Explicit ∞ -harmonic functions in high dimensions

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Abstract

The aim of this work is to derive new explicit solutions to the ∞ -Laplace equation, the fundamental PDE arising in Calculus of Variations in the space L^∞ . These solutions obey certain symmetry conditions and are derived in arbitrary dimensions, containing as particular sub-cases the already known classes of two-dimensional infinity-harmonic functions.

Keywords ∞ -Laplacian · Calculus of variations in L^∞ · Fully nonlinear systems

Mathematics Subject Classification 35D99 · 35D40 · 35J47 · 35J92 · 35J70 · 35J99

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u \in C^2(\Omega)$ a continuous twice differentiable function. In this paper we study the existence of solutions to the PDE

$$\Delta_\infty u := \sum_{i,j=1}^n D_i u D_j u D_{ij}^2 u = 0 \quad (1.1)$$

of the form

$$u(x) = \prod_{i=1}^n f_i(x_i),$$

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where f_i are possibly non-linear for $1 \leq i \leq n$, and $x = (x_1, \dots, x_n)^\top$, $x \in \Omega$. Solutions of this form are called *separated* ∞ -harmonic functions. In the above $D_i \equiv \frac{\partial}{\partial x_i}$ and

$D_{ij}^2 \equiv \frac{\partial^2}{\partial x_i \partial x_j}$. The Eq. (1.1) is called ∞ -Laplacian (being a special case of the so-called

more general the Aronsson equation) and it arises in Calculus of Variations in L^∞ as the analogue of *Euler-Lagrange* equation of the functional

$$E_\infty(u, \mathcal{O}) := \|Du\|_{L^\infty(\mathcal{O})}, \quad \mathcal{O} \Subset \Omega, \quad u \in W_{\text{loc}}^{1,\infty}(\Omega, \mathbb{R}).$$

These objects first arose in the work of G. Aronsson in the 1960s (see [3, 4]) and nowadays this is an active field of research for vectorial case $N \geq 2$ for $u \in W_{\text{loc}}^{1,\infty}(\Omega, \mathbb{R}^N)$ which has begun much more recently in 2010s (see e.g. [9]). Since then, the field has been developed enormously by N. Katzourakis in the series of papers ([11–19]) and also in collaboration with the author, Abugirda, Croce, Manfredi, Moser, Parini, Pisante and Pryer ([1, 2, 6, 20–25]). A standard difficulty of (1.1) is that it is nondivergence form equation and since in general smooth solutions do not exist, the definition of generalised solutions is an issue. To this end, the theory of viscosity solutions of Crandall–Ishii–Lions is utilised (see e.g. [10]).

In this paper all the *separated* ∞ -harmonic functions are found for $n = 2$ in polar coordinates, for $n = 3$ in spherical coordinates and for all $n \geq 2$ in cartesian coordinates. Some of these new solutions derived herein coincide with previously known classes of solutions. For instance, the well-known G. Aronsson's solution $u(x, y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$ which has a $C^{1,1/3}$ regularity, described in Remark 1. Also M.-F. Bidaut-Veron, M. Garcia Huidobro and L. Veron have found solutions in [26] which coincide with first two solutions of the theorem 1. In addition I.L. Freire, A. C. Faleiros have found solution of (1.1) in [8], but only one of their nontrivial solutions coincides with a particular case of Theorem 3 when $A = 1$. There may exist other additional solutions but this topic is not discussed herein.

The main results of this paper are contained in the following theorems.

Theorem 1 (Separated two-dimensional ∞ -harmonic functions in polar coordinates) Let $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a *separated* ∞ -harmonic function of the ∞ -Laplace equation in polar coordinates

$$u_r^2 u_{rr} + \frac{2}{r^2} u_r u_\theta u_{r\theta} + \frac{1}{r^4} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3} u_r u_\theta^2 = 0 \quad (1.2)$$

of the form $u(r, \theta) = f(r)g(\theta)$. Then, one of the following holds: either

- (i) $|f(r)| = r^A$ and $|g(\theta)| = e^{B\theta}$, where A and B are any constants, such that

$$A^2 - A + B^2 = 0$$

(ii) $|f(r)| = r^A$ and $|g(\theta)| = |g(\theta_0)|e^{\int_{\theta_0}^{\theta} G(t)dt}$, where $G(t)$ satisfies the following

$$t + c = \begin{cases} -\arctan \frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2-A}} \arctan \frac{G(t)}{\sqrt{A^2-A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan \frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}} \ln \left| \frac{G(t)-\sqrt{A-A^2}}{G(t)+\sqrt{A-A^2}} \right|, & \text{if } A^2 - A < 0 \end{cases}$$

(iii) $|g(\theta)| = e^{B\theta}$ and $|f(r)| = |f(r_0)|e^{\int_{r_0}^r \frac{\Phi(t)}{t} dt}$, where $\Phi(t)$ satisfies the following

$$\ln |t| + c = \begin{cases} \frac{1}{2} \ln \left| \frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2} \right| - \frac{1}{2} \frac{1}{\sqrt{B^2-\frac{1}{4}}} \arctan \frac{\Phi(t)-\frac{1}{2}}{\sqrt{B^2-\frac{1}{4}}}, & \text{if } B^2 - \frac{1}{4} > 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2} \right| + \frac{1}{2} \frac{1}{\Phi(t)-\frac{1}{2}}, & \text{if } B^2 - \frac{1}{4} = 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t)+B^2}{\Phi^2(t)-\Phi(t)+B^2} \right| - \frac{1}{4\sqrt{\frac{1}{4}-B^2}} \ln \left| \frac{\Phi(t)-\frac{1}{2}-\sqrt{\frac{1}{4}-B^2}}{\Phi(t)-\frac{1}{2}+\sqrt{\frac{1}{4}-B^2}} \right|, & \text{if } B^2 - \frac{1}{4} < 0, \end{cases}$$

where c is any constant.

