Indian Economic Review (2022) 57:285–300
https://doi.org/10.1007/s41775-022-00152-x

ARTICLE

Private provision of public goods: a general equilibrium analysis

Debasis Mondal[1] · **Manash Ranjan Gupta**[2]

Accepted: 14 December 2022 / Published online: 1 January 2023

Abstract
We develop a general equilibrium model of private provision of public good where capital owners contribute but others do not. It is shown that the aggregate level of provision varies positively with the number of non-contributors but may not vary positively with the number of contributors. An increase in the number of contributors raises the national income but it may lower their welfare level. Capital accumulation does not affect the public good provision but raises welfare.

Keywords Public good · Private provision · Nash equilibrium · Welfare

JEL Classification F43 · O11 · O41

1 Introduction

Public goods are often provided by governments and financed by tax revenues or borrowing. There are, however, many types of public goods that are provided privately. Common examples of private provision include donations for private charitable activities, funds for lobbying by special interest groups, political campaign funding, lifeboat services in the UK, public radio services that are funded by private donations, etc. In USA, almost 2.1% of the GDP is donated to private charities in

✉ Debasis Mondal
debasis@hss.iitd.ac.in

Manash Ranjan Gupta
manrangup@gmail.com

[1] Department of Humanities and Social Sciences, Indian Institute of Technology Delhi, Room no. MS 606, Hauz Khas, New Delhi 110016, India

[2] Economic Research Unit, Indian Statistical Institute, Kolkata 700108, India

2021 and in India, an estimated philanthropic funding is as high as USD 10 billion a year in the year 2018 (which is equivalent to INR 70,000 crores or around 0.4% of Indian GDP).[1] Theoretical literature in the areas of private provision is mostly focussed on the determinants of aggregate level of private provision. In one of the fundamental theoretical contributions, Bergstrom et al. (1986) provided a complete characterisation of the existence of equilibrium in cases of privately provided public goods. In Bergstrom et al. (1986) and in many other partial equilibrium models on private provision, the aggregate level of provision depends only on the total income of contributors but not on the income of non-contributors.[2] Income of contributors are given exogenously; and hence any change outside the set of contributors leaves the aggregate level of provision unchanged.

In this paper, we develop a general equilibrium model of privately provided public good satisfying factor markets equilibrium. Here, income of the agents as well as the aggregate level of provision are simultaneously determined and the solution depends on the level of factor endowments, which, in turn, is determined by the size of the non-contributing set as well as of the contributing set.[3]

Our model considers a society with two types of individual. Rich individuals own capital as well as labour but poor own only labour. Rich individuals have higher efficiency of labour. Rich individuals contribute to the public good provision but poor remain non-contributor. Production side is introduced in this model. Capital is made specific to the production of private goods but labour is allocated between private goods production and public good production. Income of individuals are endogenously determined here because factor prices are determined in general equilibrium.

We derive interesting results. First, an exogenous increase in the number of non-contributors raises the aggregate level of provision of public good. Second, an increase in the number of contributors may not raise its aggregate provision level. Third, capital accumulation does not affect the aggregate provision level. Existing partial equilibrium models do not point out these results, because they fail to derive the effects of factor endowment changes on factor prices. Finally, welfare level may be reduced due to increase in labour endowment but not due to capital accumulation.

This paper is organized as follows. In Sect. 2, the basic model is presented to describe the equilibrium. We derive comparative static results on public good provision in Sect. 3 and on welfare level in Sect. 4. Section 5 concludes the work.

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1 See https://www.nptrust.org/philanthropic-resources/charitable-giving-statistics/ for US data and https://www.oecd.org/development/philanthropy-centre/researchprojects/ for Indian data.

2 See Uler (2009), Tamai (2010), Itaya et al. (1997).

3 The papers within general equilibrium framework that are related to ours are Pecorino (2009), Mondal (2013) and Mondal (2015). None of these papers, however, introduces non-contributors in their model.

