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Global asymptotic stability of endemic equilibria and stability of traveling waves for a diffusive SIR epidemic model with logistic growth

Masahiko Shimojo[1] · Yuya Tanaka[2]

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Abstract
This paper deals with the asymptotic behavior of solutions for the diffusive epidemic model with logistic growth. In the first part, we consider the initial boundary value problem on the bounded domain and derive the stabilization of the solutions of the reaction–diffusion system to a constant equilibrium. In the second part, we consider the initial value problem on $\mathbb{R}$, and derive the stability of forced waves under certain perturbations of a class of initial data.

Keywords SIR epidemic model · Endemic equilibria · Logistic growth · Global stability · Asymptotic profile

Mathematics Subject Classification Primary: 35K40 · Secondary: 35B40 · 92D30

1 Introduction

In epidemiology, one of the most important questions is whether a disease spreads. There are two typical classical epidemiology models, namely, the classical Kermack–McKendrick model and the so-called endemic model (cf. [9, 11]). They are differentiated by whether the vital dynamics (births and deaths) are taken into account. The

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✉ Masahiko Shimojo
shimojo@tmu.ac.jp

1 Department of Mathematical Sciences, Tokyo Metropolitan University, 1-1 Minami-Osawa, Hachioji, Tokyo 192-0397, Japan

2 Department of Mathematics, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku, Tokyo 162-8601, Japan

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spread of infectious diseases in populations has been studied extensively. We refer the reader to, for example, [3, 9, 15] and the references therein.

We consider the following susceptible-infected-removed model:

$$\begin{cases} S_t = d_S \Delta S - \beta S I + r S \left(1 - \dfrac{S}{K}\right), & x \in \Omega,\ t > 0, \\ I_t = d_I \Delta I + \beta S I - \lambda I, & x \in \Omega,\ t > 0, \\ R_t = d_R \Delta R + \sigma I - \mu R, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = \dfrac{\partial R}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) is a bounded domain with smooth boundary $\partial\Omega$, ν denotes the outward normal vector to $\partial\Omega$. The parameters $d_S, d_I, d_R, K, r, \beta, \lambda, \sigma, \mu$ are all positive constants, in which d_S, d_I, d_R are the diffusion rates of susceptible, infected and recovered individuals, respectively; K represents the carrying capacity of susceptible population; r is the intrinsic growth rate of susceptible population; β is the infective transmission rate; λ is the sum of the death rate and the recovery rate; σ is the natural recovery rate of infected population; μ represents the natural death rate. Furthermore, $S_0, I_0 \in C(\overline{\Omega})$ with $S_0, I_0 \geq 0$. The unknown functions $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent the number of susceptible-infected-removed, respectively, at position x and time t.

Note that the simple model, the Kermack–McKendrick model in [11], is obtained by regarding $r = 0$ and the so-called endemic model in [9] is obtained by regarding $K = \infty$ in (1.1). It is natural to assume that the population of susceptible satisfies the logistic type equation when there are no infected populations. This leads us to consider the Eq. (1.1). See also [2, 4, 12] for this type of model.

Since the component $R(x,t)$ does not appear in the first two equations, we omit the third equation and focus on the following diffusive SI epidemic model:

$$\begin{cases} S_t = d_S \Delta S - \beta S I + r S \left(1 - \dfrac{S}{K}\right), & x \in \Omega,\ t > 0, \\ I_t = d_I \Delta I + \beta S I - \lambda I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The first purpose of this paper is to establish global asymptotic stability of constant equilibria in this model. The theorem reads as follows.

Theorem 1.1 *Suppose that* $S_0, I_0 \in C(\bar{\Omega})$, $S_0, I_0 \geq 0$ *and* $S_0, I_0 \not\equiv 0$.

(1) *Let* $R_0 = \frac{K\beta}{\lambda} \leq 1$. *Then the global classical solution* (S, I) *of* (1.2) *fulfills*

$$(S(\cdot,t), I(\cdot,t)) \to (K, 0) \quad \text{in } L^\infty(\Omega) \times L^\infty(\Omega)$$

as $t \to \infty$.