Theorem 2 (Separated three-dimensional ∞ -harmonic functions in spherical coordinates) Let $u : \Omega \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a *separated ∞ -harmonic function* of the ∞ -Laplace equation in spherical coordinates

$$\begin{aligned} &u_r^2 u_{rr} + \frac{2}{r^2 \sin^2 \alpha} u_r u_\theta u_{r\theta} + \frac{1}{r^4 \sin^4 \alpha} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3 \sin^2 \alpha} u_r u_\theta^2 + \\ &\frac{2}{r^2} u_r u_\alpha u_{r\alpha} + \frac{1}{r^4} u_\alpha^2 u_{\alpha\alpha} - \frac{1}{r^3} u_r u_\alpha^2 + \frac{2}{r^4 \sin^2 \alpha} u_\theta u_\alpha u_{\theta\alpha} - \frac{\cos \alpha}{r^4 \sin^3 \alpha} u_\alpha u_\theta^2 = 0 \end{aligned} \quad (1.3)$$

of the form $u(r, \theta, \alpha) = f(r)g(\theta)h(\alpha)$. Then, one of the following holds: either

(i) $|f(r)| = r^A$ and $|g(\theta)| = e^{B\theta}$, where A and B are any constants, $h(\alpha) = h(\alpha_0)e^{\int_{\alpha_0}^{\alpha} H(t)dt}$, where $H(t)$ satisfies the following

$$\left| A^2 - A + \frac{B^2}{\sin^2 t} + H^2(t) \right| = c(A, B, t_0) \left(A^2 + \frac{B^2}{\sin^2 t} + H^2(t) \right) e^{-2A \int_{t_0}^t \frac{1}{H(\lambda)} d\lambda}$$

(ii) $|f(r)| = r^A$, $|h(\alpha)| = |\sin \alpha|^A$ and $|g(\theta)| = |g(\theta_0)|e^{\int_{\theta_0}^{\theta} G(t)dt}$, where $G(t)$ satisfies the following

$$t + c = \begin{cases} -\arctan \frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2-A}} \arctan \frac{G(t)}{\sqrt{A^2-A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan \frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}} \ln \left| \frac{G(t)-\sqrt{A-A^2}}{G(t)+\sqrt{A-A^2}} \right|, & \text{if } A^2 - A < 0 \end{cases}$$

(iii) $|g(\theta)| = e^{B\theta}$, $|f(r)| = |f(r_0)|e^{\int_{r_0}^r \frac{\Phi(t)}{t} dt}$, where B is constant and $\Phi(t)$ satisfies the following

$$\ln |t| + c = \begin{cases} \frac{1}{2} \ln \left| \frac{\Phi^2(t)+C^2}{\Phi^2(t)-\Phi(t)+C^2} \right| - \frac{1}{2} \frac{1}{\sqrt{C^2-\frac{1}{4}}} \arctan \frac{\Phi(t)-\frac{1}{2}}{\sqrt{C^2-\frac{1}{4}}}, & \text{if } C^2 - \frac{1}{4} > 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t)+C^2}{\Phi^2(t)-\Phi(t)+C^2} \right| + \frac{1}{2} \frac{1}{\Phi(t)-\frac{1}{2}}, & \text{if } C^2 - \frac{1}{4} = 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t)+C^2}{\Phi^2(t)-\Phi(t)+C^2} \right| - \frac{1}{4\sqrt{\frac{1}{4}-C^2}} \ln \left| \frac{\Phi(t)-\frac{1}{2}-\sqrt{\frac{1}{4}-C^2}}{\Phi(t)-\frac{1}{2}+\sqrt{\frac{1}{4}-C^2}} \right| & \text{if } C^2 - \frac{1}{4} < 0, \end{cases}$$

and

$$|h(\alpha)| = |h(\alpha_0)|e^{B \arcsin \frac{B \cot(\alpha)}{\sqrt{C^2-B^2}} - C \arctan \frac{C \cot \alpha}{\sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}}}}$$

or

$$|h(\alpha)| = |h(\alpha_0)|e^{-B \arcsin \frac{B \cot(\alpha)}{\sqrt{C^2-B^2}} + C \arctan \frac{C \cot \alpha}{\sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}}}}$$

where c is any constant.

Theorem 3 (Separated two-dimensional ∞ -harmonic functions) Let $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a *separated ∞ -harmonic function* of the ∞ -Laplace equation

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0 \quad (1.4)$$

of the form $u(x, y) = f(x)g(y)$. Then, one of the following holds: either

(i) $|f(x)| = |f(x_0)|e^{A(x-x_0)}$ and $|g(y)| = |g(y_0)|e^{\int_{y_0}^y G(t) dt}$, where $G(t)$ satisfies

$$t + c = \begin{cases} \frac{1}{G(t)}, & \text{if } A = 0 \\ -\frac{1}{2A} \arctan \frac{G(t)}{A} + \frac{G(t)}{2(A^2 + G^2(t))}, & \text{otherwise} \end{cases}$$

