Journal of Elliptic and Parabolic Equations (2023) 9:63–92
https://doi.org/10.1007/s41808-022-00191-8

On the fundamental solution for degenerate Kolmogorov equations with rough coefficients

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Received: 13 December 2021 / Accepted: 6 September 2022 / Published online: 18 October 2022

Abstract
The aim of this work is to prove the existence of a fundamental solution associated to the Kolmogorov equation $\mathscr{L}u = f$ in the dilation invariant case, with bounded measurable first order coefficients and bounded diffusion coefficients satisfying a sort of divergence free assumption. Finally, we prove Gaussian upper and lower bounds for the fundamental solution, and other related properties, under less restrictive assumptions on the coefficients.

Keywords Kolmogorov equation · Weak regularity theory · Ultraparabolic · Fundamental solution · Potential theory

Mathematics Subject Classification 35K70 · 35E05 · 35D30 · 35Q84 · 35H20 · 35B65

1 Introduction

The aim of this work is to prove the existence of a weak fundamental solution for a second order partial differential equation of Kolmogorov type with measurable coefficients of the form

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$$\begin{aligned}\mathscr{L}u(x,t) :=& \sum_{i,j=1}^{m_0} \partial_{x_i}\Big(a_{ij}(x,t)\partial_{x_j}u(x,t)\Big) + \sum_{i=1}^{m_0} b_i(x,t)\partial_{x_i}u(x,t) \\ &+ \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + c(x,t)u(x,t) = 0,\end{aligned} \tag{1}$$

where $z = (x,t) = (x_1,\dots,x_N,t) \in \mathbb{R}^{N+1}$ and $1 \le m_0 \le N$. In particular, the matrices $A_0 = (a_{ij}(x,t))_{i,j=1,\dots,m_0}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ satisfy the following structural assumptions.

(H1) The matrix A_0 is symmetric with real measurable entries, i.e. $a_{ij}(x,t) = a_{ji}(x,t)$, for every $i,j = 1,\dots,m_0$. Moreover, there exist two positive constants λ and Λ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$. The matrix B has constant entries.

(H2) The *principal part operator* $\mathscr{K}$ *of* $\mathscr{L}$ is hypoelliptic, where $\mathscr{K}$ is defined as

$$\mathscr{K}u(x,t) := \sum_{i=1}^{m_0} \partial^2_{x_i}u(x,t) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t),$$

and it is dilation invariant with respect to the family of dilations $(\delta_r)_{r>0}$ introduced in (23).

Note that we allow operator $\mathscr{L}$ to be strongly degenerate whenever $m_0 < N$. However, it is known that the first order part of $\mathscr{L}$ may induce a strong regularizing property. Indeed, under suitable assumptions on the matrix B, the operator $\mathscr{K}$ is hypoelliptic, namely every distributional solution u to $\mathscr{K}u = f$ defined in some open set $\Omega \subset \mathbb{R}^{N+1}$ belongs to $C^\infty(\Omega)$ and it is a classical solution to $\mathscr{K}u = f$, whenever $f \in C^\infty(\Omega)$. We refer to Sect. 2 for further information on this matter. Eventually, we remark that when $\mathscr{L}$ is uniformly parabolic (i.e. $m_0 = N$ and $B \equiv \mathbb{O}$), assumption $(H2)$ is trivially satisfied. Indeed, in this case the principal part operator $\mathscr{K}$ is simply the heat operator.

In order to expose our main results, we first need to introduce some preliminary notation. From now on, we consider the strip $S_{T_0T_1} := \mathbb{R}^N \times (T_0,T_1)$, and in accordance with the scaling of the differential equation (see (23) below) we split the coordinate $x \in \mathbb{R}^N$ as

$$x = \left(x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}\right), \qquad x^{(j)} \in \mathbb{R}^{m_j}, \quad j \in \{0,\dots,\kappa\},$$

where every m_j is a positive integer such that

$$\sum_{j=0}^{\kappa} m_j = N \qquad \text{and} \qquad N \ge m_0 \ge m_1 \ge \dots \ge m_\kappa \ge 1.$$

Thus, here and in the sequel we denote by

$$D = (\partial_{x_1}, \dots, \partial_{x_N}), \quad D_{m_0} = (\partial_{x_1}, \dots, \partial_{x_{m_0}}), \quad \langle \cdot, \cdot \rangle, \quad \text{div},$$

the gradient, the partial gradient in the first m_0 components, the inner product and the divergence in $\mathbb{R}^N$, respectively. Moreover, we introduce the matrix

$$A(x,t) = \left(a_{ij}(x,t)\right)_{1 \le i,j \le N},$$

where a_{ij}, for every $i, j = 1, \dots, m_0$, are the coefficients appearing in (1), while $a_{ij} \equiv 0$ whenever $i > m_0$ or $j > m_0$. Finally, we let

$$Y := \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t) \quad \text{and} \quad b := (b_1, \dots, b_{m_0}, 0, \dots, 0). \tag{2}$$

Now, we are in a position to rewrite the operator $\mathscr{L}$ in the compact form

$$\mathscr{L}u = \text{div}(ADu) + Yu + \langle b, Du \rangle + cu$$

and we recall that its formal adjoint is defined as

$$\begin{aligned} \mathscr{L}^* v(\xi,\tau) = & \sum_{i,j=1}^{m_0} \partial_{\xi_i} \left(a_{ij}(\xi,\tau) \partial_{\xi_j} v(\xi,\tau) \right) - \sum_{i=1}^{m_0} \partial_{\xi_i} (b_i(\xi,\tau) v(\xi,\tau)) \\ & + (c - \text{Tr}(B)) v(\xi,\tau) + Y^* v(\xi,\tau) \end{aligned} \tag{3}$$

where

$$Y^* v(\xi,\tau) := - \sum_{i,j=1}^{N} b_{ij} \xi_j \partial_{\xi_i} v(\xi,\tau) + \partial_\tau v(\xi,\tau).$$

We now introduce the natural framework for studying the weak regularity theory of solutions to $\mathscr{L}u = 0$. We consider a domain $\Omega \subset \mathbb{R}^{N+1}$, where $\Omega = \Omega_{m_0} \times \Omega_{N-m_0+1}$ with $\Omega_{m_0} \subset \mathbb{R}^{m_0}$ and $\Omega_{N-m_0+1} \subset \mathbb{R}^{N-m_0+1}$. We denote by $\mathcal{D}(\Omega)$ the set of C^∞ functions compactly supported in Ω and by $\mathcal{D}'(\Omega)$ the set of distributions in Ω. From now on, $H^1_{x^{(0)}}$ denotes the Sobolev space of functions $u \in L^2(\Omega_{m_0})$ with distributional gradient $D_{m_0} u$ lying in $(L^2(\Omega_{m_0}))^{m_0}$, i.e.

$$H^1_{x^{(0)}} := \left\{ u \in L^2(\Omega_{m_0}) \, : \, D_{m_0} u \in (L^2(\Omega_{m_0}))^{m_0} \right\},$$

and we set

$$\|u\|^2_{H^1_{x^{(0)}}} := \|u\|^2_{L^2(\Omega_{m_0})} + \|D_{m_0} u\|^2_{L^2(\Omega_{m_0})}.$$

We let $H^1_{c,x^{(0)}}$ denote the closure of $C^\infty_c(\Omega_{m_0})$ in the norm of $H^1_{x^{(0)}}$ and we recall that $H^1_{c,x^{(0)}}$ is a reflexive Hilbert space and thus we may consider its dual space

$$\left(H^1_{c,x^{(0)}}\right)^* = H^{-1}_{x^{(0)}} \quad \text{and} \quad \left(H^{-1}_{x^{(0)}}\right)^* = H^1_{c,x^{(0)}},$$

where the notation we consider is the classical one. Hence, from now on we denote by $H^{-1}_{x^{(0)}}$ the dual of $H^1_{c,x^{(0)}}$ acting on functions in $H^1_{c,x^{(0)}}$ through the duality pairing $\langle \cdot, \cdot \rangle_{H^1_{x^{(0)}}, H^1_{c,x^{(0)}}}$. In a standard manner, see for instance [4, 5, 28], we let $\mathcal{W}(\Omega)$ denote the closure of $C^\infty_c(\overline{\Omega})$ in the norm

$$\|u\|^2_{\mathcal{W}(\Omega)} = \|u\|^2_{L^2(\Omega_{N-m_0+1};H^1_{x^{(0)}})} + \|Yu\|^2_{L^2(\Omega_{N-m_0+1};H^{-1}_{x^{(0)}})}, \tag{4}$$

where the previous norm can explicitly be computed as follows:

$$\|u\|^2_{\mathcal{W}(\Omega)} = \int_{\Omega_{N-m_0+1}} \|u(\cdot,\overline{x},t)\|^2_{H^1_{x^{(0)}}} \, d\overline{x}\, dt + \int_{\Omega_{N-m_0+1}} \|Yu(\cdot,\overline{x},t)\|^2_{H^{-1}_{x^{(0)}}} \, d\overline{x}\, dt,$$

where $\overline{x} = (x^{(1)}, \dots, x^{(\kappa)})$ and $x = (x^{(0)}, \overline{x}) \in \mathbb{R}^N$. In particular, $\mathcal{W}(\Omega)$ is a Banach space, and it was firstly introduced in [4] as an extension of the natural functional setting that arises in the study of the weak regularity theory for the kinetic Kolmogorov–Fokker–Planck equation [5, 17–19]. For further properties of the space $\mathcal{W}$, we refer the reader to [28], where the authors provide a characterization of this space in the kinetic Kolmogorov–Fokker–Planck setting, i.e. when $\kappa = 2$, $m_0 = m_1 = d$ and $N = m_0 + m_1$.

Definition 1.1 A function $u \in \mathcal{W}(S_{T_0T_1})$ is a weak solution to (1) if for every non-negative test function $\varphi \in \mathcal{D}(S_{T_0T_1})$, we have

$$\int_{S_{T_0T_1}} -\langle ADu, D\varphi\rangle - uY\varphi + \langle b, Du\rangle\varphi + cu\varphi = 0. \tag{5}$$

In the sequel, we will also consider weak sub-solutions to (1), namely functions $u \in \mathcal{W}(S_{T_0T_1})$ that satisfy the following inequality

$$\int_{S_{T_0T_1}} -\langle ADu, D\varphi\rangle - uY\varphi + \langle b, Du\rangle\varphi + cu\varphi \geq 0, \tag{6}$$

for every non-negative test function $\varphi \in \mathcal{D}(S_{T_0T_1})$. A function u is a weak super-solution to (1) if $-u$ is a sub-solution.

Finally, we recall the definition of weak fundamental solution for the operator $\mathscr{L}$, firstly introduced by Lanconelli et al. in [25, Definition 2.2].

