Special Issue “Affine differential geometry and Hesse geometry: a tribute and memorial to Jean–Louis Koszul”

Hideyuki Ishi¹

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Jean–Louis Koszul (3 Jan 1921–12 Jan 2018) was a French mathematician with prominent influence on a wide range of mathematical fields. He was a second-generation member of Bourbaki, with several notions in geometry and algebra named after him, such as Koszul connection, Koszul complex, Koszul cohomology, Koszul duality, and so on. He made a great contribution to the fundamental theory of differential geometry, especially the theory of linear (Koszul) connections and affinely flat manifolds, providing foundation of information geometry. This special issue is dedicated to Koszul for the mathematics he developed that bear well on information sciences.

The special issue contains six papers. They cover a wide range of topics such as affine differential geometry over statistical manifolds, Hesse geometry with Fisher information metric, Lie group representations, pre-Lie algebras and their cohomology, geometric mechanics and thermodynamics, and statistical models over homogeneous spaces. These subjects are related to each other, and have intrinsic relations to Koszul’s mathematics.

• “1-Conformal Geometry of Quasi Statistical Manifolds” [5].

A quasi statistical manifold is a generalization of a statistical manifold admitting torsion, defined by using a non-degenerate $(0, 2)$ -tensor h without the assumption of the symmetry of h . This notion is introduced in the theory of non-conservative estimating functions. Despite the difficulty caused by the non-symmetry of h , Haba [5] introduces the notion of 1-conformal equivalence for quasi statistical manifolds, and establishes a relation between affine distributions and 1-conformal flatness of quasi statistical manifolds. Moreover, a hypersurface theory of quasi statistical manifold is discussed.

• “A characterization of the alpha-connections on the statistical manifold of normal distributions” [4].

✉ Hideyuki Ishi
hideyuki@sci.osaka-cu.ac.jp

¹ Graduate School of Science, Osaka City University, Sugimoto 3-3-138, Sumiyoshi-ku, Osaka 558-8585, Japan

It is well-known that the parameter set of 1-dimensional normal distributions with the Fisher information metric is isomorphic to the upper half plane of sectional curvature -2 , and this manifold is identified with a 2-dimensional solvable Lie group with a left-invariant Riemannian metric. Furuhashi et al. [4] introduce the notion of a statistical Lie group, that is to say, a Lie group with a left-invariant statistical manifold structure. In this perspective, the space of normal distributions is shown to be a statistical Lie group. Moreover, the authors give a characterization of the Amari–Chentsov α -connections on this Lie group.

• **“Statistical submanifolds from a viewpoint of the Euler inequality” [8].**

For a surface in the Euclidean 3-space, the mean curvature squared is not less than the Gaussian curvature at each point (the Euler inequality). A totally umbilical surface, i.e., a part of a plane or a part of a round sphere, is characterized as the surface such that the equality holds at each point. Satoh et al. [8] discuss the counterpart of these facts in the statistical submanifold theory. Namely, the authors give a generalized Euler inequality for a submanifold in a statistical manifold without additional assumptions for the ambient space. Then several basic examples of doubly autoparallel statistical submanifolds in warped product spaces are described, for which the equality holds at each point.

• **“Harmonic exponential families on homogeneous spaces” [11].**

Exponential families play important roles in information geometry. Actually, many statistical models, such as gamma, Bernoulli, categorical, Wishart, von Mises, Fisher–Bingham and hyperboloid distributions, are exponential families. Furthermore, one can observe that each of these well-known exponential families admits symmetry with respect to a Lie group acting transitively on the sample space. Tojo and Yoshino [11] show that, under some mild conditions, every exponential family invariant under a natural action of a Lie group G acting transitively on the sample space is obtained as a subfamily of a *harmonic exponential family*, that is an exponential family constructed from a group representation of G by a general method introduced by the authors in the previous works.

• **“Koszul Lecture related to Geometric and Analytic Mechanics, Souriau’s Lie Group Thermodynamics and Information Geometry” [2].**

In 2019, English translation [7] of the lecture notes “Introduction to Symplectic Geometry” was published, while the original book was published in Chinese language, based on lectures that Koszul gave at Nankai University in the spring of 1983. In Chapter 4 titled ‘Symplectic G -Space’ of [7], a Hamiltonian action of a Lie group G on a symplectic manifold M is discussed. Barbaresco [2] investigates the relation of this Koszul’s argument, especially his study of the moment maps, with Lie group Thermodynamics introduced by Souriau [10], where an analog of Fisher information metric arises naturally. Besides, we can perceive Koszul’s insight into this subject from the quoted emails. The content of this article is a revision of Section 6 of the paper [1].

• **“The last formula of Jean–Louis Koszul” [3].**

A Koszul–Vinberg algebra (pre-Lie algebra, or left-symmetric algebra) is introduced by Koszul [6] and Vinberg [12] independently in their studies of affinely flat manifolds and homogeneous cones. For example, if (M, ∇) is an affinely flat manifold, then the space $\mathcal{A}(M)$ of smooth vector fields of M becomes a Koszul–Vinberg algebra by the product $X \cdot Y := \nabla_X Y \in \mathcal{A}(M)$ ($X, Y \in \mathcal{A}(M)$). Boyom [3] recalls the cohomology

theory of Koszul–Vinberg algebra, which has been developed by himself. It is shown that ‘the last formula of Koszul’ yields the KV cohomology with trivial coefficients, and the author generalizes the results to cohomology with coefficients in a non-trivial KV module. Moreover, the application of the KV cohomology for $\mathcal{A}(M)$ to statistical models is discussed in the viewpoint of information geometry.

Finally, we refer to Preamble of [2] to mention Jean–Louis Koszul’s visit to Japan in 1964. Koszul gave lectures on the theory of flat manifolds at Osaka University, and Hirohiko Shima attended the lectures as a graduate student. This was a very important encounter because it was at Shima’s work [9] that the close relation between information geometry and Hesse geometry is established. As is described in [2,3], Koszul and Shima met again at the conference GSI 2013. This special issue should be also a spot of such encounters of various researchers, from which further collaborations will arise in the future.

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