2 The model

There are two groups (sets) of individuals in the society—rich (H) and poor (L). Number of individuals in the rich (poor) group is denoted by H (L). Every individual has one unit of labour. Each member of the rich group holds $\psi > 1$ efficiency units of labour and k units of capital. The representative poor individual holds one efficiency unit of labour but does not hold any capital. Total capital endowment, rate of return on capital and wage rate are denoted by K, r and w, respectively. Income of a representative rich (poor) individual, y_H (y_L), is given by

$$y_H = \psi w + rk; \quad \text{and} \quad y_L = w.$$

Here, $k = \frac{K}{H}$. We assume that only rich individuals contribute to the provision of public good and poor remains non-contributors.[4]

2.1 The demand side

Individual preferences are given by:

$$U_{ij}(c_{ij}, G) = c_{ij}^{\alpha} G^{1-\alpha}; \ \ \alpha \in (0, 1) \tag{1}$$

Here, U_{ij} and c_{ij} represent the utility and private good consumption of a representative agent i belonging to the jth group ($j = H, L$). G is the consumption as well as the production level of the public good. Here,

$$G = \sum_{i=1}^{H} g_i;$$

where g_i is the voluntary contribution made by the ith rich individual.

Let p denote the relative price of the private good. The budgets of a representative rich and of a poor individual are given by

$$pc_{iH} + g_i = \psi w + rk; \tag{2}$$

and

$$pc_{iL} = w. \tag{3}$$

We derive following demand functions for the rich individual.[5]

$$c_{iH}^{D} = \frac{\alpha}{p} \frac{\psi H w + rK}{\alpha H + 1 - \alpha}; \tag{4}$$

[4] We derive conditions under which this will be true in the equilibrium.

[5] See Appendix A for derivation of Eqs. (4) and (5). Similar derivations appear in Pecorino (2009) when all agents contribute.

and

$$G^D = \frac{(\psi Hw + rK)(1-\alpha)}{\alpha H + 1 - \alpha}. \tag{5}$$

Here, c_{iH}^D and G^D denote the demand for private consumption good of the ith rich individual and the total public good contribution. Using Eq. (4), aggregate demand for private good of rich individuals is solved as

$$C_H^D = \sum_{i=1}^{H} c_{iH}^D = \frac{H\alpha}{p}\ \frac{\psi Hw + rK}{\alpha H + 1 - \alpha}.$$

Aggregate demand for private good of poor individual is given by

$$C_L^D = \frac{Lw}{p}.$$

Therefore, the total demand for the private good is given by

$$C^D = C_H^D + C_L^D = \frac{1}{p}\left(\frac{H\alpha(\psi Hw + rK)}{\alpha H + 1 - \alpha} + Lw\right). \tag{6}$$

2.2 The supply side

One unit of public good is produced by one efficiency unit of labour. Hence,

$$G^S = L_G, \tag{7}$$

where L_G represents efficiency units of labour employed to produce the public good.[6] The private good is produced using both capital and labour with a constant returns to scale Cobb-Douglas technology, given by

$$C^S = A(\psi H + L - L_G)^{\theta} K^{1-\theta}. \tag{8}$$

Here, A is technology parameter and $(\psi H + L - L_G)$ represents amount of labour left to produce the private good. We assume perfect competition in all markets and so each factor receives the value of its marginal product. Hence, from Eq. (8), we have

$$Ap\theta(\psi H + L - L_G)^{\theta-1} K^{1-\theta} = w, \tag{9}$$

and

$$Ap(1-\theta)(\psi H + L - L_G)^{\theta} K^{-\theta} = r. \tag{10}$$

Equations (9) and (10) together yield

[6] A superscript 'S' over a variable refers to its aggregate supply.