(2) *Let* $R_0 = \frac{K\beta}{\lambda} > 1$. *Then the global classical solution* (S, I) *of* (1.2) *fulfills*

$$(S(\cdot,t), I(\cdot,t)) \to (S^*, I^*) \quad \text{in } L^\infty(\Omega) \times L^\infty(\Omega)$$

as $t \to \infty$, *where*

$$S^* := \frac{\lambda}{\beta}, \quad I^* := \frac{r}{\beta}\left(1 - \frac{\lambda}{K\beta}\right). \tag{1.3}$$

The main idea of the proof of Theorem 1.1 is to find a suitable Lyapunov functional for the reaction–diffusion system (1.2) from a Lyapunov functional of the corresponding kinetic system:

$$\begin{cases} \frac{ds}{dt}(t) = -\beta s(t)i(t) + rs(t)\left(1 - \frac{s(t)}{K}\right), & t \in \mathbb{R}, \\ \frac{di}{dt}(t) = \beta s(t)i(t) - \lambda i(t), & t \in \mathbb{R}. \end{cases} \tag{1.4}$$

For a reaction–diffusion system (1.2) on a bounded domain with zero Neumann boundary condition, taking the integral of a Lyapunov functional for the kinetic system (1.4) over the spatial domain yields a Lyapunov functional of the reaction–diffusion system (1.2). See also [2, 4, 5, 12] for a similar calculation. Thus the typical Lyapunov functionals of (1.2) are

$$V(S(\cdot,t), I(\cdot,t)) := \int_\Omega KL\left(\frac{S(x,t)}{K}\right)dx + \int_\Omega I(x,t)\,dx, \quad \frac{K\beta}{\lambda} \le 1,$$
$$W(S(\cdot,t), I(\cdot,t)) := \int_\Omega S^*L\left(\frac{S(x,t)}{S^*}\right)dx + \int_\Omega I^*L\left(\frac{I(x,t)}{I^*}\right)dx, \quad \frac{K\beta}{\lambda} > 1,$$

where

$$L(z) := z - 1 - \ln(z), \quad z > 0. \tag{1.5}$$

In order to apply the standard theory of the dynamical system, the trajectory of the solution must be compact in a suitable function space, but this is not trivial for our problem (1.2).

To clarify the technical difficulty of the problem, we shall compare (1.2) with the related SI epidemic model with saturated incidence. The initial boundary value problem for the diffusive SI epidemic model with saturated incidence and logistic growth

$$S_t = d_S\Delta S - \frac{\beta SI}{1+\alpha I} + rS\left(1 - \frac{S}{K}\right), \quad I_t = d_I\Delta I + \frac{\beta SI}{1+\alpha I} - \lambda I,$$

is studied by [2, 4], where $\alpha > 0$ measures the saturation level [3, 15]. Their analysis of the global asymptotic stability of the constant equilibria relies on the global boundedness of the infective population [4, Proposition 2.2], which is guaranteed by the condition $\alpha > 0$ together with a comparison principle for a scalar reaction–diffusion equation. On the other hand, when $\alpha = 0$, it is impossible to derive the global boundedness of the infective population by the same comparison argument, which makes the problem involved. To overcome this difficulty, we transform (1.2) into the integral equation and apply the time exponential decay of the fundamental solution. Note that a similar result like Theorem 1.1 is obtained in [12] for a little different equation. We employ their technique to obtain our results.

Now we enter into the second part of this paper. In epidemiology, another important question is whether a disease propagates. The study of traveling waves in reaction–diffusion systems provides important insight into the spatial patterns of invading diseases. Here we consider the initial value problem of the following reaction–diffusion system on a line:

$$\begin{cases} S_t = d_S S_{xx} - \beta SI + rS\left(1 - \dfrac{S}{K}\right), & x \in \mathbb{R},\ t > 0, \\ I_t = d_I I_{xx} + \beta SI - \lambda I, & x \in \mathbb{R},\ t > 0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.6}$$

