Existence of Multiple Solutions for a Class Non-uniformly Elliptic Equations with Critical Exponential Growth

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Received: 24 January 2018 / Revised: 28 April 2018 / Accepted: 18 May 2018 /
Published online: 11 June 2018
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Abstract This paper deals with the existence and multiplicity of weak solutions to non-uniformly elliptic problem

$$-\operatorname{div}(a(x, \nabla u)) + V(x)|u|^{N-2}u = \lambda \left(\exp \left(\alpha |u|^{\frac{N}{N-1}} \right) + f(x, u) \right),$$

in $\mathbb{R}^N$, where $N \geq 2$, α is some positive constant and λ is some positive parameter. Our method is mainly based on variational arguments.

Keywords N -Laplacian · Trudinger–Moser inequality · Critical exponential growth · Variational method · Local minima

Mathematics Subject Classification 35J25 · 35J20 · 35J61

1 Introduction

In this paper, we are concerned with the existence and multiplicity of nontrivial weak solutions for the non-uniformly elliptic equations of N -Laplacian type of the form

$$-\operatorname{div}(a(x, \nabla u)) + V(x)|u|^{N-2}u = \lambda \left(\exp \left(\alpha |u|^{\frac{N}{N-1}} \right) + f(x, u) \right) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

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where $N \geq 2$, α is some positive constant and λ is some positive parameter, $V : \mathbb{R}^N \rightarrow]0, +\infty[$ is a continuous function such that $V(x) \geq V_0$ for some positive constant V_0 and $V^{-1} \in L^{\frac{1}{N-1}}(\mathbb{R}^N)$ or $\lim V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, i.e., $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$ for every $M > 0$. Motivated by [6], we assume that A be a measurable function on $\mathbb{R}^N \times \mathbb{R}$ such that $A(x, 0) = 0$ and $a(x, \tau) = \frac{\partial A(x, \tau)}{\partial \tau}$ is a Carathéodory function on $\mathbb{R}^N \times \mathbb{R}$, and there are positive real numbers c_0, c_1, k_1 and two nonnegative measurable functions h_0, h_1 on $\mathbb{R}^N$ such that $h_0 \in L^{N/(N-1)}(\mathbb{R}^N)$, $h_1 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and $h_1(x) \geq 1$, satisfying the properties:

- (A₁) $|a(x, \tau)| \leq c_0(h_0(x) + h_1(x)|\tau|^{N-1}), \quad \forall \tau \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N,$
- (A₂) $0 \leq a(x, \tau) \cdot \tau \leq NA(x, \tau) \quad \forall \tau \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N,$
- (A₃) $A(x, \tau) \geq k_0 h_1(x) |\tau|^N \quad \forall \tau \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N.$

Then A verifies the growth condition:

$$|A(x, \tau)| \leq c_0(h_0(x)|\tau| + h_1(x)|\tau|^N), \quad \forall \tau \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N.$$

Note that in the case of N -Laplacian, i.e., $A(x, \tau) = \frac{1}{N}|\tau|^N$, we choose

$$a(x, \tau) = |\tau|^{N-2}\tau, \quad k_0 = \frac{1}{N}, \quad h_1(x) = 1,$$

which has been studied extensively, both in the case $N = 2$ (i.e., Laplacian equation in $\mathbb{R}^2$) and in the case $N > 3$ (i.e., N -Laplacian equation in $\mathbb{R}^N$).

The problems of this type are important in many fields of sciences, for example, in electromagnetism, astronomy and fluid dynamics. In fact, these models describe potentials of electric, gravitation and fluid, respectively.

In the case $p < N$, by the Sobolev embedding, the critical exponent is $p^* = \frac{pN}{N-p}$. When $p = N$, one has another maximal growth for any bounded domain Ω , i.e.,

$$\sup_{u \in W_0^{1,N}(\Omega), \|\nabla u\|_{L^N} \leq 1} \frac{1}{|\Omega|} \int_{\Omega} \exp(\alpha_N |u|^{N/(N-1)}) dx < \infty,$$

where $\alpha_N = NW_{N-1}^{\frac{1}{N-1}}$ and W_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$. In this situation, the constant α_N is sharp. ([1, 4, 7])

In the case of unbounded domain, if we replace $\|\nabla u\|_{L^N}$ in the supremum by the standard Sobolev norm, then the following expression (Trudinger–Moser inequality) can be derived

$$\sup_{u \in W_0^{1,N}(\mathbb{R}^N), \|\nabla u\|_{L^N}^N + \|u\|_{L^N}^N \leq 1} \int_{\mathbb{R}^N} (\exp(\alpha |u|^{N/(N-1)}) - S_{N-2}(\alpha, u)) dx = \Xi,$$

where $\Xi \leq \infty$ if $\alpha \leq \alpha_N$ and $\Xi = \infty$ if $\alpha > \alpha_N$. In this expression, $S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |u|^{\frac{kN}{N-1}}$ (see [5, 9]).