Definition 1.2 A weak fundamental solution for $\mathscr{L}$ is a continuous positive function $\Gamma_L = \Gamma_L(x,t;\xi,\tau)$ defined for $t \in \mathbb{R}$, $0 \leq T_0 < \tau < t < T_1$ and any $x, \xi \in \mathbb{R}^N$ such that:

1. $\Gamma_L = \Gamma_L(\cdot,\cdot;\xi,\tau)$ is a weak solution to $\mathscr{L}u = 0$ in $\mathbb{R}^N \times (\tau, T_1)$ and $\Gamma_L = \Gamma_L(x,t;\cdot,\cdot)$ is a weak solution of $\mathscr{L}^* v = 0$ in $\mathbb{R}^N \times (T_0, t)$;
2. for any bounded function $\varphi \in C(\mathbb{R}^N)$ and any $x, \xi \in \mathbb{R}^N$ we have

$$\begin{cases} \mathscr{L}u(x,t) = 0 & (x,t) \in \mathbb{R}^N \times (\tau, T_1), \\ \lim\limits_{\substack{(x,t)\to(\xi,\tau) \\ t>\tau}} u(x,t) = \varphi(\xi) & \xi \in \mathbb{R}^N, \end{cases}$$
$$\begin{cases} \mathscr{L}^* v(\xi,\tau) = 0 & (\xi,\tau) \in \mathbb{R}^N \times (T_0, t), \\ \lim\limits_{\substack{(\xi,\tau)\to(x,t) \\ t>\tau}} v(\xi,\tau) = \varphi(x) & x \in \mathbb{R}^N, \end{cases} \tag{7}$$

where the above equations need to be satisfied in the weak sense and

$$u(x,t) := \int_{\mathbb{R}^N} \Gamma_L(x,t;\xi,\tau)\,\varphi(\xi)\,d\xi, \qquad v(\xi,\tau) := \int_{\mathbb{R}^N} \Gamma_L(x,t;\xi,\tau)\,\varphi(x)\,dx.$$

Now, we are in a position to state our main results. Firstly, we give answer to [25, Remark 2.3] by proving the existence of a weak fundamental solution for the operator $\mathscr{L}$ in the sense of Definition 1.2 under the following assumption for the coefficients of the operator $\mathscr{L}$.

(H3A) The coefficients $a_{ij}, b_i, c \in L^1(S_{T_0T_1}) \cap L^\infty_{loc}(S_{T_0T_1})$ for $i,j,k = 1,\dots,N$, i.e. for any given compact subset K of $S_{T_0T_1}$ there exists a positive constant M such that

$$|a_{ij}(x,t)|, |b_i(x,t)|, |c(x,t)|, \le M, \qquad \forall (x,t) \in K,\ \forall i,j = 1,\dots,N.$$

Moreover, the diffusion coefficients a_{ij} are such that

$$\mathrm{div} A_0^j = 0 \quad \forall j = 1,\dots,m_0 \text{ in the distributional sense,}$$

where A_0^j denotes the j^{th}-column of the matrix A_0 introduced in assumption **(H1)**. Note that the diverge free assumption on the columns of the matrix A_0 is required to address technical issues arising in the proof of the forthcoming Theorem 1.3. Indeed, the existence of the weak fundamental solution is achieved by combining a regularization procedure with a diagonal argument, that allows us to prove the existence of the fundamental solution by applying Theorem 2.5 to the constructed regularized operator $\mathscr{L}_\varepsilon$ under the assumption **(C)** listed in Sect. 2. It is our belief that this additional assumption can be dropped by considering more refined analytical techniques, such as the ones recently proposed in the pre-print [29] for the case of measurable in time and Hölder continuous in space diffusion coefficients. Lastly, we point out that it is possible to replace the diverge free assumption on the diffusion coefficients with the following, more restrictive, one: the coefficients a_{ij} are measurable, doubly (weakly) differentiable with respect to the first m_0 components and such that $\partial^2_{lk} a_{ij} \in L^\infty_{loc}(S_{T_0 T_1})$, for every $l, k = 2, \dots, m_0$. Indeed, this last assumption is enough to ensure that also the first order derivatives $\partial_k a_{ij}$ are Lipschitz continuous on $S_{T_0 T_1}$, with a uniform modulus of continuity not depending on the set we are considering.

Theorem 1.3 (Existence of the weak fundamental solution) *Let us consider operator* $\mathscr{L}$ *under assumptions* **(H1)–(H2)–(H3A)**. *Then there exists a fundamental solution* Γ_L *of* $\mathscr{L}$ *in the sense of Definition 1.2 and the following reproduction property holds. Indeed,*

for every $x, \xi \in \mathbb{R}^N$ *and every* $t, \tau \in \mathbb{R}$ *with* $\tau < s < t$ *such that* $\tau, t \in (T_0, T_1)$:

$$\Gamma_L(x, t; \xi, \tau) = \int_{\mathbb{R}^N} \Gamma_L(x, t; y, s)\, \Gamma_L(y, s; \xi, \tau)\, dy.$$

Moreover, the function $\Gamma^*_L(x, t; \xi, \tau) = \Gamma_L(\xi, \tau; x, t)$ *is the fundamental solution of* $\mathscr{L}^*$ *and verifies the dual properties of this statement.*

The existence of a classical fundamental solution is a problem that has been thoroughly addressed over the years. In particular, we refer to the works by Hörmander [20] and Kolmogorov [23] for the analysis of the case with constant, or smooth coefficients. Among others, we recall the paper [35] for the proof of the existence of a classical fundamental solution through the Levy parametrix method and we refer to the last part of Sect. 2 for further reference.

To our knowledge, Theorem 1.3 is the first existence result available for the weak fundamental solution to (1) in the sense of Definition 1.2. The proof we propose here is based on a limiting procedure combined with Schauder type estimates and a diagonal argument. This procedure was firstly proposed in [3] to prove the existence of a classical fundamental solution when the coefficients of (1) are locally Hölder continuous. The main difficulties we encounter when adapting this argument to the weak case are given by the low regularity of the coefficients, hence a new regularizing procedure is introduced in Section 4.

We emphasize that the PDE approach adopted in this work improves the previously known results in that it allows us to consider differential operators with bounded measurable coefficients in *both time and space*, which is a milder assumption than the ones considered in the most recent literature. Indeed, on the one hand, in [9] the authors consider the case of bounded measurable time-depending coefficients, with a proof that is based on explicit computations involving the fundamental solution. On the other hand, in [29] the case of Hölder continuous in space and bounded measurable in time coefficients is considered.

Secondly, we extend [25, Theorem 1.3] providing Gaussian upper and lower bounds for the weak fundamental solution Γ of $\mathscr{L}$ under the following more general assumption on the lower order coefficients b and c.

(H3B) The coefficients $b \in \left(L^{\infty}_{\text{loc}}(S_{T_0 T_1})\right)^{m_0}$, $c \in L^{q}_{\text{loc}}(S_{T_0 T_1})$ for some $q > \frac{Q+2}{2}$. Moreover, $c \leq 0$.

As far as we are concerned with Gaussian upper bounds for the fundamental solution Γ associated to $\mathscr{L}$ with Hölder continuous coefficients, a first result dates back to [35], where the author proves Gaussian upper bounds depending on the Hölder norm of the coefficients a, b and c. Later on, Di Francesco and Pascucci [15], Di Francesco and Polidoro [16] prove upper and lower bounds for the classical fundamental solution, where also in this case the involved constants depend on the Hölder norm of the coefficients. A first result regarding Gaussian upper bounds independent of the Hölder norm of the coefficients is due to Pascucci and Polidoro, who studied operator (1) with $b = c = 0$ (see [34, Theorem 1.1]). Later on, Lanconelli and Pascucci [24] and Lanconelli et al. [25] extended Nash upper bounds to non-homogeneous operators of the form (1) with bounded measurable coefficients.

On the other hand, if we consider Gaussian lower bounds independent of the Hölder norm of the coefficients, a first result is due to Lanconelli, Pascucci and Polidoro [25, Theorem 1.3] for the particular case of the kinetic Kolmogorov–Fokker–Planck equation. The proof of this result is based on the construction of a Harnack chain, alongside with the study of the control problem associated to the principal part operator $\mathscr{K}$. The authors of [25] already suggested this type of result could be extended to the general non-homogeneous Kolmogorov operator of step κ in (1), once a suitable Harnack inequality is established. The present work is a first step in this direction as it handles the homogeneous case, the only one for which a Harnack inequality is available, see [4, Theorem 1.3].

Theorem 1.4 (Gaussian bounds) *Let $\mathscr{L}$ be an operator of the form* (1) *under assumptions* **(H1)–(H2)–(H3B)**. *Let $I = (T_0, T_1)$ be a bounded interval, then there exist four positive constants λ^+, λ^-, C^+, C^- such that*

$$C^- \, \Gamma_K^{\lambda^-}(x,t;\xi,\tau) \;\leq\; \Gamma_L(x,t;\xi,\tau) \;\leq\; C^+ \, \Gamma_K^{\lambda^+}(x,t;\xi,\tau) \tag{8}$$

for every $(x,t),(\xi,\tau) \in \mathbb{R}^N \times (T_0,T_1)$ with $\tau < t$. The constants λ^+, λ^-, C^+, C^- only depend on B, $(T_1 - T_0)$, $\|b\|_q$, $\|c\|_q$. Note that $\Gamma_K^{\lambda^-}$ and $\Gamma_K^{\lambda^+}$ denote the fundamental solution of $\mathscr{K}_{\lambda^-}$ and $\mathscr{K}_{\lambda^+}$, where

$$\mathscr{K}^{\lambda} u(x,t) \;:=\; \frac{\lambda}{2} \sum_{i=1}^{m_0} \partial^2_{x_i} u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t), \tag{9}$$

and the explicit expression of $\Gamma_K^{\lambda^\pm}$ is given by

$$\Gamma_K^{\lambda}((x,t);(\xi,\tau)) = \Gamma_K^{\lambda}((\xi,\tau)^{-1} \circ (x,t);0,0),$$

for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$, with $(x,t) \neq (\xi,\tau)$ and

$$\Gamma_K^\lambda(x,t;0,0) = \begin{cases} \frac{(2\pi\lambda)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{2\lambda}\langle C^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B)\right), & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases} \tag{10}$$

Remark 1.5 Following the strategy proposed in Theorem 1.3, the existence of the weak fundamental solution to (1) is ensured under the more restrictive assumptions **(H1)-(H2)-(H3A)**, i.e. when $q = +\infty$. In this case, the constants appearing in the statement will only depend on $B, (T_1 - T_0), M$. Nevertheless, in the same spirit of [24] we provide Gaussian upper and lower bounds in the most general framework, that will hold true even if the existence of the fundamental solution is ensured through a different (and hopefully less resctrictive) procedure.