(ii) $|f(x)| = |f(x_0)|e^{\int_{x_0}^x F(t)dt}$ and $|g(y)| = |g(y_0)|e^{B(y-y_0)}$, where $F(t)$ satisfies

$$t + c = \begin{cases} \frac{1}{F(t)}, & \text{if } B = 0 \\ -\frac{1}{2B} \arctan \frac{F(t)}{B} + \frac{F(t)}{2(B^2 + F^2(t))}, & \text{otherwise.} \end{cases}$$

Theorem 4 (Separated n -dimensional ∞ -harmonic functions) Let $n \geq 2$ and $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a *separated ∞ -harmonic function* of the ∞ -Laplace equation

$$\sum_{i,j=1}^n D_i u D_j u D_{ij}^2 u = 0. \quad (1.5)$$

Then

$$|f_i(x_i)| = |f_i(x_i^0)| e^{A_i(x_i - x_i^0)} \text{ for } 1 \leq i \neq j \leq n$$

and

$$|f_j(x_j)| = |f_j(x_j^0)| e^{\int_{x_j^0}^{x_j} F_j(t) dt},$$

where $F_j(t)$ satisfies

$$t + c = -\frac{1}{2\left(\sum_{i \neq j} A_i^2\right)^{1/2}} \arctan \frac{F_j(t)}{\left(\sum_{i \neq j} A_i^2\right)^{1/2}} + \frac{F_j(t)}{2\left(\sum_{i \neq j} A_i^2 + F_j^2(t)\right)}.$$

2 Proofs of main results

In this section we prove our main results. The general idea of our method, which is essentially the same for all our proofs, is to use a substitution to derive a “better” PDE. Then, we take any points from the domain which are different only in one component put them to the “better” PDE and subtract these two equations from each other to get a “new” PDE.

Proof of Theorem 1 For $u \neq 0$ the Eq. (1.2) can be written as

$$\frac{u_r^2}{u^2} \frac{u_{rr}}{u} + \frac{2}{r^2} \frac{u_r}{u} \frac{u_\theta}{u} \frac{u_{r\theta}}{u} + \frac{1}{r^4} \frac{u_\theta^2}{u^2} \frac{u_{\theta\theta}}{u} - \frac{1}{r^3} \frac{u_r}{u} \frac{u_\theta^2}{u^2} = 0. \quad (2.1)$$

Let $F = \frac{u_r}{u}$ and $G = \frac{u_\theta}{u}$, then $F_r + F^2 = \frac{u_{rr}}{u}$, $G_\theta + G^2 = \frac{u_{\theta\theta}}{u}$ and $\frac{1}{2}F_\theta + \frac{1}{2}G_r + FG = \frac{u_{r\theta}}{u}$. Note that $u(r, \theta) = f(r)g(\theta)$, hence F does not depend on θ , since $F(r, \theta) = \frac{f'(r)}{f(r)}$, anal-

ogously G does not depend on r , since $G(r, \theta) = \frac{g'(\theta)}{g(\theta)}$. Thus (2.1) becomes

$$F^2 F_r + F^4 + \frac{2}{r^2} F^2 G^2 + \frac{1}{r^4} G^2 G_\theta + \frac{1}{r^4} G^4 - \frac{1}{r^3} F G^2 = 0. \quad (2.2)$$

Set $\Phi = Fr$, then $r\Phi_r - \Phi = F_r r^2$. Multiplying (2.2) by r^4 , we have

$$(\Phi^2 + G^2)(\Phi^2 + G^2 - \Phi) + r\Phi^2 \Phi_r + G^2 G_\theta = 0. \quad (2.3)$$

We have the following four cases for the functions Φ and G :

Case (A) Φ and G are constant functions.

Case (B) Φ is constant and G is non-constant functions.

Case (C) Φ is non-constant and G is constant functions.

Case (D) Φ and G are non-constant functions.

Case (A) Let $\Phi \equiv A$ and $G \equiv B$, then (2.3) gives $A \equiv B \equiv 0$ or $A^2 - A + B^2 = 0$ which can be rewritten as $(A - \frac{1}{2})^2 + B^2 = \frac{1}{4}$ and as the consequent of substitutions $f(r) = r^A$ and $g(\theta) = e^{B\theta}$ up to a constants.

Case (B) Let $\Phi \equiv A$, then G is a non-constant function satisfying (2.3)

$$(A^2 + G^2)(A^2 + G^2 - A) + G^2 G_\theta = 0. \quad (2.4)$$

Therefore

$$(A^2 + G^2)(A^2 + G^2 - A) = -G^2 \frac{dG}{d\theta}.$$

Consequently

$$\begin{aligned}\int d\theta &= \int \frac{-G^2}{(A^2 + G^2)(A^2 + G^2 - A)} dG \\ &= \int \frac{-A}{A^2 + G^2} dG - \int \frac{1 - A}{A^2 - A + G^2} dG.\end{aligned}\quad (2.5)$$

Hence

$$t + c = \begin{cases} -\arctan \frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2-A}} \arctan \frac{G(t)}{\sqrt{A^2-A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan \frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}} \ln \left| \frac{G(t)-\sqrt{A-A^2}}{G(t)+\sqrt{A-A^2}} \right|, & \text{if } A^2 - A < 0. \end{cases}$$

