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Nonlinear degenerate elliptic equations with a convection term

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Abstract
We prove some existence and regularity results for the solutions of the Dirichlet problem associated with a nonlinear degenerate elliptic equation with a convection term.

Keywords Elliptic equations · $W^{1,1}$ – solutions · Degenerate equations · Convection term

Mathematics Subject Classification 35J62 · 35J70 · 35J75

1 Introduction

This paper deals with the homogeneous Dirichlet problem associated with the non-linear second order elliptic equation

$$-\mathrm{div}\left(\frac{M(x)\mathrm{D}u}{(1+|u|)^{\theta}} - E(x)\,\mathrm{sgn}(u)|u|^{\sigma}\right) = f \quad \text{in } \Omega \tag{1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $M : \Omega \to \mathbb{R}^{N^2}$ is a matrix with measurable entries such that

$$\alpha|\xi|^2 \le M(x)\xi \cdot \xi, \quad |M(x)| \le \beta, \quad \text{for a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N, \tag{2}$$

with $\alpha, \beta > 0$ and θ and σ are two real numbers such that

$$0 \le \theta < 1 \tag{3}$$

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and $\sigma > 0$. The terms E and f are a vector field and a function, respectively, satisfying suitable summability assumptions. Already in the case $E = 0$, the main difficulty in dealing with Eq. (1) is due to the fact that the nonlinear operator

$$v \to -\mathrm{div}\left(\frac{M(x)\mathrm{D}v}{(1+|v|)^\theta}\right)$$

though well defined from $W_0^{1,2}(\Omega)$ into its dual, is not coercive when v is large. Nevertheless, some existence and regularity results of the solutions to the Dirichlet problem

$$\begin{cases} -\mathrm{div}\left(\frac{M(x)\mathrm{D}u}{(1+|u|)^\theta}\right) = f \text{ in } \Omega \\ \qquad\qquad u = 0 \text{ on } \partial\Omega \end{cases} \tag{4}$$

depending on the summability of the source term f have been proved in [1, 3, 4, 6].

On the other hand, if $E \neq 0$, $\theta = 0$ and $\sigma = 1$ the study of the boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\mathrm{D}u - E(x)\,u) = f \text{ in } \Omega \\ \qquad\qquad u = 0 \text{ on } \partial\Omega \end{cases} \tag{5}$$

goes back to the paper [16] by Stampacchia, where the existence and regularity of weak solution have been studied provided

$$E \in \left[L^N(\Omega)\right]^N \tag{6}$$

and

$$\||E|\|_{L^N(\Omega)} \text{ sufficiently small.}$$

If no smallness assumption on $\||E|\|_{L^N(\Omega)}$ is assumed, the operator

$$v \to -\mathrm{div}(M(x)\mathrm{D}v - E(x)\,v)$$

fails to be coercive and the existence and regularity of the solutions of problem (5) have been studied in [2] (see also [9–15]) by approximating the linear problem (5) with a sequence of nonlinear coercive Dirichlet problems. Dealing with the Eq. (1) and in order to follow the strategy used in the linear nondegenerate framework, the main tool is the achievement of a logarithmic estimate on the gradient of suitable approximating solutions, which we can obtain, at least formally, as follows. Assume that we can choose

$$v = \int_0^u \frac{1}{(1+|s|)^{\theta+2\sigma}}\mathrm{d}s$$

as test function in the weak formulation of the homogeneous Dirichlet problem associated with Eq. (1).

Then, easy calculations lead to the estimate

$$\int_{\Omega} \frac{|\mathrm{D}u|^2}{(1+|u|)^{2(\theta+\sigma)}} \mathrm{d}x \le c(\alpha,\sigma,\theta)\big[\||E|\|_{L^2(\Omega)} + \|f\|_{L^1(\Omega)}\big]$$

and if $\theta + \sigma \le 1$ we obtain

$$\int_{\Omega} |\mathrm{D}\log(1+|u|)|^2 \mathrm{d}x \le c(\alpha,\sigma,\theta)\big[\||E|\|_{L^2(\Omega)} + \|f\|_{L^1(\Omega)}\big].$$

At last, by Sobolev imbedding, for any $k > 0$ the N-dimensional measure of the level set $\{x \in \Omega : \quad |u(x)| > k\}$ satisfies the inequality

$$|\{x \in \Omega : \quad |u(x)| > k\}| \le \frac{c}{[\log(1+k)]^{\frac{2N}{N-2}}}$$

with $c > 0$ independent of u, which, in turn, is the starting point for obtaining suitable a priori estimates. Consequently, the most interesting choice of the parameters θ and σ is $\sigma = 1 - \theta$ and, up to now, we will concern with the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{M(x)\mathrm{D}u}{(1+|u|)^{\theta}} - E(x)\, u|u|^{-\theta}\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Namely, we prove some existence and summability results of weak or distributional solutions, according to the summability of the right hand side.

The paper is organized as follows. In the next section we state the main theorems; in Sect. 3 we introduce a sequence of approximating problems and we prove a priori estimates. At last, Sect. 4 is devoted to the proofs of the main results.

2 Statements of the main results

The first theorem we state concerns with the existence of finite energy solutions of the problem (7).

Theorem 1 *Assume that hypotheses* (2), (3), (6) *hold and let* $f \in L^m(\Omega)$ *with*

$$\frac{2N}{N+2-\theta(N-2)} \le m < \frac{N}{2}. \tag{8}$$

Then there exists $u \in W_0^{1,2}(\Omega) \cap L^{(1-\theta)m^{**}}(\Omega)$ *which is a weak solution of the problem* (7), *that is*

$$\int_{\Omega} \frac{M(x)\mathrm{D}u}{(1+|u|)^{\theta}} \mathrm{D}\varphi \,\mathrm{d}x = \int_{\Omega} E\, u|u|^{-\theta} \mathrm{D}\varphi \,\mathrm{d}x + \int_{\Omega} f\varphi \,\mathrm{d}x, \tag{9}$$

for every $\varphi \in W_0^{1,2}(\Omega)$.

$\left(\text{As usually, if } 1 < p < N \text{ we set } p^* = \dfrac{pN}{N-p}\right)$.

When the summabilities of the source f and of the convection term E are higher then $\frac{N}{2}$ and N respectively, the weak solutions of the problem (7) are bounded according to the following

Theorem 2 *Assume* (2), (3) *hold and let* $|E| \in L^r(\Omega)$ *with* $r > N$ *and* $f \in L^m(\Omega)$ *with* $m > \frac{N}{2}$.

Let $u \in W_0^{1,2}(\Omega)$ *be a weak solution of the problem* (7).

Then

$$u \in L^\infty(\Omega).$$

We point out that the number $\frac{2N}{N+2-\theta(N-2)}$ is bigger than the duality exponent $\dfrac{2N}{N+2}$, but if $f \in L^m(\Omega)$ with $\frac{2N}{N+2} \leq m < \frac{2N}{N+2-\theta(N-2)}$, then we can't expect solutions with finite energy as the following example shows.

Example 1 Let $\Omega = B(0,1)$ be the unit ball of $\mathbb{R}^N$ centered at the origin and $0 < \theta < \dfrac{N+2}{2N}$.

The function $u(x) = \dfrac{1}{|x|^\rho} - 1$, with $\dfrac{N-2}{2} < \rho < \dfrac{N-2}{2(1-\theta)}$ solves the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\mathrm{D}u}{(1+|u|)^\theta} - E(x)\, u|u|^{-\theta}\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $E(x) = \dfrac{x}{|x|^\sigma}$ with $0 < \sigma < 2$ and

$$f(x) = \rho \frac{N-2-\rho(1-\theta)}{|x|^{\rho(1-\theta)+2}} + \left(\frac{1}{|x|^\rho} - 1\right)^{1-\theta} \left[N - \sigma - \frac{\rho(1-\theta)}{1-|x|^\rho}\right] \frac{1}{|x|^\sigma}.$$

We point out that, thanks to the choice of the parameters θ, σ and ρ,

$$|E| \in L^N(\Omega) \quad \text{and } f \in L^m(\Omega)$$

with $\dfrac{2N}{N+2} < m < \min\left\{\dfrac{N}{\rho(1-\theta)+2}, \dfrac{1}{\theta}, \dfrac{2N}{N+2-\theta(N-2)}\right\}$, but $\mathrm{D}u \notin L^2(\Omega)$.