Motivated by the Trudinger–Moser inequality, we consider the following assumptions on f which allows us to treat the problem (1.1) variationally in a subspace of

$W^{1,N}(\mathbb{R}^N)$. We assume that $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function that $\forall(x, s) \in \mathbb{R}^N \times [0, +\infty]$, $f(x, s) \geq 0$, and

$$|f(x, u)| \leq b_1|u|^{\beta_1} + b_2|u|^{\beta_2}(\exp(\alpha_0|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha_0, u)), \quad \forall(x, u) \in \mathbb{R}^N \times \mathbb{R}^+, \quad (1.2)$$

for some $b_1, b_2, \alpha_0 > 0$ and $\beta_1, \beta_2 \geq 0$.

Next, we introduce some notations

$$E = \left\{ u \in W_0^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} h_1(x)|\nabla u|^N dx + \int_{\mathbb{R}^N} V(x)|u|^N dx < \infty \right\},$$

which since V is positive and bounded from below, E is a reflexive Banach space when endowed with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} (h_1(x)|\nabla u|^N + \frac{1}{k_0 N} V(x)|u|^N) dx \right)^{1/N}.$$

Then for $q \in [N, +\infty[$

$$E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N),$$

with continuous embedding. Thus, there exists a positive constant c_2 such that

$$\|u\|_{1,N} \leq c_2 \|u\|_E, \quad \forall u \in E, \quad (1.3)$$

where

$$\|u\|_{1,N} = \left(\int_{\mathbb{R}^N} (|\nabla u|^N + |u|^N) dx \right)^{\frac{1}{N}}$$

is the norm in the Sobolev space $W^{1,N}(\mathbb{R}^N)$.

Using the radial lemma which asserts

$$|u(x)| \leq |x|^{-1} \left(\frac{N}{W_{N-1}} \right)^{\frac{1}{N}} \|u\|_N \quad \forall x \neq 0,$$

for all $u \in W^{1,N}(\mathbb{R}^N)$ radially nonincreasing symmetric ([2, 5]), there exists a constant $c_3 > 0$ such that

$$\int_{\mathbb{R}^N} \left[\exp \left(\alpha |u|^{\frac{N}{N-1}} \right) - S_{N-2}(\alpha, u) \right] dx \leq c_3 \quad (1.4)$$

provided that $\|\nabla u\|_N \leq 1$, $\|u\|_N \leq M < \infty$ and $\alpha < \alpha_N$. Furthermore, there exists a constant $c_4 > 0$ such that

$$\int_{\mathbb{R}^N} |u|^q \left[\exp \left(\alpha |u|^{\frac{N}{N-1}} \right) - S_{N-2}(\alpha, u) \right] dx \leq c_4 \|u\|_{1,N}^q \quad (1.5)$$

provided that $\|u\|_{1,N} \leq M$ which $M < (\frac{\alpha_N}{\alpha})^{\frac{N-1}{N}}$.

Inequalities (1.3) and (1.5) imply that there exists a constant $c_5 > 0$ such that

$$\int_{\mathbb{R}^N} |u|^q \left[\exp \left(\alpha |u|^{\frac{N}{N-1}} \right) - S_{N-2}(\alpha, u) \right] dx \leq c_5 \|u\|_E^q \quad (1.6)$$

provided that $\|u\|_E \leq M$ with $M < \frac{1}{c_2} (\frac{\alpha_N}{\alpha})^{\frac{N-1}{N}}$.

Note that according to [9, Lemma 2.4], the assumptions $V^{-1} \in L^1(\mathbb{R}^N)$ or $\lim V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, on the potential $V(x)$ lead us to have the compactness of the imbedding $E \hookrightarrow L^q(\mathbb{R}^N)$, for all $1 \leq q < +\infty$.