A traveling wave of (1.6) connecting (S^*, I^*) and $(K, 0)$ is a solution of the form $(S, I)(x,t) = (\phi, \psi)(x - ct)$ for some constant $c \in \mathbb{R}$, where a vector-valued wave profile (ϕ, ψ) satisfies $(\phi, \psi)(-\infty) = (S^*, I^*)$ and $(\phi, \psi)(\infty) = (K, 0)$. Then, the wave profiles $\{\phi, \psi\}$ satisfy

$$\begin{cases} d_S\phi'' + c\phi' - \beta\phi\psi + r\phi(1 - \phi/K) = 0, & z \in \mathbb{R}, \\ d_I\psi_2'' + c\psi' + \beta\phi\psi - \lambda\psi = 0, & z \in \mathbb{R}, \\ (\phi, \psi)(-\infty) = (S^*, I^*), \quad (\phi, \psi)(\infty) = (K, 0). \end{cases} \tag{1.7}$$

The existence of the solution to the problem (1.7) is discussed in [6, 10].

Theorem 1.2 [6, 10] *For each $c \geq 2\sqrt{d_I(\beta K - \lambda)}$ the system* (1.6) *has a positive traveling wave solution of the form $(S, I)(x,t) = (\phi, \psi)(x - ct)$, where (ϕ, ψ) satisfies* (1.7).

Let $(S(x,t), I(x,t))$ be a positive solution of (1.6). Then, using the moving coordinate $z = x - ct$, $(S, I) = (S, I)(z, t)$ satisfies

$$\begin{cases} S_t = d_S S_{zz} + cS_z - \beta SI + rS\left(1 - \dfrac{S}{K}\right), & z \in \mathbb{R},\ t > 0, \\ I_t = d_I I_{zz} + cI_z + \beta SI - \lambda I, & z \in \mathbb{R},\ t > 0, \\ S(z,0) = S_0(z),\ I(z,0) = I_0(z), & z \in \mathbb{R}. \end{cases} \tag{1.8}$$

Note that a wave profile (ϕ, ψ) is a stationary solution of (1.8).

To prove the solution $(S, I)(x,t)$ of the problem (1.6) converges to the traveling wave $(\phi, \psi)(x - ct)$, we shall prove that the solution $(S, I)(z,t)$ of the problem (1.8) converges to $(\phi, \psi)(z)$.

Now, we state the theorem on the asymptotic stability for traveling waves in the corresponding diffusive SI model under certain perturbations of initial data.

Theorem 1.3 *Assume $d_S = d_I = 1$. Suppose $\beta K \geq \lambda + r$ and $c \geq c^* = 2\sqrt{\beta K - \lambda}$. Let (S, I) be a solution of system (1.8) with positive initial data (S_0, I_0) such that*

$$e^{\mu z}\left\{\phi L\left(\frac{S_0}{\phi}\right) + \psi L\left(\frac{I_0}{\psi}\right)\right\} \in L^1(\mathbb{R}),$$

where $\mu := (c - \sqrt{c^2 - 4(\beta K - \lambda)})/2$, and $L(\cdot)$ is a function defined by (1.5). Then $(S, I)(z, t)$ converges to $(\phi, \psi)(z)$ as $t \to +\infty$ locally uniformly for z in $\mathbb{R}$, where $\{c, (\phi, \psi)\}$ is a traveling wave obtained in [6, 10].

In most of the stability analysis, the researchers analyze the spectrum of the associated linearized operator at a given traveling wave. This approach has been widely used in the literature (see, e.g., [1, 13, 14]). However, it requires a heavy and complicated spectral analysis of the related operator. We managed to simplify this complicated approach to an elementary way from the point of a relative entropy defined by (3.1). The key significant idea is measuring the distance between the solution and a traveling wave solution through relative entropy, the analysis reduces to that of a linear heat equation, which makes the stability analysis elementary. The idea of the proof is borrowed from the general theory of [8, Theorem 1.1]. See also [7] for another application.

For the existence of traveling waves in [6, 10], the equal diffusivities condition is not required here. However, our method of deriving the convergence to traveling waves requires the equal diffusivities condition $d_S = d_I$.

The rest of this paper is organized as follows. In Sect. 2, we give the proof of Theorem 1.1. Then the proof of Theorem 1.3 is given in Sect. 3.