$$\psi H + L - L_G = \frac{\theta}{1-\theta}\frac{rK}{w}. \tag{11}$$

Using Eqs. (7), (8) and (11), we have

$$G^S = \psi H + L - \frac{\theta}{1-\theta}\frac{rK}{w}; \tag{12}$$

and

$$C^S = AK\left(\frac{\theta}{1-\theta}\right)^{\theta}\left(\frac{r}{w}\right)^{\theta}. \tag{13}$$

2.3 The equilibrium

Using Eqs. (5) and (12), and setting $G^D = G^S$, we get:

$$\frac{(\psi Hw + rK)(1-\alpha)}{\alpha H + 1 - \alpha} = \psi H + L - \frac{\theta}{1-\theta}\frac{rK}{w}. \tag{14}$$

Equation (14) shows that entire voluntary contribution of rich individuals is utilized to produce the public good. Again, setting $C^D = C^S$ and using Eqs. (6) and (13), we have

$$\frac{1}{p}\left(\frac{H\alpha(\psi Hw + rK)}{\alpha H + 1 - \alpha} + Lw\right) = AK\left(\frac{\theta}{1-\theta}\right)^{\theta}\left(\frac{r}{w}\right)^{\theta}. \tag{15}$$

Equation (15) represents equilibrium in the private goods market. Putting $w = 1$, we solve for equilibrium values of r and p as follows[7]

$$r^* = \frac{(1-\theta)[\psi\alpha H^2 + L(\alpha H + 1 - \alpha)]}{\alpha\theta H + 1 - \alpha}\frac{1}{K}; \tag{16}$$

and

$$p^* = \frac{H\alpha(\psi H + rK) + L(\alpha H + 1 - \alpha)}{AK\left(\frac{\theta}{1-\theta}\right)^{\theta}(\alpha H + 1 - \alpha)}\frac{1}{r^{*\theta}}. \tag{17}$$

r^* and p^* are relative prices normalized with respect to the wage rate. $\frac{1}{p^*}$ represents real wage and $\frac{r^*}{p^*}$ represents real rental rate on capital. Using Eqs. (12) and (16), we obtain

[7] All equilibrium values are denoted with a '*' in the superscript.

$$G^* = \frac{\psi H + L(1-\theta)}{\frac{\alpha\theta}{1-\alpha}H + 1}. \tag{18}$$

Using Eq. (16), we obtain

$$y_H^* = \frac{(\alpha H + 1 - \alpha)(\psi H + L(1-\theta))}{H(\alpha\theta H + 1 - \alpha)}; \tag{19}$$

and, using Eqs. (18) and (19), we obtain

$$G^* = \frac{y_H^*}{1 + \frac{1}{H}}. \tag{20}$$

Equation (16) shows that total rental income on capital, r^*K, is independent of capital stock. Hence the private provision of public good is also so. Equations (18) and (19) show that G^* as well as y_H^* varies positively with L.

3 Effects on public good provision

Equation (20) shows that G^* is an increasing function of y_H^* as well as of H. However, Eq. (19) shows that y_H^* is a non-monotonic function of H. Therefore, an increase in the number of contributors, H, may lower the aggregate level of provision of public good if y_H^* falls sharply outweighing the pure group size effect of an increase in H. The pure group size effect is the only effect obtained in a partial equilibrium model, because H does not affect y_H^*. We have an additional income effect in the general equilibrium model when H affects y_H^*. From Eq. (18), we have

$$\frac{dG^*}{dH} = \frac{\frac{\psi(1-\alpha)}{\alpha\theta(1-\theta)} - L}{\frac{1-\alpha}{\alpha\theta(1-\theta)}\left(\frac{\alpha\theta}{1-\alpha}H + 1\right)^2}.$$

Equation (20) shows that a change in H has an ambiguous effect on G^*. Here,

$$\frac{dG^*}{dH} > (=) < 0, \ \text{if} \ \ L < \ (=) \ > \frac{\psi(1-\alpha)}{\alpha\theta(1-\theta)}.$$