Case (C) Let $G \equiv B$, then Φ is a non-constant function satisfying (2.3)

$$(\Phi^2 + B^2)(\Phi^2 + B^2 - \Phi) + r\Phi^2\Phi_r = 0. \quad (2.6)$$

Therefore

$$(\Phi^2 + B^2)(\Phi^2 + B^2 - \Phi) = -r\Phi^2 \frac{d\Phi}{dr}.$$

Consequently

$$\begin{aligned}\int \frac{1}{r} dr &= \int \frac{-\Phi^2}{(\Phi^2 + B^2)(\Phi^2 - \Phi + B^2)} d\Phi \\ &= \int \frac{\Phi}{\Phi^2 + B^2} d\Phi - \int \frac{\Phi - \frac{1}{2}}{(\Phi - \frac{1}{2})^2 + B^2 - \frac{1}{4}} d\Phi - \int \frac{\frac{1}{2}}{(\Phi - \frac{1}{2})^2 + B^2 - \frac{1}{4}} d\Phi.\end{aligned}\quad (2.7)$$

Hence

$$\ln |t| + c = \begin{cases} \frac{1}{2} \ln \left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| - \frac{1}{2} \frac{1}{\sqrt{B^2 - \frac{1}{4}}} \arctan \frac{\Phi(t) - \frac{1}{2}}{\sqrt{B^2 - \frac{1}{4}}}, & \text{if } B^2 - \frac{1}{4} > 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| + \frac{1}{2} \frac{1}{\Phi(t) - \frac{1}{2}}, & \text{if } B^2 - \frac{1}{4} = 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t) + B^2}{\Phi^2(t) - \Phi(t) + B^2} \right| - \frac{1}{4\sqrt{\frac{1}{4} - B^2}} \ln \left| \frac{\Phi(t) - \frac{1}{2} - \sqrt{\frac{1}{4} - B^2}}{\Phi(t) - \frac{1}{2} + \sqrt{\frac{1}{4} - B^2}} \right|, & \text{if } B^2 - \frac{1}{4} < 0. \end{cases}$$

Case (D) Let Φ and G be non-constant functions, then there exist $r_1 \neq r_2$ and $\theta_1 \neq \theta_2$ such that $\Phi(r_1) \neq \Phi(r_2)$ and $G(\theta_1) \neq G(\theta_2)$ satisfying (2.3). Thus

$$r_1 \Phi(r_1)^2 \Phi_r(r_1) - \Phi^3(r_1) + \Phi(r_1)^4 + 2\Phi(r_1)^2 G(\theta)^2 + G(\theta)^2 G_\theta(\theta) + G(\theta)^4 - \Phi(r_1) G(\theta)^2 = 0 \quad (2.8)$$

$$r_2 \Phi(r_2)^2 \Phi_r(r_2) - \Phi^3(r_2) + \Phi(r_2)^4 + 2\Phi(r_2)^2 G(\theta)^2 + G(\theta)^2 G_\theta(\theta) + G(\theta)^4 - \Phi(r_2) G(\theta)^2 = 0. \quad (2.9)$$

Subtracting (2.8) and (2.9) we get for any θ

$$\begin{aligned} G^2(\theta)(\Phi(r_1) - \Phi(r_2))(2(\Phi(r_1) + \Phi(r_2)) - 1) &= r_2 \Phi^2(r_2) \Phi_r(r_2) - \Phi(r_2)^3 + \Phi(r_2)^4 \\ &\quad - r_1 \Phi^2(r_1) \Phi_r(r_1) + \Phi(r_1)^3 - \Phi(r_1)^4. \end{aligned} \quad (2.10)$$

Let's consider two cases. Case (I) If there is exists $r_1 \neq r_2$ such that $2(\Phi(r_1) + \Phi(r_2)) - 1 \neq 0$, then from (2.10) we have $G^2(\theta)$ is a constant, hence $G(\theta)$ is a step function. Case (II) For any $r_1 \neq r_2$ such that $\Phi(r_1) \neq \Phi(r_2)$ we have $2(\Phi(r_1) + \Phi(r_2)) - 1 = 0$, hence $\Phi(r)$ is a step function. For both cases we have a contradiction to $C^{1,\alpha}$ regularity for ∞ -Harmonic functions in two dimensions (see [7, 27]), since $\Phi(r) = \frac{1}{r} \frac{u_r}{u}$ and $G(\theta) = \frac{u_\theta}{u}$ have to have at least $C^{0,\alpha}$ regularity.

Finally integrating and substituting we get $|f(r)| = |f(r_0)| e^{\int_{r_0}^r \frac{\Phi(t)}{t} dt}$ and $|g(\theta)| = |g(\theta_0)| e^{\int_{\theta_0}^\theta G(t) dt}$, which completes the proof. $\square$

Remark 1 (The Arronson solution) Let $A = \frac{4}{3}$ in the Theorem 1ii, then $A^2 - A > 0$ and function $G(t)$ satisfies

$$t + c = -\arctan \frac{3}{4} G(t) + \frac{1}{2} \arctan \frac{3}{2} G(t),$$

which can be rewritten as

$$27G^3(t) + 54G^2(t) \tan 2(t + c) + 32 \tan 2(t + c) = 0.$$

Solving a third degree equation with respect to $G(t)$, we get

$$G(t) = -\frac{4}{3} \frac{\tan^{\frac{1}{3}}(t+c) + \tan^{\frac{5}{3}}(t+c) + \tan(t+c)}{1 - \tan^2(t+c)}.$$

Therefore

$$\int G(t) dt = \ln \left| \frac{(1 - \tan^{\frac{2}{3}}(t+c))(1 + \tan^{\frac{2}{3}}(t+c))^{\frac{1}{3}}}{(\tan^{\frac{4}{3}}(t+c) - \tan^{\frac{2}{3}}(t+c) + 1)^{\frac{2}{3}}} \right|.$$