2 Global asymptotic stability of constant equilibria

As a consequence of the standard well-posedness theory and the maximum principle, we first obtain the positivity of the solutions for the system (1.2).

Lemma 2.1 *Let $S_0, I_0 \in C(\bar{\Omega})$, $S_0, I_0 \geq 0$ and $S_0, I_0 \not\equiv 0$. Let $T_{\max} \in (0, \infty]$ be the maximum existence time of a classical solution $(S, I) \in C(\bar{\Omega} \times [0, T_{\max}))^2 \cap C^{2,1}(\Omega \times (0, T_{\max}))^2$. Then $S(x, t), I(x, t) > 0$ on $\Omega \times (0, T_{\max})$. In addition, if $T_{\max} < \infty$, then*

$$\lim_{t \to T_{\max}} (\|S(\cdot, t)\|_{L^\infty} + \|I(\cdot, t)\|_{L^\infty}) = \infty.$$

Proof The claim can be obtained as in [2, Section 2], so we omit the proof here. □

The next Lemma gives an L^∞ bound for the component $S(\cdot, t)$.

Lemma 2.2 *There exists* $M_1 > 0$ *such that*

$$\|S(\cdot,t)\|_{L^\infty} \le M_1 \quad \textit{for all } t \in [0, T_{\max}).$$

Proof Define a function $\hat{S}(t)$ by a solution of

$$\hat{S}'(t) = r\hat{S}(t)\Big(1 - \frac{\hat{S}(t)}{K}\Big), \quad t > 0, \qquad \hat{S}(0) = \|S_0\|_{L^\infty}.$$

By the comparison principle, we conclude

$$S(x,t) \le \hat{S}(t) \le \max\{\|S_0\|_{L^\infty}, K\} =: M_1, \qquad x \in \bar{\Omega}, \quad t \in [0, T_{\max}).$$

This proves the lemma. □

Next, we establish an L^∞ bound for the component $I(\cdot,t)$.

Proposition 2.3 *The system* (1.2) *admits a global in time solution with* $T_{\max} = \infty$. *Furthermore, there exists a constant* $C > 0$ *such that*

$$\|S(\cdot,t)\|_{L^\infty} + \|I(\cdot,t)\|_{L^\infty} \le C \quad \textit{for all } t \in [0, \infty).$$

Proof From (1.2), we have

$$\begin{aligned}(S+I)_t &= d_S\Delta S + d_I\Delta I + rS\left(1 - \frac{S}{K}\right) - \lambda I\\ &\le d_S\Delta S + d_I\Delta I + rS - \lambda I\\ &\le d_S\Delta S + d_I\Delta I + (r+\lambda)M_1 - \lambda(S+I).\end{aligned}$$

Now integrating this inequality on Ω, we obtain

$$\frac{d}{dt}\int_\Omega (S+I)(x,t)\,dx \le (r+\lambda)M_1|\Omega| - \lambda\int_\Omega (S+I)(x,t)\,dx,$$

which implies

$$\int_\Omega (S+I)(x,t)\,dx \le \max\left\{\int_\Omega (S_0+I_0)(x)\,dx, \frac{(r+\lambda)M_1|\Omega|}{\lambda}\right\}.$$

In particular,

$$\int_\Omega I(x,t)\,dx \le \max\left\{\int_\Omega (S_0+I_0)(x)\,dx, \frac{(r+\lambda)M_1|\Omega|}{\lambda}\right\} =: M_2, \quad t \ge 0. \tag{2.1}$$

Next, we improve the L^1 bound to an L^∞ bound for the component $I(\cdot,t)$. In the following, the operator Δ is provided with the Neumann boundary condition. Let $e^{t(d_I\Delta-\lambda)}$ be the Neumann heat semigroup in Ω. Then the function $I(x,t)$ satisfies

$$I(\cdot,t) = e^{t(d_I\Delta-\lambda)}I_0 + \int_0^t e^{(t-s)(d_I\Delta-\lambda)}\beta S(\cdot,s)I(\cdot,s)\,ds, \quad t \geq 0.$$

