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PREFACE

Half a century of information geometry, part 1

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This special issue has been edited to celebrate the 50th anniversary of publication of the book by Chentsov [12], which laid the foundation for information geometry; the 40 years of study of dual connections in statistics [3], which made a significant contribution to the development of this field; and the 88th birthday (called Bei-jyu in Japan) of Professor Shun-ichi Amari, who is a leading researcher in this field. The author is honored and privileged to be given the opportunity to work as the managing editor of this special issue.

Information geometry is an interdisciplinary research area that delves into the geometric approaches applicable to mathematical sciences and their underlying principles. In a narrow sense, it refers to the application of differential geometric techniques to statistics and the study of differential geometry in relation to dual affine connections.

For the relation to differential geometry and statistics, Hotelling pointed out in 1930 that a family of parametric probability density functions has a Riemannian manifold structure [15]. Although the paper is only an abstract, it is considered one of the origins of information geometry research. (There is a survey of the exchange between Hotelling and Fisher at that time [31].) Later, Rao independently published a journal paper on the Riemannian manifold structure for statistical models [27].

In the 1960s, Chentsov published many studies on the foundations of information geometry. He showed that the Fisher metric and a 1-parameter family of affine connections, which is called α-connections today, are a unique invariant Riemannian metric and invariant affine connections under the categorical mapping on the manifold of the set of all probability functions on a finite sample space [12, Theorem 11.1 and Theorem 12.2]. See also the survey by Fujiwara [14] in this special issue. Furthermore, Chentsov elucidated the relationship between the bi-orthogonality of

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coordinates and the Legendre duality [12, Lemma 19.5], and showed the generalized Pythagorean theorem with respect to the Kullback–Leibler divergence [11], [12, Theorem 22.1]. These series of studies were published in Russian as a monograph in 1972 [12]. After that, Chentsov and his wife, Morozova, contributed to the study of invariant Riemannian metrics in the quantum setting [19]. (Based on their study, Petz obtained a characterization theorem for monotone metrics [25].)

The duality of geometric structures has one of its origins in affine differential geometry. In 1906, Tzitzeica introduced the notion of "surfaces polaires réciproques" [32], and these surfaces have a structure called dual statistical manifold in differential geometry [16]. (Originally, the notion of statistical manifolds was introduced by Lauritzen [17].) Around 1920, Blaschke and his colleagues systematically studied affine differential geometry [9]. From the viewpoint of information geometry, their research also included the study of fundamental structures of statistical manifolds such as cubic forms (also called Amari–Chentsov tensors on statistical models). In 1937, Norden studied a pair of affine connections in which the Riemannian metric is preserved by parallel translations [23]. (See also [24].) These connections are called dual (or conjugate) connections. Sen also independently introduced the notion of dual connections [28–30]. Coincidentally, Rao's paper [27] and Sen's paper [29] are published in the same volume of the same journal, *Bulletin of the Calcutta Mathematical Society*.

In 1982, Amari studied dual affine connections, independently of the researches mentioned above, and applied them to statistics [3]. Amari also introduced the α-connections, which coincide with the α-connection introduced by Chentsov on the manifold of the set of all probability functions on a finite sample space. Nagaoka's contribution to the study of dual affine connections is also important [21], although the paper is not published as a journal article, unfortunately.

Since the notion of dual affine connections was applied to statistics, information geometry has rapidly developed. In particular, a canonical divergence was defined on a dually flat space, which is a purely differential geometric object, and the generalized Pythagorean theorem with respect to the canonical divergence was proved [7, Chapter 3]. As a result, information geometry has become applicable not only to the geometry of probability density functions but also to many fields of mathematical science. Today, information geometry is studied in a wide variety of fields such as mathematics, physics, statistics, and machine learning theory. For an overview of information geometry, refer to the book by Amari [5] and survey papers [6, 18, 22]. In addition, many books such as textbooks and monographs related to information geometry from various perspectives such as mathematics and statistics have been published. (For example [4, 7, 8, 10, 13, 20, 26].) Amari has been applying differential geometry to mathematical engineering already before the 1970s [1, 2]. Those early studies also contributed to the development of information geometry. The word "information geometry" can be found in the Japanese textbook "Information theory" by Amari published in 1970 [2].

This special issue includes many surveys and perspectives, in addition to research papers by leading researchers. The editorial committee hopes that this special issue will provide readers with extensive knowledge about information geometry, as well as contribute to its further development.

During the editing of this special issue, one of the above-mentioned founders of information geometry, Calyampudi Radhakrishna Rao, passed away. The author expresses his deepest condolences.

Articles in Part 1

Half a Century of Information Geometry consists of two issues. Here is a list of articles for Part 1 in the order of submission date.