Equation (18) shows that G^* is increasing in terms of L. From Eq. (19), we have

$$\frac{dy_H^*}{dH} = \frac{H^2(1-\alpha)\alpha\psi(1-\theta) - 2\alpha\theta(1-\theta)(1-\alpha)LH - (1-\alpha)^2(1-\theta)L - \alpha^2\theta(1-\theta)LH^2}{H^2(\alpha\theta H + 1 - \alpha)^2};$$

and this implies that

$$\frac{dy_H^*}{dH} \leq 0 \quad \text{if,} \ \ \psi \leq \frac{\alpha\theta L}{1-\alpha} + \frac{(1-\alpha)L}{\alpha H^2} + \frac{2\theta L}{H}. \tag{21}$$

However, $\frac{dG^*}{dH} > 0$ if $\psi > L\frac{\alpha\theta(1-\theta)}{1-\alpha}$. Therefore, the condition for $\frac{dG^*}{dH} > 0$ and $\frac{dy_H^*}{dH} < 0$ to hold simultaneously is given by[8]

$$L\frac{\alpha\theta(1-\theta)}{1-\alpha} < \psi < \frac{\alpha\theta L}{1-\alpha} + \frac{(1-\alpha)L}{\alpha H^2} + \frac{2\theta L}{H}. \tag{22}$$

ψ represents the relative wage (efficiency) gap between two groups of workers; and inequality (22) is satisfied when this relative wage gap is neither very low nor very high. We now summarise these results.

Proposition 1 (a) *An increase in the number of contributors lowers (raises) the aggregate level of provision of the public good if* $L > (<) \frac{\psi(1-\alpha)}{\alpha\theta(1-\theta)}$. (*b*) *An increase in the number of non-contributors always raises the aggregate level of provision.* (*c*) *An increase in the number of contributors lowers their per-capita income but raises the aggregate level of provision of public good when inequality* (22) *is satisfied.*

An increase in the number of non-contributors raises the aggregate demand for the private good. This, in turn, raises the demand for capital and hence its rate of return. Therefore, capital owners become richer and contribute more to provide the public good. This positive income effect due to an increase in the number of non-contributors is absent in the existing partial equilibrium literature.

A decline in the number of contributors reduces their aggregate income but may raise their per capita income. Therefore, they contribute more and hence the aggregate level of provision need not fall. In fact, the aggregate provision will increase due to a decline in the number of contributors if the number of non-contributors is very high.

4 Effects on welfare

Here, we consider a special case with $\alpha = \theta = 0.5$ for the sake of simplicity and without loss of generality.[9] Then, from Eqs. (16), (17) and (18), we have

$$r^* = \frac{1}{K}\frac{\psi H^2 + L(H+1)}{H+2}; \tag{23}$$

$$p^* = \frac{2\sqrt{r^*}}{A}; \tag{24}$$

[8] We assume $\psi > 1$ to prevent non-contributors from contributing toward the public good. Therefore, we need $L < \frac{1-\alpha}{\alpha\theta(1-\theta)}$ as a sufficient condition for $\frac{dG^*}{dH} > 0$ to hold true. We assume that this is always satisfied.

[9] This means that the utility function as well as the production function of private good is symmetric in terms of arguments.

and

$$G^* = \frac{2\psi H + L}{H + 2}. \tag{25}$$

Using Eqs. (1), (23), (24) and (25), we have

$$U_{iH} = \frac{G^*}{\sqrt{p^*}}; \quad \text{and} \quad U_{iL} = \left(\frac{G^*}{p^*}\right)^{\frac{1}{2}}.$$

Equations (23) and (24) show that r^* as well as p^* varies positively with H and L. Therefore, welfare level must fall with an increase in H if G^* varies inversely with H; and this happens if $L \geq 4\psi$.[10] When $L < 4\psi$, G^* rises with an increase in H but welfare level still may fall if the negative effect of the increase in the relative price of private good dominates the positive effect of public good provision. However, Eq. (25) shows that G^* always varies positively with L. The national income, Y, is given by

$$\begin{aligned} Y =& (\psi H + L) + r^* K, \\ \Rightarrow \quad Y =& (\psi H + L) + \frac{(1-\theta)[\psi \alpha H^2 + L(\alpha H + 1 - \alpha)]}{\alpha \theta H + (1-\alpha)}. \end{aligned}$$

