^{p^-} - \lambda \int_{\partial\Omega} \max_{|s| \leq \delta} |f(x, s)| |u| d\sigma,$$

where c is a positive constant from the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^-}(\partial\Omega)$. Choose now ϵ small enough such that $\frac{1}{p^+} - \frac{\lambda c \epsilon}{p^-} > 0$. Moreover, due to the continuity of f and to the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^1(\partial\Omega)$, there exists $k'' > 0$ such that

$$\psi_{\lambda,f}(u) \geq \left(\frac{1}{p^+} - \frac{\lambda c \epsilon}{p^-} \right) \|u\|^{p^-} - \lambda k'' \|u\|, \quad \forall u \in W^{1,p(x)}(\Omega) \quad \text{with } \|u\| > 1.$$

Thus, $\psi_{\lambda,f}$ is coercive. Let us focus now on proving that $\psi_{\lambda,f}$ satisfies the Palais–Smale condition. Let $c \geq 0$ and $(u_n) \subset W^{1,p(x)}(\Omega)$ be such that $|\psi_{\lambda,f}(u_n)| < c$ and $\psi'_{\lambda,f}(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $\psi(\lambda, f)$ is coercive, (u_n) is bounded. To prove that (u_n) contains a subsequence converging to a critical point of $\psi_{\lambda,f}$, we follow the same steps as in Sect. 3 of the proof of Lemma 3.5, using (4.3) instead of (I1) and $\psi_{\lambda,f}$ instead of $\phi_{\lambda,f}$. Hence, $\psi_{\lambda,f}$ satisfies the Palais–Smale condition. $\square$

We are now in a position to provide the proof of Theorem 1.6. By Theorem 2.5 and by Lemmas 4.1 and 4.2, we deduce that there exists a global minimizer u_1 to $\psi_{\lambda,f}$ in $W^{1,p(x)}(\Omega)$, and implicitly, a weak solution to problem(1.1) case (ii). A first question comes to mind: is u_1 nontrivial? Let us see what guarantees the nontriviality of u_1 . It is clear that, for all $\lambda > 0$,

$$\psi_{\lambda,f}(u_1) \leq \psi_{\lambda,f}(u) \quad \text{for all } u \in W^{1,p(x)}(\Omega). \quad (4.6)$$

Using (H5) and (I4), we can choose $\lambda_0 > 0$ as follows:

$$\lambda_0 = \left[\sup_{u \in W^{1,p(x)}(\Omega), u \neq 0} \left(\frac{\int_{\partial\Omega} F(x, u) d\sigma}{\int_{\Omega} [A(x, \nabla u) + \frac{|u|^{p(x)}}{p(x)}] dx} \right) \right]^{-1}.$$

Note that assumption (I4) implies that for all $\lambda > \lambda_0$, there exists $\omega \in W^{1,p(x)}(\Omega)$ such that

$$\psi_{\lambda,f}(\omega) = \int_{\Omega} \left[A(x, \nabla \omega) + \frac{|\omega|^{p(x)}}{p(x)} \right] dx - \lambda \int_{\partial\Omega} F(x, \omega) d\sigma < 0. \quad (4.7)$$

Under (4.6) and (4.7), we obtain

$$\psi_{\lambda,f}(u_1) \leq \psi_{\lambda,f}(\omega) < 0 \quad \text{for all } \lambda > \lambda_0. \quad (4.8)$$

Since by (H2) we have $\psi_{\lambda,f}(0) = 0$, relation (4.8) yields that $u_1 \neq 0$. Thus, for $\lambda > \lambda_0$, problem (1.1) the case (ii) admits a nontrivial weak solution. Hence, we have finished the proof of Theorem 1.6.

4.2 Proof of Theorem 1.7

Our proof is based on Theorems 1.6 and 2.3.

Theorem 1.6 assures the existence of a nontrivial weak solution u_1 to problem 1.1 in case (ii), for all $\lambda > \lambda_0$. To find a second weak solution for all $\lambda > \lambda_0$, we turn to Theorem 2.3. By Lemma 4.2, we know that $\psi_{\lambda,f}$ satisfies the Palais–Smale condition for all $\lambda > 0$. Also $\psi_{\lambda,f}(0) = 0$ and u_1 is nontrivial with $\psi_{\lambda,f}(u_1) < 0$, by (4.8). Therefore, in order to prove that $\psi_{\lambda,f}$ has a mountain pass geometry for all $\lambda > \lambda_0$; it remains to show that there exist two positive constants, ρ and $r \leq \|u_1\|$, such that $\psi_{\lambda,f}(u) \geq \rho$ whenever $\|u\| = r$.

We take $0 < \epsilon$ and $s \in C(\overline{\Omega}, \mathbb{R})$, $p^+ < s^- < s^+ < p^\partial$. By (I5) and (I6), there exist $\delta_1 \geq 1$ and $\delta_2 > 0$ such that

$$\begin{aligned} |f(x, t)| &\leq \epsilon |t|^{s(x)-1} \quad \text{for all } |t| > \delta_1 \quad \text{and a.e. } x \in \partial\Omega, \\ |f(x, t)| &\leq \epsilon |t|^{p(x)-1} \quad \text{for all } |t| < \delta_2 \quad \text{and a.e. } x \in \partial\Omega. \end{aligned}$$

Thus, there exists a constant $k_1 > 0$ chosen sufficiently big to have the following inequality:

$$|F(x, t)| \leq \epsilon |t|^{p(x)} + k_1 |t|^{s(x)} \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \partial\Omega.$$

This leads to

$$\psi_{\lambda,f}(u) \geq \frac{1}{p^+} \int_{\Omega} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx - \lambda k_1 \int_{\partial\Omega} |u|^{s(x)} d\sigma - \lambda \epsilon \int_{\partial\Omega} |u|^{p(x)} d\sigma$$

$$\geq \frac{1}{p^+} \int_{\Omega} [|\nabla u|^{p(x)} + |u|^{p(x)}] dx - c_1 \lambda k_1 \|u\|^{s^-} - c_2 \lambda \epsilon \|u\|^{p^+}.$$

By Proposition 2.2, we have that, for all $\lambda > \lambda_0$ and all $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = r < 1$,

$$\psi_{\lambda,f}(u) \geq \left(\frac{1}{p^+} - c_2 \lambda \epsilon \right) \|u\|^{p^+} - c_1 \lambda k_1 \|u\|^{s^-}.$$

Then, since $p^+ < s^-$ and choose ϵ small enough such that $(\frac{1}{p^+} - \lambda c_2 \epsilon) > 0$, for r sufficiently small, there exists $\rho > 0$ such that $\psi_{\lambda,f} \geq \rho > 0$ for all $\lambda > \lambda_0 > 0$ and all $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = r < \min(1, \|u_1\|)$. Thus, we can apply now Theorem 2.3 to find a second critical point $u_2 \in W^{1,p(x)}(\Omega)$ with

$$\psi_{\lambda,f}(u_2) = \inf_{\gamma \in P} \sup_{u \in \gamma} \psi_{\lambda,f}(u) \geq \rho > 0.$$

In this way, we have obtained a second nontrivial weak solution of the problem (1.1) in the case (ii) and the proof of Theorem 1.7 is complete.

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