It is known that there exists $C > 0$ such that

$$\|e^{t(d_I\Delta-\lambda)}\|_{\mathcal{L}(L^\infty,L^\infty)} \leq Ce^{-\lambda t}, \quad \|e^{t(d_I\Delta-\lambda)}\|_{\mathcal{L}(L^1,L^\infty)} \leq Ce^{-\lambda t}, \qquad t \geq 0.$$

See [12, Appendix]. Thus we see that

$$\begin{aligned}
\|I(\cdot,t)\|_{L^\infty} &\leq Ce^{-\lambda t}\|I_0\|_{L^\infty} + \int_0^t \|e^{(t-s)(d_I\Delta-\lambda)}\beta S(\cdot,s)I(\cdot,s)\|_{L^\infty}\,ds \\
&\leq Ce^{-\lambda t}\|I_0\|_{L^\infty} + C\int_0^t e^{-\lambda(t-s)}\|\beta S(\cdot,s)I(\cdot,s)\|_{L^1}\,ds \\
&\leq Ce^{-\lambda t}\|I_0\|_{L^\infty} + C\beta \sup_{t\geq 0}\|S(\cdot,s)\|_{L^\infty}\int_0^t e^{-\lambda(t-s)}\|I(\cdot,s)\|_{L^1}\,ds \\
&\leq Ce^{-\lambda t}\|I_0\|_{L^\infty} + C\beta \sup_{t\geq 0}\|S(\cdot,s)\|_{L^\infty}\ \sup_{t\geq 0}\|I(\cdot,s)\|_{L^1}\int_0^t e^{-\lambda(t-s)}\,ds.
\end{aligned}$$

Now we apply Lemma 2.2 and (2.1) to obtain

$$\|I(\cdot,t)\|_{L^\infty} \leq Ce^{-\lambda t}\|I_0\|_{L^\infty} + C\beta M_1 M_2 \frac{1-e^{-\lambda t}}{\lambda}.$$

Therefore, the proposition is proved. □

We will now prove the global stability of the equilibrium $(K,0)$ for the case $R_0 \leq 1$.

Proposition 2.4 *Let $R_0 \leq 1$. Then*

$$\lim_{t\to\infty}(\|S(\cdot,t)-K\|_{L^\infty} + \|I(\cdot,t)\|_{L^\infty}) = 0.$$

Proof By a simple calculation,

$$\begin{aligned}
\frac{d}{dt}V(S(\cdot,t),I(\cdot,t)) &= \int_\Omega \left(1-\frac{K}{S}\right)S_t + \int_\Omega I_t \\
&= \int_\Omega \left(1-\frac{K}{S}\right)\left[d_S\Delta S - \beta SI + rS\left(1-\frac{S}{K}\right)\right] + \int_\Omega (\beta SI - \lambda I) \\
&= -Kd_S\int_\Omega \frac{|\nabla S|^2}{S^2} - \frac{r}{K}\int_\Omega (K-S)^2 - (\lambda - K\beta)\int_\Omega I. \qquad (2.2)
\end{aligned}$$

Since $R_0 = K\beta/\lambda \le 1$ implies $\lambda - K\beta \ge 0$, we have

$$\frac{d}{dt}V(S(\cdot,t), I(\cdot,t)) \le 0.$$

Note that $\{(K,0)\}$ is the largest positive invariant set included in $\{(S,I) \mid \frac{dV}{dt}(S,I) = 0\}$. By the standard theory of Lyapunov functional with the LaSalle's invariance principle together with (2.2) and the standard parabolic regularity theory, we conclude that $(S(\cdot,t), I(\cdot,t)) \to (K,0)$ as $t \to \infty$. This ends the proof of the proposition. □

Similarly, we can obtain the following global stability of the equilibrium (S^*, I^*) for the case when $R_0 > 1$.