Therefore, Y must vary positively with H and L. Therefore, an increase in the number of contributors or of non-contributors must always raise the level of national income but may lower the welfare level. Thus, a possibility of immiserizing growth arises when the real national income, $\frac{Y}{p^*}$, is also improved.[11]

An increase in the number of contributors raises the relative price of the private goods. Therefore, the cost of consumption of private good is increased. When the number of contributors is very high, their per capita real income goes down. Therefore, they contribute less to the public good provision; and thus the amount of public good is reduced. Therefore, welfare level falls. In a partial equilibrium model, there is no relative price effect due to change in group size. Also, the effect on public good provision is always positive. Therefore, the possibility of welfare loss does not arise there.

Equations (23) and (24) show that an increase in capital stock lowers the relative price of private goods. Capital is specific to private goods production. Therefore, an increase in capital stock raises the supply of private good and thus lowers its relative price. Therefore, welfare level is increased because G^* is independent of K; and this is true for both types of individuals.

Unlike capital, an increase in the number of non-contributors, L, raises both G^* and p^*. Therefore, its net impact on welfare level involves ambiguity. Here,

[10] Note that, using proposition 1(a), $\frac{dG^*}{dH} < 0$, if $L > 4\psi$ for $\alpha = \theta = 0.5$.

[11] In Appendix C, we show that $\frac{Y}{p^*}$ always improves with an increase in H, L and K for a wide range of parameter values.

$$\frac{\partial U_{iH}}{\partial L} = \frac{U_{iH}}{4} \frac{2\psi H(H-1) + 3L(H+1)}{(2\psi H + L)(\psi H^2 + L(H+1))} > 0 \quad \text{for} \quad H \geq 0.$$

Therefore, the welfare level of rich individuals (contributors) always improves with an increase in L. However, this is not true for the poor individuals. Here[12]

$$\frac{\partial U_{iL}}{\partial L} = \frac{U_{iL}}{4} \frac{L(H+1) - 2\psi H}{(2\psi H + L)(\psi H^2 + L(H+1))} \gtreqless 0 \quad \text{if} \quad \psi \lesseqgtr \frac{L(H+1)}{2H}. \tag{26}$$

Therefore, due to an increase in L, non-contributors are worse off when the relative labour efficiency of contributors is very high and when the number of non-contributors (contributors) is very low (high). We now summarize these results.

Proposition 2

(a) *An increase in the number of contributors always lowers the welfare level of contributors and non-contributors if the number of non-contributors is very high. Otherwise, the effect involves ambiguity.*
(b) *Capital accumulation always leads to welfare improvement of both contributors and non-contributors.*
(c) *An increase in the number of non-contributors always raises the welfare of the contributors but may lower the welfare of the non contributors if the relative labour efficiency of the contributors is very high and if the number of non-contributors (contributors) is very low (high).*

5 Conclusion

We provide a simple general equilibrium model of private provision of public good when both contributors and non-contributors exist and where relative product prices and factor prices are endogenously determined. The presence of non-contributors plays a critical role to derive many interesting results. The increase in the number of contributors does not necessarily raise the level of provision of public good. There is a possibility of welfare loss of individuals when the group size is expanded.

In our model, capital is specific to the production of private goods. Our results are sensitive to this assumption. If capital is used in both public and private goods production, relative factor intensity in production will become crucial in determining the aggregate provision of the public goods. Our model is over simplified in many other dimensions. We do not consider capital allocation to the production of public goods and the role of income inequality in determining the aggregate level of private provision. When capital is used to produce public good, capital accumulation may

[12] See Appendix B for the derivation of Eq. (26).

not lower the relative price of the private good. Income inequality can be measured by Gini coefficient following the approach adopted in our paper. It is then possible to study the joint evolution of income inequality and aggregate level of voluntary provision in an economy. We plan to work on these areas in our future research.

