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REVIEW PAPER

Collision term for uniformly magnetized plasmas

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Abstract

Collision process is crucial to the transport in magnetized plasmas. This article reviews the three typical approaches, i.e. the Fokker-Planck (FP) approach, the Bogoliubov-Born-Green-Kirwood-Yvon (BBGKY) approach, and the quasilinear (QL) approach, to deriving the kinetic equation for weakly coupled uniformly magnetized plasmas. The collision terms derived based on these three approaches are shown to be identical and satisfy the conservation laws and *H* theorem. Relatively speaking, the BBGKY and QL approaches are more systematic and readily to be generalized from weakly magnetized plasmas to strongly magnetized plasmas. The FP approach is pretty simple for weakly magnetized plasmas and has the advantage that the collision term derived based on it can be naturally separated into two parts, one part arising from the polarization and the other from the correlation of the fluctuating electrostatic field. However, the usual form of the FP equation is not suitable for strongly magnetized plasmas. To derive the magnetized collision term based on the FP approach, a general form of the FP equation for magnetized plasmas has to be found first.

Keywords Magnetized plasmas · Kinetic equation · Collision term · Fokker-Planck equation · BBGKY hierarchy of equations · Klimontovich equation · Conservation laws · *H* theorem

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1 Introduction

Kinetic equation describing the evolution of particle distribution functions is central to plasma physics, in which the collision term plays a very important role. It can serve not only as the starting point of the kinetic description of plasmas but also as the basis to derive the fluid equations for the fluid description of plasmas. Therefore, kinetic equation is fundamental for the study and understanding of various plasma processes including waves (Stix 1992), instabilities (Goldston and Rutherford 1995), transport (Spitzer and Härm 1953; Braginskii 1965), and so on. Derivation of a kinetic equation as accurate as possible is thus a longstanding subject in plasma physics.

The first kinetic equation was developed by Boltzmann (1872) using the binary collision (BC) model for molecular gases which have short-range interactions. The Boltzmann collision term provides a valid description for well-separated BCs. When applied to charged particles interacting through long-range Coulomb forces, it breaks down in the sense that it is divergent logarithmically when the impact parameter $b \to \infty$. This divergence is due to the omission of the collective interactions since many particles interact simultaneously for the collisions with b larger than the mean interparticle spacing, and can be cured by introducing for the impact parameter an upper cutoff at the Debye length λ_D by taking account of the Debye screening (Montgomery and Tidman 1964). For the weakly coupled plasmas with $\Gamma \ll 1$, where Γ is the Coulomb coupling parameter defined as the ratio of the mean interaction Coulomb potential energy between the charged particles to the mean particle kinetic energy, the distant collisions with b larger than the Landau length λ_L producing weak deflections play the dominant role (Cohen et al. 1950). λ_L corresponds to the interparticle spacing at which the Coulomb interaction energy becomes of the order of the mean particle kinetic energy. By expanding the Boltzmann collision term in powers of the velocity change in a BC and retaining terms up to the second order, the Landau collision term could be derived (Montgomery and Tidman 1964; Landau 1965). Besides the divergence as $b \to \infty$, the Landau collision term also diverges logarithmically as $b \to 0$. This divergence is due to the unperturbed rectilinear particle trajectory approximation made for the BC process and can be removed by introducing for the impact parameter a lower cutoff at λ_L (Landau 1965). It seems that the divergence issues appearing in the Boltzmann and Landau collision terms can be healed through introducing appropriate cutoffs for the impact parameter. However, the simple semi-intuitive argument of scattering particles into or out of a volume element in the velocity space based on which the Boltzmann collision term was derived is not appropriate for the weakly coupled plasmas in which each particle undergoes many distant collisions during the time in which the particle travels over its mean free path $\lambda_{mfp} \sim \Gamma^{-3/2}\lambda_D$ (Cohen et al. 1950).

For the weakly coupled plasmas, there are three typical approaches to deriving the kinetic equation: the Fokker-Planck (FP) approach, the Bogoliubov-Born-Green-Kirwood-Yvon (BBGKY) approach, and the quasilinear (QL) approach. The FP equation (Fokker 1914; Planck 1917) was originally developed to describe the Brownian motion (Einstein 1906). It is well-suited to handle the physical processes in which the interactions are very frequent but most are quite weak. This is exactly the main feature of the collisions in the weakly coupled plasmas. The key point of the FP approach is to calculate the FP coefficients. In the BC model, usually only the first and second order FP coefficients referred to as the dynamical friction and diffusion coefficients, respectively, are retained and calculated within the logarithmic accuracy (Rosenbluth et al. 1957). When higher accuracy is required for the moderately-coupled plasmas, higher order FP coefficients need to be kept as well (Li and Petrasso 1993a, b). Since the collective interactions are not taken into account, a natural upper cutoff at λ_D for the impact parameter is usually introduced to remove the divergence within the BC model (Rosenbluth et al. 1957; Li and Petrasso 1993a, b). This difficulty can be overcome by calculating the FP coefficients based on the wave theory (Thompson and Hubbard 1960; Hubbard 1961a, b). In the wave theory, the fluctuating electric field can be first evaluated by combining the linearized Vlasov equation and Poisson equation in the electrostatic approximation, and then used to calculate the FP coefficients. In this way, the collective interactions are properly considered, making it unnecessary to artificially introduce a cutoff for the collisions with $b > \lambda_D$. However, the close collisions with $b < \lambda_L$ are not treated appropriately in the wave theory. Consequently, a upper cutoff for the wavenumber k at $1/\lambda_L$ has to be introduced to suppress the divergence of the FP coefficients (Thompson and Hubbard 1960; Hubbard 1961a). A satisfactory synthesis of the BC theory and wave theory has been made by Hubbard (1961b), in which both the collective interactions and the contribution from the close collisions are considered in a proper manner. In such a theory, no artificial cutoffs need to be introduced.

Compared to the FP approach, the BBGKY approach is a more systematic way to derive the kinetic equation. Its starting point is the first two equations of the BBGKY hierarchy for the evolution of the one-particle and two-particle distribution functions, respectively (Montgomery and Tidman 1964; Ichimaru 1992). These two equations can be made to be a closed set using the Mayer cluster expansions and neglecting the three-particle correlation (Lenard 1960; Montgomery and Tidman 1964; Ichimaru 1992). In this case, the collision term is determined by the two-particle correlation functions. In principle, the BBGKY approach can simultaneously take into account the collective interactions and the contribution from the close collisions. However, in such a case, the evolution equation of the two-particle correlation function becomes rather complex. It is usually assumed that the two-particle correlation function is small compared to the product of the corresponding two one-particle distribution functions, which is untenable for the close collisions. Under this assumption, using the Bogoliubov's adiabatic hypothesis that the two-particle correlation functions relax much faster than the one-particle distribution functions, the two-particle correlation function can be analytically solved (Lenard 1960; Montgomery and Tidman 1964; Ichimaru 1992). Then, the collision term can be obtained commonly referred to as the Balescu-Lenard-Guernsey (BLG) collision term (Lenard 1960; Balescu 1960; Guernsey 1962). The BLG collision term takes account of the collective interactions and thus shows no divergence as $k \to 0$ corresponding to $b \to \infty$. However, it diverges as $k \to \infty$ corresponding to $b \to 0$ since the close collisions are not treated properly. Consequently, an upper cutoff for the wavenumber at $k_{max} = 1/\lambda_L$ is generally introduced in the BLG collision term.

The QL approach is another systematic but much simpler method than the BBGKY approach to derive the kinetic equation. Through taking the statistical average of the Klimontovich equation, the kinetic equation can be obtained with the collision term expressed as the divergence with respect to the velocity of the statistical average of the product of the electric field and distribution function fluctuations in the electrostatic approximation (Lifshitz and Pitaevskii 1981; Klimontovich 1982; Chavanis 2012; Schlickeiser and Yoon 2022). In the polarization approximation (Klimontovich 1982), the electric field and distribution function fluctuations can be determined through combining the linearized Vlasov equation and Poisson equation as the wave theory used to calculate the FP coefficients. In this way, the collective interactions are included automatically but the close collisions are not treated properly as the nonlinear terms involving the product of the electric field and distribution function fluctuations are neglected in the fluctuation part of the Klimontovich equation. As a result, a cutoff at large wavenumber corresponding to small impact parameter has to be introduced to eliminate the divergence.

The three approaches described above are equivalent in deriving the BLG collision term for plasmas and are also generalized to systems with power-law potentials in different dimensions of space (Chavanis 2012, 2013a, b). Relatively speaking, the FP approach is rather straightforward while the BBGKY and QL approaches are more systematic. When the collective effects are neglected, the BLG collision term reduces to the Landau collision term.

The collision terms described above do not take account of the magnetic field effects and thus applies only to unmagnetized plasmas and weakly magnetized plasmas in which all the plasma species' gyration periods are much longer than the duration of the collision process. For the strongly magnetized plasmas where there exist particle species with gyration period shorter than the collision duration, the magnetic field affects the species' trajectories remarkably in the collision process and its effects on the collision term have to be considered. The strong magnetization condition can alternatively be expressed as the particle species' thermal gyro-radius smaller than λ_D or the species' gyro-frequency larger than its plasma frequency. Plasmas with a strongly magnetized component are found in many instances such as the antimatter traps (Fajans and Surko 2020), non-neutral plasmas (Anderegg et al. 1997; Hollmann et al. 1999; Affolter et al. 2016, 2018), ultracold neutral plasmas (Zhang et al. 2008; Gorman et al. 2021), tokamak scrape-off layer plasmas (Greenwald et al. 2014; Creely et al. 2020), astrophysical plasmas (Harding and Lai 2006; Valyavin et al. 2014), laser plasmas (Wilks et al. 1992; Mason and Tabak 1998), strongly anisotropic plasmas (Kennedy and Helander 2021a, b) due to cyclotron emission and so on. In such strongly magnetized plasmas, consideration of the magnetic field in the collisions will significantly affect the collisional transport processes, such as the friction force (Matsuda 1982; Ware 1989; Lafleur and Baalrud 2019; Bernstein et al. 2020; Bernstein and Baarlrud 2021), electron cooling (Sørensen and Bonderup 1983; Nersisyan et al. 2007; Men'shikov 2008; Nersisyan and Zwicknagel 2013; Evans et al. 2018; Cohen et al. 2019), resistivity (Daybelge 1969; Ghendrih et al. 1987; Baalrud and Lafleur 2021; Dong et al. 2022), temperature relaxation (Aliev and Silin 1963; Silin 1963; Ichimaru and Rosenbluth 1970; Montgomery et al. 1974; Øien 1995; Dong et al. 2013a, b; Yoon 2016), perpendicular particle (Aliev and Shister 1964; Ichimaru and Tange 1974; Matsuda 1983; Øien 1995; Anderegg et al. 1997; Dubin 1997) and thermal transport (Aliev and Shister 1964; Psimopoulos and Li 1992; Dubin and O'Neil 1997; Hollmann et al. 1999), etc. All the above discussed three approaches have been successfully generalized to deriving the kinetic equation for the strongly magnetized plasmas. Apart from the more complex derivation, there is no essential difference in applying the BBGKY (Rostoker and Rosenbluth 1960; Rostoker 1960; Silin 1963; Hassan and Watson 1977; Øien 1995) and QL (Klimontovich 1982; Yoon 2016) approaches to deriving the magnetized collision term. Relatively speaking, the generalization of the FP approach to strong magnetization cases is not so smooth. In strongly magnetized plasmas, the particle's velocity change arising from the gyromotion during the collision process is comparable to the velocity itself. This makes the contribution to the collision term from the high order moments of the velocity change cannot be neglected. As a result, the usual form of the FP equation becomes inapplicable in this case. This issue was addressed by Dong et al. (2016) using the coordinate transform method (Newman 1973). They derived a general form of the FP equation for spatially homogeneous magnetized plasmas. Employing the BC theory and wave theory to calculate the magnetized FP coefficients, respectively, the magnetized Landau collision term (Dong et al. 2016) and BLG collision term (Dong et al. 2017) were reproduced.