Proposition 2.5 *Let* $R_0 > 1$. *Then*

$$\lim_{t\to\infty} (\|S(\cdot,t) - S^*\|_{L^\infty} + \|I(\cdot,t) - I^*\|_{L^\infty}) = 0.$$

Proof By a simple calculation, we get

$$\begin{aligned}
\frac{d}{dt}W(S(\cdot,t), I(\cdot,t)) &= \int_\Omega \left(1 - \frac{S^*}{S}\right) S_t + \int_\Omega \left(1 - \frac{I^*}{I}\right) I_t \\
&= \int_\Omega \left(1 - \frac{S^*}{S}\right)\left[d_S \Delta S - \beta S I + r S\left(1 - \frac{S}{K}\right)\right] \\
&\quad + \int_\Omega \left(1 - \frac{I^*}{I}\right)\left[d_I \Delta I + \beta S I - \lambda I\right] \\
&= -S^* d_S \int_\Omega \frac{|\nabla S|^2}{S^2} - I^* d_I \int_\Omega \frac{|\nabla I|^2}{I^2} \\
&\quad + \int_\Omega \left[\beta S^* I - \frac{r}{K}(S - S^*)(S - K)\right] \\
&\quad + \int_\Omega \left[-\beta S I^* - \lambda I + \lambda I^*\right].
\end{aligned}$$

Since the definition of S^* in (1.3) implies $\beta S^* = \lambda$, it follows that

$$\begin{aligned}
\frac{d}{dt}W(S(\cdot,t), I(\cdot,t)) &= -S^* d_S \int_\Omega \frac{|\nabla S|^2}{S^2} - I^* d_I \int_\Omega \frac{|\nabla I|^2}{I^2} \\
&\quad - \int_\Omega \frac{r}{K}(S - S^*)(S - K) - \int_\Omega \beta I^*(S - S^*).
\end{aligned}$$

Moreover, the definitions of S^* and I^* in (1.3) yield $\beta I^* = (r/K)(K - S^*)$. Therefore,

$$\begin{aligned}\frac{d}{dt}W(S(\cdot,t),I(\cdot,t)) &= -S^* d_S\int_\Omega \frac{|\nabla S|^2}{S^2} - I^* d_I\int_\Omega \frac{|\nabla I|^2}{I^2} \\ &\quad - \int_\Omega \frac{r}{K}(S-S^*)(S-K) - \int_\Omega \frac{r}{K}(S-S^*)(K-S^*) \\ &= -S^* d_S\int_\Omega \frac{|\nabla S|^2}{S^2} - I^* d_I\int_\Omega \frac{|\nabla I|^2}{I^2} - \frac{r}{K}\int_\Omega (S-S^*)^2 \le 0.\end{aligned}$$

Note that $\{(S^*, I^*)\}$ is the largest positive invariant set included in $\{(S, I) \mid \frac{dW}{dt}(S, I) = 0\}$. The result can be proved by a similar argument to that of the proof of Proposition 2.4. See [5, Theorem 3.1] or [12, Theorem 3.2] for the detail. □

Proof of Theorem 1.1 We only need to collect Propositions 2.4 and 2.5. □

3 Stability of traveling wave

In this section, we prove Theorem 1.3. We apply the general theory of [8] to prove Theorem 1.3. Here we write the detailed calculation for the reader's convenience.

Proof First, we check that the solution is global in time and uniformly bounded. The function $U = S + I$ satisfies

$$\begin{aligned}U_t \le U_{zz} + cU_z + rS\Big(1 - \frac{S}{K}\Big) - \lambda I &\le U_{zz} + cU_z + (r+\lambda)S - \lambda U \\ &\le U_{zz} + cU_z + (r+\lambda)M_1 - \lambda U.\end{aligned}$$

Hence by comparing the solution with the solution of the following ordinary differential equation

$$\frac{d}{dt}P(t) = (r+\lambda)M_1 - \lambda P(t), \quad P(0) = \max_{z\in\mathbb{R}} U(z,0),$$

we obtain $U(z,t) \le P(t)$ for all $t \ge 0$. It is easy to check that $P(t)$ is uniformly bounded as $t \to \infty$. Therefore, the solution (S, I) exists globally in time. Define

$$F(z,t) := \phi(z)L\left(\frac{S(z,t)}{\phi(z)}\right) + \psi(z)L\left(\frac{I(z,t)}{\psi(z)}\right). \tag{3.1}$$