Appendices

Appendix A: Derivation of Eqs. (4) and (5)

We define

$$G_{-i} = \sum_{\substack{j=1 \\ j \neq i}}^{H} g_j;$$

and adding G_{-i} to both sides of Eq. (2), we have

$$pc_{iH} + G = \psi w + r\frac{K}{H} + G_{-i}.$$

Maximising

$$U_{iH}(c_{iH}, G) = c_{iH}^{\alpha} G^{1-\alpha}$$

subject to this budget constraint, we have following optimality conditions:

$$\alpha c_{iH}^{\alpha-1} G^{1-\alpha} = \lambda p; \tag{A.1}$$

and

$$(1-\alpha) c_{iH}^{\alpha} G^{-\alpha} = \lambda. \tag{A.2}$$

Here, λ is the Lagrangian multiplier. Using Eqs. (A.1) and (A.2), we have

$$\frac{\alpha G}{1-\alpha} = pc_{iH}. \tag{A.3}$$

Using Eqs. (A.3) and (2), we have

$$\begin{aligned} G_{-i} &= \frac{G}{1-\alpha} - \left(\psi w + r\frac{K}{H}\right); \\ \Rightarrow \quad \sum_{i=1}^{H} G_{-i} &= \sum_{i=1}^{H}\left[\frac{G}{1-\alpha} - \left(\psi w + r\frac{K}{H}\right)\right]; \\ \Rightarrow \ (H-1)G &= \frac{HG}{1-\alpha} - (H\psi w + rK); \\ \Rightarrow \ G &= \frac{(H\psi w + rK)(1-\alpha)}{H\alpha + 1 - \alpha}. \end{aligned} \tag{A.4}$$

This Eq. (A.4) is identical to Eq. (5) in the text. Next using Eqs. (A.1), (A.2) and (A.4), we have

$$c_{iH} = \frac{\alpha}{p}\frac{\psi Hw + rK}{\alpha H + 1 - \alpha}. \tag{A.5}$$

This Eq. (A.5) is identical to Eq. (4) in the text.

Appendix B: Derivation of $\frac{\partial U_{iL}}{\partial L}$ as in Eq. (26)

Here,

$$U_{iL} = \left(\frac{G^*}{p^*}\right)^{\frac{1}{2}}.$$

Hence,

$$\frac{1}{U_{iL}}\frac{\partial U_{iL}}{\partial L} = \frac{1}{2G}\frac{\partial G}{\partial L} - \frac{1}{2p}\frac{\partial p}{\partial L}. \tag{B.1}$$

Using Eq. (25), we have

$$\frac{\partial G}{\partial L} = \frac{1}{H+2}. \tag{B.2}$$

Using Eqs. (23) and (24), we have

$$\frac{\partial p}{\partial L} = \frac{1}{A\sqrt{r}}\frac{\partial r}{\partial L} = \frac{1}{AK\sqrt{r}}\frac{H+1}{H+2}. \tag{B.3}$$

Using Eqs. (B.1), (B.2) and (25), we have

$$\frac{1}{U_{iL}}\frac{\partial U_{iL}}{\partial L} = \frac{1}{2}\frac{1}{2\psi H + L} - \frac{1}{2p}\frac{1}{AK\sqrt{r}}\frac{H+1}{H+2}. \tag{B.4}$$

Using Eqs. (23), (24) and (B.4), we have

$$
\begin{aligned}
\frac{2}{U_{iL}}\frac{\partial U_{iL}}{\partial L} &= \frac{1}{2\psi H + L} - \frac{H+1}{2\psi H^2 + 2LH + 2L};\\
\Rightarrow \frac{\partial U_{iL}}{\partial L} &= \frac{U_{iL}}{4}\,\frac{L(H+1) - 2\psi H}{(2\psi H + L)(\psi H^2 + L(H+1))}.
\end{aligned} \tag{B.5}
$$

This Eq. (B.5) is identical to Eq. (26) in the text.