Now from (1.2), the imbedding $E \hookrightarrow L^q(\mathbb{R}^N)$ for $q = \beta_1 + 1$ and (1.5) and (1.6) for $q = \beta_2 + 1$, there exist $c_6, c_7 > 0$ such that for all $(x, u) \in \mathbb{R}^N \times E$,

$$\begin{aligned} \int_{\mathbb{R}^N} |F(x, u)| dx &= \int_{\mathbb{R}^N} \left| \int_0^{u(x)} f(x, s) ds \right| dx \\ &\leq \frac{b_1}{\beta_1 + 1} \int_{\mathbb{R}^N} |u|^{\beta_1+1} dx \\ &\quad + \frac{b_2}{\beta_2 + 1} \int_{\mathbb{R}^N} |u|^{\beta_2+1} \left(\exp \left(\alpha |u|^{\frac{N}{N-1}} \right) - S_{N-2}(\beta, u) \right) dx \\ &\leq \frac{b_1}{\beta_1 + 1} c_6 \|u\|_E^{\beta_1+1} + \frac{b_2}{\beta_2 + 1} c_7 \|u\|_E^{\beta_2+1} \end{aligned} \quad (1.7)$$

provided that $\|u\|_E \leq \frac{1}{c_2} (\frac{\alpha_N}{\alpha})^{\frac{N-1}{N}}$.

Thus, by (1.4), we have $F(x, u) \in L^1(\mathbb{R}^N)$ for all $u \in E$. Therefore, the functional $I_\lambda : E \rightarrow \mathbb{R}$ given by

$$I_\lambda(u) = \phi(u) - \lambda \psi(u),$$

for

$$\begin{aligned} \phi(u) &= \int_{\mathbb{R}^N} A(x, \nabla u) dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |u|^N dx \\ &\geq \int_{\mathbb{R}^N} k_0 h_1(x) |\nabla u|^N + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |u|^N dx \\ &= k_0 \|u\|^N \end{aligned}$$

and

$$\psi(u) = \int_{\mathbb{R}^N} \left(\int_0^{u(x)} \exp \left(\alpha |s|^{\frac{N}{N-1}} \right) ds + F(x, u) \right) dx,$$

is well defined. Moreover, I_λ is a C^1 functional on E and for $u, v \in E$

$$\begin{aligned} I'_\lambda(u)v &= \int_{\mathbb{R}^N} a(x, \nabla u) \nabla v \, dx + \int_{\mathbb{R}^N} V(x) |u|^{N-2} uv \, dx \\ &\quad - \lambda \int_{\mathbb{R}^N} \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) v \, dx - \lambda \int_{\mathbb{R}^N} f(x, u) v \, dx. \end{aligned}$$

Consequently, the critical points of the functional I_λ are precisely nontrivial weak solutions of the problem (1.1).

In the present paper, first we obtain the existence of a nontrivial weak solution for the problem (1.1), in a certain range of λ (in Theorem 3.1). Then we prove the existence of the second weak solution for the problem (1.1) distinct from the first one (in Theorem 4.1).

2 Preliminaries

Our main tool is Theorem 2.2, consequence of a local minimum theorem [3, Theorem 5.1] which is inspired by Ricceri's variational principle.

For a given non-empty set X , and two functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$, we define the following functions:

$$\begin{aligned} \varsigma(r_1, r_2) &= \inf_{v \in \Phi^{-1}([r_1, r_2])} \frac{\sup_{u \in \Phi^{-1}([r_1, r_2])} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \\ \rho_1(r_1, r_2) &= \sup_{v \in \Phi^{-1}([r_1, r_2])} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1} \end{aligned}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Definition 2.1 Let ϕ_0 and ψ_0 be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$. We say that the functional $I_0 = \phi_0 - \psi_0$ verifies the Palais–Smale condition cut off lower at r_1 and upper at r_2 (in short $^{[r_1]}(PS)^{[r_2]}$) if any sequence $(u_n) \subset X$ such that

- (i) $(I_0(u_n))$ is bounded,
- (ii) if $I'_0(u_n) \rightarrow 0$ in X^* (where X^* denotes the topological dual of X),
- (iii) $r_1 < \phi_0(u_n) < r_2$, $\forall n \in \mathbb{N}$

has a convergent subsequence. Clearly, if $r_1 = -\infty$ and $r_2 = +\infty$ it coincides with the classical (PS) condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(PS)^{[r_2]}$.