Hence

$$\begin{aligned} e^{\int_{\theta_0}^{\theta} G(t) dt} &= \frac{|1 - \tan^{\frac{2}{3}}(\theta+c)| |1 + \tan^{\frac{2}{3}}(\theta+c)|^{\frac{1}{3}}}{|\tan^{\frac{4}{3}}(\theta+c) - \tan^{\frac{2}{3}}(\theta+c) + 1|^{\frac{2}{3}}} \cdot c(\theta_0) \\ &= \frac{|1 - \tan^{\frac{4}{3}}(\theta+c)|}{|1 + \tan^2(\theta+c)|^{\frac{2}{3}}} \cdot c(\theta_0) \\ &= |\cos^{\frac{4}{3}}(\theta+c) - \sin^{\frac{4}{3}}(\theta+c)| \cdot c(\theta_0). \end{aligned}$$

Finally

$$\begin{aligned} |g(\theta)| &= |g(\theta_0)| e^{\int_{\theta_0}^{\theta} G(t) dt} \\ &= |g(\theta_0)| (|\cos^{\frac{4}{3}}(\theta+c) - \sin^{\frac{4}{3}}(\theta+c)|), \\ |f(r)| &= r^{\frac{4}{3}}. \end{aligned}$$

Thus, one of the possible solutions is

$$\begin{aligned} u(r, \theta) &= f(r)g(\theta) \\ &= r^{\frac{4}{3}} (\cos^{\frac{4}{3}}(\theta+c) - \sin^{\frac{4}{3}}(\theta+c)). \\ u(x, y) &= |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}. \end{aligned}$$

Remark 2 (The Aronsson solution) Let $A = -\frac{1}{3}$ in the Theorem 1ii, then $A^2 - A > 0$ and function $G(t)$ satisfies

$$t+c = \arctan 3G(t) - 2\arctan \frac{3}{2}G(t).$$

Carrying out a similar series of calculations as in Remark 1 we can find that possible solutions are $f(r) = r^{-\frac{1}{3}}$ and $g(\theta) = \cos^{\frac{4}{3}}(\frac{\theta+c}{2}) - \sin^{\frac{4}{3}}(\frac{\theta+c}{2})$, hence $u(r, \theta) = r^{-\frac{1}{3}}(\cos^{\frac{4}{3}}(\frac{\theta+c}{2}) - \sin^{\frac{4}{3}}(\frac{\theta+c}{2}))$ is the solution of the ∞ -Laplace equation which was described in [5] as $u(r, \theta) = r^{-\frac{1}{3}}g(\theta)$, where

$$g(\theta) = \frac{\cos t}{(1 + 3 \cos^2 t)^{\frac{2}{3}}}, \quad \theta = t - 2 \arctan \left(\frac{\tan t}{2} \right), \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

The key fact these two solutions are identically equal is $\tan \frac{\theta}{2} = -\tan^3 \frac{t}{2}$.

Proof of Theorem 3 It is particular case of the Theorem 4. $\square$

Proof of Theorem 4 For $u \neq 0$ Eq. (1.5) can be written as

$$\sum_{i,j=1}^n \frac{D_i u}{u} \frac{D_j u}{u} \frac{D_{ij}^2 u}{u} = 0. \quad (2.11)$$

Let $F_i = \frac{D_i u}{u}$ then $D_i F_i + F_i^2 = \frac{D_{ii} u}{u}$ and $F_i F_j = \frac{D_{ij} u}{u}$. Thus (2.11) becomes

$$\left(\sum_{i=1}^n F_i^2(x_i) \right)^2 + \sum_{i=1}^n F_i^2(x_i) D_i F_i(x_i) = 0. \quad (2.12)$$

Since $u(x) = \prod_{i=1}^n f_i(x_i)$, then F_i depends only on x_i , consequently $D_i F_i(x_i) = F_i'(x_i)$. Set $x^1, x^2 \in \Omega$ such that $x^1 = (x_1, x_2, \dots, x_j^1, \dots, x_n)$ and $x^2 = (x_1, x_2, \dots, x_j^2, \dots, x_n)$,

where $x_j^1 \neq x_j^2$ in (2.12) and subtract these two equations. We find

$$\left(F_j^2(x_j^1) - F_j^2(x_j^2) \right) \left(2 \sum_{i \neq j} F_i^2(x_i) + 2F_j^2(x_j^1) + 2F_j^2(x_j^2) \right) + F_j^2(x_j^1) F_j'(x_j^1) - F_j^2(x_j^2) F_j'(x_j^2) = 0,$$

assuming $F_j^2(x_j^1) \neq F_j^2(x_j^2)$, we have

$$2 \sum_{i \neq j} F_i^2(x_i) = - \frac{F_j^2(x_j^1) F_j'(x_j^1) - F_j^2(x_j^2) F_j'(x_j^2)}{F_j^2(x_j^1) - F_j^2(x_j^2)} - 2F_j^2(x_j^1) - 2F_j^2(x_j^2). \quad (2.13)$$