Appendix C: Change in Real National Income

We define real national income as

$$z = \frac{Y}{p};$$

where

$$
\begin{aligned}
Y &= \psi H + L + rK;\\
p^2 &= \frac{4r}{A^2};
\end{aligned}
$$

and

$$rK(H+2) = \psi H^2 + LH + L.$$

We have, $ln(z) = ln(Y) - ln(p)$. This gives

$$
\begin{aligned}
\frac{1}{z}\frac{dz}{dH} &= \frac{1}{Y}\frac{dY}{dH} - \frac{1}{p}\frac{dp}{dH};\\
&= \frac{1}{Y}\left(\psi + K\frac{dr}{dH}\right) - \frac{2}{A^2p^2}\frac{dr}{dH}; \quad [\text{ using, } \frac{dp}{dH} = \frac{2}{A^2p}\frac{dr}{dH} \text{ and } \frac{dY}{dH} = \psi + K\frac{dr}{dH}]\\
&= \frac{\psi}{Y} - \frac{dr}{dH}\left(\frac{2}{A^2p^2} - \frac{K}{Y}\right);\\
&= \frac{\psi}{Y} - \frac{dr}{dH}\left(\frac{1}{2r} - \frac{K}{Y}\right);\\
&= \frac{\psi}{Y} - \frac{dr}{dH}\left(\frac{\psi H + L - rK}{2rY}\right).
\end{aligned}
$$

Therefore, $\frac{dz}{dH} \geq 0$ if and only if

$$\frac{dr}{dH} \leq \frac{2\psi r}{\psi H + L - rK}. \tag{C.1}$$

We have

$$rK(H+2) = \psi H^2 + LH + L.$$

Using this, we get

$$\frac{dr}{dH} = \frac{2\psi H + L - rK}{K(H+2)}.$$

Therefore, condition (C.1) becomes,

$$\frac{2\psi H + L - rK}{K(H+2)} \leq \frac{2\psi r}{\psi H + L - rK};$$

$$\text{or,} \quad (2\psi H + L - rK)\,(\psi H + L - rK)\,\frac{1}{2\psi}\,\frac{1}{rK(H+2)} \leq 1. \tag{C.2}$$

Replacing the expression for

$$rK = \frac{\psi H^2 + LH + L}{H+2}$$

in inequality (C.2), we have,

$$\frac{\psi H^2 + 4\psi H + L}{H+2}\,\frac{2\psi H + L}{H+2}\,\frac{1}{2\psi}\,\frac{1}{\psi H^2 + LH + L} \leq 1.$$

Rearranging terms, the above expression becomes

$$\underbrace{\frac{\psi H^2 + 4\psi H + L}{\psi H^2 + LH + L}}_{\text{term 1}}\,\underbrace{\frac{2\psi H + L}{(H+2)^2}}_{term2}\,\underbrace{\frac{1}{2\psi}}_{\text{term 3}} \leq 1. \tag{C.3}$$

$\frac{dz}{dH} \geq 0$ if and only if Eq. (C.3) is satisfied. For all $\psi \geq 1$, (*term* 3) ≤ 1. *Term* 2 and *term* 1 are less than unity if

$$4\psi \leq L \leq (H+2)^2 - 2\psi H;$$

and this range of L is not void if

$$H \geq 2(\psi - 1).$$

Therefore,

$$\frac{dz}{dH} \geq 0 \quad \text{if} \quad 4\psi \leq L \leq (H+2)^2 - 2\psi H \quad \text{and} \quad \psi \geq 1. \tag{C.4}$$

As an example, take $\psi = 2$, and $H = 3$; then for all L such that $8 \leq L \leq 13$, we have $\frac{dz}{dH} \geq 0$.