Theorem 2.2 ([3] Theorem 5.1) *Let X be a real Banach space; $\Phi : X \rightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^* , $\Psi : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\varsigma(r_1, r_2) < \rho_1(r_1, r_2),$$

and for each $\lambda \in \Lambda =]\frac{1}{\rho_1(r_1, r_2)}, \frac{1}{\varsigma(r_1, r_2)}[$ the functional $I_\lambda = \phi - \lambda\psi$ satisfies $[r_1](PS)^{[r_2]}$ condition.

Then, for each $\lambda \in \Lambda$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Lemma 2.3 Assume that (A_1) , (A_2) , (A_3) and (f_1) hold. Then for each

$$0 < r < k_0 \inf \left(1, \left(\frac{1}{2c_2} \right)^N \left(\frac{(N-1)\alpha_N}{N\alpha_0} \right) \right)^{N-1}$$

and $\lambda > 0$, the functional I_λ satisfies $(PS)^{[r]}$.

Proof Let $(u_n) \subset E$ be such that $(I_\lambda(u_n))$ is bounded, $I'_\lambda(u_n) \rightarrow 0$ and $\phi(u_n) < r$, for all $n \in \mathbb{N}$. Since $k_0 \|u_n\|_E^N \leq \phi(u_n)$, so $\|u_n\|_E < (\frac{r}{k_0})^{\frac{1}{N}}$. Therefore, there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E . Thanks to Lemma 2.1. of [9] and using the assumption (1.2), one has

$$\begin{aligned} & \int_{\mathbb{R}^N} |f(x, u_n)|^{\frac{N}{N-1}} dx \\ & \leq b_3 \left(\int_{\mathbb{R}^N} |u_n|^{\frac{\beta_1 N}{N-1}} dx + \int_{\mathbb{R}^N} |u_n|^{\frac{\beta_2 N}{N-1}} \left(\exp \left(\frac{N}{N-1} \alpha_0 |u|^{\frac{N}{N-1}} \right) \right. \right. \\ & \quad \left. \left. - S_{N-2} \left(\frac{N\alpha_0}{N-1}, u \right) \right) dx \right), \end{aligned}$$

where $b_3 = \max\{b_1, b_2\}$. Since $\|u_n\|_E \leq (\frac{r}{k_0})^{\frac{1}{N}} < \frac{1}{2c_2} (\frac{\alpha_N}{\frac{N}{N-1}\alpha_0})^{\frac{N-1}{N}}$, using (1.6) for $q = \frac{\beta_2 N}{N-1}$, we deduce that

$$\sup_{n \in \mathbb{N}} \left(\int_{\mathbb{R}^N} |f(x, u_n)|^{\frac{N}{N-1}} dx \right) < +\infty.$$

This fact together with the compactness of the imbedding $E \hookrightarrow L^N(\mathbb{R}^N)$ implies

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} f(x, u_n)(u_n - u) dx = 0. \quad (2.1)$$

On the other hand, using the definition of S_{N-2} , one has

$$\begin{aligned} \int_{\mathbb{R}^N} \exp \left(\alpha |u_n|^{\frac{N}{N-1}} \right) (u_n - u) dx &= \int_{\mathbb{R}^N} \exp \left(\alpha |u_n|^{\frac{N}{N-1}} - S_{N-2}(\alpha, u_n) \right) (u_n - u) dx \\ &\quad + \int_{\mathbb{R}^N} S_{N-2}(\alpha, u_n)(u_n - u) dx. \end{aligned}$$

Using the compact imbedding $E \hookrightarrow L^1(\mathbb{R}^N)$ together with (1.4) for the first part of the above inequality and $E \hookrightarrow L^2(\mathbb{R}^N)$ together with the fact that u_n is converges

and so bounded in L^q for any $q \geq 1$, arguing as before, one has

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \exp\left(\alpha |u_n|^{\frac{N}{N-1}}\right) (u_n - u) \, dx = 0. \quad (2.2)$$