LHS of (2.13) does not depend on x_j^1 and x_j^2 so

$$\sum_{i \neq j} F_i^2(x_i) \equiv c$$

for all x_i . Then $F_i(x_i) = A_i$, where A_i is a constant for all $1 \leq i \neq j \leq n$ and hence $|f_i(x_i)| = |f_i(x_i^0)| e^{A_i(x_i - x_i^0)}$. Thus (2.12) gives

$$\left(\sum_{i \neq j} A_i^2 + F_j^2(x_j) \right)^2 + F_j(x_j)^2 F_j'(x_j) = 0,$$

consequently

$$dx_j = - \frac{F_j^2}{\left(\sum_{i \neq j} A_i^2 + F_j^2 \right)^2} dF_j,$$

hence $|f_j(x_j)| = |f_j(x_j^0)| e^{\int_{x_j^0}^{x_j} F_j(t) dt}$, where $F_j(t)$ satisfies

$$t + c = - \frac{1}{2 \sqrt{\sum_{i \neq j} A_i^2}} \arctan \frac{F_j(t)}{\sqrt{\sum_{i \neq j} A_i^2}} + \frac{F_j(t)}{2 \left(\sum_{i \neq j} A_i^2 + F_j^2(t) \right)}, \text{ if } \sum_{i \neq j} A_i^2 \neq 0.$$

Otherwise

$$F_j^4(x_j) + F_j^2(x_j) F_j'(x_j) = 0,$$

so

$$F_j^2(x_j) + F_j'(x_j) = 0, \text{ since we assume } F_j^2(x_j^1) \neq F_j^2(x_j^2). \quad (2.14)$$

Solving (2.14) we get $F_j(x_j) = \frac{1}{x_j + c}$. Hence $|f_i(x_i)| = c_i$ for all $i \neq j$ and

$|f_j(x_j)| = c_j(|x_j + c|)$, where c and c_i are constants for all $1 \leq i \leq n$.

If there is no j such that $F_j^2(x_j^1) \neq F_j^2(x_j^2)$ then $F_j^2(x_j) \equiv c_j$ for all $1 \leq j \leq n$ and

(2.12) gives that $c_j = 0$ for all $1 \leq j \leq n$. So $|f_i(x_i)| = C_i$, where C_i is constant for all i . $\square$

Proof of Theorem 2 For $u \neq 0$ we can rewrite (1.3) as

$$\begin{aligned} & \frac{u_r^2}{u^2} \frac{u_{rr}}{u} + \frac{2}{r^2 \sin^2 \alpha} \frac{u_r}{u} \frac{u_\theta}{u} \frac{u_{r\theta}}{u} + \frac{1}{r^4 \sin^4 \alpha} \frac{u_\theta^2}{u^2} \frac{u_{\theta\theta}}{u} - \frac{1}{r^3 \sin^2 \alpha} \frac{u_r}{u} \frac{u_\theta^2}{u^2} \\ & + \frac{2}{r^2} \frac{u_r}{u} \frac{u_\alpha}{u} \frac{u_{r\alpha}}{u} + \frac{1}{r^4} \frac{u_\alpha^2}{u^2} \frac{u_{\alpha\alpha}}{u} - \frac{1}{r^3} \frac{u_r}{u} \frac{u_\alpha^2}{u^2} \\ & + \frac{2}{r^4 \sin^2 \alpha} \frac{u_\theta}{u} \frac{u_\alpha}{u} \frac{u_{\theta\alpha}}{u} - \frac{\cos \alpha}{r^4 \sin^3 \alpha} \frac{u_\alpha}{u} \frac{u_\theta^2}{u^2} = 0. \end{aligned} \quad (2.15)$$

Let

$$F = \frac{u_r}{u}, \quad G = \frac{u_\theta}{u} \text{ and } H = \frac{u_\alpha}{u}$$

Then, we have

$$F' + F^2 = \frac{u_{rr}}{u}, \quad G' + G^2 = \frac{u_{\theta\theta}}{u} \text{ and } H' + H^2 = \frac{u_{\alpha\alpha}}{u}.$$

Also,

$$\frac{u_{r\theta}}{u} = F'G', \quad \frac{u_{r\alpha}}{u} = F'H' \text{ and } \frac{u_{\theta\alpha}}{u} = G'H'.$$

Note that F , G and H depend on only r , θ and α respectively, since $\frac{u_r}{u} = \frac{f'}{f}$, $\frac{u_\theta}{u} = \frac{g'}{g}$

and $\frac{u_\alpha}{u} = \frac{h'}{h}$. Thus (2.14) becomes

$$\begin{aligned} & F^2(F' + F^2) + \frac{2}{r^2 \sin^2 \alpha} F^2 G^2 + \frac{1}{r^4 \sin^4 \alpha} G^2(G^2 + G') - \frac{1}{r^3 \sin^2 \alpha} F G^2 + \\ & + \frac{2}{r^2} F^2 H^2 + \frac{1}{r^4} H^2(H^2 + H') - \frac{1}{r^3} F H^2 + \frac{2}{r^4 \sin^2 \alpha} G^2 H^2 - \frac{\cos \alpha}{r^4 \sin^3 \alpha} H G^2 = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \left(F^2 + \frac{1}{r^2 \sin^2 \alpha} G^2 + \frac{1}{r^2} H^2 \right)^2 + F^2 F' + \frac{1}{r^4 \sin^4 \alpha} G^2 G' + \frac{1}{r^4} H^2 H' - \\ & - \frac{\cos \alpha}{r^4 \sin^3 \alpha} H G^2 - \frac{1}{r^3 \sin^2 \alpha} F G^2 - \frac{1}{r^3} F H^2 = 0. \end{aligned} \quad (2.16)$$