In this paper, we shall attempt to present a systematic survey of the theoretical knowledge of the collision term for magnetized plasmas. Two cases are distinguished. One is the weakly magnetized plasmas where the thermal gyro-radii of all the plasma species are smaller than λ_{mfp} but larger than λ_D. In this case, the collisions can be viewed as occurring in the absence of a magnetic field. Thus, the collision term is the same as that of unmagnetized plasmas. The other is the strongly magnetized plasmas where the thermal gyro-radii of all the plasma species are much larger than λ_L but at least one of them is smaller than λ_D. If the collisions involve one species with thermal gyro-radius smaller than λ_D, the magnetic field effects have to be taken into account. In most instances, the collision term can be derived for a magnetized plasma without distinguishing whether it is weakly magnetized or strongly magnetized. To better describe the approaches and processes of deriving the collision terms and show the differences between the collision terms for weak and strong magnetization, the two cases without magnetic field and with a uniform magnetic field are actually treated in the paper. For the no magnetic field case, spatially inhomogeneous plasmas are considered but the collisions are treated in a local approximation, while for the case with a uniform magnetic field, only spatially homogeneous plasmas are considered. Most of the discussion in the paper adopts the perturbation method, i.e., integration along the unperturbed orbits, or the linear response theory. The novel effects from reflection (O'Neil 1983; Psimopoulos and Li 1992; Dong et al. 2019) and "collisional caging" (Dubin 1997, 2014) due to velocity diffusion in guiding center collisions are set aside, partly because the relevant research is not well systematic and flawless from the viewpoint of obtaining a collision term. Readers interested in these respects can refer to the series of work of the UCSD group (O'Neil 1983; Anderegg et al. 1997; Dubin 1997; Dubin and O'Neil 1997; Hollmann et al. 1999; Dubin 2014; Affolter et al. 2016, 2018).

The rest of the paper is organized as follows: Sects. 2, 3, and 4 focus on the derivation of the collision term based, respectively, on the FP, BBGKY, and QL approaches with one subsection discussing the case without magnetic field and the other subsection the case with a uniform magnetic field. Sect. 5 presents some properties of the magnetized collision term. Conservation of particle number, momentum, and energy is proven. It is also shown that the magnetized collision term satisfies the H theorem and ensures the distribution function be nonnegative. Finally, a conclusion is given in Sect. 6.

2 Derivation of the collision term based on the FP approach

2.1 The case without B

Each particle experiences abundant collisions during the time in which it travels over λ_{mfp} in a weakly coupled plasma. It has been demonstrated that the distant collisions are more important than the close collisions (Cohen et al. 1950). The particle's diffusion in the velocity space caused by the collisions closely resembles that of a Brownian particle in the configuration space and is well described by the FP equation (Chandrasekhar 1943). For conciseness, a six-dimensional vector $\mathbf{X}_\alpha \equiv (\mathbf{r}_\alpha, \mathbf{v}_\alpha)$ consisting of the position vector $\mathbf{r}_\alpha$ and velocity vector $\mathbf{v}_\alpha$ is introduced for the α ($\alpha, \beta, \gamma \ldots$ denote particle species) particle. Under the Markoffian hypothesis, the α particle distribution function $f_\alpha(\mathbf{X}_\alpha, t+\Delta t)$ at time $t+\Delta t$ in the phase space $\mathbf{X}_\alpha$ can be determined from its value $f_\alpha(\mathbf{X}_\alpha, t)$ at time t by the following integral equation (Chandrasekhar 1943):

$$f_\alpha(\mathbf{X}_\alpha, t+\Delta t) = \int p_\alpha(\mathbf{X}_\alpha, t+\Delta t; \mathbf{X}_{\alpha 0}, t) f_\alpha(\mathbf{X}_{\alpha 0}, t)\, \mathrm{d}^6\mathbf{X}_{\alpha 0}, \tag{1}$$

where $p_\alpha(\mathbf{X}_\alpha, t+\Delta t; \mathbf{X}_{\alpha 0}, t)$ is the transition probability giving the probability that the α particle is at $\mathbf{X}_\alpha$ at $t+\Delta t$ when it is at $\mathbf{X}_{\alpha 0}$ at t, and satisfies the normalization condition

$$\int p_\alpha(\mathbf{X}_\alpha, t+\Delta t; \mathbf{X}_{\alpha 0}, t)\, \mathrm{d}^6\mathbf{X}_\alpha = 1. \tag{2}$$

The time interval Δt should be chosen to be much larger than the correlation time τ_c of the fluctuations. For a quiescent plasma, τ_c is approximately of the order of the plasma oscillation period $\omega_{p\alpha}^{-1} = \sqrt{\varepsilon_0 m_\alpha / (n_\alpha q_\alpha^2)}$ with ε_0 being the permittivity of the vacuum and q_α, m_α, and n_α the charge, mass, and density of the α particle, respectively. By means of the variable substitution $\Delta\mathbf{X}_\alpha = \mathbf{X}_\alpha - \mathbf{X}_{\alpha 0}$, we get from Eq. (1)

$$f_\alpha(\mathbf{X}_\alpha, t+\Delta t) = \int p_\alpha(\mathbf{X}_\alpha, t+\Delta t; \mathbf{X}_\alpha - \Delta\mathbf{X}_\alpha, t) f_\alpha(\mathbf{X}_\alpha - \Delta\mathbf{X}_\alpha, t)\, \mathrm{d}^6\Delta\mathbf{X}_\alpha. \tag{3}$$

Taylor expansion of both $p_\alpha(\mathbf{X}_\alpha, t+\Delta t; \mathbf{X}_\alpha - \Delta\mathbf{X}_\alpha, t)$ and $f_\alpha(\mathbf{X}_\alpha - \Delta\mathbf{X}_\alpha, t)$ over $\Delta\mathbf{X}_\alpha$ in the above equation and invoking Eq. (2), we obtain after some rearrangements

$$\frac{f_\alpha(\mathbf{X}_\alpha, t+\Delta t) - f_\alpha(\mathbf{X}_\alpha, t)}{\Delta t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n [\langle \Delta X_{\alpha i_1} \cdots \Delta X_{\alpha i_n} \rangle f_\alpha(\mathbf{X}_\alpha, t)]}{\partial X_{\alpha i_1} \cdots \partial X_{\alpha i_n}}, \tag{4}$$

where the summation convention over repeated indices i_n is assumed in the phase space and $\langle (\Delta \mathbf{X}_\alpha)^n \rangle$ defined by

$$\langle (\Delta \mathbf{X}_\alpha)^n \rangle \equiv \frac{1}{\Delta t} \int (\Delta \mathbf{X}_\alpha)^n p_\alpha(\mathbf{X}_\alpha + \Delta \mathbf{X}_\alpha, t + \Delta t; \mathbf{X}_\alpha, t)\, \mathrm{d}^6 \Delta \mathbf{X}_\alpha \tag{5}$$

is the nth order FP coefficient denoting the time and statistical average of $(\Delta \mathbf{X}_\alpha)^n$. When the characteristic time scale τ_s of change of f_α is much larger than τ_c, one can find Δt satisfying $\tau_c \ll \Delta t \ll \tau_s$. In this case,

$$\frac{f_\alpha(\mathbf{X}_\alpha, t+\Delta t) - f_\alpha(\mathbf{X}_\alpha, t)}{\Delta t} \approx \frac{\partial f_\alpha(\mathbf{X}_\alpha, t)}{\partial t}. \tag{6}$$

Eq. (4) thus becomes

$$\frac{\partial f_\alpha}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n [\langle \Delta X_{\alpha i_1} \cdots \Delta X_{\alpha i_n} \rangle f_\alpha]}{\partial X_{\alpha i_1} \cdots \partial X_{\alpha i_n}}. \tag{7}$$

The above equation includes all the moments of $\Delta \mathbf{X}_\alpha$. Recall that $\Delta \mathbf{X}_\alpha \equiv (\Delta \mathbf{r}_\alpha, \Delta \mathbf{v}_\alpha)$ is a six-dimensional vector. Since the average of $\Delta \mathbf{v}_\alpha$ is much smaller than $\mathbf{v}_\alpha$, $\Delta \mathbf{r}_\alpha$ is approximately given by

$$\Delta \mathbf{r}_\alpha \approx \mathbf{v}_\alpha \Delta t. \tag{8}$$

Assuming the collisions occur locally and neglecting the effects on the particles' trajectories during the collision process of the macroscopic electric field $\mathbf{E} = \langle \mathbf{E}^M \rangle$ which is the statistical average of the microscopic electric field $\mathbf{E}^M$, $\Delta \mathbf{v}_\alpha$ can be divided into two parts,

$$\Delta \mathbf{v}_\alpha = \frac{q_\alpha}{m_\alpha} \mathbf{E} \Delta t + \Delta \mathbf{v}_\alpha^c. \tag{9}$$