Now using (A_1) , it follows that

$$\begin{aligned} |\langle I'_\lambda(u_n), u_n - u \rangle| &= \left| \int_{\mathbb{R}^N} (a(x, u_n)(u_n - u) + V(x)|u_n - u|^{N-2}(u_n - u)) \, dx \right. \\ &\quad \left. - \lambda \int_{\mathbb{R}^N} \left(\exp\left(\alpha |u_n|^{\frac{N}{N-1}}\right) (u_n - u) + f(x, u_n)(u_n - u) \right) \, dx \right| \\ &\leq \int_{\mathbb{R}^N} (|a(x, u_n)||u_n - u| + V(x)|u_n|^{N-2}|u_n - u|) \, dx \\ &\quad + \lambda \int_{\mathbb{R}^N} \left(\left| \exp\left(\alpha |u_n|^{\frac{N}{N-1}}\right) \right| |u_n - u| + |f(x, u_n)(u_n - u)| \right) \, dx \\ &\leq c_0 \int_{\mathbb{R}^N} ((h_0(x)|u_n - u| + h_1(x)|u_n|^{N-1}|u_n - u|) \\ &\quad + V(x)|u_n|^{N-2}|u_n - u|) \, dx \\ &\quad + \lambda \int_{\mathbb{R}^N} (|\exp\left(\alpha |u_n|^{\frac{N}{N-1}}\right)| |u_n - u| + |f(x, u_n)(u_n - u)|) \, dx. \end{aligned}$$

This inequality together with $h_0 \in L^{N/(N-1)}(\mathbb{R}^N)$ and $h_1 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ (u_n is converges to u in L^1 and L^N) and using the fact

$$\langle I'_\lambda(u_n), u_n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

it can be derived that (u_n) is strongly convergent to u in E . $\square$

3 Existence of the First Solution

Chosen large enough $R > 0$ such that

$$\begin{aligned} (4)^{\frac{1+N}{N}} c_0 \left(\int_{R \leq |x| \leq R+1} h_0 \, dx \right) &\left(C^* + \frac{b_1}{\beta_1 + 1} c_6 + \frac{b_2}{\beta_2 + 1} c_7 \right) \\ &< k_0^{\frac{1}{N}} B \left(k_0 \int_{R \leq |x| \leq R+1} h_1(x) \, dx + \frac{1}{N} \Gamma_V \right)^{\frac{N-1}{N}}, \end{aligned} \quad (3.1)$$

where

$$\Gamma_V = \int_{R \leq |x| \leq R+1} V(x) \, dx + \int_{R \leq |x| \leq R+1} V(x)(R+1-|x|)^N \, dx$$

and

$$B = W_{N-1} \frac{R^N}{N}$$

and the constant c_0, c_6, c_7 introduced in (A_1) , (1.6) and (1.7).

Now we are ready to present our first main result

Theorem 3.1 *Assume that (A_1) , (A_2) , (A_3) and (f_1) hold. Then there exists $\lambda^* = \lambda^*(r) > 0$ such that for all $0 < \lambda < \lambda^*$ the functional I_λ admits a nontrivial critical point u_1 satisfying*

$$0 < \phi(u_1) < r \quad \text{and} \quad I_\lambda(u_1) \leq I_\lambda(w) \quad \text{for all } w \in \phi^{-1}([0, r]),$$

for each $r > 0$ with the property

$$r \leq \frac{k_0}{4} \inf \left(1, \left(\frac{1}{c_2} \right)^N \left(\frac{(N-1)\alpha_N}{N\alpha_0} \right)^{N-1}, \left(\frac{1}{c_2} \right)^N \left(\frac{\alpha_N}{\alpha} \right)^{N-1} \right). \quad (3.2)$$

Proof For $\lambda > 0$ and $R > 0$ as in (3.1), define the function

$$\theta_\lambda = \begin{cases} \eta_\lambda & \text{if } |x| < R, \\ \eta_\lambda(R+1-|x|) & \text{if } R \leq |x| \leq R+1, \\ 0 & \text{if } |x| > R+1, \end{cases}$$

with η_λ is a real number satisfying

$$0 < \eta_\lambda < \inf \left(Z_\lambda, T_r \right), \quad (3.3)$$

where

$$Z_\lambda = \left(\frac{\lambda B - c_0 \int_{R \leq |x| \leq R+1} h_0}{c_0 \int_{R \leq |x| \leq R+1} h_1 + \frac{1}{N} \Gamma_V} \right)^{\frac{1}{N-1}}$$

and

$$T_r = \left(\frac{r}{k_0 \int_{R \leq |x| \leq R+1} h_1(x) + \frac{1}{N} \Gamma_V} \right)^{\frac{1}{N}}.$$

First we note that $\theta_\lambda \in E$.