Remark 1.6 As pointed out in [4, Remark 1.7], we can replace assumption **(H3B)** with the one firstly considered by Wang and Zhang in [39, 40]: $b \in L^q_{loc}(S_{T_0T_1})$ for some $q > (Q+2)$ and $c \in L^q_{loc}(S_{T_0T_1})$ for some $q > \frac{Q+2}{2}$, with the additional requirement of $c \le 0$. Note that the last assumption on the sign of c is necessary to handle unbounded coefficients, and thus when $q = +\infty$, i.e. assumption **(H3A)** is in place, we are able to drop it.

Remark 1.7 Since the proof of the upper bound in (8) does not rely on the Harnack inequality stated in Theorem 3.1, the rightmost inequality of (8) holds true for the more general operator

$$\begin{aligned} \widetilde{\mathscr{L}}u(x,t) := & \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i=1}^{m_0} b_i(x,t)\partial_{x_i}u(x,t) \\ & - \sum_{i=1}^{m_0} \partial_{x_i} a_i(x,t) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + c(x,t)u(x,t), \end{aligned} \tag{11}$$

with $a \in \left(L^q_{\mathrm{loc}}(S_T)\right)^{m_0}$ and $\mathrm{div}\, a \ge 0$.

1.1 Motivation and background

Kolmogorov equations appear in the theory of stochastic processes as linear second order parabolic equations with non-negative characteristic form. In its simplest form, if $\left(W_t\right)_{t\ge 0}$ denotes a real Brownian motion, the density $p = p(t, v, y, v_0, y_0)$ of the stochastic process $(V_t, Y_t)_{t\ge 0}$

$$\begin{cases} V_t = v_0 + \sigma W_t \\ Y_t = y_0 + \int_0^t V_s\, ds \end{cases} \tag{12}$$

is a solution to one of the simplest strongly degenerate Kolmogorov equation, that is

$$\frac{1}{2}\sigma^2\partial_{vv}p - v\partial_y p = \partial_t p, \qquad t \ge 0, \qquad (v,y) \in \mathbb{R}^2. \tag{13}$$

In 1934 Kolmogorov provided us with the explicit expression of the density $p = p(t, v, y; v_0, y_0)$ of the above equation (see [23]) that, when $(v_0, y_0) = (0, 0)$, reads as

$$p(t, v, y; 0, 0) = \frac{\sqrt{3}}{\pi t^2 \sigma^2} \exp\left(-\frac{2}{\sigma^2}\left(\frac{v^2}{t} - 3\frac{vy}{t^2} + 3\frac{y^2}{t^3}\right)\right) \quad t > 0,$$

and pointed out it is a smooth function despite the strong degeneracy of (13). This immediately suggested that the operator $\mathscr{L}$ associated to Eq. (13)

$$\mathscr{L} := \frac{1}{2}\sigma^2 \partial_{vv} - v\partial_y - \partial_t,$$

is hypoelliptic. Indeed, later on Hörmander considered this operator as a prototype for the family of hypoelliptic operators studied in his seminal work [20].

Kolmogorov equations find their application in different research fields. First of all, the process in (12) is the solution to the Langevin equation

$$\begin{cases} dV_t = dW_t \\ dY_t = V_t\, dt, \end{cases}$$

and several mathematical models involving linear and non linear Kolmogorov type equations have also appeared in finance, see for instance [1, 6, 8, 14]. Indeed, equations of the form (13) appear in the Black–Scholes model for the pricing of geometric averaged options (see for instance, [7, 33] and the references therein). For example, equation

$$\partial_t P + \frac{1}{2}\sigma^2 S^2 \partial_S^2 P + (\log S)\partial_A P + r(S\partial_S P - P) = 0, \qquad S > 0,\ A, t \in \mathbb{R},$$

arises in the Black and Scholes option pricing problem

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dA_t = S_t\, dt, \end{cases}$$

where σ is the volatility of the stock price S, r is the interest rate of a riskless bond and $P = P(S, A, t)$ is the price of the Asian option depending on the price of the stock S, the geometric average A of the past price and the time to maturity t.

In this framework, knowing that the fundamental solution to the Kolmogorov equation exists is helpful for the study of the option pricing problem and allows us to have various advantages when dealing with numerical simulations. For further information on this topic, we refer to [7]. It is worth noting that, when dealing with the theory of stochastic processes, Kolmogorov equations arise either in non-divergence form (backward Kolmogorov) or in super-divergence form (forward Kolmogorov). These models can be written as (1) only when the coefficients are regular enough, see assumption **(C)** below. Despite these limitations, our analysis is useful especially from the weak regularity theory point of view, where it is fundamental to lower the regularity assumptions on the coefficients as much as possible. As a byproduct, our results may find application in numerical analysis, for example when studying a priori well-posedness of a numerical method suitable for the analysis of the pricing problem.

Moreover, we recall that the Kolmogorov equation is the prototype for a family of evolution equations arising in kinetic theory of gas, which take the following general form

$$Yu = \mathcal{J}(u). \tag{14}$$

In this case, we have that $u = u(v, y, t)$ is the density of particles with velocity $v = (v_1, \dots, v_n)$ and position $y = (y_1, \dots, y_n)$ at time t. Moreover,

$$Yu := \sum_{j=1}^{n} v_j \partial_{y_j} u + \partial_t u$$

is the so called total derivative with respect to time in the phase space $\mathbb{R}^{2n+1}$, and $\mathcal{J}(u)$ is the collision operator, which can be either linear or non-linear. For instance, in the usual Fokker–Planck equation (cf. [13, 37]) we have a linear collision operator of the form

$$\mathcal{J}(u) = \sum_{i,j=1}^{n} a_{ij}\, \partial^2_{v_i, v_j} u + \sum_{i=1}^{n} a_i\, \partial_{v_i} u + au$$

where a_{ij}, a_i and a are functions of (y, t); $\mathcal{J}(u)$ can also occur in divergence form

$$\mathcal{J}(u) = \sum_{i,j=1}^{n} \partial_{v_i} (a_{ij}\, \partial_{v_j} u + b_i u) + \sum_{i=1}^{n} a_i \partial_{v_i} u + au.$$

We also mention the following non-linear collision operator of the Fokker-Planck-Landau type

$$\mathcal{J}(u) = \sum_{i,j=1}^{n} \partial_{v_i} \left(a_{ij}(z, u) \partial_{v_j} u + b_i(z, u) \right),$$

where the coefficients a_{ij} and b_i depend both on $z \in \mathbb{R}^{2n+1}$ and the unknown functions u through some integral expression. Moreover, this last operator is studied as a simplified version of the Boltzmann collision operator (see for instance [11, 27]). For the description of wide classes of stochastic processes and kinetic models leading to equations of the previous type, we refer to the classical monographies [10, 11] and [12].

1.2 Plan of the paper

This work is organized as follows. In Sect. 2 we recall the properties of the geometrical structure associated to operator $\mathscr{L}$. In Sect. 3 we prove Gaussian lower bounds for the fundamental solution associated to operator $\mathscr{L}$ under the assumption **(H3B)**. In Sect. 4 we prove the existence of a weak fundamental solution for operator $\mathscr{L}$ under the assumption **(H3A)**.

2 Preliminaries

In this section we recall some notation and known results about the non-Euclidean geometry underlying the operators $\mathscr{L}$ and $\mathscr{K}$. We refer to the survey paper [2] and the references therein for a comprehensive treatment of this subject.

As first observed by Lanconelli and Polidoro in [26], the principal part operator $\mathscr{K}$ is invariant with respect to left translations in the group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$, where the group law is defined by

$$(x,t)\circ(\xi,\tau) = (\xi + E(\tau)x, t+\tau), (x,t), (\xi,\tau) \in \mathbb{R}^{N+1}, \tag{15}$$

and

$$E(s) = \exp(-sB), \qquad s \in \mathbb{R}. \tag{16}$$

Then $\mathbb{K}$ is a non-commutative group with zero element $(0,0)$ and inverse

$$(x,t)^{-1} = (-E(-t)x, -t).$$

For a given $\zeta \in \mathbb{R}^{N+1}$ we denote by ℓ_ζ the left translation on $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$ defined as follows

$$\ell_\zeta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}, \quad \ell_\zeta(z) = \zeta\circ z.$$

Then the operator $\mathscr{K}$ is left invariant with respect to the Lie product $\circ$, that is

$$\mathscr{K}\circ\ell_\zeta = \ell_\zeta\circ\mathscr{K} \qquad \textit{or, equivalently,} \qquad \mathscr{K}(u(\zeta\circ z)) = (\mathscr{K}u)(\zeta\circ z),$$

for every u sufficiently smooth.

We recall that, by [26] (Propositions 2.1 and 2.2), the dilation invariance in **(H2)** is equivalent to assume that, for some basis on $\mathbb{R}^N$, the matrix B takes the following form

$$B = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \mathbb{O} \end{pmatrix} \tag{17}$$

where every B_j is a $m_j \times m_{j-1}$ matrix of rank m_j, $j = 1, 2, \dots, \kappa$ with

$$m_0 \geq m_1 \geq \cdots \geq m_\kappa \geq 1 \quad \text{and} \quad \sum_{j=0}^{\kappa} m_j = N.$$

Hence, in the sequel we will assume that B has the canonical form (17). We remark that the first part of assumption **(H2)**, i.e. the hypoellipticity of $\mathscr{K}$, implies the condition introduced by Hörmander in [20] applied to $\mathscr{K}$:

$$\text{rank Lie}\Big(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\Big)(x,t) = N+1, \qquad \forall\,(x,t) \in \mathbb{R}^{N+1}, \tag{18}$$

where $\mathrm{Lie}\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)$ denotes the Lie algebra generated by the first order differential operators $\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)$ computed at (x, t). Yet another condition equivalent to (18), (see [26], Proposition A.1), is that

$$C(t) > 0, \quad \text{for every } t > 0, \tag{19}$$

where

$$C(t) = \int_0^t \quad E(s)\, A_0\, E^T(s)\, ds, \tag{20}$$

and $E(\cdot)$ is the matrix defined in (16). Lastly, we recall that Hörmander explicitely constructed in [20] the fundamental solution of $\mathscr{K}$ as

$$\Gamma_K(z, \zeta) = \Gamma_K(\zeta^{-1} \circ z, 0), \quad \forall z, \zeta \in \mathbb{R}^{N+1}, \quad z \neq \zeta, \tag{21}$$

where

$$\Gamma_K((x,t),(0,0)) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\langle C^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B)\right), & \text{if} \quad t > 0, \\ 0, & \text{if} \quad t \le 0. \end{cases}$$

In particular, condition (19) implies that Γ_K is well-defined.