Let $\Phi = rF$. Then, $r\Phi' - \Phi = r^2 F'$. Multiplying (2.16) by r^4 we have

$$\begin{aligned} & \left(\Phi(r)^2 + \frac{1}{\sin^2 \alpha} G(\theta)^2 + H(\alpha)^2 \right) \left(\Phi(r)^2 - \Phi(r) + \frac{1}{\sin^2 \alpha} G(\theta)^2 + H(\alpha)^2 \right) - \\ & \frac{\cos \alpha}{\sin^3 \alpha} H(\alpha) G(\theta)^2 + r \Phi(r)^2 \Phi'(r) + \frac{1}{\sin^4 \alpha} G(\theta)^2 G'(\theta) + H(\alpha)^2 H'(\alpha) = 0. \end{aligned} \quad (2.17)$$

Setting $(r_1, \theta, \alpha), (r_2, \theta, \alpha) \in \Omega$ such that $r_1 \neq r_2$ in (2.17) and subtracting the two equations we get

$$\begin{aligned} & \left(2\Phi^2(r_1) - \Phi(r_1) - 2\Phi^2(r_2) + \Phi(r_2) \right) \left(H^2(\alpha) + \frac{1}{\sin^2 \alpha} G^2(\theta) \right) \\ & + \Phi^4(r_1) - \Phi^4(r_2) - \Phi^3(r_1) + \Phi^3(r_2) + r_1 \Phi^2(r_1) \Phi'(r_1) - r_2 \Phi^2(r_2) \Phi'(r_2) = 0. \end{aligned}$$

Case 1. If $\frac{2\Phi^2(r) - \Phi(r)}{\sin^2 \alpha} \neq c$, then $H^2(\alpha) + \frac{1}{\sin^2 \alpha} G^2(\theta) = C^2$, hence $G(\theta) \equiv B$, $H(\alpha) = \pm \sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}}$ and (2.17) becomes

$$\left(\Phi^2(r) + C^2 \right) \left(\Phi^2(r) - \Phi(r) + C^2 \right) + r \Phi^2(r) \Phi'(r) = H(\alpha) B^2 \frac{\cos \alpha}{\sin^3 \alpha} - H^2(\alpha) H'(\alpha). \quad (2.18)$$

LHS in (2.18) depends on r only and RHS in (2.18) depends on α only, so we have the system

$$\begin{cases} \left(\Phi^2(r) + C^2 \right) \left(\Phi^2(r) - \Phi(r) + C^2 \right) + r \Phi^2(r) \Phi'(r) = c_1 \\ H(\alpha) B^2 \frac{\cos \alpha}{\sin^3 \alpha} - H^2(\alpha) H'(\alpha) = c_1 \\ H(\alpha) = \pm \sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}} \\ G(\theta) \equiv B. \end{cases} \quad (2.19)$$

Solving (2.19) we get $c_1 = 0$, $G(\theta) \equiv B$ and $H(\alpha) = \pm \sqrt{C^2 - \frac{B^2}{\sin^2 \alpha}}$. Since $c_1 = 0$, Φ

is the solution of the PDE which coincides with (2.6), hence Φ satisfies the following

$$\ln |t| + c = \begin{cases} \frac{1}{2} \ln \left| \frac{\Phi^2(t) + C^2}{\Phi^2(t) - \Phi(t) + C^2} \right| - \frac{1}{2} \frac{1}{\sqrt{C^2 - \frac{1}{4}}} \arctan \frac{\Phi(t) - \frac{1}{2}}{\sqrt{C^2 - \frac{1}{4}}}, & \text{if } C^2 - \frac{1}{4} > 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t) + C^2}{\Phi^2(t) - \Phi(t) + C^2} \right| + \frac{1}{2} \frac{1}{\Phi(t) - \frac{1}{2}}, & \text{if } C^2 - \frac{1}{4} = 0 \\ \frac{1}{2} \ln \left| \frac{\Phi^2(t) + C^2}{\Phi^2(t) - \Phi(t) + C^2} \right| - \frac{1}{4\sqrt{\frac{1}{4} - C^2}} \ln \left| \frac{\Phi(t) - \frac{1}{2} - \sqrt{\frac{1}{4} - C^2}}{\Phi(t) - \frac{1}{2} + \sqrt{\frac{1}{4} - C^2}} \right|, & \text{if } C^2 - \frac{1}{4} < 0. \end{cases}$$

If $B = 0$, then $C = 0$, $\Phi = 0$ is also a solution of (2.19).

Case 2. If $2\Phi^2(r) - \Phi(r) \equiv c$, then $\Phi(r) \equiv A$. Setting $(r, \theta_1, \alpha), (r, \theta_2, \alpha) \in \Omega$ such that $\theta_1 \neq \theta_2$ in (2.17) and subtracting two equations, we get

$$\begin{aligned} & \left(G^2(\theta_1) - G^2(\theta_2) \right) \left(2H^2(\alpha) - \cot(\alpha)H(\alpha) + 2A^2 - A \right) \frac{1}{\sin^2 \alpha} + \\ & + \frac{1}{\sin^4 \alpha} \left(G^2(\theta_1)G'(\theta_1) - G^2(\theta_2)G'(\theta_2) + G^4(\theta_1) - G^4(\theta_2) \right) = 0. \end{aligned}$$