However, if the summability of the datum is lower then $\dfrac{2N}{N+2-\theta(N-2)}$ we can prove the existence of distributional solutions of the problem (7), i.e. functions belonging to $W_0^{1,1}(\Omega)$ (infinite energy solutions) such that the following integral identity

$$\int_\Omega \frac{M(x)\mathrm{D}u}{(1+|u|)^\theta}\mathrm{D}\varphi\,\mathrm{d}x = \int_\Omega E\,u|u|^{-\theta}\,\mathrm{D}\varphi\,\mathrm{d}x + \int_\Omega f\varphi\,\mathrm{d}x \tag{10}$$

holds for every $\varphi \in C_0^\infty(\Omega)$.

Actually, if the summability of f is sufficiently larger than 1, we can prove the existence of distributional solutions of the problem (7) belonging to $W_0^{1,q}(\Omega)$, with suitable $1 < q < 2$, as the following theorem states:

Theorem 3 *Assume that hypotheses (2), (3), (6) hold and let $f \in L^m(\Omega)$ with*

$$\max\left\{1, \frac{N}{N+1-\theta(N-1)}\right\} < m < \frac{2N}{N+2-\theta(N-2)}. \tag{11}$$

Then there exists $u \in W_0^{1,q}(\Omega) \cap L^{(1-\theta)m^{**}}(\Omega)$, with $q = \dfrac{(1-\theta)mN}{N-m(1+\theta)}$, which is a distributional solution to the problem (7).

Remark 1 The results stated above coincide with the classical regularity results for uniformly elliptic equations if $\theta = 0$ (see [2, 16]), and with those obtained for degenerate elliptic equation without convection term if $0 < \theta < 1$ (see [1, 3, 6]).

Now, we focus our attention on the existence of distributional solutions when the right–hand side has the borderline summability

$$f \in L^m(\Omega) \text{ with } m = \max\left\{1, \frac{N}{N+1-\theta(N-1)}\right\},$$

that is

$$m = \begin{cases} 1 & \text{if } 0 \le \theta \le \frac{1}{N-1} \\ \frac{N}{N+1-\theta(N-1)} & \text{if } \frac{1}{N-1} < \theta < 1. \end{cases}$$

Namely, we will prove the following theorems.

Theorem 4 *Assume that hypotheses (2), (6) hold and let $0 \le \theta < \frac{1}{N-1}$ and $f \in L^1(\Omega)$*
.

Then there exists $u \in W_0^{1,q}(\Omega)$, with $1 \le q < \dfrac{(1-\theta)N}{N-(1+\theta)}$, which is a distributional solution of the problem (7).

Theorem 5 *Assume that hypotheses (2), (6) hold and let $\frac{1}{N-1} < \theta < 1$ and*

$$f \in L^m(\Omega) \text{ with } \quad m = \frac{N}{N+1-\theta(N-1)}.$$

Then there exists $u \in W_0^{1,1}(\Omega)$ *which is a distributional solution of the problem* (7).

Theorem 6 *Assume that hypotheses* (2), (6) *hold and let* $\theta = \frac{1}{N-1}$ *and*

$$f\ \log(1+|f|) \in L^1(\Omega).$$

Then there exists $u \in W_0^{1,1}(\Omega)$ *which is a distributional solution of the problem* (7).

Remark 2 If $E = 0$ the existence of $W_0^{1,1}$-solutions was first studied in [5] in the case $\theta = 0$ and in [4] in the degenerate framework (see also [7, 8]). Our results retrieve the results already known in these cases.

3 A priori estimates

In this section we follow the approach used in [2, 6] (see also [1, 3]).

As usual, for any $k > 0$ we denote by $T_k(s)$ the standard truncation function defined by

$$T_k(s) = \max\{-k, \min\{k, s\}\}, \quad \forall s \in \mathbb{R}$$

and we set

$$G_k(s) = s - T_k(s), \quad \forall s \in \mathbb{R}.$$

For every $n \in \mathbb{N}$ and for a.e. $x \in \Omega$, let us introduce the bounded functions

$$f_n(x) = \frac{f(x)}{1+\frac{1}{n}|f(x)|} \qquad \text{and} \qquad E_n(x) = \frac{E(x)}{1+\frac{1}{n}|E(x)|}$$

and let us consider the following approximating problems:

$$\begin{cases} -\operatorname{div}\left(\dfrac{M(x)\mathrm{D}u_n}{(1+|u_n|)^\theta}\right) \\ = -\operatorname{div}\left(E_n(x)\,\dfrac{u_n}{\frac{1}{n}+|u_n|}\,\dfrac{|u_n|^{1-\theta}}{1+\frac{1}{n}|u_n|^{1-\theta}}\right)+f_n(x) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Thanks to the Schauder's Theorem and using the existence and boundedness results of [6], for every $n \in \mathbb{N}$, there exists $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ which is a weak solution to the problem (12).

For every $n \in \mathbb{N}$ and $k > 0$ we denote by

$$A_n(k) = \big\{x \in \Omega \ : \ |u_n(x)| > k\big\}.$$

the level set of u_n and by $|A_n(k)|$ its $N-$ dimensional measure. Next Lemma shows that, under minimal assumptions on the summabilities of E and f, the $N-$ dimensional measure of the set $A_n(k)$ is small enough as k goes to infinity, uniformly with respect to n.

Lemma 1 *Assume that hypotheses (2), (3) hold and let $|E| \in L^2(\Omega)$ and $f \in L^1(\Omega)$. Let $u_n \in W_0^{1,2}(\Omega)$ be a weak solution of the problem (12). Then, for any $\varepsilon > 0$ there exists k_ε such that*

$$\big|A_n(k)\big|^{\frac{2}{2^*}} \le \varepsilon, \quad \text{for all } k > k_\varepsilon \tag{13}$$

uniformly with respect to n.

Proof We take $v = \left[1 - \dfrac{1}{(1+|u_n|)^{1-\theta}}\right]\mathrm{sgn}(u_n)$ as test function in (12), we use the assumption (2) and Young's inequality and we get

$$\begin{aligned}\int_\Omega \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^2}\,\mathrm{d}x &\le c\left\{\int_\Omega |E|\,\frac{|\mathrm{D}u_n|}{1+|u_n|}\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x\right\}\\ &\le c\left\{\tau\int_\Omega \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^2}\,\mathrm{d}x + C(\tau)\int_\Omega |E|^2\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x\right\}\end{aligned} \tag{14}$$

with $\tau > 0$ to be choosen later. Here and in the sequel we will denote by c various positive constants, whose values may depend on $\alpha, N, \mathrm{meas}(\Omega), \theta, \||E|\|_{L^N(\Omega)}$ and on the $L^m(\Omega)$ norm of the source f.

Choosing a suitable $0 < \tau < 1$, from the above relation it follows

$$\int_\Omega \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^2}\,\mathrm{d}x \le c\left\{\int_\Omega |E|^2\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x\right\}$$

and using Sobolev's inequality in the left-hand side we get the following estimate

$$\left(\int_\Omega |\log(1+|u_n|)|^{2^*}\,\mathrm{d}x \right)^{\frac{2}{2^*}} \leq \mathcal{S}^2 c \left\{ \int_\Omega |E|^2\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x \right\}. \tag{15}$$

Therefore,

$$\left| A_n(k) \right|^{\frac{2}{2^*}} \leq \frac{\mathcal{S}^2 c}{\log^2(1+k)} \left\{ \int_\Omega |E|^2\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x \right\} \tag{16}$$

and, consequently, the Lemma follows. □

Lemma 2 *Assume that hypotheses* (2), (3) *hold and let* $|E| \in L^2(\Omega)$ *and* $f \in L^1(\Omega)$. *Let* $u_n \in W_0^{1,2}(\Omega)$ *be a weak solution of the problem* (12). *Then, there exists a positive constant* c, *only depending on* α *such that*

$$\int_\Omega |DT_k(u_n)|^2\,\mathrm{d}x \leq c(1+k)^2 \left[\int_\Omega |E|^2\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x \right] \tag{17}$$

for any $k > 0$.