By a simple calculation, we have

$$\begin{aligned}F_t &= S_t\left(1 - \frac{\phi}{S}\right) + I_t\left(1 - \frac{\psi}{I}\right), \\ F_z &= \left\{S_z\left(1 - \frac{\phi}{S}\right) - \phi' \ln\frac{S}{\phi}\right\} + \left\{I_z\left(1 - \frac{\psi}{I}\right) - \psi' \ln\frac{I}{\psi}\right\},\end{aligned}$$

and

$$F_{zz} = \left\{ S_{zz} \left(1 - \frac{\phi}{S}\right) - \phi'' \ln \frac{S}{\phi} + \left[\frac{\sqrt{\phi} S_z}{S} - \frac{\phi'}{\sqrt{\phi}} \right]^2 \right\}$$
$$+ \left\{ I_{zz} \left(1 - \frac{\psi}{I}\right) - \psi'' \ln \frac{I}{\psi} + \left[\frac{\sqrt{\psi} I_z}{I} - \frac{\psi'}{\sqrt{\psi}} \right]^2 \right\}.$$

Thus, by substituting (1.8), we obtain

$$\begin{aligned} F_t - F_{zz} - cF_z \le\ & (S-\phi)\left\{ -\beta I + r\left(1 - \frac{S}{K}\right)\right\} + (I-\psi)(\beta S - \lambda) \\ & + \left[\beta\phi\psi - r\phi\left(1 - \frac{\phi}{K}\right)\right] \ln \frac{S}{\phi} + [-\beta\phi\psi + \lambda\psi] \ln \frac{I}{\psi} \\ =\ & (S-\phi)\left\{ -\beta I + r\left(1 - \frac{S}{K}\right)\right\} + (I-\psi)(\beta S - \lambda) \\ & + \left[\beta\psi - r\left(1 - \frac{\phi}{K}\right)\right]\left((S-\phi) - \phi L\left(\frac{S}{\phi}\right)\right) \\ & + (-\beta\phi + \lambda)\left((I-\psi) - \psi L\left(\frac{I}{\psi}\right)\right) \\ =\ & -\frac{r}{K}(S-\phi)^2 + \left[r\left(1 - \frac{\phi}{K}\right) - \beta\psi\right]\phi L\left(\frac{S}{\phi}\right) \\ & + (\beta\phi - \lambda)\psi L\left(\frac{I}{\psi}\right). \end{aligned}$$

From the construction of the traveling wave, it is known that $\phi(z) \le K$ holds for all $z \in \mathbb{R}$ (see [6, Lemma 15]). Thus

$$F_t - F_{zz} - cF_z \le -\frac{r}{K}(S-\phi)^2 + r\phi L\left(\frac{S}{\phi}\right) + (\beta K - \lambda)\psi L\left(\frac{I}{\psi}\right).$$

Now we assume $\beta K \ge \lambda + r$, then we conclude

$$F_t - F_{zz} - cF_z \le (\beta K - \lambda)F - \frac{r}{K}(S-\phi)^2 \le (\beta K - \lambda)F. \tag{3.2}$$

Let us define

$$G(z,t) := e^{\mu z} F(z,t), \quad \mu := \frac{c - \sqrt{c^2 - 4(\beta K - \lambda)}}{2}.$$

Then the (3.2) is transformed to the equation

$$G_t \le G_{zz} + (c - 2\mu)G_z.$$

Hence we obtain

$$0 \leq G(z,t) \leq \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\left(- \frac{(z+(c-2\mu)t-y)^2}{4t} \right) G(y,0)\,dy$$
$$\leq \frac{\|G(\cdot,0)\|_{L^1(\mathbb{R})}}{\sqrt{4\pi t}} \to 0$$

uniformly on $\mathbb{R}$ as $t \to \infty$. Thus for any $l > 0$

$$\lim_{t\to\infty} \|F(\cdot,t)\|_{L^\infty(-l,l)} = e^{\mu l} \lim_{t\to\infty} \|G(\cdot,t)\|_{L^\infty(-l,l)} = 0.$$

Hence the theorem is proved. □

Remark 3.1 The boundedness of the solution to the Cauchy problem does not follow immediately when $d_S \neq d_I$, since we can not apply the comparison argument as above. We leave the problem of the uniform boundedness of the solution, and its convergence to a traveling wave when $d_S \neq d_I$, as an open problem.

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