Let us now consider the second part of assumption **(H2)**. We say that $\mathscr{K}$ is invariant with respect to $(\delta_r)_{r>0}$ if

$$\mathscr{K}\left(u \circ \delta_r\right) = r^2 \delta_r(\mathscr{K}u), \quad \text{for every} \quad r > 0, \tag{22}$$

for every function u sufficiently smooth. It is known (see Proposition 2.2 of [26]) that it is possible to read this dilation invariance property in the expression of the matrix B in (17). More precisely, $\mathscr{K}$ satisfies (22) if and only if the matrix B takes the form (17). In this case, we have

$$\delta_r = (\delta_r^0, r^2), \qquad\qquad r > 0, \tag{23}$$

where

$$\delta_r^0 = \mathrm{diag}(r\mathbb{I}_{m_0}, r^3\mathbb{I}_{m_1}, \dots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}), \qquad\qquad r > 0. \tag{24}$$

Furthermore, we introduce the family of slanted cylinders on which we usually study the local properties of the Kolmogorov equation starting from the unit past cylinder

$$\mathcal{Q}_1 := B_1 \times B_1 \times \cdots \times B_1 \times (-1, 0), \tag{25}$$

defined through the open balls

$$B_1 = \{x^{(j)} \in \mathbb{R}^{m_j} \,:\, |x| \le 1\}, \tag{26}$$

where $j = 0, \dots, \kappa$ and $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^{m_j}$. Now, for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$\mathcal{Q}_r(z_0) := z_0 \circ \big(\delta_r\big(\mathcal{Q}_1\big)\big) = \{z \in \mathbb{R}^{N+1} \ : \ z = z_0 \circ \delta_r(\zeta), \zeta \in \mathcal{Q}_1\} \tag{27}$$

the cylinder centered at an arbitrary point $z_0 \in \mathbb{R}^{N+1}$ and of radius r. We next introduce a homogeneous norm of degree 1 with respect to the dilations $(\delta_r)_{r>0}$ and a corresponding quasi-distance which is invariant with respect to the group operation (15).

Definition 2.1 (Homogeneous norm) Let

$$\alpha_1, \dots, \alpha_{m_0} = 1, \alpha_{m_0+1}, \dots, \alpha_{m_0+m_1} = 3, \alpha_{N-m_\kappa}, \dots, \alpha_N = 2\kappa + 1$$

be the positive integers defined as in (24).

If $\|z\| = 0$ we set $z = 0$ while, if $z \in \mathbb{R}^{N+1} \setminus \{0\}$ we define $\|z\| = r$ where r is the unique positive solution to the equation

$$\frac{x_1^2}{r^{2\alpha_1}} + \frac{x_2^2}{r^{2\alpha_2}} + \dots + \frac{x_N^2}{r^{2\alpha_N}} + \frac{t^2}{r^4} = 1.$$

Accordingly, we define the quasi-distance d by

$$d(z, w) = \|z^{-1} \circ w\|, \qquad z, w \in \mathbb{R}^{N+1}. \tag{28}$$

As $\det E(t) = e^{t \operatorname{trace} B} = 1$, the Lebesgue measure is invariant with respect to the translation group associated to $\mathscr{K}$. Moreover, since $\det \delta_r = r^{Q+2}$, we also have

$$\operatorname{meas}\big(\mathcal{Q}_r(z_0)\big) = r^{Q+2} \operatorname{meas}\big(\mathcal{Q}_1(z_0)\big), \qquad \forall\, r > 0, z_0 \in \mathbb{R}^{N+1},$$

where $\mathcal{Q}_r(z_0)$ is defined as in (27) and

$$Q = m_0 + 3m_1 + \dots + (2\kappa + 1) m_\kappa. \tag{29}$$

The natural number $Q + 2$ is called the *homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $(\delta_r)_{r>0}$. This denomination is proper since the Jacobian determinant of δ_r equals to r^{Q+2}.

Since $\mathscr{K}$ is dilation invariant with respect to $(\delta_r)_{r>0}$, also its fundamental solution Γ_K is a homogeneous function of degree $-Q$, namely

$$\Gamma_K\big(\delta_r(z), 0\big) = r^{-Q} \quad \Gamma_K(z, 0), \qquad \forall z \in \mathbb{R}^{N+1} \setminus \{0\}, r > 0.$$

We now recall the definition of Hölder continuous function in this framework.

Definition 2.2 (Hölder continuity) Let α be a positive constant, $\alpha \in (0, 1]$, and let Ω be an open subset of $\mathbb{R}^{N+1}$. We say that a function $f : \Omega \longrightarrow \mathbb{R}$ is Hölder continuous with exponent α in Ω with respect to the group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$, defined in (15), (in short: Hölder continuous with exponent α, $f \in C^\alpha_K(\Omega)$) if there exists a positive constant $C > 0$ such that

$$|f(z) - f(\zeta)| \leq C\, d(z, \zeta)^{\alpha} \qquad \text{for every}\, z, \zeta \in \Omega,$$

where d is the distance defined in (28). Moreover, we associate to every bounded function $f \in C^{\alpha}_{K}(\Omega)$ the semi-norm

$$[f]_{C^{\alpha}(\Omega)} = \sup_{\substack{z, \zeta \in \Omega \\ z \neq \zeta}} \frac{|f(z) - f(\zeta)|}{d(z, \zeta)^{\alpha}}.$$

Eventually, we say a function f is locally Hölder continuous, and we write $f \in C^{\alpha}_{K,\text{loc}}(\Omega)$, if $f \in C^{\alpha}_{K}(\Omega')$ for every compact subset Ω' of Ω.

We conclude this section by presenting an overview of results regarding the *classical* theory and the corresponding definition of fundamental solution for the operator $\mathscr{L}$ under the following assumption on the coefficients a, b and c.

(C) The matrix A_0 satisfies assumption **(H1)**, while the matrix B has constant entries. The principal part operator $\mathscr{K}$ satisfies assumption **(H2)**. Finally, the coefficients a_{ij}, b_i, c, and $\partial_{x_k} a_{ij}$, for $i, j, k = 1, \ldots, N$, are bounded and Hölder continuous of exponent $\alpha \in (0, 1]$.

Note that the Hölder continuity of the derivatives of a_{ij}, $i, j = 1, \ldots, N$, is required since the operator we consider in (1) is in divergence form, whereas the results we will present below (see Theorem 2.7) are proved for a trace form Kolmogorov operator.

First of all, let us recall the notion of Lie derivative Yu of a function u with respect to the vector field Y defined in (2). A function u is Lie differentiable with respect to Y at the point (x, t) if there exists and is finite

$$Yu(x, t) := \lim_{s \to 0} \frac{u(\gamma(s)) - u(\gamma(0))}{s}, \qquad \gamma(s) = (E(-s)x, t - s). \tag{30}$$

Note that γ is the integral curve of Y, i.e. $\dot{\gamma}(s) = Y(\gamma(s))$. Clearly, if $u \in C^1(\Omega)$, with Ω open subset of $\mathbb{R}^{N+1}$, then $Yu(x, y, t)$ agrees with $\sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x, t) - \partial_t u(x, t)$ considered as a linear combination of the derivatives of u. Then we are in a position to introduce the notion of classical solution to $\mathscr{L}u = 0$ under the assumptions **(C)**.

Definition 2.3 A function u is a *classical solution* to equation $\mathscr{L}u = 0$ in a domain Ω of $\mathbb{R}^{N+1}$ under the assumptions **(C)** if the derivatives $\partial_{x_i} u, \partial^2_{x_i x_j} u$, for $i, j = 1, \ldots, m_0$, and the Lie derivative Yu exist as continuous functions in Ω, and the equation $\mathscr{L}u(x, t) = 0$ is satisfied at any point $(x, t) \in \Omega$. Finally, we say that u is a *classical super-solution* to $\mathscr{L}u = 0$ if $\mathscr{L}u \leq 0$. We say that u is a *classical sub-solution* if $-u$ is a classical supersolution.

A fundamental tool in the classical regularity theory for PDEs are Schauder type estimates. In particular, when considering the case of bounded and Hölder continuous coefficients, we recall the result proved by Manfredini in [31] (see Theorem 1.4) for classical solutions to $\mathscr{L}u = 0$, where the natural functional setting is

$$C^{2+\alpha}(\Omega) = \left\{ u \in C^{\alpha}(\Omega) \mid \partial_{x_i} u, \partial^2_{x_i x_j} u, Yu \in C^{\alpha}(\Omega), \, for \, i,j = 1, \dots, m_0 \right\},$$

and $C^{\alpha}(\Omega)$ is given in Definition 2.2. Moreover, if $u \in C^{2+\alpha}(\Omega)$ then we define the norm

$$|u|_{2+\alpha,\Omega} := |u|_{\alpha,\Omega} + \sum_{i=1}^{m_0} |\partial_{x_i} u|_{\alpha,\Omega} + \sum_{i,j=1}^{m_0} |\partial^2_{x_i x_j} u|_{\alpha,\Omega} + |Yu|_{\alpha,\Omega}.$$

Clearly, the definition of $C^{2+\alpha}_{\mathrm{loc}}(\Omega)$ follows straightforwardly from the definition of $C^{\alpha}_{\mathrm{loc}}(\Omega)$. Finally, we write $u \in C^2(\Omega)$ if u, its derivatives $\partial_{x_i} u, \partial^2_{x_i x_j} u$, for $i,j = 1, \dots, m_0$, and the Lie derivative Yu exist as continuous functions in Ω. In the framework of semigroups, Schauder estimates where proved by Lunardi in [30]. Moreover, a complete characterization of the intrinsic Hölder spaces is provided by Pagliarani, Pascucci and Pignotti in [32]. Finally, Schauder estimates for the Boltzmann fractional framework were recently proved by Imbert and Silvestre in [22]. For a comparison between the different types of Hölder spaces considered in literature we refer to [22, 32].

As we work with first order coefficients which are not Hölder continuous but only measurable, we now introduce the Schauder type estimates proved in [36]. First of all, we recall that the modulus of continuity of a function f on any set $H \subset \mathbb{R}^{N+1}$ is defined as follows

$$\omega_f(r) := \sup_{\substack{(x,t),(\xi,\tau) \in H \\ d((x,t),(\xi,\tau)) < r}} |f(x,t) - f(\xi,\tau)|.$$

Definition 2.4 A function f is said to be Dini-continuous in H if

$$\int_0^1 \frac{\omega_f(r)}{r} dr < +\infty.$$

We are now in position to state the following result (see [36, Theorem 1.6]).

Theorem 2.5 *Let $\mathscr{L}$ be an operator in the form* (1) *satisfying hypothesis* **(H1)-(H2)-(H3A)**. *Let u be a classical solution to $\mathscr{L}u = f$. Suppose that f is Dini continuous. Then there exists a positive constant c, only depending on the operator $\mathscr{L}$, such that*:

(i)

$$|\partial^2 u(0,0)| \le c\left(\sup_{\mathcal{Q}_1(0,0)} |u| + |f(0,0)| + \int_0^1 \frac{\omega_f(r)}{r} dr \right);$$

(ii) for any points (x, t) and $(\xi, \tau) \in \mathcal{Q}_{\frac{1}{4}}(0,0)$ we have

$$\begin{aligned}&|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \\ &\le c\left(d \sup_{\mathcal{Q}_1(0,0)} |u| + d \sup_{\mathcal{Q}_1(0,0)} |f| + \int_0^d \frac{\omega_f(r)}{r} dr + d \int_d^1 \frac{\omega_f(r)}{r^2} dr \right).\end{aligned}$$

where $d := d((x,t),(\xi,\tau))$ and ∂^2 stands either for $\partial^2_{x_i x_j}$, with $i, j = 1, \dots, m$, or for Y.