Owing to our assumptions, one has

$$\phi(\theta_\lambda) = \int_{\mathbb{R}^N} A(x, \nabla \theta_\lambda) dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |\theta_\lambda|^N dx$$

$$\begin{aligned}
 &= \int_{R \leq |x| \leq R+1} A(x, \nabla \eta_\lambda(R+1-|x|)) dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |\theta_\lambda|^N dx \\
 &\leq \int_{R \leq |x| \leq R+1} c_0(h_0(x)|\eta_\lambda| + h_1(x)|\eta_\lambda|^N) dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |\theta_\lambda|^N dx \\
 &= c_0 \eta_\lambda \int_{R \leq |x| \leq R+1} h_0(x) dx + c_0 \delta_\lambda^N \int_{R \leq |x| \leq R+1} h_1(x) dx + \frac{1}{N} \delta_\lambda^N \Gamma_V := G.
 \end{aligned} \tag{3.4}$$

So using the fact that $F(x, \theta_\lambda) \geq 0$, and $\exp(\alpha|s|^{\frac{N}{N-1}}) \geq 1$, it follows that

$$\psi(\theta_\lambda) \geq \int_{|x| < R} \left(\int_0^{\theta_\lambda} \exp\left(\alpha|s|^{\frac{N}{N-1}}\right) ds \right) dx \geq \int_{|x| < R} \theta_\lambda dx = B\eta_\lambda.$$

Thanks to (3.2) and (3.3), we get

$$\frac{\psi(\theta_\lambda)}{\phi(\theta_\lambda)} \geq \frac{B\delta_\lambda}{G} \geq \frac{1}{\lambda}. \tag{3.5}$$

On the other hand, if $u \in E$ has the property $\phi(u) < r$, then

$$\begin{aligned}
 k_0 \int_{R \leq |x| \leq R+1} h_1(x) |\nabla u|^N dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |u|^N dx &< \phi(u) \\
 &= \int_{\mathbb{R}^N} A(x, \nabla u) + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |u|^N dx < r,
 \end{aligned}$$

and so

$$k_0 \|u\|_E^N \leq \phi(u) < r,$$

i.e.,

$$\|u\| < \left(\frac{r}{k_0} \right)^{\frac{1}{N}}.$$

By (3.1) and (1.7), there exists $C^* > 0$ such that

$$\begin{aligned}
 \psi(u) &= \int_{\mathbb{R}^N} \left(\int_0^{u(x)} \exp\left(\alpha|s|^{\frac{N}{N-1}}\right) ds + F(x, u) \right) dx \\
 &\leq \int_{\mathbb{R}^N} \left(\exp\left(\alpha|u|^{\frac{N}{N-1}}\right) |u| ds + F(x, u) \right) dx \\
 &\leq \int_{\mathbb{R}^N} \exp\left(\alpha|u|^{\frac{N}{N-1}}\right) - S_{N-2}(a, u) |u| dx \\
 &\quad + \int_{\mathbb{R}^N} S_{N-2}(a, u) |u| dx + \int_{\mathbb{R}^N} F(x, u) dx
 \end{aligned}$$

$$\begin{aligned}
&\leq C^* \|u\|_E + \left(\frac{b_1}{\beta_1 + 1} c_6 + \frac{b_2}{\beta_2 + 1} c_7 \right) \|u\|_E \\
&\leq \left(C^* + \frac{b_1}{\beta_1 + 1} c_6 + \frac{b_2}{\beta_2 + 1} c_7 \right) \left(\frac{r}{k_0} \right)^{\frac{1}{N}}.
\end{aligned} \quad (3.6)$$

Notice that in (3.5) we use the property $\|u\| \leq 1$.

Set

$$\lambda^* = \frac{r^{\frac{N-1}{N}}}{\left(\frac{4}{k_0} \right)^{\frac{1}{N}} \left(C^* + \frac{b_1}{\beta_1 + 1} c_6 + \frac{b_2}{\beta_2 + 1} c_7 \right)}. \quad (3.7)$$

Using the definition of T_r and (3.4), we have $0 < \phi(\theta_\lambda) < r$. Therefore, one has

$$\begin{aligned}
\zeta(0, r) &= \inf_{v \in \Phi^{-1}([0, r])} \frac{\sup_{u \in \Phi^{-1}([0, r])} \Psi(u) - \Psi(v)}{r - \Phi(v)} \\
&\leq \frac{\sup_{u \in \Phi^{-1}([0, r])} \Psi(u) - \Psi(\theta_\lambda)}{r - \Phi(\theta_\lambda)} \\
&\leq \frac{\sup_{u \in \Phi^{-1}([0, r])} \Psi(u)}{r} \leq \frac{1}{\lambda^*} < \frac{1}{\lambda} < \frac{\psi(\theta_\lambda)}{\phi(\theta_\lambda)} \\
&< \sup_{v \in \Phi^{-1}([0, r])} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([-\infty, 0])} \Psi(u)}{\Phi(v) - 0} \\
&= \rho_1(0, r),
\end{aligned}$$

for all $0 < \lambda < \lambda^*$. (Note that $\Psi(0) = 0$)