- [Research Paper] H.S. Battey, D.R. Cox & S.H. Lee: On partial likelihood and the construction of factorisable transformations https://doi.org/10.1007/s41884-022-00068-8
- [Research Paper] F.M. Ciaglia, F. Di Cosmo, A. Ibort & G. Marmo: G-dual Teleparallel Connections in Information Geometry https://doi.org/10.1007/s41884-023-00117-w
- [Research Paper] P. Vos: Generalized estimators, slope, efficiency, and fisher information bounds https://doi.org/10.1007/s41884-022-00085-7
- [Survey] H.V. Lê: Natural differentiable structures on statistical models and the Fisher metric https://doi.org/10.1007/s41884-022-00090-w
- [Research Paper] S. Luo & Y. Sun: Gram matrices of quantum channels via quantum Fisher information with applications to decoherence and uncertainty https://doi.org/10.1007/s41884-023-00096-y
- [Survey] H. Hino, S. Akaho & N. Murata: Geometry of EM and related iterative algorithms https://doi.org/10.1007/s41884-022-00080-y
- [Research Paper] F.M. Ciaglia, F. Di Nocera, J. Jost & L. Schwachhöfer: Parametric models and information geometry on W^*-algebras https://doi.org/10.1007/s41884-022-00094-6
- [Survey] S. Watanabe: Recent advances in algebraic geometry and Bayesian statistics https://doi.org/10.1007/s41884-022-00083-9
- [Research Paper] A.S. Kainth, T.K.L. Wong & F. Rudzicz: Conformal mirror descent with logarithmic divergences. https://doi.org/10.1007/s41884-022-00089-3
- [Perspective] H. Furuhata: Toward differential geometry of statistical submanifolds https://doi.org/10.1007/s41884-022-00075-9
- [Survey] T. Sei: Conditional inference of Poisson models and information geometry: an ancillary review https://doi.org/10.1007/s41884-022-00082-w
- [Research Paper] P.E. Jupp: A parameterisation-invariant modification of the score test https://doi.org/10.1007/s41884-023-00101-4
- [Research Paper] S. Eguchi: Minimum information divergence of Q-functions for dynamic treatment resumes https://doi.org/10.1007/s41884-022-00084-8
- [Research Paper] K. Ishige, P. Salani & A. Takatsu: Hierarchy of deformations in concavity https://doi.org/10.1007/s41884-022-00088-4
- [Research Paper] S. Ito: Geometric thermodynamics for the Fokker-Planck equation:

stochastic thermodynamic links between information geometry and optimal transport https://doi.org/10.1007/s41884-023-00102-3
- [Research Paper] N.J. Newton: A two-parameter family of non-parametric, deformed exponential manifolds https://doi.org/10.1007/s41884-022-00079-5
- [Research Paper] J. Armstrong, D. Brigo & E. Ferrucci: Projections of SDEs onto submanifolds https://doi.org/10.1007/s41884-022-00093-7
- [Survey] P. Marriott: Geometry and applied statistics https://doi.org/10.1007/s41884-022-00086-6
- [Research Paper] A. Ohara, H. Ishi & T. Tsuchiya: Doubly autoparallel structure and curvature integrals https://doi.org/10.1007/s41884-023-00116-x
- [Perspective] J.F. Cardoso: Independent component analysis in the light of Information Geometry https://doi.org/10.1007/s41884-022-00073-x
- [Survey] A. Fujiwara: Hommage to Chentsov's theorem https://doi.org/10.1007/s41884-022-00077-7
- [Research Paper] A. Jenčová: The exponential Orlicz space in quantum information geometry https://doi.org/10.1007/s41884-023-00097-x
- [Survey] G. Pistone: Affine statistical bundle modeled on a Gaussian Orlicz–Sobolev space https://doi.org/10.1007/s41884-022-00078-6
- [Perspective] P. Gibilisco: Uncertainty and Quantum Variance at the light of Quantum Information Geometry https://doi.org/10.1007/s41884-022-00087-5
- [Research Paper] J. Müller & G. Montúfar: Geometry and convergence of natural policy gradient methods https://doi.org/10.1007/s41884-023-00106-z
- [Research Paper] T. Kurose: A certain ODE-system defining the geometric divergence https://doi.org/10.1007/s41884-023-00110-3
- [Survey] J. Armstrong, D. Brigo & B. Hanzon: Optimal projection filters with information geometry https://doi.org/10.1007/s41884-023-00108-x
- [Research Paper] K. Fukumizu: Infinite dimensional exponential family for statistical inference. https://doi.org/10.1007/s41884-023-00122-z
- [Survey] J. Naudts & J. Zhang: Convex duality in Information Geometry https://doi.org/10.1007/s41884-023-00121-0
- [Research Paper] H. Nagaoka: The Fisher metric as a metric on the cotangent bundle https://doi.org/10.1007/s41884-023-00126-9

Handling Editors

Each paper in this special issue was handled by one of the following editors:

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Some of the editors are also authors of papers in this special issue. Needless to say, editors were not involved in the peer review or the handling of their own manuscripts. The name of handling editor for each paper is indicated with "Communicated by" in a footnote to the PDF version of the article and at the bottom of its HTML version.

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Data availability Data sharing is not applicable to this article as no data sets were generated or analyzed during the current editorial.

Declarations

Conflict of interest The author is currently a Co-Editor of *Information Geometry*. He was not involved in the peer review or handling of this preface. The author states that there is no other conflict of interest.

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