Remark 2.6 The proof of the above statement is derived applying the techniques of [36, Section 6] combined with the proof of the Dini continuity of the coefficients following the lines of Sect. 4, provided that **(H3A)** holds true.

Lastly, we recall that the existence of a fundamental solution Γ for the operator $\mathscr{L}$ under the regularity assumptions **(C)** has widely been investigated over the years, and the Levy parametrix method provides us with a classic fundamental solution. Among the first results of this type we recall [21, 38, 41] and we remark that this method was firstly considered in this setting by Polidoro in [35], and then later on extended in the works [15, 16]. In particular, we report here the existence result for a classical fundamental solution for $\mathscr{L}$ proved in [15, Theorem 1.4-1.5].

Theorem 2.7 *Let us consider an operator $\mathscr{L}$ of the form* (1) *under the assumption* **(C)**. *Then there exists a fundamental solution $\Gamma : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to \mathbb{R}$ for $\mathscr{L}$ with the following properties*:

1. $\Gamma(\cdot,\cdot;\xi,\tau) \in L^1_{\text{loc}}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \backslash \{(\xi,\tau)\}$ *for every* $(\xi,\tau) \in \mathbb{R}^{N+1}$;
2. $\Gamma(\cdot,\cdot;\xi,\tau)$ *is a classical solution of* $\mathscr{L}u = 0$ *in* $\mathbb{R}^{N+1} \backslash \{(\xi,\tau)\}$ *for every* $(\xi,\tau) \in \mathbb{R}^{N+1}$ *in the sense of Definition* 2.3;
3. *let* $\varphi \in C(\mathbb{R}^N)$ *such that for some positive constant* c_0 *we have*

$$|\varphi(x)| \le c_0 e^{c_0 |x|^2} \qquad \text{for every } x \in \mathbb{R}^N, \tag{31}$$

then there exists

$$\lim_{\substack{(x,t) \to (x_0,\tau) \\ t > \tau}} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\varphi(\xi) d\xi = \varphi(x_0) \qquad \text{for every } x_0 \in \mathbb{R}^N;$$

4. *let* $\varphi \in C(\mathbb{R}^N)$ *verifying* (31). *Then there exists* $T \in (T_0, T_1]$ *such that the function*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,T_0)\,\varphi(\xi)\; d\xi\, d\tau \tag{32}$$

is a classical solution to the Cauchy problem

$$\begin{cases} \mathscr{L}u = 0 & \text{in } S_{T_0,T}, \\ u(\cdot,T_0) = \varphi & \text{in } \mathbb{R}^N; \end{cases} \tag{33}$$

5. *the reproduction property holds. Indeed, for every* $x, \xi \in \mathbb{R}^N$ *and* $t, \tau \in \mathbb{R}$ *with* $\tau < s < t$:

$$\Gamma(x,t;\xi,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t;y,s)\,\Gamma(y,s;\xi,\tau)\,dy;$$

6. *for every* $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$ *with* $t \le \tau$ *we have that* $\Gamma(x,t;\xi,\tau) = 0$;
7. *if* $c(x,t) = c$ *is constant, then*

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\,d\xi = e^{-c(t-\tau)}, \qquad \forall x \in \mathbb{R}^N,\;\; \tau < t;$$

8. *for every* $\lambda^+ > \lambda$ *and for every positive* $0 < T_0 < T_1$, *there exists a constant* C^+ , *only dependent on* λ, B *and* T *such that*

$$\begin{aligned} \Gamma(x,t;\xi,\tau) \;&\le\; C^+\,\Gamma^+(x,t;\xi,\tau) \\ |\partial_{x_i}\Gamma(x,t;\xi,\tau)| \;&\le\; \frac{C^+}{\sqrt{t-\tau}}\,\Gamma^+(x,t;\xi,\tau) \\ |\partial^2_{x_i x_j}\Gamma(x,t;\xi,\tau)| \;&\le\; \frac{C^+}{t-\tau}\,\Gamma^+(x,t;\xi,\tau) \\ |Y\Gamma(x,t;\xi,\tau)| \;&\le\; \frac{C^+}{t-\tau}\,\Gamma^+(x,t;\xi,\tau) \end{aligned} \tag{34}$$

for any $i,j = 1,\dots,m_0$ and $(x,t),(\xi,\tau) \in \mathbb{R}^N \times (T_0,T_1)$, and where Γ^+ denotes the fundamental solution of $\mathscr{K}_{\lambda^+}$, defined in (10) and (9) respectively.

Moreover, there exists a fundamental solution Γ^* to $\mathscr{L}^*$ verifying the dual properties of this statement and $\Gamma^*(x,t;\xi,\tau) = \Gamma(\xi,\tau;x,t)$ for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$, $(x,t) \neq (\xi,\tau)$.

We observe that in [15] the authors considered a Kolmogorov-type operator in trace form. As we work with operators in divergence form, we were forced to require in **(C)** an additional regularity assumption on the derivatives of the coefficients a_{ij}, for $i,j = 1,\dots,N$. This also reflects in **(H3A)**.

3 Proof of Theorem 1.4

This section is devoted to the proof of Gaussian bounds (Theorem 1.4) for the weak fundamental solution defined in Definition 1.2. All the results proved in this section are obtained under the less restrictive assumption **(H3B)** and, when possible, for the more general operator $\widetilde{\mathscr{L}}$.

Now, given the notation of (27), we introduce the upper and lower cylinders

$$\begin{aligned} \mathcal{Q}_+ &= \delta_\omega\left(\widetilde{\mathcal{Q}}_1\right) = B_\omega \times B_{\omega^3} \times \cdots \times B_{\omega^{2\kappa+1}} \times (-\omega^2, 0], \\ \mathcal{Q}_- &= (0, \ldots, 0, -1 + 2\rho^2) \circ \delta_\rho\left(\mathcal{Q}_1\right) \\ &= B_\rho \times \cdots \times B_{\rho^{2\kappa+1}} \times (-1 + \rho^2, -1 + 2\rho^2), \end{aligned}$$

and we introduce some preliminary results. Quite recently, there have been various developments in the study of weak solutions to $\mathscr{L}u = 0$, and in particular the following Harnack inequality proved in [4, Theorem 1.3], the first one in the weak framework considered in this work, holds true in our setting.

Theorem 3.1 (Harnack inequality) *Let* $\mathcal{Q}_1 := B_1 \times B_1 \times \cdots \times B_1 \times (-1, 0]$ *and let u be a non-negative weak solution to* $\mathscr{L}u = 0$ *in* $\Omega \supset \widetilde{\mathcal{Q}}_1$ *under assumptions* **(H1)**–**(H2)**–**(H3B)**. *Then we have*

$$\sup_{\mathcal{Q}_-} u \leq C \inf_{\mathcal{Q}_+} u, \tag{35}$$

where $0 < \omega < 1$ *and* $0 < \rho < \frac{\omega}{\sqrt{2}}$. *Finally, the constants* C, ω, ρ *only depend on the homogeneous dimension* Q *defined in* (29), q *and on the ellipticity constants* λ *and* Λ *in* **(H1)**.

Remark 3.2 When considering assumption **(H3A)** the constants appearing in the above statement only depend on M, since we assume $|b(x,t)| \leq M$, $|c(x,t)| \leq M$ for every $(x,t) \in \mathbb{R}^{N+1}$.

We recall the following result, which will be useful in the proof of the upcoming Lemma 3.4.

Remark 3.3 Let u be a weak solution to $\mathscr{L}u = 0$ and $r > 0$. Then $v := u \circ \delta_\rho$ solves equation $\mathscr{L}^{(r)} v = 0$, where

$$\mathscr{L}^{(r)} v := \mathrm{div}(A^{(r)} Du) - \mathrm{div}(a^{(r)} v) + \langle b^{(r)}, Dv \rangle + c^{(r)} v + \langle Bx, Dv \rangle - \partial_t v$$

with $A^{(r)} = A \circ \delta_r$, $a^{(r)} = r(a \circ \delta_r)$, $b^{(r)} = r(b \circ \delta_r)$ and $c^{(r)} = r^2 (c \circ \delta_r)$.

Moreover, if u is a solution to $\mathscr{L}u = 0$, then, for any $\zeta \in \mathbb{R}^{N+1}$, $v := u \circ \ell_z$ solves equation $(\mathscr{L} \circ \ell_z) v = 0$, where $\mathscr{L} \circ \ell_z$ is the operator obtained by $\mathscr{L}$ via a ℓ_z-translation of the coefficients.

For $\beta, r, R > 0$ and $z_0 \in \mathbb{R}^{N+1}$, we define the cones

$$P_{\beta,r,R} := \{z \in \mathbb{R}^{N+1} : z = \delta_\rho(\xi, \beta), |\xi| < r, 0 < \rho \le R\},$$

and we set $P_{\beta,r,R}(z_0) := z_0 \circ P_{\beta,r,R}$. We are now in a position to derive the following Lemma, which is a consequence of Theorem 3.1.

Lemma 3.4 *Let $z \in \mathbb{R}^{N+1}$ and $R \in (0, 1]$. Moreover, let u be a continuous and non-negative weak solution to $\mathscr{L}u = 0$ in $\mathcal{Q}_R(z)$ under the assumptions* **(H1)-(H2)-(H3B)**. *Then we have*

$$\sup_{P_{1,\omega,R/R_0}(z)} u \le Cu(z),$$

where C, R_0 and ω are the constants appearing in Theorem 3.1 and they only depend on Q, λ, Λ and q.

Proof Let $w \in P_{1,\omega,R}(z)$, i.e. $w = z \circ \delta_\sigma(\xi, 1)$ for some $\sigma \in (0, R]$ and $|\xi| < \omega$. We now define the function $u_{z,\sigma} := u \circ \ell_z \circ \delta_\sigma$, which is a continuous and non-negative solution to $\mathscr{L}^{(\sigma)} u_{z,\sigma} = 0$ in $\mathcal{Q}_{R_0}(0, 0) \subset \mathcal{Q}_{R/\sigma}(0, 0)$ in virtue of Remark 3.3. Thus, we can apply the Harnack inequality (35) and infer

$$u(w) = u_{z,\sigma}(\xi, 1) \le \sup_{\mathcal{Q}_-} u_{z,\sigma} \le C \inf_{\mathcal{Q}_+} u_{z,\sigma} \le Cu_{z,\sigma}(0, 0) = Cu(z).$$

□

We next state a global version of the Harnack inequality, which is a crucial step in proving the Gaussian lower bound (see Theorem 3.8 below).