Hence, employing Theorem 2.2, for each $0 < \lambda < \lambda^*$, the functional $\Phi - \lambda \Psi$ admits at least one critical point $u_1 \in E$ such that $0 < \phi(u_1) < r$, that is $\|u\|_E < \left(\frac{r}{k_0}\right)^{1/N}$, and $I_\lambda(u_1) \leq I_\lambda(w)$, for any $w \in \phi^{-1}([0, r])$. This completes the proof. $\square$

4 Existence of the Second Solution

Now, in this section, we prove the existence of the second local minimum distinct from the first one. For this purpose, we verify the hypotheses of Theorem 2.2. for another range of r .

Theorem 4.1 Assume that (A_1) , (A_2) , (A_3) and (f_1) hold. Then there exists $\lambda_* \in]0, \lambda^*[$ such that for $\lambda_* < \lambda < \lambda^*$, the functional I_λ admits a critical point u_2 which satisfies

$$r < \phi(u_2) < 2r, \quad \text{and} \quad I_\lambda(u_2) \leq I_\lambda(w), \quad \forall w \in \phi^{-1}([r, 2r]).$$

Proof First set

$$\lambda_* = \left(\frac{2r}{k_0 \int_{R \leq |x| \leq R+1} h_1(x) + \frac{1}{N} \Gamma_V} \right)^{\frac{N-1}{N}} \left(\frac{2c_0 \int_{R \leq |x| \leq R+1} h_0}{B} \right).$$

From assumption (3.1), we get

$$\lambda_* < \lambda^*.$$

For $\lambda_* < \lambda < \lambda^*$, we take the function θ_λ as before but in the following different condition:

$$T_r < \eta_\lambda < \inf\{T_{2r}, Z_{\lambda/2}\}. \quad (4.1)$$

Therefore,

$$r < k_0 \int_{R \leq |x| \leq R+1} h_1(x) |\nabla \theta_\lambda|^N dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |\theta_\lambda|^N \leq \phi(\theta_\lambda) \leq G < 2r,$$

so 4.1 yields that

$$\frac{\psi(\theta_\lambda)}{2\phi(\theta_\lambda)} > \frac{B\eta_\lambda}{2(c_0\eta_\lambda \int_{R \leq |x| \leq R+1} h_0 + c_0\delta_\lambda^N \int_{R \leq |x| \leq R+1} h_1 + \frac{1}{N}\delta_\lambda^N \Gamma_V)} > \frac{1}{\lambda}.$$

Similar to the arguments in the last section, by replacing r by $2r$, we have

$$\begin{aligned} \varsigma(r, 2r) &= \inf_{v \in \Phi^{-1}([r, 2r])} \frac{\sup_{u \in \Phi^{-1}([r, 2r])} \Psi(u) - \Psi(v)}{2r - \Phi(v)} \\ &\leq \frac{\sup_{u \in \Phi^{-1}([r, 2r])} \Psi(u) - \Psi(\theta_\lambda)}{2r - \Phi(\theta_\lambda)} \\ &\leq \frac{\sup_{u \in \Phi^{-1}([r, 2r])} \Psi(u)}{2r} \\ &\leq \left(\frac{1}{k_0}\right)^{\frac{1}{N}} \left(C^* + \frac{b_1}{\beta_1 + 1} c_6 + \frac{b_2}{\beta_2 + 1} c_7\right) (2r)^{-\frac{N-1}{N}} \\ &\leq \left(\frac{1}{k_0}\right)^{\frac{1}{N}} \left(C^* + \frac{b_1}{\beta_1 + 1} c_6 + \frac{b_2}{\beta_2 + 1} c_7\right) r^{-\frac{N-1}{N}} \\ &< \frac{1}{\lambda^*} < \frac{1}{\lambda} < \frac{\psi(\theta_\lambda) - \sup_{u \in \Phi^{-1}([-\infty, 0])} \Psi(u)}{\phi(\theta_\lambda) - r} \\ &< \sup_{v \in \Phi^{-1}([r, 2r])} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{\Phi(v) - r} \\ &= \rho_1(r, 2r), \end{aligned}$$