Proof Let $k > 0$. We take $v = T_k(u_n)$ as test function in (12). By using Young's inequality we have

$$\begin{aligned} \alpha \int_{\{|u_n|\leq k\}} \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^\theta}\,\mathrm{d}x &\leq k^{1-\theta} \int_{\{|u_n|\leq k\}} |E|\,|\mathrm{D}u_n|\,\mathrm{d}x + k\int_\Omega |f|\,\mathrm{d}x \\ &\leq \left\{ \frac{1}{2\alpha} k^{2(1-\theta)}(1+k)^\theta \int_\Omega |E|^2 + \frac{\alpha}{2} \int_{\{|u_n|\leq k\}} \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^\theta} dx + k \int_\Omega |f|\,\mathrm{d}x \right\}. \end{aligned} \tag{18}$$

Easy calculations lead to

$$\frac{\alpha}{2} \int_\Omega |\mathrm{D}T_k(u_n)|^2\,\mathrm{d}x \leq (1+k)^\theta \left\{ \frac{1}{2\alpha} k^{2(1-\theta)}(1+k)^\theta \int_\Omega |E|^2 + k\int_\Omega |f|\,\mathrm{d}x \right\}$$

which implies estimate (17). □

Now, we state some a priori estimates on u_n depending on the summability of the source term f.

Lemma 3 *Assume that hypotheses (2), (3), (6) hold and let $f \in L^m(\Omega)$ with*

$$m \geq \frac{2N}{N+2-\theta(N-2)}. \tag{19}$$

Then the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$.

Proof In the following, as usual, we set $[s]_+ = \max\{0, s\}, \forall s \in \mathbb{R}$. We choose $v = \left[|u_n|^{1+\theta} - k^{1+\theta}\right]_+ \operatorname{sgn}(u_n)$ as test function in (12). By using (2) and taking into account the estimate

$$\frac{|u_n(x)|^\theta}{(1+|u_n(x)|)^\theta} \geq \left(\frac{k}{1+k}\right)^\theta \qquad \text{for every } x \in A_n(k)$$

we have

$$\begin{aligned}
&\left(\frac{k}{1+k}\right)^\theta \int_{A_n(k)} |\mathrm{D}u_n|^2 \,\mathrm{d}x \\
&\leq c\left\{\int_{A_n(k)} |E|\,|u_n|\,|\mathrm{D}u_n| \,\mathrm{d}x + \int_{A_n(k)} |f|\,|u_n|^{1+\theta} \,\mathrm{d}x + k^{1+\theta}\int_\Omega |f| \,\mathrm{d}x\right\}.
\end{aligned}$$

Let $\tau > 0$. Noting that $|u_n| \leq |G_k(u_n)| + k$ and applying Hölder's and Young's inequalities from the previous estimate we deduce

$$\begin{aligned}
&\left(\frac{k}{1+k}\right)^\theta \int_{A_n(k)} |\mathrm{D}u_n|^2 \,\mathrm{d}x \\
&\leq c\left\{\left(\int_\Omega |G_k(u_n)|^{2^*} \,\mathrm{d}x\right)^{\frac{1}{2^*}}\left(\int_{A_n(k)} |E|^N \,\mathrm{d}x\right)^{\frac{1}{N}}\left(\int_{A_n(k)} |\mathrm{D}u_n|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\right. \\
&\quad +\tau \int_{A_n(k)} |\mathrm{D}u_n|^2 \,\mathrm{d}x + C(\tau)k^2 \int_\Omega |E|^2 \,\mathrm{d}x \\
&\quad \left. +\left(\int_\Omega |f|^m \,\mathrm{d}x\right)^{\frac{1}{m}}\left(\int_\Omega |G_k(u_n)|^{2^*} \,\mathrm{d}x\right)^{\frac{1+\theta}{2^*}} + k^{1+\theta}\int_\Omega |f| \,\mathrm{d}x\right\}
\end{aligned} \tag{20}$$

Thanks to the hypotheses $|E| \in L^N(\Omega)$ and (19), and using Sobolev's and Young's inequalities, we derive

$$\left(\left(\frac{k}{1+k}\right)^{\theta} - c\mathcal{S}\left(\int_{A_n(k)} |E|^N \,\mathrm{d}x\right)^{\frac{1}{N}} - 2c\tau\right)\int_{\Omega} |\mathrm{D}G_k(u_n)|^2 \,\mathrm{d}x$$
$$\le c(1+k)^2\left\{\int_{\Omega} |E|^2 \,\mathrm{d}x + \left(\int_{\Omega} |f|^m \,\mathrm{d}x\right)^{\frac{1-\theta}{2m}} + \int_{\Omega} |f| \,\mathrm{d}x\right\} \tag{21}$$

Therefore, thanks to Lemma 1 we can choose suitable $\tau < 1$ and $k > k^*$ such that

$$\left(\frac{k}{1+k}\right)^{\theta} - c\mathcal{S}\left(\int_{A_n(k)} |E|^N \,\mathrm{d}x\right)^{\frac{1}{N}} - 2c\tau > \frac{1}{2}$$

hence

$$\int_{\Omega} |\mathrm{D}G_k(u_n)|^2 \,\mathrm{d}x \le c(1+k)^2\left\{\int_{\Omega} |E|^2 \,\mathrm{d}x + \left(\int_{\Omega} |f|^m \,\mathrm{d}x\right)^{\frac{1-\theta}{2m}} + \int_{\Omega} |f| \,\mathrm{d}x\right\}. \tag{22}$$

From (22) and (17), we get:

$$\int_{\Omega} |\mathrm{D}u_n|^2 \,\mathrm{d}x \le c(1+k)^2\left\{\int_{\Omega} |E|^2 \,\mathrm{d}x + \left(\int_{\Omega} |f|^m \,\mathrm{d}x\right)^{\frac{1-\theta}{2m}} + \int_{\Omega} |f| \,\mathrm{d}x\right\}, \tag{23}$$

and the Lemma follows. □

Lemma 4 *Assume that hypothesis* (2), (3), (6) *hold and let* $f \in L^m(\Omega)$ *with* $1 < m < \frac{N}{2}$. *Then the sequence* $\{u_n\}$ *is bounded in* $L^{m^{**}(1-\theta)}(\Omega)$.

Proof We take as test function in (12) $v = \left[(1+|u_n|)^{2\gamma-1+\theta} - 1\right]\mathrm{sgn}(u_n)$, with $2\gamma - 1 + \theta > 0$ and we give

$$\alpha(2\gamma - 1 + \theta)\int_{\Omega} (1+|u_n|)^{2\gamma-2}|\mathrm{D}u_n|^2 \,\mathrm{d}x$$
$$\le c\left\{\int_{\Omega} |E|(1+|u_n|)^{2\gamma-1}|\mathrm{D}u_n| \,\mathrm{d}x + \int_{\Omega} |f|\left[(1+|u_n|)^{2\gamma-1+\theta} - 1\right]\mathrm{d}x\right\}.$$

Let $\tau > 0$. Applying Young's and Hölder's inequalities in the right-hand side of the above inequality we obtain

$$
\begin{aligned}
&\int_\Omega (1+|u_n|)^{2\gamma-2}|\mathrm{D}u_n|^2\,\mathrm{d}x \\
&\le c\Bigg\{C(\tau)\Bigg(\int_{\{|u_n|>k\}}|E|^N\,\mathrm{d}x\Bigg)^{\frac{2}{N}}\Bigg(\int_\Omega (1+|u_n|)^{2^*\gamma}\,\mathrm{d}x\Bigg)^{\frac{2}{2^*}} \\
&\quad + C(\tau)(1+k)^{2\gamma}\int_\Omega |E|^2\,\mathrm{d}x + \tau\int_\Omega (1+|u_n|)^{2\gamma-2}|\mathrm{D}u_n|^2\,\mathrm{d}x \\
&\quad + \|f\|_{L^m(\Omega)}\Bigg(\int_\Omega (1+|u_n|)^{(2\gamma-1+\theta)m'}\Bigg)^{\frac{1}{m'}}\Bigg\}.
\end{aligned}
\tag{24}
$$