The first part represents the change of $\mathbf{v}_\alpha$ due to $\mathbf{E}$ which is treated as time independent during the collision process. The second part represents the change of $\mathbf{v}_\alpha$ due to the fluctuating electric field $\delta \mathbf{E} = \mathbf{E}^M - \mathbf{E}$ and corresponds to the collisions. $\langle (\Delta \mathbf{X}_\alpha)^n \rangle$ can be obtained using Eqs. (8) and (9). Keeping only the zeroth order terms in Δt, we get

$$\langle \Delta \mathbf{X}_\alpha \rangle \approx \left(\mathbf{v}_\alpha, \frac{q_\alpha}{m_\alpha} \mathbf{E} + \langle \Delta \mathbf{v}_\alpha^c \rangle \right), \tag{10}$$

$$\langle (\Delta \mathbf{X}_\alpha)^n \rangle \approx \langle (\Delta \mathbf{v}_\alpha^c)^n \rangle, \quad n \geq 2. \tag{11}$$

Substituting the above two equations into Eq. (7) gives

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}_\alpha} + \frac{q_\alpha}{m_\alpha}\mathbf{E} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n[\langle \Delta v_{\alpha j_1} \cdots \Delta v_{\alpha j_n} \rangle f_\alpha]}{\partial v_{\alpha j_1} \cdots \partial v_{\alpha j_n}}, \tag{12}$$

where the summation over repeated indices j_n is performed in the velocity space and the superscript c in $\Delta \mathbf{v}_\alpha^c$ is omitted without causing any ambiguity. For the weakly coupled plasmas with the Coulomb logarithm $\ln \Lambda \sim \ln(\Gamma^{-3/2})$ larger than 10, usually only the terms involving the friction coefficient $\langle \Delta \mathbf{v}_\alpha \rangle$ and diffusion coefficient $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ are retained within the logarithmic accuracy. In this case, Eq. (12) reduces to

$$\begin{aligned} \frac{\partial f_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}_\alpha} + \frac{q_\alpha}{m_\alpha}\mathbf{E} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} = & -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{v}_\alpha \rangle f_\alpha \right] \\ & + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v}_\alpha \partial \mathbf{v}_\alpha} : \left[\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle f_\alpha \right]. \end{aligned} \tag{13}$$

This is the famous FP equation for unmagnetized plasmas. The right-hand side (RHS) is the FP collision term comprising a convective term and a diffusion term. $\langle \Delta \mathbf{v}_\alpha \rangle$ and $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ can be calculated based either on the BC theory or on the wave theory. The detailed calculation procedures are reviewed below.

2.1.1 Calculation of $\langle \Delta \mathbf{v}_\alpha \rangle$ and $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ based on the BC theory

In the BC theory, $\langle \Delta \mathbf{v}_\alpha \rangle$ and $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ result from the successive BCs and in a local approximation are given by

$$\langle \Delta \mathbf{v}_\alpha \rangle = \sum_\beta \int \mathrm{d}^3 \mathbf{v}_\beta \int \mathrm{d}^2 \mathbf{b}\, \Delta \mathbf{v}_{\overline{\alpha\beta}} v_{\alpha\beta} f_\beta(\mathbf{v}_\beta), \tag{14}$$

$$\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle = \sum_\beta \int \mathrm{d}^3 \mathbf{v}_\beta \int \mathrm{d}^2 \mathbf{b}\, \Delta \mathbf{v}_{\overline{\alpha\beta}} \Delta \mathbf{v}_{\overline{\alpha\beta}} v_{\alpha\beta} f_\beta(\mathbf{v}_\beta), \tag{15}$$

where $\Delta \mathbf{v}_{\overline{\alpha\beta}}$ is the velocity change of the α particle during a BC specified by a relative velocity $\mathbf{v}_{\alpha\beta} \equiv \mathbf{v}_\alpha - \mathbf{v}_\beta$ and a vector $\mathbf{b}$ with the β particle. $v_{\alpha\beta} \equiv |\mathbf{v}_{\alpha\beta}|$. The length of $\mathbf{b}$ is the impact parameter b which is the distance of closet approach between the two colliding particles when their mutual interaction is not considered, and its direction is pointing from the β particle to the α particle when the distance of closest approach is reached. $\mathrm{d}^2\mathbf{b} = b\mathrm{d}b\mathrm{d}\phi$ with ϕ being the angle between $\mathbf{b}$ and a fixed plane containing $\mathbf{v}_{\alpha\beta}$. $\Delta \mathbf{v}_{\overline{\alpha\beta}}$ is directly related to the relative velocity change $\Delta \mathbf{v}_{\alpha\beta}$ by

$$\Delta \mathbf{v}_{\overline{\alpha\beta}} = \frac{m_{\alpha\beta}}{m_\alpha} \Delta \mathbf{v}_{\alpha\beta}, \tag{16}$$

where $m_{\alpha\beta} \equiv m_\alpha m_\beta/(m_\alpha + m_\beta)$ is the reduced mass. Expressed in terms of the scattering angle θ for $\mathbf{v}_{\alpha\beta}$, $\Delta \mathbf{v}_{\overline{\alpha\beta}}$ is given by

$$\Delta\mathbf{v}_{\overline{\alpha\beta}} = \frac{m_{\alpha\beta}}{m_\alpha}\Big[s_{\alpha\beta}v_{\alpha\beta}\sin\theta\hat{\mathbf{b}} + v_{\alpha\beta}(\cos\theta - 1)\hat{\mathbf{v}}_{\alpha\beta}\Big], \tag{17}$$

where $\hat{\mathbf{b}} \equiv \mathbf{b}/b$, $\hat{\mathbf{v}}_{\alpha\beta} \equiv \mathbf{v}_{\alpha\beta}/v_{\alpha\beta}$, and $s_{\alpha\beta} = 1$ for the repulsive interactions and -1 for the attractive interactions. Using the Rutherford scattering formula

$$\tan\frac{\theta}{2} = \frac{b_0}{b} \tag{18}$$

for the bare Coulomb interaction, where $b_0 \equiv |q_\alpha q_\beta|/(4\pi\varepsilon_0 m_{\alpha\beta}v_{\alpha\beta}^2)$ is the impact parameter for which $\theta = 90°$, $\Delta\mathbf{v}_{\overline{\alpha\beta}}$ can be expressed as

$$\Delta\mathbf{v}_{\overline{\alpha\beta}} = \frac{2m_{\alpha\beta}}{m_\alpha}\frac{b_0(s_{\alpha\beta}v_{\alpha\beta}\mathbf{b} - b_0\mathbf{v}_{\alpha\beta})}{b^2 + b_0^2}. \tag{19}$$

Substituting the above expression for $\Delta\mathbf{v}_{\overline{\alpha\beta}}$ into Eqs. (14) and (15) and carrying out the integrals over ϕ gives

$$\langle\Delta\mathbf{v}_\alpha\rangle = -\sum_\beta\frac{4\pi m_{\alpha\beta}}{m_\alpha}\int \mathrm{d}^3\mathbf{v}_\beta\int_0^\infty \mathrm{d}b\,\frac{b_0^2 b}{b^2 + b_0^2}\mathbf{v}_{\alpha\beta}v_{\alpha\beta}f_\beta(\mathbf{v}_\beta), \tag{20}$$

$$\begin{aligned}\langle\Delta\mathbf{v}_\alpha\Delta\mathbf{v}_\alpha\rangle = &\sum_\beta\frac{4\pi m_{\alpha\beta}^2}{m_\alpha^2}\int \mathrm{d}^3\mathbf{v}_\beta\int_0^\infty \mathrm{d}b\,\frac{b_0^2 b[v_{\alpha\beta}^2 b^2(\mathrm{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}) + 2b_0^2\mathbf{v}_{\alpha\beta}\mathbf{v}_{\alpha\beta}]}{(b^2 + b_0^2)^2}\\ &\times v_{\alpha\beta}f_\beta(\mathbf{v}_\beta),\end{aligned} \tag{21}$$

where I is the unit dyadic. The integrals over b in the above two equations are divergent as $b \to \infty$. The divergence arises from the neglect of collective interactions in the BC theory. To cure the divergence, an upper cutoff at $\lambda_D = (\sum_\gamma \lambda_{D\gamma}^{-2})^{-1/2}$ is introduced for b where $\lambda_{D\gamma} = \sqrt{\varepsilon_0 k_B T_\gamma/(n_\gamma q_\gamma^2)}$ is the γ particle Debye screening length with k_B being the Boltzmann constant and T_γ the temperature of the γ particle. Carrying out the integrals over b in Eqs. (20) and (21) thus gives

$$\langle\Delta\mathbf{v}_\alpha\rangle = -\sum_\beta\frac{q_\alpha^2 q_\beta^2}{4\pi\varepsilon_0^2 m_\alpha m_{\alpha\beta}}\int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta)\frac{\mathbf{v}_{\alpha\beta}}{v_{\alpha\beta}^3}\ln\sqrt{\frac{\lambda_D^2}{b_0^2} + 1}, \tag{22}$$

$$\begin{aligned}\langle\Delta\mathbf{v}_\alpha\Delta\mathbf{v}_\alpha\rangle = &\sum_\beta\frac{q_\alpha^2 q_\beta^2}{4\pi\varepsilon_0^2 m_\alpha^2}\int \mathrm{d}^3\mathbf{v}_\beta\frac{f_\beta(\mathbf{v}_\beta)}{v_{\alpha\beta}}\Bigg[\frac{\lambda_D^2}{\lambda_D^2 + b_0^2}\hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}\\ &+\left(\ln\sqrt{\frac{\lambda_D^2}{b_0^2} + 1} - \frac{1}{2}\frac{\lambda_D^2}{\lambda_D^2 + b_0^2}\right)\big(\mathrm{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}\big)\Bigg].\end{aligned} \tag{23}$$