Theorem 3.5 (Global Harnack inequality) *Let $t_0 \in \mathbb{R}$ and $\tau \in (0, 1]$. If u is a continuous and non-negative weak solution to $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1} \times (\tau - t_0, \tau + t_0)$ under the assumptions* **(H1)-(H2)-(H3B)**, *then we have*

$$u(\xi, t) \le c_0 e^{c_0 \langle C^{-1}(t-t_0)\left(\xi - e^{(t-t_0)B}x\right), \xi - e^{(t-t_0)B}x\rangle} u(x, t_0),$$

where $t \in (t_0, \tau + t_0)$, $x, \xi \in \mathbb{R}^N$, C is the matrix introduced in (20) and c_0 is a positive constant only depending on Q, λ, Λ and q.

The proof of Theorem 3.5 is based on a classical argument that makes use of the so-called Harnack chains, alongside with control theory. Moreover, the proof of this theorem follows the one of [25, Theorem 3.6], with the only difference that we here apply Theorem 3.1 and Lemma 3.4 instead of Theorem 3.1 and Lemma 3.5 of [25]. Indeed, the method we rely on has the advantage of highlighting the geometric structure of the operator $\mathscr{L}$ and can be therefore automatically extended to more general operators. For this reason, we here do not show the derivation of Theorem 3.5 and we refer the reader to [25] for the details.

Moreover, in proving our main result Theorem 3.8, we will also make use of the following estimate, which provides an upper bound for the fundamental solution. Let us remark that in the upcoming theorem we consider operators of the form (11) satisfying the less restrictive assumptions **(H1)–(H2)–(H3B)**.

Theorem 3.6 (Gaussian upper bound) *Let $\widetilde{\mathscr{L}}$ be an operator of the form* (11) *satisfying assumptions* **(H1)–(H2)–(H3B)**. *Then there exists a positive constant c_1, only dependent on Q, λ, Λ and q, such that*

$$\Gamma(x,t;y,t_0) \le \frac{c_1}{\left(t-t_0\right)^{\frac{Q}{2}}} \exp\left(-\frac{1}{c_1}|\delta^0_{(t-t_0)^{-\frac{1}{2}}}\left(y-e^{(t-t_0)B}x\right)|^2\right) \tag{36}$$

for any $0 < t - t_0 \le 1$ *and* $x, y \in \mathbb{R}^N$.

Proof The Gaussian upper bound (36) was proved in [25, Theorem 4.1] under the stronger assumption that the coefficients a_i, b_i, with $i = 1, \dots, m_0$, and c are bounded measurable functions of (x, t) and in [24, Theorem 1.4] under the additional hypothesis that the coefficients b_i, with $i = 1, \dots, m_0$, are null. The more general case where the first order coefficients satisfy assumption **(H3B)** can be treated similarly. Thus, here we just sketch the few adjustments required to adapt the proof of [25] to the present case.

The argument relies on the combination of a Caccioppoli type inequality and a Sobolev type inequality. In order to handle the more general case, we need to replace the Caccioppoli and the Sobolev inequality contained in [24] (Theorem 2.3 and 2.5, respectively) with the ones given in [4]. More precisely, we consider the Caccioppoli inequality [4, Theorem 3.4] and we focus on the new term involving the coefficient $a \in \left(L^q_{\mathrm{loc}}(\Omega)\right)^{m_0}$, which is handled as follows

$$\begin{aligned}
&\frac{(2p-1)}{2}\int_{\mathcal{Q}_r} \langle a, D_{m_0} v^2\rangle \psi^2 \; + 2p\int_{\mathcal{Q}_r} \langle a v^2, D_{m_0}\psi\rangle\psi \\
&= -\frac{2p-1}{2}\int_{\mathcal{Q}_r} \mathrm{div}\cdot a\, v^2\psi^2 - (2p-1)\int_{\mathcal{Q}_r} \langle a, D_{m_0}\psi\rangle\psi v^2 + 2p\int_{\mathcal{Q}_r}\langle a v^2, D_{m_0}\psi\rangle\psi \\
&\le |2p-1|\int_{\mathcal{Q}_r} |a||\psi||D_{m_0}\psi|v^2 + 2|p|\int_{\mathcal{Q}_r}|a||D_{m_0}\psi||\psi|v^2 \\
&\le \frac{C|2p-1|}{(r-\rho)} \parallel a \parallel_{L^q(\mathcal{Q}_r)} \parallel v \parallel^2_{L^{2\beta}(\mathcal{Q}_r)} + \frac{\lambda|2p|\, c_1}{(r-\rho)} \parallel a \parallel_{L^q(\mathcal{Q}_r)} \parallel v \parallel^2_{L^{2\beta}(\mathcal{Q}_r)},
\end{aligned}$$

where $\beta = \frac{q}{q-1}$, and q is the integrability exponent introduced in **(H3B)**. From this point, we obtain the Caccioppoli inequality reasoning as in the proof of [4, Theorem 3.4].

As far as the Sobolev inequality is concerned, we find two extra terms in the representation formula of sub-solutions. More precisely, following the notation of [4, Theorem 3.3], the term $I_0(z)$ here becomes

$$I_0(z) = \int_{Q_r} \left[-\langle a, D(\psi\Gamma(z,\cdot))\rangle v \right](\zeta) d\zeta + \int_{Q_r} \left[\langle b, Dv\rangle \Gamma(z,\cdot)\psi \right](\zeta) d\zeta + \int_{Q_r} [cv\Gamma(z,\cdot)\psi](\zeta) d\zeta.$$

Since

$$\langle a, Dv\rangle \in L^{2\frac{q}{q+2}} \quad \text{for } a \in L^q, \quad q > \frac{Q+2}{2} \quad \text{and } v \in L^2,$$

reasoning as in [4, Theorem 3.3] we infer

$$\begin{aligned} \| I_0(\zeta) \|_{L^{2\alpha}(Q_\rho)} &\leq \| \Gamma * \left(\langle a, D_{m_0} v\rangle \psi\right) + \Gamma * \left(\langle b, D_{m_0} v\rangle \psi\right) + \Gamma * (cv\psi) \|_{L^{2\alpha}(Q_\rho)} \\ &\leq C \cdot (\| a \|_{L^q(Q_\rho)} + \| b \|_{L^q(Q_\rho)} \| D_{m_0} v \|_{L^2(Q_\rho)} \\ &\quad + \| c \|_{L^q(Q_\rho)} \| v \|_{L^2(Q_\rho)})), \end{aligned}$$

where

$$\alpha = \frac{q(Q+2)}{q(Q-2) + 2(Q+2)}.$$

In addition, the term $I_3(z)$ here becomes

$$\begin{aligned} I_3(z) = &\int_{Q_r} \left[\langle ADv, D(\Gamma(z,\cdot)\psi)\rangle \right](\zeta) d\zeta - \int_{Q_r} [(\Gamma(z,\cdot)\psi) Yv](\zeta) d\zeta \\ &+ \int_{Q_r} \left[\langle a, D(\Gamma(z,\cdot)\psi)\rangle v \right](\zeta) d\zeta - \int_{Q_r} \left[\langle b, Dv\rangle \Gamma(z,\cdot)\psi \right](\zeta) d\zeta \\ &- \int_{Q_r} [cv\Gamma(z,\cdot)\psi](\zeta) d\zeta \end{aligned}$$

and can be treated exactly as the analogous one in [4]. The rest of the proof of the Sobolev inequality follows the one contained in [4, Theorem 3.3].

Lastly, when considering our case, the proof of inequality (3.4) in [24, Theorem 3.3] needs to be treated slightly differently. In particular, inequality (3.2) in [24] becomes

$$\begin{aligned} &\int_{\mathbb{R}^N} u^2 \gamma_R^2 e^{2h} u^2 \|_{t=\tau} dx - 2 \int\int_{[\tau,\eta]\times\mathbb{R}^N} e^{2h} u^2 \gamma_R^2 (3\langle AD_{m_0} h, D_{m_0} h\rangle - Yh \\ &\quad - \mathrm{Tr}B + \langle a, D_{m_0} h\rangle + \langle b, D_{m_0} h\rangle) dxdt \\ &\leq \int_{\mathbb{R}^N} u^2 \gamma_R^2 e^{2h} u^2 \|_{t=\eta} dx + 2 \int\int_{[\tau,\eta]\times\mathbb{R}^N} e^{2h} u^2 (3\Lambda |D_{m_0} \gamma_R| + |Y\gamma_R|^2 \\ &\quad - \mathrm{Tr}B + \langle a, D_{m_0} \gamma_R\rangle \gamma_R + \langle b, D_{m_0} \gamma_R\rangle \gamma_R) dxdt. \end{aligned}$$

Thus, inequality (3.4) in [24] can be rewritten as

$$3\langle AD_{m_0}h, D_{m_0}h\rangle - Yh - \mathrm{Tr}B + \langle a, D_{m_0}h\rangle + \langle b, D_{m_0}h\rangle \le 0, \tag{37}$$

where $\eta - \frac{\eta-s}{k} \le t \le \eta, x \in \mathbb{R}^N$, $h(x,t) = -\frac{|x|^2}{\delta} + \alpha(\eta - t)$, with $\delta = 2(\eta - s) - k(\eta - t)$. Hence, inequality (37) becomes

$$\begin{aligned}
&3\langle AD_{m_0}h, D_{m_0}h\rangle - Yh - \mathrm{Tr}B + \langle a, D_{m_0}h\rangle + \langle b, D_{m_0}h\rangle \\
&\le \frac{12\Lambda|x|^2}{\delta^2} + \frac{2\|B\|\,|x|^2}{\delta} - \frac{k|x|^2}{\delta^2} - \alpha - \mathrm{Tr}B + \frac{2}{\delta}\left(\|a\|_2^2 + \|b\|_2^2\right) + \frac{4|x|^2}{\delta} \\
&\le \frac{|x|^2}{\delta^2}(12\Lambda + 2\delta\|B\| - k + 4\delta) - \alpha - \mathrm{Tr}B + \frac{2}{\delta}\left(\|a\|_2^2 + \|b\|_2^2\right) \\
&\le \frac{|x|^2}{\delta^2}(12\Lambda + 4\|B\| - k + 8) - \alpha - \mathrm{Tr}B + \frac{2}{\delta}\left(\|a\|_2^2 + \|b\|_2^2\right)
\end{aligned}$$

and therefore inequality (37) holds true provided that we choose $\alpha = \frac{2}{\delta}\left(\|a\|_2^2 + \|b\|_2^2\right) - \mathrm{Tr}B$ and k big enough. The rest of the proof follows the same lines of the one of [25, Theorem 4.1]. □

Lemma 3.7 *Let $\mathscr{L}$ be an operator of the form* (1) *satisfying assumptions* **(H1)**–**(H2)**–**(H3B)**. *Then there exist two positive constants R and c_2, only dependent on Q and B, such that*

$$\int_{|\delta^0_{(\sqrt{t-t_0})}(y - e^{(t-t_0)B}x)| \le R} \Gamma(x, t; y, t_0)dx \ge c_2, \tag{38}$$

for any $0 < t - t_0 \le 1$ *and* $y \in \mathbb{R}^N$.