Choosing a suitable $0<\tau<1$, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^{2(1-\gamma)}}\,\mathrm{d}x \\
&\le c\Bigg\{\Bigg(\int_{\{|u_n|>k\}}|E|^N\,\mathrm{d}x\Bigg)^{\frac{2}{N}}\Bigg(\int_\Omega \Big[(1+|u_n|)^{\gamma}-1\Big]^{2^*}\,\mathrm{d}x+|\Omega|\Bigg)^{\frac{2}{2^*}} \\
&\quad +(1+k)^{2\gamma}\int_\Omega |E|^2\,\mathrm{d}x + \|f\|_{L^m(\Omega)}\Bigg(\int_\Omega (1+|u_n|)^{(2\gamma-1+\theta)m'}\Bigg)^{\frac{1}{m'}}\Bigg\}.
\end{aligned}
\tag{25}
$$

By using Sobolev's inequality in the left-hand side of (25) we get

$$
\begin{aligned}
&\Bigg[\frac{1}{\gamma^2\mathcal{S}^2}-c\Bigg(\int_{\{|u_n|>k\}}|E|^N\,\mathrm{d}x\Bigg)^{\frac{2}{N}}\Bigg]\Bigg(\int_\Omega \Big[(1+|u_n|)^{\gamma}-1\Big]^{2^*}\,\mathrm{d}x\Bigg)^{\frac{2}{2^*}} \\
&\le c\Bigg\{(1+k)^{2\gamma}\int_\Omega |E|^2\,\mathrm{d}x + \|\,|E|\,\|^2_{L^N(\Omega)}|\Omega|^{\frac{2}{2^*}} \\
&\quad +\|f\|_{L^m(\Omega)}\Bigg(\int_\Omega (1+|u_n|)^{(2\gamma-1+\theta)m'}\Bigg)^{\frac{1}{m'}}\Bigg\}.
\end{aligned}
\tag{26}
$$

Assumption $|E|\in L^N(\Omega)$ and Lemma 1 imply that there exists $\bar{k}>0$ such that, if $k\ge\bar{k}$, we have

$$\frac{1}{\gamma^2\mathcal{S}^2} - c\left(\int\limits_{\{|u_n|>k\}} |E|^N \,\mathrm{d}x\right)^{\frac{2}{N}} \geq \frac{1}{2}.$$

Therefore,

$$\begin{aligned}&\left(\int\limits_{\Omega} (1+|u_n|)^{2^*\gamma}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\quad \leq c\left\{(1+k)^{2\gamma}\int\limits_{\Omega} |E|^2\,\mathrm{d}x + \|f\|_m \left(\int\limits_{\Omega} (1+|u_n|)^{(2\gamma-1+\theta)m'}\right)^{\frac{1}{m'}} + 1\right\}.\end{aligned} \tag{27}$$

Now, we choose γ such that $2^*\gamma = (2\gamma - 1 + \theta)m'$, that is

$$\gamma = \frac{(1-\theta)m^{**}}{2^*}. \tag{28}$$

Note that $\frac{1}{m'} < \frac{2}{2^*}$, since $m < \frac{N}{2}$, hence, for $k > \bar{k}$, inequality (27) gives

$$\left(\int\limits_{\Omega} (1+|u_n|)^{(1-\theta)m^{**}}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leq c\left\{(1+k)^{2\gamma}\int\limits_{\Omega} |E|^2\,\mathrm{d}x + \|f\|_m + 1\right\} \tag{29}$$

and the boundedness of $\{u_n\}$ in $L^{(1-\theta)m^{**}}(\Omega)$ is achieved. At last, going back to (25) the following inequality holds too

$$\int\limits_{\Omega} \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^{2(1-\gamma)}}\,\mathrm{d}x \leq c \tag{30}$$

with γ defined as in (28). ☐

Lemma 5 *Assume that hypotheses (2), (3), (6) hold and let $f \in L^m(\Omega)$ with*

$$\max\left\{1, \frac{N}{N+1-\theta(N-1)}\right\} < m < \frac{2N}{N+2-\theta(N-2)}.$$

Then the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$, with $q = \frac{mN(1-\theta)}{N-m(1+\theta)}$.

Proof Let $q = \dfrac{(1-\theta)mN}{N-m(1+\theta)} < 2$ and γ as in (28). We write

$$\int\limits_{\Omega} |\mathrm{D}u_n|^q\,\mathrm{d}x = \int\limits_{\Omega} \frac{|\mathrm{D}u_n|^q}{(1+|u_n|)^{q(1-\gamma)}} \cdot (1+|u_n|)^{q(1-\gamma)}\,\mathrm{d}x. \tag{31}$$

The Hölder's inequality with exponent $\frac{2}{q}$ and $\frac{2}{2-q}$ together with the estimates (29) and (30) gives

$$\int_\Omega |\mathrm{D}u_n|^q \,\mathrm{d}x \leq \left(\int_\Omega \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^{2(1-\gamma)}}\,\mathrm{d}x\right)^{\frac{q}{2}}\left(\int_\Omega (1+|u_n|)^{\frac{2q}{2-q}(1-\gamma)}\,\mathrm{d}x\right)^{\frac{2-q}{2}} \leq c \tag{32}$$

since $\dfrac{2q}{2-q}(1-\gamma) = m^{**}(1-\theta)$.

This estimate implies that the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$. □

Lemma 6 *Assume that hypotheses* (2), (6) *hold and let* $0 \leq \theta < \frac{1}{N-1}$ *and* $f \in L^1(\Omega)$. *Then the sequence* $\{u_n\}$ *is bounded in* $W_0^{1,q}(\Omega)$, *for any* $1 \leq q < \frac{N(1-\theta)}{N-(1+\theta)}$

Proof We take as test function in (12) the following function

$$v = \frac{1-(1+|u_n|)^{1-2\gamma+\theta}}{2\gamma-1-\theta}\,\mathrm{sgn}(u_n)$$

where γ is a positive number that we will choose later in order to have $2\gamma - 1 - \theta > 0$. Using Young's and Sobolev's inequalities, we obtain

$$\begin{aligned}\frac{1}{(1-\gamma)^2S^2}\left(\int_\Omega [(1+|u_n|)^{1-\gamma}-1]^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} &\leq \int_\Omega \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^{2\gamma}}\,\mathrm{d}x\\ &\leq c\left\{\left(\int_{\{|u_n|>k\}} |E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}\left(\int_\Omega (1+|u_n|)^{2^*(1-\gamma)}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\right.\\ &\quad\left. +(1+k)^{2(1-\gamma)}\int_\Omega |E|^2\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x\right\}\end{aligned} \tag{33}$$

Assumption $|E| \in L^N(\Omega)$ and Lemma 1 imply that there exists $\tilde{k}$ such that, if $k \geq \tilde{k}$, we have

$$\frac{1}{(1-\gamma)^2S^2} - c\left(\int_{\{|u_n|>k\}} |E|^N\,\mathrm{d}x\right)^{\frac{2}{N}} \geq \frac{1}{2}.$$

Therefore,

$$\left(\int_\Omega (1+|u_n|)^{2^*(1-\gamma)}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \\ \le c\left\{(1+k)^{2(1-\gamma)}\int_\Omega |E|^2\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x + \left(\int_\Omega |E|^N\,\mathrm{d}x\right)^{\frac{2}{N}} + 1\right\}. \tag{34}$$

Let $1 \le q < \dfrac{(1-\theta)N}{N-(1+\theta)}$ and $\gamma = \frac{N(2-q)}{2(N-q)}$ (note that $q<2$ and $2\gamma-1-\theta>0$ thanks to the choice of q). We write

$$\int_\Omega |\mathrm{D}u_n|^q\,\mathrm{d}x = \int_\Omega \frac{|\mathrm{D}u_n|^q}{(1+|u_n|)^{q\gamma}}\cdot(1+|u_n|)^{q\gamma}\,\mathrm{d}x \tag{35}$$

The Hölder's inequality with exponent $\frac{2}{q}$ and $\frac{2}{2-q}$ together with the estimates (34) and (33) gives

$$\int_\Omega |\mathrm{D}u_n|^q\,\mathrm{d}x \\ \le \left(\int_\Omega \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^{2\gamma}}\,\mathrm{d}x\right)^{\frac{q}{2}}\left(\int_\Omega (1+|u_n|)^{\frac{2q}{2-q}\gamma}\,\mathrm{d}x\right)^{\frac{2-q}{2}} \le c \tag{36}$$

since $\dfrac{2q}{2-q}\gamma = 2^*(1-\gamma)$.