Approximating λ_D^2/b_0^2+1 to be λ_D^2/b_0^2 and $\lambda_D^2/(\lambda_D^2+b_0^2)$ to be 1, Eqs. (22) and (23) are identical with the results of Li and Petrasso (1993b). For the weakly coupled plasmas, the ratio of λ_D to the average value of b_0 is a very large number in the sense that its logarithm is much larger than 1. Within the logarithmic accuracy, the terms of the order of unity can be neglected compared to the big logarithm and the weak dependence of the logarithm on $v_{\alpha\beta}$ can be eliminated by choosing $v_{\alpha\beta}^2 \sim v_{th\alpha}^2 + v_{th\beta}^2$ where $v_{th\alpha(\beta)} \equiv \sqrt{k_B T_{\alpha(\beta)}/m_{\alpha(\beta)}}$ is the $\alpha\,(\beta)$ particle thermal velocity. In this way, $\langle \Delta\mathbf{v}_\alpha \rangle$ and $\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha \rangle$ are simplified to

$$\langle \Delta\mathbf{v}_\alpha \rangle = -\sum_\beta \Gamma_{\alpha\beta} \frac{m_\alpha}{m_{\alpha\beta}} \int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta) \frac{\mathbf{v}_{\alpha\beta}}{v_{\alpha\beta}^3}, \tag{24}$$

$$\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha \rangle = \sum_\beta \Gamma_{\alpha\beta} \int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta) \frac{\mathbf{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}}, \tag{25}$$

where $\Gamma_{\alpha\beta} \equiv q_\alpha^2 q_\beta^2 \ln\Lambda/(4\pi\varepsilon_0^2 m_\alpha^2)$, $\ln\Lambda = \ln(\lambda_D/\lambda_L)$, and $\lambda_L = |q_\alpha q_\beta|/[4\pi\varepsilon_0 m_{\alpha\beta}(v_{th\alpha}^2 + v_{th\beta}^2)]$. It is clear that λ_L is approximately the average of b_0 over the particle velocity distributions and inversely proportional to the plasma temperature. It is generally regarded as the impact parameter value that distinguishes the close and distant collisions.

In terms of the Rosenbluth-Trubnikov potentials (Rosenbluth et al. 1957; Trubnikov 1965):

$$h_\beta(\mathbf{v}) \equiv \int \mathrm{d}^3\mathbf{v}_\beta \frac{f_\beta(\mathbf{v}_\beta)}{|\mathbf{v}_\beta - \mathbf{v}|}, \tag{26}$$

$$g_\beta(\mathbf{v}) \equiv \int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta)|\mathbf{v}_\beta - \mathbf{v}|, \tag{27}$$

$\langle \Delta\mathbf{v}_\alpha \rangle$ and $\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha \rangle$ can be rewritten as

$$\langle \Delta\mathbf{v}_\alpha \rangle = \sum_\beta \Gamma_{\alpha\beta} \frac{m_\alpha}{m_{\alpha\beta}} \frac{\partial h_\beta(\mathbf{v}_\alpha)}{\partial \mathbf{v}_\alpha}, \tag{28}$$

$$\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha \rangle = \sum_\beta \Gamma_{\alpha\beta} \frac{\partial^2 g_\beta(\mathbf{v}_\alpha)}{\partial \mathbf{v}_\alpha \partial \mathbf{v}_\alpha}, \tag{29}$$

which can be verified by directly substituting h_β and g_β given in Eqs. (26) and (27) into the above two equations and invoking the following two identities:

$$\frac{\partial}{\partial \mathbf{v}_\alpha}\frac{1}{v_{\alpha\beta}} = -\frac{\mathbf{v}_{\alpha\beta}}{v_{\alpha\beta}^3}, \tag{30}$$

$$\frac{\partial^2 v_{\alpha\beta}}{\partial \mathbf{v}_\alpha \partial \mathbf{v}_\alpha} = \frac{\mathtt{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}}. \tag{31}$$

h_β and g_β are called the potential functions as they satisfy the differential equations of similar form as the Poisson equation for the electrostatic potential,

$$\nabla_{\mathbf{v}}^2 h_\beta(\mathbf{v}) = -4\pi f_\beta(\mathbf{v}), \tag{32}$$

$$\nabla_{\mathbf{v}}^2 g_\beta(\mathbf{v}) = 2h_\beta(\mathbf{v}). \tag{33}$$

Using $\langle \Delta\mathbf{v}_\alpha\rangle$ and $\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha\rangle$ given in Eqs. (28) and (29), the collision term $\mathcal{C}_\alpha$ can be expressed in a very compact form:

$$\mathcal{C}_\alpha(f_\alpha) = -\sum_\beta \Gamma_{\alpha\beta}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left[f_\alpha \frac{m_\alpha}{m_{\alpha\beta}}\frac{\partial h_\beta}{\partial \mathbf{v}_\alpha} - \frac{1}{2}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left(f_\alpha\frac{\partial^2 g_\beta}{\partial \mathbf{v}_\alpha\partial \mathbf{v}_\alpha}\right)\right]. \tag{34}$$

Another very useful form of $\mathcal{C}_\alpha$ is the Landau form. Using the identity:

$$\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{\mathtt{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}} = -\frac{2\mathbf{v}_{\alpha\beta}}{v_{\alpha\beta}^3} \tag{35}$$

and $\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha\rangle$ in Eq. (25), $\langle \Delta\mathbf{v}_\alpha\rangle$ in Eq. (24) can be re-expressed as

$$\langle \Delta\mathbf{v}_\alpha\rangle = \sum_\beta \Gamma_{\alpha\beta}\frac{m_\alpha}{2m_\beta}\int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta)\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{\mathtt{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}} + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha\rangle. \tag{36}$$

Noting that

$$\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{\mathtt{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}} = -\frac{\partial}{\partial \mathbf{v}_\beta}\cdot\frac{\mathtt{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}} \tag{37}$$

and integrating by parts over $\mathbf{v}_\beta$, we find from Eq. (36)

$$\langle \Delta\mathbf{v}_\alpha\rangle = \sum_\beta \Gamma_{\alpha\beta}\frac{m_\alpha}{2m_\beta}\int \mathrm{d}^3\mathbf{v}_\beta \frac{\mathtt{I} - \hat{\mathbf{v}}_{\alpha\beta}\hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}}\cdot\frac{\partial f_\beta(\mathbf{v}_\beta)}{\partial \mathbf{v}_\beta} + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha\rangle. \tag{38}$$

Using the above equation for $\langle \Delta \mathbf{v}_\alpha \rangle$ and Eq. (25) for $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$, $\mathcal{C}_\alpha$ in the Landau form can be obtained,

$$\begin{aligned} \mathcal{C}_\alpha(f_\alpha) &= -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\left(\langle \Delta \mathbf{v}_\alpha \rangle - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle \right) f_\alpha \right] \\ &\quad + \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \right] \\ &= -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \Gamma_{\alpha\beta} \frac{m_\alpha}{2} \int \mathrm{d}^3 \mathbf{v}_\beta \, \frac{\mathbf{I} - \hat{\mathbf{v}}_{\alpha\beta} \hat{\mathbf{v}}_{\alpha\beta}}{v_{\alpha\beta}} \\ &\quad \cdot \left(\frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}_\beta} - \frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}_\alpha} \right) f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta). \end{aligned} \tag{39}$$

$\mathcal{C}_\alpha$ in the above equation is identical to that obtained by Landau (1965), which demonstrates the equivalence (Enoch 1960) of the Landau and FP collision terms on one hand and the correctness (Allis 1949; Landau 1950; Cohen et al. 1950) of the Landau collision term on the other hand.

2.1.2 Calculation of $\langle \Delta \mathbf{v}_\alpha \rangle$ and $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ based on the wave theory

Using the BC theory to calculate the FP coefficients, it is necessary to truncate the integrals over b at an upper limit to remove the divergence due to the long-range nature of the Coulomb interactions. This issue naturally disappears when calculating the FP coefficients based on the wave theory which was originated by Thompson and Hubbard (Thompson and Hubbard 1960; Hubbard 1961a, b) and treats the collective interactions in a proper way. Central to the wave theory is to derive the fluctuating electric field $\delta \mathbf{E}$. In Thompson and Hubbard's work, the spectral function (Thompson and Hubbard 1960) of $\delta \mathbf{E}$ or the dynamically screened potential (Hubbard 1961a) were employed directly. Here, we give the detailed derivation of $\delta \mathbf{E}$. In the electrostatic approximation, $\delta \mathbf{E}$ is determined by the Poisson equation:

$$\nabla \cdot \delta \mathbf{E}(\mathbf{r}, t) = \frac{1}{\varepsilon_0} \sum_\beta q_\beta \int \delta N_\beta(\mathbf{r}, \mathbf{v}, t) \, \mathrm{d}^3 \mathbf{v}, \tag{40}$$

where $\delta N_\beta \equiv N_\beta - f_\beta$,

$$N_\beta(\mathbf{X}, t) \equiv \sum_j \delta\big(\mathbf{X} - \mathbf{X}_{\beta j}(t)\big) \tag{41}$$

is the well-known Klimontovich distribution function with $\mathbf{X}_{\beta j}$ representing the phase space coordinate of the jth β particle, and $\delta(x)$ is the delta function. As can be seen, N_β depends on the microscopic motion states of all the β particles. Its statistical average is f_β,

$$f_\beta(\mathbf{X}, t) = \langle N_\beta(\mathbf{X}, t)\rangle. \tag{42}$$

In the QL approximation, δN_β obeys the following linearized Vlasov equation without considering the influence of $\mathbf{E}$:

$$\frac{\partial \delta N_\beta}{\partial t} + \mathbf{v}_\beta \cdot \frac{\partial \delta N_\beta}{\partial \mathbf{r}_\beta} = -\frac{q_\beta}{m_\beta}\delta\mathbf{E} \cdot \frac{\partial f_\beta}{\partial \mathbf{v}_\beta}. \tag{43}$$

Integrating the above equation along the unperturbed linear orbit:

$$\mathbf{v}_\beta^{(0)}(t) = \mathbf{v}_\beta(0), \quad \mathbf{r}_\beta^{(0)}(t) = \mathbf{r}_\beta(0) + \mathbf{v}_\beta(0)t, \tag{44}$$

we obtain

$$\delta N_\beta(\mathbf{X}_\beta, t) = \delta N_\beta(\mathbf{r}_\beta - \mathbf{v}_\beta t, \mathbf{v}_\beta, 0) - \frac{q_\beta}{m_\beta}\int_0^t \mathrm{d}t'\, \delta\mathbf{E}(\mathbf{r}_\beta - \mathbf{v}_\beta(t-t'), t') \cdot \frac{\partial f_\beta}{\partial \mathbf{v}_\beta}. \tag{45}$$