Proof We first notice that for a small enough constant c_3, which depends only on Q and B, the function

$$v(y, t_0) := \int_{\mathbb{R}^N} \Gamma(x, t; y, t_0) - e^{-c_3(t-t_0)}, \quad t > t_0, \quad y \in \mathbb{R}^N,$$

is a weak super-solution of the Cauchy problem

$$\begin{cases} \mathscr{L}^* v(y, t_0) = -e^{-c(t-t_0)}(c - \mathrm{Tr}(B) + c_3) \le 0, & t > t_0, \quad y \in \mathbb{R}^N, \\ v(y, t) = 0, & y \in \mathbb{R}^N, \end{cases}$$

where $\mathscr{L}^*$ is the adjoint operator defined in (3). Hence, in virtue of the maximum principle we infer $v \ge 0$, that is

$$\int_{\mathbb{R}^N} \Gamma(x, t; y, t_0) \ge e^{-c_3(t-t_0)}, \quad t > t_0, \quad y \in \mathbb{R}^N. \tag{39}$$

We now observe that

$$
\begin{aligned}
&\int_{|\delta^0_{(\sqrt{t-t_0})}(y-e^{(t-t_0)B}x)|\geq R} \Gamma(x,t;y,t_0)dx \\
&\leq \frac{c_1}{\left(t-t_0\right)^{\frac{Q}{2}}} \int_{|\delta^0_{(\sqrt{t-t_0})}(y-e^{(t-t_0)B}x)|\geq R} \exp\left(-\frac{1}{c_1}|\delta^0_{(t-t_0)^{-\frac{1}{2}}}\left(y-e^{(t-t_0)B}x\right)|^2\right)dx \\
&= c_1 \int_{|z|\geq R} \exp\left(-\frac{1}{c_1}|z|^2\right)dz,
\end{aligned}
\tag{40}
$$

where in the second line we have used the upper bound (36) and in the third line we have performed the change of variables $z = \delta^0_{(\sqrt{t-t_0})}\left(y - e^{(t-t_0)B}x\right)$. Combining (39) and (40) and choosing c_3 small enough we obtain the thesis. □

We are now in a position to state and prove the following result concerning the Gaussian lower bound of the fundamental solution.

Theorem 3.8 (Gaussian lower bound) *Let $\mathscr{L}$ be an operator of the form* (1) *satisfying assumptions* **(H1)–(H2)–(H3B)**. *Then there exists a positive constant c_4, only dependent on Q, λ, Λ and q, such that*

$$
\Gamma(x,t;y,t_0) \geq \frac{c_4}{\left(t-t_0\right)^{\frac{Q}{2}}} e^{-c_4\langle C^{-1}(t-t_0)\left(y-e^{(t-t_0)B}x\right), y-e^{(t-t_0)B}x\rangle}
\tag{41}
$$

for any $0 < t - t_0 \leq 1$ *and* $x, y \in \mathbb{R}^N$.

Proof We restrict ourselves to the case where $x = 0$, as the general statement can be obtained from the dilation and translation-invariance of the operator $\mathscr{L}$. Then, for every $y \in \mathbb{R}^N$ and $R > 0$, we set

$$
D_R := \left\{ \xi \in \mathbb{R}^N \,:\, |\delta^0_{\sqrt{\tau}}(y - e^{\tau B}\xi)| \leq R \right\}
$$

and we compute

$$
\begin{aligned}
\text{meas}(D_R) &= \int_{D_R} d\xi = R^Q \int_{|\delta^0_{\sqrt{\tau}}(y-e^{\tau B}\xi)|\leq 1} d\xi = R^Q \tau^{Q/2} \int_{|(y-e^{\tau B}\xi)|\leq 1} d\xi \\
&= R^Q \tau^{Q/2} \int_{|\xi|\leq 1} d\xi = R^Q \tau^{Q/2}\text{meas}(B_1(0)) = c_5 \tau^{Q/2},
\end{aligned}
$$

where the constant c_5 only depends on B and R. We also note that the function $\langle C^{-1}(t-t_0)\left(y-e^{(t-t_0)B}x\right), y-e^{(t-t_0)B}x\rangle$ is bounded by a constant M in D_R (see [26, Lemma 3.3]). Lastly, we now set $\tau = \frac{t-t_0}{2}$ and apply to Γ the global Harnack inequality stated in Theorem 3.5, which yields

$$
\Gamma(y,t;y,t_0) \geq c_0 e^{-c_0\langle C^{-1}(\tau)\left(\xi-e^{\tau B}x\right), \xi-e^{\tau B}x\rangle} \Gamma(\xi, t+\tau; y, t_0), \quad y, \xi \in \mathbb{R}^N.
\tag{42}
$$

Hence, integrating inequality (42) over D_R, we infer

$$\begin{aligned}\Gamma(y,t;y,t_0) &= \frac{c_6}{\tau^{Q/2}}\int_{D_R}\Gamma(y,t;y,t_0)d\xi\\ &\geq \frac{c_6\,c_0}{\tau^{Q/2}}\int_{D_R} e^{-c_0\langle C^{-1}(\tau)(\xi-e^{\tau B}x),\xi-e^{\tau B}x\rangle}\Gamma(\xi,t+\tau;y,t_0)d\xi\\ &\geq \frac{c_6\,c_0}{\tau^{Q/2}}\int_{D_R} e^{-M}\Gamma(\xi,t+\tau;y,t_0)d\xi\\ &\geq \frac{c_7}{(t-t_0)^{Q/2}},\end{aligned}$$

where the last inequality is a direct consequence of Lemma 3.7 and the constant c_7 only depends on Q, λ, Λ, q and B.

Setting $\tau = \frac{3}{4}(t-t_0)$ and $x=0$, we apply once again Theorem 3.5 and we get

$$\begin{aligned}\Gamma(0,t;y,t_0) &\geq c_0 e^{-c_0\langle C^{-1}(\tau)y,y\rangle}\Gamma(y,t+\tau;y,t_0)\\ &\geq \frac{c_8}{(t-t_0)^{Q/2}}e^{-c_0\langle C^{-1}(\tau)y,y\rangle} \geq \frac{c_9}{(t-t_0)^{Q/2}}e^{-c_9\langle C^{-1}(t-t_0)y,y\rangle},\end{aligned}$$

where the last inequality is a consequence of a property of the covariance matrix C (see [25, Remark 4.5]). This concludes the proof. □

4 Proof of Theorem 1.3

This section is devoted to the proof of our existence result, Theorem 1.3, under assumptions **(H1)–(H2)–(H3A)**. Our idea is to adapt the limiting procedure proposed in [3] to the case of our interest.

Let us consider the operator $\mathscr{L}$ under the assumption **(H1)-(H3A)**. Our first aim is to build a sequence of operators $\left(\mathscr{L}_\varepsilon\right)_\varepsilon$ satisfying the assumption **(C)** of Theorem 2.7. Without loss of generality we restrict ourselves to the case of $(T_0,T_1)=(0,T)$, with $T>1$ and hence we denote $S_T := S_{0T}$. Thus, we may consider $\rho\in C_0^\infty(\mathbb{R})$ and $\psi\in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}}\rho(t)\,dt = 1, \quad \operatorname{supp}\rho\subset B\Big(\frac{T}{2},\frac{T}{4}\Big), \quad \text{and}$$
$$\int_{\mathbb{R}^N}\psi(x)\,dx = 1, \quad \operatorname{supp}\psi\subset B(0,1),$$

where by abuse of notation $B\Big(\frac{T}{2},\frac{T}{4}\Big)$ denotes the Euclidean ball on $\mathbb{R}$ of radius $\frac{T}{4}$ and center $\frac{T}{2}$ of suitable dimension and $B(0,1)$ denotes the Euclidean ball of $\mathbb{R}^N$ of radius

1 and center 0 of suitable dimension. Then, for every $\varepsilon \in (0,1]$ we classically construct two families of mollifiers

$$\rho_\varepsilon(t) = \frac{1}{\varepsilon}\,\rho\left(\frac{t-\frac{T}{2}}{\varepsilon}\right), \qquad \psi_\varepsilon(x) = \frac{1}{\varepsilon^N}\,\psi\left(\frac{x}{\varepsilon}\right).$$

Lastly, for every $\varepsilon \in (0,1]$, for every $t \in (0,T)$ and $x \in \mathbb{R}^N$ we define

$$\begin{aligned}
(a_{ij})_\varepsilon(x,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^N} a_{ij}(x-y,(1-\varepsilon)t+\tau)\psi_\varepsilon(y)\,\rho_\varepsilon(\tau)\,dy\,d\tau, \quad \forall i,j=1,\dots,N,\\
(b_i)_\varepsilon(x,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^N} b_i(x-y,(1-\varepsilon)t+\tau)\psi_\varepsilon(y)\,\rho_\varepsilon(\tau)\,dy\,d\tau, \qquad \forall i=1,\dots,N,\\
c_\varepsilon(x,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^N} c(x-y,(1-\varepsilon)t+\tau)\psi_\varepsilon(y)\,\rho_\varepsilon(\tau)\,dy\,d\tau.
\end{aligned}$$

These newly defined coefficients are smooth and such that $(a_{ij})_\varepsilon \to a_{ij}$, $(b_i)_\varepsilon \to b_i$, $(c)_\varepsilon \to c$ in $L^1(S_T)$. Hence, the L^1 convergence implies the pointwise convergence a.e. Moreover, for every $\varepsilon \in (0,1]$ the coefficients $(a_{ij})_\varepsilon$, $(b_i)_\varepsilon$ and $(c)_\varepsilon$ are bounded from above by the same constant appearing in assumption **(H3A)**. Indeed, for every $(x,t) \in K$, with $K \subset \mathbb{R}^{N+1}$ compact,

$$|(a_{ij})_\varepsilon(x,t)| \le \sup_{(x,t)\in K} |a_{ij}(x,t)| \le M,$$

for every $i,j = 1,\dots,N$. The same statement holds true for the coefficients c_ε and $(b_i)_\varepsilon$, with $i = 1,\dots,N$ and $\varepsilon \in (0,1]$.