□

4 Proofs of the main results

Proof of Theorems 1 and 3.

By Lemma 4 the sequence $\{u_n\}$ is bounded in $L^{m^{**}(1-\theta)}(\Omega)$. Moreover, thanks to Lemmas 3 and 5 $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$, with $q=2$ or $q=\frac{mN(1-\theta)}{N-m(1+\theta)}$ depending on whether the hypotheses of Theorem 1 or 3, respectively, are verified. Then up to subsequences, $\{u_n\}$ converges weakly to some function u which belongs to the same spaces. Moreover, $\{u_n\}$ converges to u a.e. in Ω.

We fix $\varphi \in C_0^\infty(\Omega)$ and we take $v=\varphi$ in (12). We obtain

$$\int_\Omega \frac{M(x)\mathrm{D}u_n}{(1+|u_n|)^\theta}\mathrm{D}\varphi\,\mathrm{d}x = \int_\Omega E_n\frac{u_n}{\frac{1}{n}+|u_n|}\frac{|u_n|^{1-\theta}}{1+\frac{1}{n}|u_n|^{1-\theta}}\mathrm{D}\varphi\,\mathrm{d}x + \int_\Omega f_n\varphi\,\mathrm{d}x.$$

We define

$$Y_n(x) \equiv E_n(x)\, \frac{u_n(x)}{\frac{1}{n} + |u_n(x)|}\, \frac{|u_n(x)|^{1-\theta}}{1 + \frac{1}{n}|u_n(x)|^{1-\theta}}. \tag{37}$$

Thanks to the convergence of $\{u_n\}$ to u a.e. in Ω, the sequence $\{Y_n\}$ converges to $E(x)\, u(x)|u(x)|^{-\theta}$ a.e. $x \in \Omega$.

We note that

$$\left| \frac{u_n(x)}{\frac{1}{n} + |u_n(x)|}\, \frac{|u_n(x)|^{1-\theta}}{1 + \frac{1}{n}|u_n(x)|^{1-\theta}} \right| \le |u_n(x)|^{1-\theta} \quad \forall x \in \Omega$$

then, by using Hölder's inequality, for any measurable subset $A \subseteq \Omega$ we have

$$\int_A \left|Y_n(x)\right| \mathrm{d}x \le \| |E| \|_{L^N(\Omega)} \| |u_n|^{1-\theta} \|_{L^{m^{**}}(\Omega)}\, |A|^{1-1/m+1/N}. \tag{38}$$

At last, the boundedness of the sequence $\{u_n\}$ in $L^{m^{**}(1-\theta)}(\Omega)$ implies the equi-integrability of Y_n and therefore

$$Y_n \to E\, u|u|^{-\theta} \quad \text{in } L^1(\Omega).$$

Moreover

$$\frac{M(x)}{(1+|u_n|)^\theta} \to \frac{M(x)}{(1+|u|)^\theta} \qquad *\text{-weakly in } L^\infty(\Omega)\text{and a.e. in}\Omega$$

and, thanks to the boundedness of u_n in $W_0^{1,q}(\Omega)$

$$\mathrm{D}u_n \rightharpoonup \mathrm{D}u \quad \text{weakly in } L^q(\Omega, \mathbb{R}^N),$$

so that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \frac{M(x)\mathrm{D}u_n}{(1+|u_n|)^\theta} \mathrm{D}\varphi\, \mathrm{d}x \\ &\quad = \lim_{n\to\infty} \int_\Omega \frac{M(x)}{(1+|u_n|)^\theta} \left[\mathrm{D}u_n - \mathrm{D}u\right] \mathrm{D}\varphi\, \mathrm{d}x \\ &\qquad + \lim_{n\to\infty} \int_\Omega \frac{M(x)}{(1+|u_n|)^\theta} \mathrm{D}u\, \mathrm{D}\varphi\, \mathrm{d}x = \int_\Omega \frac{M(x)\mathrm{D}u}{(1+|u|)^\theta} \mathrm{D}\varphi\, \mathrm{d}x. \end{aligned}$$

Finally, the sequence $\{f_n\}$ converges to f in $L^m(\Omega)$, therefore passing to the limit as $n \to +\infty$ in (12), u satysfies the integral identity (10).

Then, if the assumptions of Theorem 3 hold we conclude that u is a distributional solution of (7) and the thesis follows.

Instead, under the assumptions of Theorem 1, by standard density arguments, we deduce that the integral identity (10) holds also for any function $\varphi \in W_0^{1,2}(\Omega)$, which means that u is a weak solution of (7).

Proof of Theorem 4 Thanks to Lemma 6 the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$, with $1 \le q < \frac{N(1-\theta)}{N-(1+\theta)}$. Then there exists a subsequence, not relabelled, which converges to some function u weakly in $W_0^{1,q}(\Omega)$, strongly in $L^p(\Omega)$, for any $1 \le p < q^*$ and a.e. in Ω. We note that, using Hölder's and Sobolev's inequalities, for any measurable subset $A \subseteq \Omega$ we have

$$\int_A \left|Y_n(x)\right| \mathrm{d}x \le \|E\|_{L^N(\Omega)} \| |Du_n| \|^{1-\theta}_{L^q(\Omega)} |A|^{1-\frac{1}{N}-\frac{1-\theta}{q^*}}$$

with Y_n defined by (37), and the boundedness of the sequence $\{Du_n\}$ in $L^q(\Omega)$ implies the equi-integrability of Y_n. Up to now, we can argue as in the previous proof and we get the thesis. □

Proof of Theorem 5 Here we follow the outlines of [4, 5].

Since $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$ up to a subsequence still denoted by $\{u_n\}$, it converges to some function u strongly in $L^r(\Omega)$, for any $1 \le r < \frac{N}{N-1}$ and a.e. in Ω.

Moreover, since $1 < m < \frac{N}{2}$ the assumptions of Lemma 4 are satisfied and estimate (30) holds with $\gamma = \frac{N}{2^*(N-1)}$ (note that now by the assumption on m it results $(1-\theta)m^{**} = \frac{N}{N-1}$). Then, the function

$$v_n = (1 + |u_n|)^\gamma - 1)\mathrm{sgn}(u_n)$$

is bounded in $W_0^{1,2}(\Omega)$. Thus, there exist a subsequence, not relabelled, and a function $v \in W_0^{1,2}(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{weakly in } W_0^{1,2}(\Omega)$$
$$v_n \to v \quad \text{strongly in } {}^{\checkmark 2}(\Omega) \text{and a.e. in} \Omega.$$

Thanks to the almost everywhere convergence of $\{u_n\}$ to u we deduce that $v = (1 + |u|)^\gamma - 1)\mathrm{sgn}(u)$.