Case 2a. If $G^2(\theta) \neq c$, then $\left(2H^2(\alpha) - \cot(\alpha)H(\alpha) + 2A^2 - A \right) \sin^2 \alpha \equiv C$ and (2.17)

becomes

$$\begin{aligned} & \frac{G^4(\theta)}{\sin^4 \alpha} + \frac{G^2(\theta)}{\sin^2 \alpha} \left(2H^2(\alpha) - \cot(\alpha)H(\alpha) + 2\Phi(r)^2 - \Phi(r) \right) + \frac{G^2(\theta)G'(\theta)}{\sin^4 \alpha} = \\ & \Phi^3(r) - \Phi^4(r) - 2H^2(\alpha)\Phi^2(\theta) + \Phi(r)H^2(\alpha) - H^4(\alpha) - H^2(\alpha)H'(\alpha), \end{aligned}$$

thus

$$\begin{aligned} G^4(\theta) + G^2(\theta)C + G^2(\theta)G'(\theta) &= \sin^4 \alpha \left(\Phi^3(r) - \Phi^4(r) - 2H^2(\alpha)\Phi^2(\theta) \right. \\ & \left. + \Phi(r)H^2(\alpha) - H^4(\alpha) - H^2(\alpha)H'(\alpha) \right). \end{aligned} \quad (2.20)$$

The LHS of (2.20) depends on θ only and the RHS depends on r and α only, hence we have a system

$$\begin{cases} G^4(\theta) + G^2(\theta)C + G^2(\theta)G'(\theta) = c_1 \\ \sin^4 \alpha \left(\Phi^3(r) - \Phi^4(r) - 2H^2(\alpha)\Phi^2(\theta) + \Phi(r)H^2(\alpha) - H^4(\alpha) - H^2(\alpha)H'(\alpha) \right) = c_1 \\ \Phi(r) \equiv A \\ \left(2H^2(\alpha) - \cot(\alpha)H(\alpha) + 2A^2 - A \right) \sin^2 \alpha \equiv C. \end{cases} \quad (2.21)$$

Solving (2.21) we get $C = 2A^2 - A$, $c_1 = -A^4 + A^3$, $\Phi(r) = A$, $H(\alpha) = A \cot(\alpha)$ and $G(\theta)$ is the solution of the following PDE

$$\left(G^2(\theta) + A^2\right)\left(G^2(\theta) + A^2 - A\right) + G^2(\theta)G'(\theta) = 0,$$

which coincides with (2.4) and hence $G(\theta)$ satisfies the following

$$t + c = \begin{cases} -\arctan \frac{G(t)}{A} + \frac{A-1}{\sqrt{A^2-A}} \arctan \frac{G(t)}{\sqrt{A^2-A}}, & \text{if } A^2 - A > 0 \\ \frac{1}{G(t)}, & \text{if } A = 0 \\ -\arctan G(t), & \text{if } A = 1 \\ -\arctan \frac{G(t)}{A} + \frac{A-1}{2\sqrt{A-A^2}} \ln \left| \frac{G(t)-\sqrt{A-A^2}}{G(t)+\sqrt{A-A^2}} \right|, & \text{if } A^2 - A < 0. \end{cases}$$

If A is equal to 0 or 1, then $G = 0$ is also a solution of (2.21).

Case 2b. If $G^2(\theta) \equiv c$, then $G(\theta) = B$. Thus (2.17) gives

$$\left(A^2 - A + \frac{B^2}{\sin^2 \alpha} + H^2(\alpha)\right)\left(A^2 + \frac{B^2}{\sin^2 \alpha} + H^2(\alpha)\right) - \frac{B^2 \cos \alpha}{\sin^3 \alpha} H(\alpha) + H^2(\alpha)H'(\alpha) = 0. \quad (2.22)$$

Let $\Psi(\alpha) = \frac{B^2}{\sin^2 \alpha} + H^2(\alpha)$, then (2.22) becomes

$$(A^2 - A + \Psi(\alpha))(A^2 + \Psi(\alpha)) + H(\alpha)\frac{1}{2}\Psi'(\alpha) = 0,$$

which gives us

$$1 + \frac{H(\alpha)}{2A} \left(\ln \left| \frac{A^2 - A + \Psi(\alpha)}{A^2 + \Psi(\alpha)} \right| \right)_{\alpha} = 0,$$

consequently

$$|A^2 - A + \Psi(t)| = c(A, \Psi(t_0))(A^2 + \Psi(t))e^{-2A \int_{t_0}^t \frac{1}{H(\lambda)} d\lambda},$$

Fig. 1 The approximation to u of the Theorem 1i, depending on the parameters A and B

so that

$$\left| A^2 - A + \frac{B^2}{\sin^2 t} + H^2(t) \right| = c(A, B, H(t_0)) \left(A^2 + \frac{B^2}{\sin^2 t} + H^2(t) \right) e^{-2A \int_{t_0}^t \frac{1}{H(\lambda)} d\lambda}.$$

Finally integrating and substituting we get $|f(r)| = |f(r_0)| e^{\int_{r_0}^r \frac{\Phi(t)}{t} dt}$, $|g(\theta)| = |g(\theta_0)| e^{\int_{\theta_0}^{\theta} G(t) dt}$ and $h(\alpha) = h(\alpha_0) e^{\int_{\alpha_0}^{\alpha} H(t) dt}$, which completes the proof. $\square$

3 Numerical approximations of ∞ -harmonic functions

In this section we illustrate the ∞ -harmonic functions derived earlier, depending on the parameter(s) (see Figs. 1, 2, 3, 4). The results illustrate that we may have a family of solutions depending on the 2π -interval even if the parameter(s) is/are

Fig. 2 The approximation to u of the Theorem 1ii, depending on the parameter A

fixed. For example: the solution on Fig. 2h is a combination of those in Fig. 2i, j when θ belongs to 1st and 2nd 2π - interval of the domain respectively.

Fig. 3 The approximation to u of the Theorem 1iii, depending on the parameter B

Fig. 4 The approximation to u of the Theorem 3i, depending on the parameter A

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Compliance with ethical standards

Conflict of interest The author has no conflict of interest to declare.

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