Next, to find the effect of L on real national income (z), we have,

$$
\begin{aligned}
\frac{1}{z}\frac{dz}{dL} &= \frac{1}{Y}\frac{dY}{dL} - \frac{1}{p}\frac{dp}{dL};\\
&= \frac{1}{Y}\left(1 + K\frac{dr}{dL}\right) - \frac{2}{A^2p^2}\frac{dr}{dL}; \quad [\text{ using, } \quad \frac{dp}{dL} = \frac{2}{A^2p}\frac{dr}{dL} \quad \textit{and} \quad \frac{dY}{dL} = 1 + K\frac{dr}{dL}]\\
&= \frac{1}{Y} - \frac{dr}{dL}\left(\frac{2}{A^2p^2} - \frac{K}{Y}\right);\\
&= \frac{1}{Y} - \frac{dr}{dL}\left(\frac{1}{2r} - \frac{K}{Y}\right);\\
&= \frac{1}{Y} - \frac{dr}{dL}\left(\frac{Y - 2rK}{2rY}\right).
\end{aligned} \tag{C.5}
$$

Using the expression

$$rK(H + 2) = \psi H^2 + LH + L;$$

we have

$$\frac{dr}{dL} = \frac{H + 1}{K(H + 2)}.$$

Then

$$\frac{1}{z}\frac{dz}{dL} = \frac{1}{Y}\left(1 - \underbrace{\frac{H + 1}{H + 2}\frac{Y - 2rK}{2rK}}_{\text{term 1}}\right). \tag{C.6}$$

Term 1 in Eq. (C.6) is less than unity for all $H \geq 2$. $\frac{H+1}{H+2} < 1$ for all $H \geq 1$; and $\frac{Y-2rK}{2rK} \leq 1$ if $Y \leq 4rK$. Replacing,

$$Y = \psi H + L + rK$$

and

$$rK = \frac{\psi H^2 + LH + L}{H + 2};$$

we have $Y \leq 4rK$ for all $H \geq 2$. Therefore, using Eq. (C.6) we have,

$$\frac{dz}{dL} \geq 0 \quad \text{if} \quad H \geq 2. \tag{C.7}$$

Next, to find the effect of K on real national income (z), we have,

$$\begin{aligned}\frac{1}{z}\frac{dz}{dK} =& \frac{1}{Y}\frac{dY}{dK} - \frac{1}{p}\frac{dp}{dK};\\ =& \frac{1}{Y}\left(r + K\frac{dr}{dK}\right) - \frac{1}{2r}\frac{dr}{dK}; \quad [\text{ using, } \quad \frac{1}{p}\frac{dp}{dL} = \frac{1}{2r}\frac{dr}{dK} \quad and \quad \frac{dY}{dK} = r + K\frac{dr}{dK}]\\ =& \frac{r}{Y} + \frac{dr}{dK}\left(\frac{K}{Y} - \frac{1}{2r}\right);\\ =& \frac{r}{Y} + \underbrace{\frac{dr}{dK}}_{\text{term 1}}\left(\underbrace{\frac{2rK - Y}{2rY}}_{\text{term 2}}\right).\end{aligned} \tag{C.8}$$

Replacing

$$Y = \psi H + L + rK$$

and

$$rK = \frac{\psi H^2 + LH + L}{H + 2}$$

, we have

$$\frac{2rK - Y}{2rY} = -\frac{2\psi H + L}{H + 2}\,\frac{1}{2rY} < 0$$

and

$$\frac{dr}{dK} = -\frac{r}{K} < 0.$$

Term 1 and *term* 2 in Eq. (C.8) are negative. So $\frac{dz}{dK} > 0$.

Acknowledgements The authors would like to thank an anonymous referee, the editor and seminar participants at the Delhi School of Economics and at the Indian Institute of Management, Kozhikode for helpful comments. Any remaining errors are of ours.

Funding The authors did not receive funding or grants directly related to the research presented in this manuscript.

Declarations

Conflict of Interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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