The first term on the RHS of the above equation is the spontaneous fluctuation denoted by δN_β^S, arising from the random thermal motion of the discrete particles. It exists even in a fictitious noninteracting system. The second term is the induced fluctuation denoted by δN_β^I, arising from the interactions between the particles. Since the characteristic spatio-temporal scales of change of f_β is substantially greater than those of δN_β and $\delta\mathbf{E}$, f_β can be viewed as uniform and time-independent as far as the fluctuating quantities are concerned. Substituting Eq. (45) into Eq. (40) and expressing $\delta\mathbf{E}$ in terms of the fluctuating electrostatic potential $\delta\phi$ through $\delta\mathbf{E} = -\nabla\delta\phi$ yields

$$\begin{aligned}\nabla^2\delta\phi = -\frac{1}{\varepsilon_0}\sum_\beta q_\beta \int \mathrm{d}^3\mathbf{v}\, \Bigg[&\delta N_\beta(\mathbf{r} - \mathbf{v}t, \mathbf{v}, 0) \\ &+ \frac{q_\beta}{m_\beta}\int_0^t \mathrm{d}t'\, \nabla\delta\phi(\mathbf{r} - \mathbf{v}(t-t'), t') \cdot \frac{\partial f_\beta(\mathbf{v})}{\partial \mathbf{v}}\Bigg].\end{aligned} \tag{46}$$

The above equation can be manipulated into a physically more intuitive form by making the Fourier transform with respect to $\mathbf{r}$ and Laplace transform with respect to t:

$$\tilde{A}(\mathbf{k}, \mathbf{v}, \omega) = \frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}t \int \mathrm{d}^3\mathbf{r}\, A(\mathbf{r}, \mathbf{v}, t)\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{47}$$

It follows that

$$\delta\tilde{\phi}(\mathbf{k}, \omega) = \frac{1}{\varepsilon_0 \varepsilon(\mathbf{k}, \omega)k^2}\sum_\beta q_\beta \int \delta\tilde{N}_\beta^S(\mathbf{k}, \mathbf{v}, \omega)\, \mathrm{d}^3\mathbf{v}, \tag{48}$$

where

$$\delta\tilde{N}^S_\beta(\mathbf{k},\,\mathbf{v},\,\omega)=\frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{r}\int_0^\infty \mathrm{d}t\,\delta N_\beta(\mathbf{r}-\mathbf{v}t,\,\mathbf{v},\,0)\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{49}$$

is the Fourier-Laplace transform of δN^S_β and

$$\varepsilon(\mathbf{k},\,\omega)=1-\sum_\beta \frac{q_\beta^2}{\varepsilon_0 m_\beta k^2}\int \frac{\mathbf{k}\cdot\partial f_\beta(\mathbf{v})/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-\omega}\,\mathrm{d}^3\mathbf{v} \tag{50}$$

is the dielectric response function. $\varepsilon(\mathbf{k},\,\omega)$ is defined in the upper half ω plane. It can be continued analytically to the real axis and lower half plane by deforming the v_k integration contour in Eq. (50) to lie below the pole at $v_k=\omega/k$as that shown in Fig. 1, where $v_k\equiv\mathbf{k}\cdot\mathbf{v}/k$ is the component of $\mathbf{v}$ along the $\mathbf{k}$ direction. It can be seen clearly from Eq. (48) that the induced fluctuation leads to the dynamic screening of the fluctuating electrostatic potential generated by the spontaneous fluctuation.

Now, we are in a position to calculate $\langle\Delta\mathbf{v}_\alpha\rangle$ and $\langle\Delta\mathbf{v}_\alpha\Delta\mathbf{v}_\alpha\rangle$ based on the wave theory. Without considering the influence of $\mathbf{E}$ on the collisions, $\langle\Delta\mathbf{v}_\alpha\rangle$ and $\langle\Delta\mathbf{v}_\alpha\Delta\mathbf{v}_\alpha\rangle$ can be calculated as if $\mathbf{E}$ were absent. Under this condition, the α particle motion equation is

$$\frac{\mathrm{d}\mathbf{v}_\alpha(t)}{\mathrm{d}t}=-\frac{q_\alpha}{m_\alpha}\nabla\delta\phi(\mathbf{r}_\alpha(t),\,t). \tag{51}$$

Integrating the above equation over t gives the α particle's trajectory,

$$\mathbf{v}_\alpha(t)=\mathbf{v}_\alpha(0)-\frac{q_\alpha}{m_\alpha}\int_0^t \nabla\delta\phi(\mathbf{r}_\alpha(t'),\,t')\,\mathrm{d}t', \tag{52}$$

$$\mathbf{r}_\alpha(t)=\mathbf{r}_\alpha^{(0)}(t)-\frac{q_\alpha}{m_\alpha}\int_0^t (t-t')\nabla\delta\phi(\mathbf{r}_\alpha(t'),\,t')\,\mathrm{d}t'. \tag{53}$$

$\Delta\mathbf{v}_\alpha\equiv\mathbf{v}_\alpha(t+\Delta t)-\mathbf{v}_\alpha(t)$ can be found from Eq. (52) to be

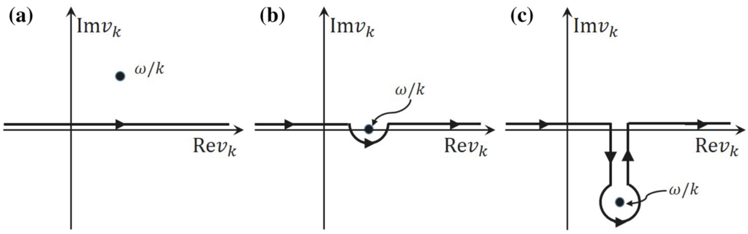

Fig. 1 The contour of the $v_k\equiv\mathbf{k}\cdot\mathbf{v}/k$ integral in Eq. (50) for the cases: **a** Im$\omega>0$; **b** Im$\omega=0$; and **c** Im$\omega<0$

$$\Delta \mathbf{v}_\alpha = -\frac{q_\alpha}{m_\alpha} \int_t^{t+\Delta t} \nabla \delta\phi(\mathbf{r}_\alpha(t'), t')\, \mathrm{d}t'. \tag{54}$$

Substituting Eq. (53) for $\mathbf{r}_\alpha(t)$ into the above equation, expanding $\Delta \mathbf{v}_\alpha$ with respect to $\delta\phi$, and retaining terms up to the second order in $\delta\phi$ yields

$$\begin{aligned} \Delta \mathbf{v}_\alpha = &-\frac{q_\alpha}{m_\alpha} \int_t^{t+\Delta t} \nabla \delta\phi(\mathbf{r}_\alpha^{(0)}(t'), t')\, \mathrm{d}t' \\ &+ \frac{q_\alpha^2}{m_\alpha^2} \int_t^{t+\Delta t} \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\, (t'-t'') \nabla \delta\phi(\mathbf{r}_\alpha^{(0)}(t''), t'') \cdot \nabla\nabla \delta\phi(\mathbf{r}_\alpha^{(0)}(t'), t'). \end{aligned} \tag{55}$$

Using the Fourier-Laplace inversion transform:

$$A(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2\pi} \int \mathrm{d}^3\mathbf{k} \int_C \mathrm{d}\omega\, \tilde{A}(\mathbf{k}, \mathbf{v}, \omega) \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{56}$$

where the integral over ω is performed along a horizontal line in the upper half plane above all the poles of $\tilde{A}(\mathbf{k}, \mathbf{v}, \omega)$, $\Delta \mathbf{v}_\alpha$ can be re-expressed to be

$$\begin{aligned} \Delta \mathbf{v}_\alpha = &-\frac{q_\alpha}{2\pi m_\alpha} \mathrm{i} \int_t^{t+\Delta t} \mathrm{d}t' \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega'\, \mathbf{k}' \delta\tilde{\phi}(\mathbf{k}', \omega') \mathrm{e}^{\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t') - \omega' t'\right]} \\ &- \frac{q_\alpha^2}{(2\pi)^2 m_\alpha^2} \mathrm{i} \int_t^{t+\Delta t} \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega' \int \mathrm{d}^3\mathbf{k}'' \int_{C''} \mathrm{d}\omega''\, (t'-t'') \\ &\times \mathbf{k}'\mathbf{k}' \cdot \mathbf{k}'' \delta\tilde{\phi}(\mathbf{k}', \omega') \delta\tilde{\phi}(\mathbf{k}'', \omega'') \mathrm{e}^{\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t') - \omega' t' + \mathbf{k}''\cdot\mathbf{r}_\alpha^{(0)}(t'') - \omega'' t''\right]}. \end{aligned} \tag{57}$$

Using $\Delta \mathbf{v}_\alpha$ given in the above equation, $\langle \Delta \mathbf{v}_\alpha \rangle$ can be calculated according to its definition. $\langle \Delta \mathbf{v}_\alpha \rangle$ can be separated into two parts, $\langle \Delta \mathbf{v}_\alpha \rangle = \langle \Delta \mathbf{v}_\alpha \rangle_p + \langle \Delta \mathbf{v}_\alpha \rangle_c$, where $\langle \Delta \mathbf{v}_\alpha \rangle_p$ and $\langle \Delta \mathbf{v}_\alpha \rangle_c$ correspond, respectively, to the first and second terms on the RHS of Eq. (57). $\langle \Delta \mathbf{v}_\alpha \rangle_p$ is given by

$$\langle \Delta \mathbf{v}_\alpha \rangle_p = -\frac{q_\alpha}{2\pi m_\alpha} \frac{\mathrm{i}}{\Delta t} \int_t^{t+\Delta t} \mathrm{d}t' \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega'\, \mathbf{k}' \langle \delta\tilde{\phi}(\mathbf{k}', \omega') \rangle \mathrm{e}^{\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t') - \omega' t'\right]}. \tag{58}$$

$\langle \delta\tilde{\phi} \rangle$ in the above equation cannot be taken for granted to be 0 since the fluctuating electrostatic potential contributed by the α particle itself survives the statistical average (Chavanis 2012). Using Eqs. (48), (49), and (41), $\langle \delta\tilde{\phi} \rangle$ can be obtained,

$$\begin{aligned} \langle \delta\tilde{\phi}(\mathbf{k}', \omega') \rangle &= \frac{q_\alpha}{(2\pi)^3 \varepsilon_0 \varepsilon(\mathbf{k}', \omega') k'^2} \int \mathrm{d}^6\mathbf{X} \int_0^\infty \mathrm{d}t''\, \delta\left(\mathbf{X} - \mathbf{X}_\alpha^{(0)}(t'')\right) \mathrm{e}^{-\mathrm{i}(\mathbf{k}'\cdot\mathbf{r} - \omega' t'')} \\ &= \frac{q_\alpha}{(2\pi)^3 \varepsilon_0 \varepsilon(\mathbf{k}', \omega') k'^2} \int_0^\infty \mathrm{d}t''\, \mathrm{e}^{-\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t'') - \omega' t''\right]}. \end{aligned} \tag{59}$$