for all $\lambda_* < \lambda < \lambda^*$. Note that in the second inequality we use the fact that if $\frac{t_1}{t_2} < s < \frac{t_3}{t_4}$, then $\frac{t_1 - t_3}{t_2 - t_4} < s$, and the following inequalities:

$$\frac{\psi(\theta_\lambda)}{2\phi(\theta_\lambda)} > \frac{1}{\lambda} > \frac{\sup_{\phi(u) < 2r} \psi(u)}{2r} > \frac{\sup_{r < \phi(u) < 2r} \psi(u)}{2r}$$

for all $\lambda_* < \lambda < \lambda^*$.

Since

$$0 < 2r < k_0 \inf \left(1, \left(\frac{1}{2c_2} \right)^N \left(\frac{\alpha_N}{\frac{N}{N-1}\alpha_0} \right)^{N-1} \right),$$

then by Lemma 2.3, the functional I_λ satisfies $^{[r]}(PS)^{[2r]}$. Hence, employing Theorem 2.2, we conclude that the functional I_λ admits a critical point u_2 satisfying

$$r < \phi(u_2) < 2r, \quad \text{and} \quad I_\lambda(u_2) \leq I_\lambda(w), \quad \forall \quad w \in \phi^{-1}([r, 2r]),$$

for each $\lambda_* < \lambda < \lambda^*$. Since $\phi(u_1) < r$, then $u_1 \neq u_2$. $\square$

Remark 4.2 To prove the existence of another local minimum distinct from the first and second ones, we can assume an algebraic condition on f with the classical Ambrosetti–Rabinowitz condition: there exist $\nu > N$ and $R' > 0$ such that

$$0 < \nu F(x, t) \leq tf(x, t) \quad \text{for all } |t| > R', x \in \mathbb{R}^N.$$

Clearly, the functional $\Phi - \lambda\Psi$ is of class C^1 and $(\Phi - \lambda\Psi)(0) = 0$.

The first part of proof guarantees that $u_2 \in E$ is a local nontrivial local minimum for $\Phi - \lambda\Psi$ in E . Therefore, there is $s > 0$ such that

$$\inf_{\|u-u_2\|=s} (\Phi - \lambda\Psi)(u) > (\Phi - \lambda\Psi)(u_2).$$

So the condition [8, Theorem 2.2, (I_1)] is verified. Now choosing $u \neq 0$, from Ambrosetti–Rabinowitz condition, one has

$$\begin{aligned} (\Phi - \lambda\Psi)(tu) &\leq c_0 t \int_{\mathbb{R}^N} h_0(x) |\nabla u| dx + c_0 t^N \int_{\mathbb{R}^N} h_1(x) |\nabla u|^N dx \\ &\quad + \frac{t^N}{N} \int_{\mathbb{R}^N} V(x) |u|^N dx \\ &\quad - \lambda \int_{\mathbb{R}^N} \left(\int_0^{tu} \exp \left(\alpha |s|^{\frac{N}{N-1}} \right) ds + F(x, tu) \right) dx \\ &\leq c_0 t \int_{\mathbb{R}^N} h_0(x) |\nabla u| dx + c_0 t^N \int_{\mathbb{R}^N} h_1(x) |\nabla u|^N dx \\ &\quad + \frac{t^N}{N} \int_{\mathbb{R}^N} V(x) |u|^N dx \\ &\quad - \lambda t \int_{\mathbb{R}^N} |u| dx - \lambda a_3 t^\nu \int_{\mathbb{R}^N} |u|^\nu dx + \lambda a_4 \longrightarrow -\infty \end{aligned}$$

as $t \rightarrow \infty$, since $\nu > N$. So the condition [8, Theorem 2.2, (I_1)] is fulfilled.

Moreover, by standard computations, the functional $\Phi - \lambda\Psi$ satisfies (PS) condition. Hence, the classical theorem of Ambrosetti and Rabinowitz gives a critical point u_3

of $\Phi - \lambda\Psi$ such that $(\Phi - \lambda\Psi)(u_3) > (\Phi - \lambda\Psi)(u_2)$. So u_1 , u_2 and u_3 are distinct weak solutions of the problem (1.1).

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