Now, we will prove that up to subsequences, $\{u_n\}$ is weakly convergent to u in $W_0^{1,1}(\Omega)$. Given $k > 0$, let us take $[|u_n|^{2\gamma-1+\theta} - k^{2\gamma-1+\theta}]_+ \mathrm{sgn}(u_n)$ as test function in (12) (note that $2\gamma - 1 + \theta > 0$ since $\frac{1}{N-1} < \theta < 1$) and we deduce

$$\begin{aligned} &\alpha \int_{\{|u_n|>k\}} \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^\theta} |u_n|^{2(\gamma-1)+\theta} \,\mathrm{d}x \\ &\quad \le \int_{\{|u_n|>k\}} |E|\, |u_n|^\gamma |u_n|^{\gamma-1} |\mathrm{D}u_n| \,\mathrm{d}x + \int_{\{|u_n|>k\}} |f|\, |u_n|^{2\gamma-1+\theta} \,\mathrm{d}x. \end{aligned} \tag{39}$$

Note that in the set $\{|u_n| > k\}$ it results $\left(\frac{|u_n|}{1+|u_n|}\right)^\theta \ge \left(\frac{k}{1+k}\right)^\theta$ and using the Hölder inequality in the right–hand side of (39) we obtain

$$
\begin{aligned}
&\alpha\left(\frac{k}{1+k}\right)^{\theta}\int\limits_{\left\{|u_n|>k\right\}}|\mathrm{D}u_n|^2|u_n|^{2(\gamma-1)}\,\mathrm{d}x\\
&\quad\le\left(\int\limits_{\left\{|u_n|>k\right\}}|E|^N\,\mathrm{d}x\right)^{\frac{1}{N}}\left(\int\limits_{\left\{|u_n|>k\right\}}|u_n|^{2^*\gamma}\,\mathrm{d}x\right)^{\frac{1}{2^*}}\\
&\qquad\times\left(\int\limits_{\left\{|u_n|>k\right\}}|u_n|^{2(\gamma-1)}|\mathrm{D}u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\qquad+\left(\int\limits_{\left\{|u_n|>k\right\}}|f|\,\mathrm{d}x\right)^{\frac{1}{m}}\left(\int\limits_{\left\{|u_n|>k\right\}}|u_n|^{(2\gamma-1+\theta)m'}\,\mathrm{d}x\right)^{\frac{1}{m'}}.
\end{aligned}
\tag{40}
$$

Thanks to the choice of m and γ it results $2^*\gamma=(2\gamma-1+\theta)m'=\frac{N}{N-1}$ and using the boundedness of $\|u_n\|_{L^{\frac{N}{N-1}}}(\Omega)$ together with Young's inequality we obtain

$$
\begin{aligned}
&\alpha\left(\frac{k}{1+k}\right)^{\theta}\int\limits_{\left\{|u_n|>k\right\}}|\mathrm{D}u_n|^2|u_n|^{2(\gamma-1)}\,\mathrm{d}x\\
&\quad\le c\left[\frac{1}{2\alpha}\left(\int\limits_{\left\{|u_n|>k\right\}}|E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}+\left(\int\limits_{\left\{|u_n|>k\right\}}|f|\,\mathrm{d}x\right)^{\frac{1}{m}}\right]\\
&\qquad+\frac{\alpha}{2}\int\limits_{\left\{|u_n|>k\right\}}|u_n|^{2(\gamma-1)}|\mathrm{D}u_n|^2\,\mathrm{d}x
\end{aligned}
\tag{41}
$$

which, in turn implies

$$
\alpha\left[\left(\frac{k}{1+k}\right)^{\theta}-\frac{1}{2}\right]\int\limits_{\left\{|u_n|>k\right\}}|\mathrm{D}u_n)|^2|u_n|^{2(\gamma-1)}\,\mathrm{d}x
\le c\left[\frac{1}{2\alpha}\left(\int\limits_{\left\{|u_n|>k\right\}}|E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}+\left(\int\limits_{\left\{|u_n|>k\right\}}|f|\,\mathrm{d}x\right)^{\frac{1}{m}}\right]. \tag{42}
$$

For any $\epsilon>0$ let us choose $\bar{k}>0$ such that $\forall k>\bar{k}$ the following inequalities hold

$$
\left(\frac{k}{1+k}\right)^{\theta}-\frac{1}{2}>\frac{1}{4}
$$

and

$$
c\left[\frac{1}{2\alpha}\left(\int\limits_{\left\{|u_n|>k\right\}}|E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}+\left(\int\limits_{\left\{|u_n|>k\right\}}|f|\,\mathrm{d}x\right)^{\frac{1}{m}}\right]\le\frac{\epsilon}{4}.
$$

From (42) we get

$$
\alpha\int\limits_{\left\{|u_n|>k\right\}}\frac{|\mathrm{D}u_n|^2}{|u_n|^{2(1-\gamma)}}\,\mathrm{d}x\le\epsilon\qquad\forall k>\bar{k}
$$

and then

$$
\int\limits_{\left\{|u_n|>k\right\}}|\mathrm{D}u_n|\,\mathrm{d}x=\int\limits_{\left\{|u_n|>k\right\}}\frac{|\mathrm{D}u_n|}{|u_n|^{1-\gamma}}|u_n|^{1-\gamma}\,\mathrm{d}x
\le\left(\int\limits_{\left\{|u_n|>k\right\}}\frac{|\mathrm{D}u_n|^2}{|u_n|^{2(1-\gamma)}}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\Omega}|u_n|^{\frac{N}{N-1}}\,\mathrm{d}x\right)^{\frac{1}{2}}\le c\sqrt{\epsilon},\qquad\forall k>\bar{k} \tag{43}
$$

since $2(1-\gamma)=\frac{N}{N-1}$. Let $A\subseteq\Omega$ be a measurable set and $k>\bar{k}$. We have

$$\int_A |\mathrm{D}u_n|\,\mathrm{d}x \le \int_{\{|u_n|>k\}} |\mathrm{D}u_n| + \int_A |\mathrm{D}T_k(u_n)|\,\mathrm{d}x \le c\sqrt{\epsilon} + |A|^{\frac{1}{2}}\left(\int_\Omega |\mathrm{D}T_k(u_n)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \tag{44}$$

Using the estimate (17) we deduce that holds

$$\limsup_{|A|\to 0} \int_A |\mathrm{D}u_n|\,\mathrm{d}x \le c\sqrt{\epsilon}, \qquad \forall \epsilon > 0$$

uniformly with respect to n and, at last, we conclude that the sequence $\{\mathrm{D}u_n\}$ is equiintegrable. By the Dunford Pettis Theorem, for any $i = 1, \dots, N$ there exists $y_i \in L^1(\Omega)$ such that

$$\frac{\partial u_n}{\partial x_i} \rightharpoonup y_i \quad \text{weakly in } L^1(\Omega).$$

Moreover, since $\frac{\partial u_n}{\partial x_i}$ is the distributional derivative of u_n we have

$$\int_\Omega \frac{\partial u_n}{\partial x_i}\varphi\,\mathrm{d}x = -\int_\Omega u_n \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega)$$

and by virtue of the strong convergence on $\{u_n\}$ to u in $L^r(\Omega)$, with $1 \le r < \frac{N}{N-1}$ we can pass to the limit as $n \to +\infty$ in the previous equality, obtaining

$$\int_\Omega y_i\varphi\,\mathrm{d}x = -\int_\Omega u \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega)$$

that is, $y_i = \frac{\partial u}{\partial x_i}$, $\forall i = 1, \dots, N$ and thus $u \in W_0^{1,1}(\Omega)$.