This is precisely the electrostatic potential produced by the α particle itself including the dynamic screening effect of the plasma, implying that $\langle \Delta \mathbf{v}_\alpha \rangle_p$ comes from the polarization. Substituting the above equation for $\langle \delta\tilde{\phi} \rangle$ into Eq. (58) gives

$$\langle \Delta \mathbf{v}_\alpha \rangle_p = -\frac{q_\alpha^2}{(2\pi)^4 \varepsilon_0 m_\alpha} \frac{\mathrm{i}}{\Delta t} \int_t^{t+\Delta t} \mathrm{d}t' \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega' \int_0^\infty \mathrm{d}t'' \frac{\mathbf{k}'}{\varepsilon(\mathbf{k}', \omega') k'^2} \times \mathrm{e}^{\mathrm{i}(\mathbf{k}'\cdot\mathbf{v}_\alpha - \omega')(t'-t'')}. \tag{60}$$

Carrying out the integral over t'', $\langle \Delta \mathbf{v}_\alpha \rangle_p$ becomes

$$\langle \Delta \mathbf{v}_\alpha \rangle_p = \frac{q_\alpha^2}{(2\pi)^4 \varepsilon_0 m_\alpha} \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathrm{d}t' \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega' \frac{\mathbf{k}'}{\varepsilon(\mathbf{k}', \omega') k'^2} \frac{\mathrm{e}^{\mathrm{i}(\mathbf{k}'\cdot\mathbf{v}_\alpha - \omega')t'}}{\omega' - \mathbf{k}'\cdot\mathbf{v}_\alpha}. \tag{61}$$

Assuming the system considered is Vlasov stable, i.e., all the zeros of $\varepsilon(\mathbf{k}', \omega')$ lie in the lower half ω' plane, and displacing the contour of the ω' integral in the above equation into the lower half plane going around the pole at $\omega' = \mathbf{k}' \cdot \mathbf{v}_\alpha$ and the poles ω'_k $(k = 1, 2, \cdots)$ corresponding to the zeros of $\varepsilon(\mathbf{k}', \omega')$ that are encountered in the manner shown in Fig. 2, it is readily to find that when t' is very large the contribution from the residue of the pole at $\omega' = \mathbf{k}' \cdot \mathbf{v}_\alpha$ independent of t' is dominant, and the contributions from the other parts of the contour including the horizontal lines and circles around the poles ω'_k encountered decaying exponentially with t' can be discarded. We thus have

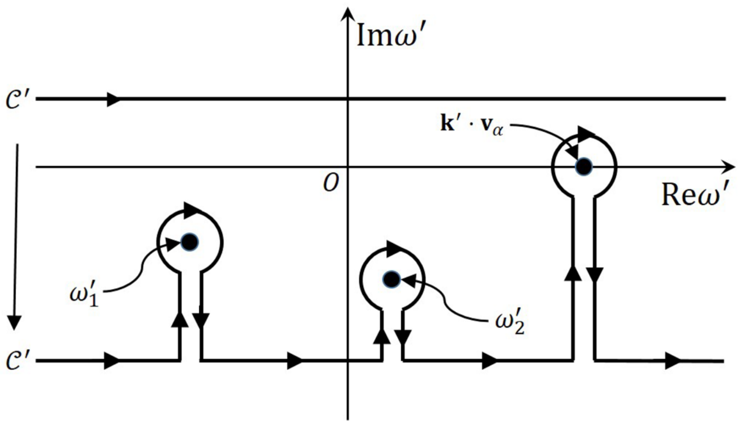

Fig. 2 The contour C' of the ω' integral in Eq. (61) is moved into the lower half plane going around the pole at $\omega' = \mathbf{k}' \cdot \mathbf{v}_\alpha$ and the poles ω'_k corresponding to the zeros of $\varepsilon(\mathbf{k}', \omega')$ that are encountered clockwise

$$
\begin{aligned}
\langle \Delta \mathbf{v}_\alpha \rangle_p &= -\frac{q_\alpha^2}{(2\pi)^3 \varepsilon_0 m_\alpha} \mathrm{i} \int \mathrm{d}^3\mathbf{k} \, \frac{\mathbf{k}}{\varepsilon(\mathbf{k},\, \mathbf{k}\cdot\mathbf{v}_\alpha) k^2} \\
&= -\frac{q_\alpha^2}{(2\pi)^3 \varepsilon_0 m_\alpha} \int \mathrm{d}^3\mathbf{k} \, \frac{\mathbf{k}\varepsilon_i(\mathbf{k},\, \mathbf{k}\cdot\mathbf{v}_\alpha)}{|\varepsilon(\mathbf{k},\, \mathbf{k}\cdot\mathbf{v}_\alpha)|^2 k^2},
\end{aligned} \tag{62}
$$

where $\varepsilon_i(\mathbf{k},\, \omega)$ is the imaginary part of $\varepsilon(\mathbf{k},\, \omega)$. From Eq. (50), it follows that

$$
\varepsilon_i(\mathbf{k},\, \omega) = -\sum_\beta \frac{\pi q_\beta^2}{\varepsilon_0 m_\beta k^2} \int \mathrm{d}^3\mathbf{v}\, \mathbf{k}\cdot \frac{\partial f_\beta(\mathbf{v})}{\partial \mathbf{v}} \delta(\mathbf{k}\cdot\mathbf{v} - \omega) \tag{63}
$$

for real ω. Substituting the above equation for ε_i into Eq. (62) gives

$$
\langle \Delta \mathbf{v}_\alpha \rangle_p = \sum_\beta \frac{q_\alpha^2 q_\beta^2}{8\pi^2 \varepsilon_0^2 m_\alpha m_\beta} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{v} \, \frac{\mathbf{k}\mathbf{k}\cdot \partial f_\beta(\mathbf{v})/\partial \mathbf{v}}{|\varepsilon(\mathbf{k},\, \mathbf{k}\cdot\mathbf{v}_\alpha)|^2 k^4} \delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}_\alpha). \tag{64}
$$

$\langle \Delta \mathbf{v}_\alpha \rangle_c$ can be obtained by substituting Eqs. (48) and (49) into the second term on the RHS of Eq. (57),

$$
\begin{aligned}
\langle \Delta \mathbf{v}_\alpha \rangle_c = -\sum_{\beta,\gamma} & \frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^8 \varepsilon_0^2 m_\alpha^2} \frac{\mathrm{i}}{\Delta t} \int_t^{t+\Delta t} \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \\
& \times \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega' \int \mathrm{d}^3\mathbf{k}'' \int_{C''} \mathrm{d}\omega'' \int \mathrm{d}^6\mathbf{X}' \int \mathrm{d}^6\mathbf{X}'' \\
& \times \frac{(t'-t'')\mathbf{k}'\mathbf{k}'\cdot\mathbf{k}''}{\varepsilon(\mathbf{k}',\, \omega')\varepsilon(\mathbf{k}'',\, \omega'') k'^2 k''^2} \langle \delta N_\beta(\mathbf{X}',\, 0) \delta N_\gamma(\mathbf{X}'',\, 0) \rangle \\
& \times \mathrm{e}^{-\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}' + (\mathbf{k}'\cdot\mathbf{v}' - \omega')t_1 + \mathbf{k}''\cdot\mathbf{r}'' + (\mathbf{k}''\cdot\mathbf{v}'' - \omega'')t_2\right]} \\
& \times \mathrm{e}^{\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t') - \omega' t' + \mathbf{k}''\cdot\mathbf{r}_\alpha^{(0)}(t'') - \omega'' t''\right]}.
\end{aligned} \tag{65}
$$

Carrying out the integrals over t_1 and t_2 yields

$$
\begin{aligned}
\langle \Delta \mathbf{v}_\alpha \rangle_c = \sum_{\beta,\gamma} & \frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^8 \varepsilon_0^2 m_\alpha^2} \frac{\mathrm{i}}{\Delta t} \int_t^{t+\Delta t} \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega' \int \mathrm{d}^3\mathbf{k}'' \int_{C''} \mathrm{d}\omega'' \\
& \times \int \mathrm{d}^6\mathbf{X}' \int \mathrm{d}^6\mathbf{X}'' \frac{(t'-t'')\mathbf{k}'\mathbf{k}'\cdot\mathbf{k}''}{\varepsilon(\mathbf{k}',\, \omega')\varepsilon(\mathbf{k}'',\, \omega'') k'^2 k''^2} \frac{1}{\omega' - \mathbf{k}'\cdot\mathbf{v}'} \frac{1}{\omega'' - \mathbf{k}''\cdot\mathbf{v}''} \\
& \times \langle \delta N_\beta(\mathbf{X}',\, 0) \delta N_\gamma(\mathbf{X}'',\, 0) \rangle \mathrm{e}^{-\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}' + \mathbf{k}''\cdot\mathbf{r}'')} \\
& \times \mathrm{e}^{\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t') - \omega' t' + \mathbf{k}''\cdot\mathbf{r}_\alpha^{(0)}(t'') - \omega'' t''\right]}.
\end{aligned} \tag{66}
$$