In addition, given assumption **(H3A)** and our definition of the family of mollifiers we have

$$\begin{aligned}
\left|\frac{\partial}{\partial x_k}(b_i)_\varepsilon(x,t)\right| &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}^N} b_i(y,(1-\varepsilon)t+\tau)\frac{\partial\psi_\varepsilon}{\partial x_k}(x-y)\,\rho_\varepsilon(\tau)\,dy\,d\tau\right|\\
&\le M\int_{B(\frac{T}{2},\varepsilon)} |\rho_\varepsilon(\tau)|d\tau \int_{B(0,\varepsilon)} \left|\frac{\partial\psi_\varepsilon}{\partial x_k}(x-y)\right| dy \le MC_1
\end{aligned}$$

for every $i = 1,\dots,m_0$ and for every $k = 1,\dots,m_0$, where C_1 is a constant depending on ψ. Indeed, for every $y \in B(0,\varepsilon)$, we have

$$\left|\frac{\partial\psi_\varepsilon}{\partial x_k}(x-y)\right| \le \frac{1}{\varepsilon^{N+1}}\left\|\frac{\partial\psi}{\partial x_k}\right\|_{L^\infty(B(0,\varepsilon))} \le \frac{1}{\varepsilon^{N+1}}\left(\frac{2}{e}\right)^2 \sup_{[-\varepsilon,\varepsilon]}|2y| \le \frac{C_1}{\varepsilon^N},$$

where C_1 is a constant that does not dependent on ε.

The same statement holds true also for $\partial_{x_k}(a_{ij})_\varepsilon$ and $\partial_{x_k}c_\varepsilon$, with $k = 1,\dots,m_0$ and $\varepsilon \in (0,1]$. Hence, thanks to the mean value theorem along the direction of the vector fields ∂_{x_k}, the coefficients $(a_{ij})_\varepsilon$,c_ε and $(b_i)_\varepsilon$, with $i = 1, \dots, N$ and $\varepsilon \in (0,1]$, are uniformly Lipschitz continuous (i.e. Hölder continuous of exponent $\alpha = 1$), and therefore Dini continuous.

Hence, we can apply Theorem 2.5 to $(\Gamma_L^\varepsilon)_\varepsilon$ for every $\varepsilon \in (0,1]$. Thus, there exists a sequence of equibounded fundamental solutions $(\Gamma_L^\varepsilon)_\varepsilon$, in the sense that each of them satisfies Theorem 1.4, i.e. for every $(x,t),(\xi,\tau) \in S_T$, with $0 < \tau < t < T$

$$C^- \, \Gamma_K^{\lambda^-}(x,t;y,\tau) \;\le\; \Gamma_L^\varepsilon(x,t;y,\tau) \;\le\; C^+ \, \Gamma_K^{\lambda^+}(x,t;y,\tau).$$

We point out that, since the coefficients of $\mathscr{L}_\varepsilon$ are uniformly bounded by M, the coefficients of Theorem 1.3 do not depend on ε.

First of all, for every fixed $(\xi,\tau) \in S_T$ our aim is to show there exists a converging subsequence $\big(\Gamma_L^\varepsilon(\cdot,\cdot;\xi,\tau)\big)_\varepsilon$, from now on simply $(\Gamma_L^\varepsilon)_\varepsilon$, in every compact subset of $\big(\mathbb{R}^N \backslash \{\xi\}\big) \times (\tau,T)$.

For this reason, we define a sequence of open subsets $(\Omega_p)_{p\in\mathbb{N}}$ of S_T

$$\Omega_p \;:=\left\{ x \in \mathbb{R}^N \;:\; |x|^2 < p^2, |x-\xi|^2 > \frac{1}{2p} \right\} \times \left(\tau + \frac{1}{p}, T - \frac{1}{p} \right).$$

Note that $\Omega_p \subset\subset \Omega_{p+1}$ for every $p \in \mathbb{N}$. Moreover, $\cup_{p=1}^{+\infty} \Omega_p = \big(\mathbb{R}^N \backslash \{\xi\}\big) \times (\tau,T)$. Since $\Gamma_L^{\lambda^+}$ is a bounded function in Ω_p, we have that $(\Gamma_L^\varepsilon)_\varepsilon$ is an equibounded sequence in every Ω_p. Then, as the sequence $(\Gamma_L^\varepsilon)_\varepsilon$ is equibounded in Ω_2, it is equicontinuous in Ω_1 thanks to Theorem 2.5. Moreover, by Theorem 2.7 and Theorem 2.5, we also have that

$$\left(\frac{\partial \Gamma_L^\varepsilon}{\partial x}\right)_\varepsilon, \quad \left(\frac{\partial \Gamma_L^\varepsilon}{\partial \xi}\right)_\varepsilon, \quad \left(\frac{\partial^2 \Gamma_L^\varepsilon}{\partial x^2}\right)_\varepsilon, \quad \left(\frac{\partial^2 \Gamma_L^\varepsilon}{\partial \xi^2}\right)_\varepsilon, \quad \big(Y\Gamma_L^\varepsilon\big)_\varepsilon,$$

are bounded sequences in $C^0(\Omega_1)$, where Y is the Lie derivative defined in (30). Thus, there exists a subsequence $(\Gamma_L^{1,\varepsilon_1})_{\varepsilon_1}$ that converges uniformly to some function Γ_1 that satisfies (1.4) in Ω_1. Moreover, $\Gamma_1 \in C^2(\Omega_1)$ and the function $u(x,t) := \Gamma_1(x,t;\xi,\tau)$ is a.e. a classical solution to $\mathscr{L}u = 0$ in Ω_1, and hence a weak solution in the set Ω_1.

We next apply the same argument to the sequence $(\Gamma_L^{1,\varepsilon_1})_{\varepsilon_2}$ on the set Ω_2, and obtain a subsequence $(\Gamma_L^{2,\varepsilon_2})_{\varepsilon_2}$ that converges in $C^2(\Omega_2)$ to some function Γ_2, that belongs to $C^2(\Omega_2)$ and satisfies the bounds of Theorem 1.4 in Ω_2. Moreover, the function $u(x,t) := \Gamma_2(x,t;\xi,\tau)$ is a. e. a classical solution to $\mathscr{L}u = 0$ in the set Ω_2, and hence a weak solution, to $\mathscr{L}u = 0$ in the set Ω_2.

We next proceed by induction. Let us assume that the sequence $(\Gamma_L^{q-1,\varepsilon_{q-1}})_{\varepsilon_{q-1}}$ on the set Ω_q has been defined for some $q \in \mathbb{N}$. We extract from it a subsequence $(\Gamma_L^{q,\varepsilon_q})_{\varepsilon_q}$ converging in $C^2(\Omega_q)$ to some function Γ_q, satisfying Theorem 1.4 in Ω_q and it agrees with Γ_{q-1} on the set Ω_{q-1}.

Next, we define a function Γ_L in the following way: for every $\big(\mathbb{R}^N \backslash \{\xi\}\big) \times (\tau,T)$ we choose $q \in \mathbb{N}$ such that $(x,t) \in \Omega_q$ and we set $\Gamma_L(x,t;\xi,\tau) := \Gamma_q(x,t;\xi,\tau)$.

This argument can be repeatedly applied to any choice of $(\xi,\tau) \in S_T$. Hence, it provides us with a non ambiguous definition of Γ_L. Indeed, for any given choice of $(\xi,\tau) \in S_T$, if $(x,t) \in \Omega_p$, then $\Gamma_p(x,t;\xi,\tau) = \Gamma_q(x,t;\xi,\tau)$ for every choice of

$p, q \in \mathbb{N}$. In particular, we proved that Γ_L^ε converges compactly uniformly on S_T to a function Γ on a compactly generated space. Hence, $\Gamma(\cdot, \cdot; \xi, \tau)$ is continuous on $\mathbb{R}^N \times (\tau, T) \backslash \{(\xi, \tau)\}$ and a weak solution to $\mathscr{L}u = 0$ on $\mathbb{R}^N \times (\tau, T)$. Finally, Theorem 1.4 holds true for Γ_L because it is a weak solution to (1) in the sense of Definition 1.1.

Secondly, we verify that for any bounded function $\varphi \in C(\mathbb{R}^N)$ and any $x, \xi \in \mathbb{R}^N$ the function

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma_L(x,t;\xi,\tau)\, \varphi(\xi)\, d\xi \tag{43}$$

verifies the corresponding weak Cauchy problem in (7), hence it is a weak solution to (1) in $\mathbb{R}^N \times (\tau, T)$ and takes the initial datum when $t \to \tau$, with $t > \tau$. Note that u is well-defined given the Gaussian bounds of Theorem 1.4 and property 7. of Theorem 2.7. Then, considering that for every $\varepsilon \in (0, 1]$

$$u_\varepsilon(x,t) := \int_{\mathbb{R}^N} \Gamma_L^\varepsilon(x,t;\xi,t_0)\, \varphi(\xi)\, d\xi. \tag{44}$$

satisfies $\mathscr{L}_\varepsilon u_\varepsilon = 0$ in the classical sense, see Theorem 2.7, thanks to the Dominated Lebesgue convergence theorem we get u defined in (43) is a weak solution to (1) in $\mathbb{R}^N \times (\tau, T)$.

Thus, we are left with the proof of the limiting property. By applying property 3. of Theorem 2.7 to the regularized operator $\mathscr{L}_\varepsilon$, we have that for every $\varepsilon \in (0, 1]$ and for every $(\xi, \tau) \in \mathbb{R}^N \times (0, T)$ the following holds

$$\lim_{\substack{(x,t) \to (\xi,\tau) \\ t > \tau}} u_\varepsilon(x,t) = \varphi(\xi),$$

where u_ε is defined as above in (44). Now, thanks to Theorem 1.4 we are able to apply the Lebesgue Dominated Convergence Theorem, and thus for every $(\xi, \tau) \in \mathbb{R}^N \times (0, T)$ we have

$$\varphi(\xi) = \lim_{\varepsilon \to 0} \lim_{\substack{(x,t) \to (\xi,\tau) \\ t > \tau}} u_\varepsilon(x,t) = \lim_{\substack{(x,t) \to (\xi,\tau) \\ t > \tau}} u(x,t).$$

Finally, we are left with the proof the reproduction property listed in Theorem 1.3.

For every $\varepsilon > 0$, $x, \xi \in \mathbb{R}^N$ and $0 < \tau < s < t < T$ we get

$$\Gamma_L^\varepsilon(x,t;y,s)\Gamma_L^\varepsilon(y,s;\xi,\tau) \le C^+\, \Gamma_L^{\lambda^+}(x,t;y,s) C^+\, \Gamma_L^{\lambda^+}(y,s;\xi,\tau),$$

where the right-most inequality is obtained by applying the Gaussian upper bound in Theorem 1.4 to the fundamental solution Γ_ε. Hence, by applying the reproduction property of the fundamental solution $\Gamma_L^{\lambda^+}$ we get

$$\int_{\mathbb{R}^N} \Gamma_L^{\lambda^+}(x,t;y,s)\Gamma_L^{\lambda^+}(y,s;\xi,\tau)\, d\xi\, d\tau = \Gamma_L^{\lambda^+}(x,t;\xi,\tau),$$

which allows us to use the Lebesgue Dominated Convergence theorem. Thus, the property holds true.

We complete the proof by adapting these arguments to the adjoint operator $\mathscr{L}^*$ when considering the function v. □

Funding Open access funding provided by Università Politecnica delle Marche within the CRUI-CARE Agreement.

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