Since $\gamma < 1$ the function $g(s) = (1 + |s|)^\gamma - 1)\mathrm{sgn}(s)$ is Lipschitz continuous and by the chain rule it results

$$\mathrm{D}v = \mathrm{D}((1 + |u|)^\gamma - 1)\mathrm{sgn}(u)) = \gamma \frac{\mathrm{D}u}{(1 + |u|)^{1-\gamma}}$$

almost everywhere in Ω. At last, using the weak convergence of $\{\mathrm{D}v_n\}$ to $\mathrm{D}v$ in $L^2(\Omega)$ we conclude that

$$\frac{\mathrm{D}u_n}{(1 + |u_n|)^{1-\gamma}} \rightharpoonup \frac{\mathrm{D}u}{(1 + |u|)^{1-\gamma}}, \quad \text{weakly in } L^2(\Omega). \tag{45}$$

Now, we can pass to the limit in the approximate problems (12). Let $\varphi \in C_0^\infty(\Omega)$.We note that

$$\int_\Omega M(x)\frac{\mathrm{D}u_n}{(1 + |u_n|)^\theta}\mathrm{D}\varphi\,\mathrm{d}x = \int_\Omega M(x)\frac{\mathrm{D}u_n}{(1 + |u_n|)^{1-\gamma}}(1 + |u_n|)^{1-\gamma-\theta}\mathrm{D}\varphi\,\mathrm{d}x$$

and the integral in the right–hand side converges to

$$\int_\Omega M(x)\frac{\mathrm{D}u}{(1+|u|)^\theta}\mathrm{D}\varphi\,\mathrm{d}x$$

since (45) holds and

$$(1+|u_n|)^{1-\gamma-\theta} \to (1+|u|)^{1-\gamma-\theta} \quad \text{strongly in } L^2(\Omega)$$

because we can write

$$(1+|u_n|)^{1-\gamma-\theta} = \frac{(1+|u_n|)^{1-\gamma-\sigma}}{(1+|u_n|)^{\theta-\sigma}} \le (1+|u_n|)^{1-\gamma-\sigma}$$

for any $0<\sigma<\theta$ and $\left\{(1+|u_n|)^{1-\gamma-\sigma}\right\}$ strongly converges in $L^2(\Omega)$ if σ satisfies $1 \le 2(1-\gamma-\sigma) < \frac{N}{N-1}$. Finally, as in the proof of Theorem 4 the function Y_n defined by (37) verifies

$$\int_A \left|Y_n(x)\right| \mathrm{d}x \le \|E\|_{L^N(\Omega)}\|Du_n\|^{1-\theta}_{L^1(\Omega)} \, |A|^{(1-\frac{1}{N})\theta}$$

or any measurable subset $A \subseteq \Omega$ and the boundedness of the sequence $\left\{Du_n\right\}$ in $L^1(\Omega)$ implies the equi-integrability of Y_n. Then, passing to the limit as $n \to +\infty$ in the approximating problems (12) we obtain that u is a distributional solution of the problem(7). □

Proof of Theorem 6 Let $k \ge 0$, we take

$$v = \left[\log(1+|u_n|) - \log(1+k)\right]_+ \mathrm{sgn}(u_n)$$

as test function in (12). Thanks to the assumption (2) and Young's inequality we obtain

$$\begin{aligned} &\frac{\alpha}{2}\int_{\{|u_n|\ge k\}} \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^{\theta+1}}\,\mathrm{d}x \\ &\quad\le \frac{1}{2\alpha}\int_{\{|u_n|\ge k\}} |E|^2\,(1+|u_n|)^{1-\theta}\,\mathrm{d}x + 2\int_{\{|u_n|\ge k\}} |f|\,\log(1+|u_n|)\,\mathrm{d}x \end{aligned} \tag{46}$$

We observe that the following inequality holds

$$t\log(1+s) \le \frac{t}{\rho}\log\left(1+\frac{t}{\rho}\right) + (1+s)^\rho$$

for all s, t positive real numbers and $0<\rho<\frac{N-2}{N-1}$. Now, using Hölder's inequality, from (46) for any $k \ge 0$ we deduce

$$\begin{aligned} &\frac{\alpha}{2}\int_{\{|u_n|\geq k\}} \frac{|\mathrm{D}u_n|^2}{(1+|u_n|)^{\theta+1}}\,\mathrm{d}x \\ &\quad\leq \frac{1}{2\alpha}\left(\int_{\{|u_n|\geq k\}} |E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}\left(\int_{\{|u_n|\geq k\}} (1+|u_n|)^{(1-\theta)\frac{2^*}{2}}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\qquad+2\int_{\{|u_n|\geq k\}} \frac{|f|}{\rho}\log\left(1+\frac{|f|}{\rho}\right)\mathrm{d}x+2\int_{\{|u_n|\geq k\}} (1+|u_n|)^{\rho}\,\mathrm{d}x \end{aligned} \tag{47}$$

In particular, for $k \geq 1$ we have

$$\begin{aligned} &\frac{\alpha}{2^{\theta+2}}\int_{\{|u_n|\geq k\}} \frac{|\mathrm{D}u_n|^2}{|u_n|^{\theta+1}}\,\mathrm{d}x \\ &\quad\leq \frac{2^{1-\theta}}{2\alpha}\left(\int_{\{|u_n|\geq k\}} |E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}\left(\int_{\{|u_n|\geq k\}} |u_n|^{(1-\theta)\frac{2^*}{2}}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\qquad+2\int_{\{|u_n|\geq k\}} \frac{|f|}{\rho}\log\left(1+\frac{|f|}{\rho}\right)\mathrm{d}x+2^{\rho+1}\int_{\{|u_n|\geq k\}} |u_n|^{\rho}\,\mathrm{d}x \end{aligned} \tag{48}$$

Taking into account the above relation and using Sobolev's inequality, for any $k \geq 1$ we have

$$\begin{aligned} &\left(\int_\Omega \left|\left[|u_n|^{\frac{1-\theta}{2}}-k^{\frac{1-\theta}{2}}\right]_+\right|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\quad\leq c\left\{\left(\int_{\{|u_n|\geq k\}} |E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}\left(\int_\Omega \left|\left[|u_n|^{\frac{1-\theta}{2}}-k^{\frac{1-\theta}{2}}\right]_+\right|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}\right. \\ &\qquad\left.+k^{1-\theta}\|E\|^2_{L^N(\Omega)}+\int_{\{|u_n|\geq k\}} \frac{|f|}{\rho}\log\left(1+\frac{|f|}{\rho}\right)\mathrm{d}x+\int_{\{|u_n|\geq k\}} |u_n|^{\rho}\,\mathrm{d}x\right\} \end{aligned} \tag{49}$$

hence

$$\left[1-c\left(\int_{\{|u_n|\geq k\}}|E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}\right]\left(\int_\Omega\left|\left[|u_n|^{\frac{1-\theta}{2}}-k^{\frac{1-\theta}{2}}\right]_+\right|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}$$
$$\leq c\left\{k^{1-\theta}+\int_{\{|u_n|\geq k\}}\frac{|f|}{\rho}\log\left(1+\frac{|f|}{\rho}\right)\mathrm{d}x+\int_{\{|u_n|\geq k\}}|u_n|^\rho\,\mathrm{d}x\right\} \tag{50}$$

As usual, by assumption $|E|\in L^N(\Omega)$ and Lemma 1 we can choose $\bar{k}\geq 1$ such that

$$1-c\left(\int_{\{|u_n|\geq \bar{k}\}}|E|^N\,\mathrm{d}x\right)^{\frac{2}{N}}>\frac{1}{2}$$

therefore

$$\left(\int_\Omega\left|\left[|u_n|^{\frac{1-\theta}{2}}-\bar{k}^{\frac{1-\theta}{2}}\right]_+\right|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}$$
$$\leq c\left\{\bar{k}^{1-\theta}+\int_{\{|u_n|\geq \bar{k}\}}\frac{|f|}{\rho}\log\left(1+\frac{|f|}{\rho}\right)\mathrm{d}x+\int_{\{|u_n|\geq \bar{k}\}}|u_n|^\rho\,\mathrm{d}x\right\} \tag{51}$$

By using Hölder's inequality with exponent $\frac{(1-\theta)2^*}{2\rho}$ on the last term of the right hand side, we get

$$\left(\int_\Omega|u_n|^{\frac{(1-\theta)2^*}{2}}\,\mathrm{d}x\right)^{\frac{1}{2^*}}\leq c\left\{\left(\int_{\{|u_n|\geq \bar{k}\}}\frac{|f|}{\rho}\log\left(1+\frac{|f|}{\rho}\right)\mathrm{d}x\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\int_\Omega|u_n|^{\frac{(1-\theta)2^*}{2}}\,\mathrm{d}x\right)^{\frac{\rho}{(1-\theta)2^*}}+\bar{k}^{\frac{1-\theta}{2}}\right\} \tag{52}$$