Moving the contours C' and C'' of the ω' and ω'' integrals in the above equation into the lower half planes in a similar manner as that shown in Fig. 2 and retaining only the contributions from the residues of the poles at $\omega' = \mathbf{k}'\cdot\mathbf{v}'$ and $\omega'' = \mathbf{k}''\cdot\mathbf{v}''$ which do not decay with t' and t'', $\langle \Delta \mathbf{v}_\alpha \rangle_c$ becomes

$$
\begin{aligned}
\langle\Delta\mathbf{v}_\alpha\rangle_c = & -\sum_{\beta,\gamma}\frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^6\varepsilon_0^2 m_\alpha^2}\frac{\mathrm{i}}{\Delta t}\int_t^{t+\Delta t}\mathrm{d}t'\int_0^{t'}\mathrm{d}t''\int\mathrm{d}^3\mathbf{k}'\int\mathrm{d}^3\mathbf{k}''\int\mathrm{d}^6\mathbf{X}'\int\mathrm{d}^6\mathbf{X}'' \\
& \times\frac{(t'-t'')\mathbf{k}'\mathbf{k}'\cdot\mathbf{k}''}{\varepsilon(\mathbf{k}',\mathbf{k}'\cdot\mathbf{v}')\varepsilon(\mathbf{k}'',\mathbf{k}''\cdot\mathbf{v}'')k'^2k''^2}\langle\delta N_\beta(\mathbf{X}',0)\delta N_\gamma(\mathbf{X}'',0)\rangle \\
& \times\mathrm{e}^{-\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}'+\mathbf{k}''\cdot\mathbf{r}'')}\mathrm{e}^{\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t')-\mathbf{k}'\cdot\mathbf{v}'t'+\mathbf{k}''\cdot\mathbf{r}_\alpha^{(0)}(t'')-\mathbf{k}''\cdot\mathbf{v}''t''\right]}.
\end{aligned}
\tag{67}
$$

Note that when neglecting the influence of the polarization force on the α particle's trajectory

$$
\langle\delta N_\beta(\mathbf{X}',t)\delta N_\gamma(\mathbf{X}'',t)\rangle = \delta_{\beta\gamma}f_\beta(\mathbf{v}',t)\delta(\mathbf{X}'-\mathbf{X}'') + g_{\beta\gamma}(\mathbf{X}',\mathbf{X}'',t), \tag{68}
$$

where $g_{\beta\gamma}(\mathbf{X}',\mathbf{X}'',t)$ is the pair correlation function and assumed to be smooth. Substituting the above equation into Eq. (67), the term corresponding to $g_{\beta\gamma}$ can be neglected, which decays exponentially with t' and t'' when the integrals over $\mathbf{v}'$ and $\mathbf{v}''$ are carried out by recalling that all the zeros of $\varepsilon(\mathbf{k},\omega)$ lie in the lower half ω plane. For the remaining part, carrying out the integral over $\mathbf{X}''$, we obtain

$$
\begin{aligned}
\langle\Delta\mathbf{v}_\alpha\rangle_c = & -\sum_{\beta}\frac{q_\alpha^2 q_\beta^2}{(2\pi)^6\varepsilon_0^2 m_\alpha^2}\frac{\mathrm{i}}{\Delta t}\int_t^{t+\Delta t}\mathrm{d}t'\int_0^{t'}\mathrm{d}t''\int\mathrm{d}^3\mathbf{k}'\int\mathrm{d}^3\mathbf{k}''\int\mathrm{d}^6\mathbf{X}' \\
& \times f_\beta(\mathbf{v}')\frac{(t'-t'')\mathbf{k}'\mathbf{k}'\cdot\mathbf{k}''}{\varepsilon(\mathbf{k}',\mathbf{k}'\cdot\mathbf{v}')\varepsilon(\mathbf{k}'',\mathbf{k}''\cdot\mathbf{v}')k'^2k''^2}\mathrm{e}^{-\mathrm{i}(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{r}'} \\
& \times\mathrm{e}^{\mathrm{i}\left[\mathbf{k}'\cdot\mathbf{r}_\alpha^{(0)}(t')-\mathbf{k}'\cdot\mathbf{v}'t'+\mathbf{k}''\cdot\mathbf{r}_\alpha^{(0)}(t'')-\mathbf{k}''\cdot\mathbf{v}'t''\right]}.
\end{aligned}
\tag{69}
$$

The integral over $\mathbf{r}'$ gives $(2\pi)^3\delta(\mathbf{k}'+\mathbf{k}'')$. Proceeding to carry out the integral over $\mathbf{k}''$ yields

$$
\begin{aligned}
\langle\Delta\mathbf{v}_\alpha\rangle_c = & \sum_{\beta}\frac{q_\alpha^2 q_\beta^2}{(2\pi)^3\varepsilon_0^2 m_\alpha^2}\frac{\mathrm{i}}{\Delta t}\int_t^{t+\Delta t}\mathrm{d}t'\int_0^{t'}\mathrm{d}t''\int\mathrm{d}^3\mathbf{k}\int\mathrm{d}^3\mathbf{v}f_\beta(\mathbf{v}) \\
& \times\frac{(t'-t'')\mathbf{k}}{|\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2k^2}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{v}_\alpha-\mathbf{v})(t'-t'')},
\end{aligned}
\tag{70}
$$

where $\varepsilon(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v}) = \varepsilon^\star(\mathbf{k},\mathbf{k}\cdot\mathbf{v})$ is used with $\varepsilon^\star(\mathbf{k},\mathbf{k}\cdot\mathbf{v})$ being the complex conjugate of $\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})$. Making variable substitution $t''\to t'-t''$ and noticing that the upper limit of the t'' integral can be extended to ∞ since Δt is much larger than the correlation time, we have

$$
\begin{aligned}
\langle\Delta\mathbf{v}_\alpha\rangle_c = & \sum_{\beta}\frac{q_\alpha^2 q_\beta^2}{(2\pi)^3\varepsilon_0^2 m_\alpha^2}\mathrm{i}\int_0^\infty\mathrm{d}t\int\mathrm{d}^3\mathbf{k}\int\mathrm{d}^3\mathbf{v}f_\beta(\mathbf{v})\frac{t\mathbf{k}}{|\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2k^2}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{v}_\alpha-\mathbf{v})t} \\
= & \frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\sum_\beta\frac{q_\alpha^2 q_\beta^2}{8\pi^2\varepsilon_0^2 m_\alpha^2}\int\mathrm{d}^3\mathbf{k}\int\mathrm{d}^3\mathbf{v}f_\beta(\mathbf{v})\frac{\mathbf{k}\mathbf{k}}{|\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2k^4}\delta(\mathbf{k}\cdot\mathbf{v}_\alpha-\mathbf{k}\cdot\mathbf{v}).
\end{aligned}
\tag{71}
$$

According to the definition of $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$, we get from Eq. (57) by keeping terms up to the second order in $\delta\tilde{\phi}$

$$\begin{aligned}\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle = & -\frac{q_\alpha^2}{(2\pi)^2 m_\alpha^2} \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathrm{d}t' \int_t^{t+\Delta t} \mathrm{d}t'' \int \mathrm{d}^3\mathbf{k}' \int_{C'} \mathrm{d}\omega' \\ & \times \int \mathrm{d}^3\mathbf{k}'' \int_{C''} \mathrm{d}\omega'' \, \mathbf{k}'\mathbf{k}'' \langle \delta\tilde{\phi}(\mathbf{k}', \omega') \delta\tilde{\phi}(\mathbf{k}'', \omega'') \rangle \\ & \times \mathrm{e}^{\mathrm{i}\left[\mathbf{k}' \cdot \mathbf{r}_\alpha^{(0)}(t') - \omega' t' + \mathbf{k}'' \cdot \mathbf{r}_\alpha^{(0)}(t'') - \omega'' t''\right]}.\end{aligned} \tag{72}$$

Substituting Eqs. (48) and (49) into the above equation and ignoring the contribution fromthe polarizationforce as $(\langle \Delta \mathbf{v}_\alpha \rangle_p)^2 \Delta t \ll \langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ justified by Hubbard (1961a), we obtain through similar procedures as calculating $\langle \mathbf{v}_\alpha \rangle_c$

$$\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle = \sum_\beta \frac{q_\alpha^2 q_\beta^2}{4\pi^2 \varepsilon_0^2 m_\alpha^2} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v}) \frac{\mathbf{k}\mathbf{k}}{|\varepsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2 k^4} \delta(\mathbf{k}\cdot\mathbf{v}_\alpha - \mathbf{k}\cdot\mathbf{v}). \tag{73}$$

It is evident that both $\langle \Delta \mathbf{v}_\alpha \rangle_c$ and $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ result from the correlation of the fluctuations at different phase space points. They satisfy the following relationship:

$$\langle \Delta \mathbf{v}_\alpha \rangle_c = \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle. \tag{74}$$

In this case, the form of the FP collision term is simplified to

$$\mathcal{C}_\alpha(f_\alpha) = -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{v}_\alpha \rangle_p f_\alpha \right] + \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \right]. \tag{75}$$

As can be seen, the FP collision term consists of a convective term due to the polarization and a diffusion term due to the correlation of the fluctuations. Inserting Eqs. (64) and (73) into the above equation, we obtain

$$\begin{aligned}\mathcal{C}_\alpha(f_\alpha) = & -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha^2 q_\beta^2}{8\pi^2 \varepsilon_0^2 m_\alpha} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{v}_\beta \frac{\delta(\mathbf{k}\cdot\mathbf{v}_\alpha - \mathbf{k}\cdot\mathbf{v}_\beta)}{|\varepsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_\beta)|^2 k^4} \\ & \times \mathbf{k}\mathbf{k} \cdot \left(\frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}_\beta} - \frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}_\alpha} \right) f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta).\end{aligned} \tag{76}$$

This is the BLG collision term (Lenard 1960; Balescu 1960; Guernsey 1962). It automatically includes the Debye screening and the effects of plasma oscillations through introducing $\varepsilon(\mathbf{k}, \omega)$, and thus does not diverge as $k \to 0$ corresponding to $b \to \infty$. However, as $\Delta \mathbf{v}_\alpha$ is calculated in a perturbative manner, the close collisions are not considered appropriately. As a result, the integral over k is divergent as $k \to \infty$ corresponding to $b \to 0$ in the BLG collision term. Usually, an upper cutoff at $1/\lambda_L$ is introduced for k to remove the divergence, corresponding to a lower cutoff at λ_L for b. In the approximation $\varepsilon(\mathbf{k}, \omega) = 1$, the BLG collision term reduces to the

Landau collision term by carrying out the integral over $\mathbf{k}$ with a lower cutoff at $1/\lambda_D$.