By the choice of ρ

the above inequality implies that

$$\int_\Omega|u_n|^{\frac{(1-\theta)2^*}{2}}\,\mathrm{d}x\leq c \tag{53}$$

Now, we can argue as in [4] (see also [5]) and we deduce that $u_n \rightharpoonup u$ weakly in $W_0^{1,1}(\Omega)$ and

$$\frac{\mathrm{D}u_n}{(1+|u_n|)^{\frac{\theta+1}{2}}} \rightharpoonup \frac{\mathrm{D}u}{(1+|u|)^{\frac{\theta+1}{2}}} \quad \text{weakly in } [L^2(\Omega)]^N. \tag{54}$$

Let $\varphi \in C_0^\infty(\Omega)$. We note that

$$\int_\Omega M(x)\frac{\mathrm{D}u_n}{(1+|u_n|)^\theta}\mathrm{D}\varphi\,\mathrm{d}x = \int_\Omega M(x)\frac{\mathrm{D}u_n}{(1+|u_n|)^{\frac{1+\theta}{2}}}\,(1+|u_n|)^{\frac{1-\theta}{2}}\mathrm{D}\varphi\,\mathrm{d}x$$

and the integral in the right–hand side converges to

$$\int_\Omega M(x)\frac{\mathrm{D}u}{(1+|u|)^\theta}\mathrm{D}\varphi\,\mathrm{d}x$$

since (54) holds and

$$M(x)(1+|u_n|)^{\frac{1-\theta}{2}}\mathrm{D}\varphi \to M(x)(1+|u|)^{\frac{1-\theta}{2}}\mathrm{D}\varphi \quad \text{strongly in } [L^2(\Omega)]^N$$

because $1-\theta < \frac{N}{N-1}$ and $|M(x)\mathrm{D}\varphi|$ is bounded.

Due to (53), inequality (38) is also valid for $m=1$ therefore we deduce

$$Y_n \to E\,u|u|^{1-\theta} \quad \text{in } L^1(\Omega).$$

Finally, the sequence $\{f_n\}$ converges to f in $L^1(\Omega)$, therefore passing to the limit as $n \to \infty$ in (12), u satysfies the integral identity (10), which means that u is a distributional solution of (7) and thesis of the Theorem 6 holds. □

Proof of Theorem 2 Let be $u \in W_0^{1,2}(\Omega)$ a weak solution of problem (7).

Let us take as test function in (7) $v = \left[\frac{1}{(1+k)^{1-\theta}} - \frac{1}{(1+|u|)^{1-\theta}}\right]_+ \mathrm{sgn}(u)$. Since $|v| \le 1$, by using hypothesis (2) and Young's inequality with $\tau > 0$, we have

$$\begin{aligned}\alpha(1-\theta)\int_{\{|u|>k\}}\frac{|\mathrm{D}u|^2}{(1+|u|)^2}\,\mathrm{d}x &\le \int_{\{|u|>k\}}|E|\frac{|\mathrm{D}u|}{1+|u|}\,\mathrm{d}x + \int_{\{|u|>k\}}|f|\,\mathrm{d}x\\ &\le c\left\{\tau\int_{\{|u|>k\}}\frac{|\mathrm{D}u|^2}{(1+|u|)^2}\,\mathrm{d}x + C(\tau)\int_{\{|u|>k\}}|E|^2\,\mathrm{d}x + \int_{\{|u|>k\}}|f|\,\mathrm{d}x\right\}\end{aligned} \tag{55}$$

Therefore, choosing a suitable $0 < \tau < 1$:

$$\int_{\{|u|>k\}} \frac{|Du|^2}{(1+|u|)^2}\,dx \le c\left\{\int_{\{|u|>k\}} |E|^2\,dx + \int_{\{|u|>k\}} |f|\,dx\right\}. \tag{56}$$

Now, if $k = e^h - 1$ the above inequality implies

$$\begin{aligned}&\int_{\{\log(1+|u|)>h\}} |D\log(1+|u|)|^2\,dx \\ &\le c\left\{\int_{\{\log(1+|u|)>h\}} |E|^2\,dx + \int_{\{\log(1+|u|)>h\}} |f|\,dx\right\}.\end{aligned} \tag{57}$$

Since, $|E|^2 + |f| \in L^r(\Omega)$, with $r > \frac{N}{2}$, from Stampacchia's iterating lemma (cfr [16, Lemma 4.1]) it follows that there exists a positive constant M such that

$$\|\log(1+|u|)\|_{L^\infty(\Omega)} \le M,$$

which implies

$$\|u\|_{L^\infty(\Omega)} \le e^M - 1.$$

□

Remark 3 We point out that if $f \in L^m(\Omega)$, $m \ge 1$, and $f(x) \ge 0$ then the solution u of the problem (7) obtained as limit of the solutions of the approximating problems (12) satisfies $u \ge 0$ a.e. in Ω. As a matter of the fact, we can prove that the weak solution u_n of the problem (12) satisfies $u_n \ge 0$ a.e. in Ω.

Let $\delta, \varepsilon > 0$ such that $0 < \varepsilon < \delta$ and we set $u_n^-(x) = \min\{0, u_n(x)\}$.

We choose $v(x) = T_\varepsilon(u_n^-)$ as test function in (12).

Consequently

$$DT_\varepsilon(u_n^-) = Du_n \chi_{\{-\varepsilon<u_n<0\}},$$

where $\chi_{\{-\varepsilon<u_n<0\}}$ is the characteristic function of the set $\{x \in \Omega : -\varepsilon < u_n(x) < 0\}$. Taking into account that $f\,T_\varepsilon(u_n^-) \le 0$, using the hypothesis (2) and Young's inequality, we obtain

$$
\begin{aligned}
&\frac{\alpha}{(1+n)^{\theta}} \int_{\Omega} |\mathrm{D}T_{\varepsilon}(u_n^-)|^2 \,\mathrm{d}x \\
&\le n \int_{\{|u_n^-|<\varepsilon \,\cap\, u_n<0\}} |E|\,|u_n|^{2-\theta}\, \mathrm{D}T_{\varepsilon}(u_n^-)\,\mathrm{d}x + \int_{\Omega} f\, T_{\varepsilon}(u_n^-)\,\mathrm{d}x \\
&\le \frac{(1+n)^{2+\theta}\,\varepsilon^{2(2-\theta)}}{2\alpha} \int_{\Omega} |E|^2\,\mathrm{d}x + \frac{\alpha}{2(1+n)^{\theta}} \int_{\Omega} |\mathrm{D}T_{\varepsilon}(u_n^-)|^2\,\mathrm{d}x.
\end{aligned} \tag{58}
$$

Therefore

$$
\int_{\Omega} |\mathrm{D}T_{\varepsilon}(u_n^-)|^2\,\mathrm{d}x \le c(n)\varepsilon^{2(2-\theta)} \int_{\Omega} |E|^2\,\mathrm{d}x \tag{59}
$$

From (59) and using Poincarè inequality, we have

$$
\int_{|u_n^-|>\delta} |T_{\varepsilon}(u_n^-)|^2\,\mathrm{d}x \le \int_{\Omega} |T_{\varepsilon}(u_n^-)|^2\,\mathrm{d}x \le C_P c(n)\varepsilon^{2(2-\theta)} \int_{\Omega} |E|^2\,\mathrm{d}x \tag{60}
$$

hence

$$
\varepsilon^2\,\mathrm{mis}\big\{|u_n^-(x)| > \delta\big\} \le \varepsilon^{2+2(1-\theta)} \int_{\Omega} |E|^2\,\mathrm{d}x \tag{61}
$$

or

$$
\mathrm{mis}\big\{|u_n^-(x)| > \delta\big\} \le \varepsilon^{2(1-\theta)} \int_{\Omega} |E|^2\,\mathrm{d}x. \tag{62}
$$

The right-hand side of (62) goes to zero as $\varepsilon \to 0$, hence

$$
\mathrm{mis}\big\{x \in \Omega \;:\; |u_n^-(x)| > \delta\big\} = 0 \qquad \text{for any } \delta > 0
$$

and, because of $u_n^-(x) = 0$ a.e. in Ω.

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