2.2 The case with a uniform B

2.2.1 The general form of the FP equation in the presence of a uniform magnetic field

This subsection deals with the case with a uniform $\mathbf{B} = B\hat{\mathbf{e}}_z$ along the z direction. For spatially homogeneous plasmas with $\mathbf{E} = 0$, the α particle obeys the following Newton motion equation:

$$\frac{\mathrm{d}\mathbf{v}_\alpha(t)}{\mathrm{d}t} = \frac{q_\alpha}{m_\alpha}\delta\mathbf{E}\big(\mathbf{r}_\alpha(t),\, t\big) + \Omega_\alpha \mathbf{v}_\alpha(t) \times \hat{\mathbf{e}}_z, \tag{77}$$

where $\Omega_\alpha \equiv q_\alpha B/m_\alpha$ is the gyro-frequency including the sign of the charge. From the above equation, it follows that

$$\Delta\mathbf{v}_\alpha = \Delta\mathbf{v}_\alpha^{\mathbf{B}} + \Delta\mathbf{v}_\alpha^{c}, \tag{78}$$

where

$$\Delta\mathbf{v}_\alpha^{\mathbf{B}} = \big[\mathsf{T}_\alpha(\Delta t) - \mathtt{I}\big] \cdot \mathbf{v}_\alpha(t) \tag{79}$$

with

$$\mathsf{T}_\alpha(t) = \begin{pmatrix} \cos(\Omega_\alpha t) & \sin(\Omega_\alpha t) & 0 \\ -\sin(\Omega_\alpha t) & \cos(\Omega_\alpha t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{80}$$

is the velocity change along the unperturbed helical orbit, i.e., when $\delta\mathbf{E}$ is absent, and

$$\Delta\mathbf{v}_\alpha^{c} = \frac{q_\alpha}{m_\alpha}\int_t^{t+\Delta t} \mathsf{T}_\alpha(t + \Delta t - t') \cdot \delta\mathbf{E}(\mathbf{r}_\alpha(t'),\, t')\,\mathrm{d}t' \tag{81}$$

is the velocity change due to $\delta\mathbf{E}$ which also includes the magnetic field influence.

For weak magnetic field when all the particles' gyration periods are much longer than τ_c, considering τ_c of the order of the plasma oscillation period that is the particles' thermal gyro-radius $\rho_{th\alpha} \equiv v_{th\alpha}/|\Omega_\alpha|$ much larger than $\lambda_{D\alpha}$, it is easy to find Δt satisfying $\tau_c \ll \Delta t \ll \tau_s,\ |\Omega_\alpha|^{-1}$. In this case,

$$\big|\Delta\mathbf{v}_\alpha^{\mathbf{B}}\big| \approx |\Omega_\alpha \mathbf{v}_\alpha \times \hat{\mathbf{e}}_z \Delta t| \ll |\mathbf{v}_\alpha|. \tag{82}$$

Furthermore, $\mathsf{T}_\alpha(t + \Delta t - t')$ in Eq. (81) can be approximated by $\mathtt{I}$ and the magnetic field impact on the particles' trajectories and $\delta\mathbf{E}$ is trivial, implying that the magnetic field effects on the collisions can be neglected. Under this condition, we have

$$\langle \Delta \mathbf{v}_\alpha \rangle \approx \Omega_\alpha \mathbf{v}_\alpha \times \hat{\mathbf{e}}_z + \left\langle \Delta \mathbf{v}_\alpha^c \right\rangle. \tag{83}$$

For $\langle (\Delta \mathbf{v}_\alpha)^n \rangle$ $(n \geq 2)$, we use the binomial expansion

$$\begin{aligned} \langle \left(\Delta \mathbf{v}_\alpha\right)^n \rangle = & \frac{1}{\Delta t}\left(\Delta \mathbf{v}_\alpha^{\mathbf{B}}\right)^n + C_n^1 \left\langle \Delta \mathbf{v}_\alpha^c \right\rangle \left(\Delta \mathbf{v}_\alpha^{\mathbf{B}}\right)^{n-1} + C_n^2 \left\langle \left(\Delta \mathbf{v}_\alpha^c\right)^2 \right\rangle \left(\Delta \mathbf{v}_\alpha^{\mathbf{B}}\right)^{n-2} \\ & + \sum_{m=3}^{n} C_n^m \left\langle \left(\Delta \mathbf{v}_\alpha^c\right)^m \right\rangle \left(\Delta \mathbf{v}_\alpha^{\mathbf{B}}\right)^{n-m}, \end{aligned} \tag{84}$$

where $C_n^m = n!/[m!(n-m)!]$ is the binomial coefficient. As a matter of fact, there are C_n^m different terms in the term $C_n^m \left\langle \left(\Delta \mathbf{v}_\alpha^c\right)^m \right\rangle \left(\Delta \mathbf{v}_\alpha^{\mathbf{B}}\right)^{n-m}$ in the above equation according to different permutations of $\Delta \mathbf{v}_\alpha^c$ and $\Delta \mathbf{v}_\alpha^{\mathbf{B}}$. Since they give rise to the same result when substituted into the FP equation, all these C_n^m terms are represented by $\left\langle \left(\Delta \mathbf{v}_\alpha^c\right)^m \right\rangle \left(\Delta \mathbf{v}_\alpha^{\mathbf{B}}\right)^{n-m}$ in Eq. (84) for brevity. Retaining only the zeroth order terms in δt, we have

$$\langle (\Delta \mathbf{v}_\alpha)^n \rangle \approx \left\langle \left(\Delta \mathbf{v}_\alpha^c\right)^n \right\rangle, \quad (n \geq 2). \tag{85}$$

Substituting Eqs. (83) and (85) into the usual form of the FP equation, we find that the only change compared to the no magnetic field case is the appearance of the convective term $\Omega_\alpha \mathbf{v}_\alpha \times \hat{\mathbf{e}}_z \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha$ due to the Lorentz force $q_\alpha \mathbf{v}_\alpha \times \mathbf{B}$. The FP collision term can be calculated as if there were no magnetic field.

When the magnetic field becomes strong in the sense that $|\Omega_\alpha| \tau_c > 1$, the situation is totally changed. In this case, $\Delta t \gg \tau_c$ implies $|\Omega_\alpha| \Delta t \gg 1$. This indicates that during the collision process the α particle's velocity and the Lorentz force $q_\alpha \mathbf{v}_\alpha \times \mathbf{B}$ acting on it vary notably due to the rapid gyration induced by the magnetic field. Consequently, $\Delta \mathbf{v}_\alpha^{\mathbf{B}}$ is comparable to $\mathbf{v}_\alpha$ itself and cannot be approximated as $\Omega_\alpha \mathbf{v}_\alpha \times \hat{\mathbf{e}}_z \Delta t$ any longer. In light of this, Eq. (83) does not hold any more and one is unable to obtain the term $\Omega_\alpha \mathbf{v}_\alpha \times \hat{\mathbf{e}}_z \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha$ from the term involving $\langle \Delta \mathbf{v}_\alpha \rangle$ in the FP equation as the weak magnetic field case. What is worse, the second and higher order moments $\langle (\Delta \mathbf{v}_\alpha)^n \rangle$ $(n \geq 2)$ are more tricky. As $\Delta \mathbf{v}_\alpha^{\mathbf{B}} \sim \mathbf{v}_\alpha$, every term in Eq. (84) cannot be ignored when required to retain the zeroth order terms in Δt. Even keeping terms only up to the second order in $\delta \mathbf{E}$, the first three terms have to be retained. As a consequence, the terms involving the third and higher order moments $\langle (\Delta \mathbf{v}_\alpha)^n \rangle$ $(n \geq 3)$ cannot be discarded in the FP equation even for distant collisions. In addition, when f_α is not gyrotropic there exists the possibility that $\tau_c > \tau_s$. In this case, one cannot find Δt satisfying $\tau_c \ll \Delta t \ll \tau_s$. In view of the above analyses, it is clear that the usual form of the FP equation without magnetic field is not applicable to strongly magnetized plasmas. The general form of the FP equation in the presence of a uniform magnetic field of arbitrary strength has to be found.

Newman (1973) has demonstrated that the difficulties resulting from the presence of a macroscopic field can be overcome by the coordinate transformation approach. For the case of a uniform magnetic field we are concerned with here, the transformation was found by Dong et al. (2016) to be

$$\mathbf{v}_\alpha = \mathsf{T}_\alpha(t-\tau) \cdot \mathbf{V}_\alpha. \tag{86}$$

The parameter τ is introduced into the transformation just to simplify the appearance of the resultant FP equation and will not change the final result. Inserting Eq. (86) into Eq. (77) yields the motion equation governing the evolution of $\mathbf{V}_\alpha(t)$:

$$\frac{\mathrm{d}\mathbf{V}_\alpha(t)}{\mathrm{d}t} = \frac{q_\alpha}{m_\alpha}\mathsf{T}_\alpha^{-1}(t-\tau)\cdot\delta\mathbf{E}(\mathbf{r}_\alpha(t),\, t), \tag{87}$$

where $\mathsf{T}_\alpha^{-1}(t)$ is the inverse of $\mathsf{T}_\alpha(t)$. Integrating the above equation over t, the change of $\mathbf{V}_\alpha$ during the time interval Δt can be obtained,

$$\Delta\mathbf{V}_\alpha \equiv \mathbf{V}_\alpha(t+\Delta t) - \mathbf{V}_\alpha(t) = \frac{q_\alpha}{m_\alpha}\int_t^{t+\Delta t}\mathsf{T}_\alpha^{-1}(t'-\tau)\cdot\delta\mathbf{E}(\mathbf{r}_\alpha(t'),\, t')\,\mathrm{d}t'. \tag{88}$$

As T_α^{-1} is orthonormal, it will not change the magnitude of $\delta\mathbf{E}$ when acting on it but just makes it rotate counterclockwise around the magnetic field with $\mathbf{\Omega}_\alpha$ as the angular frequency. Under this condition, one can generally find Δt satisfying $\tau_c \ll \Delta t \ll \tau_S$ and $|\Delta\mathbf{V}_\alpha| \ll |\mathbf{V}_\alpha(t)|$, where τ_S is the characteristic time scale of change of the α particle distribution function $F_\alpha(\mathbf{V}_\alpha,\, t)$ in the $\mathbf{V}_\alpha$ coordinate system. Using the relations:

$$F_\alpha(\mathbf{V}_\alpha,\, t) = f_\alpha(\mathbf{v}_\alpha,\, t), \tag{89}$$

$$P_\alpha(\mathbf{V}_\alpha,\, t+\Delta t;\mathbf{V}_{\alpha 0},\, t) = \int p_\alpha(\mathbf{X}_\alpha,\, t+\Delta t;\mathbf{X}_{\alpha 0},\, t)\,\mathrm{d}^3\mathbf{r}_{\alpha 0}, \tag{90}$$

where $P_\alpha(\mathbf{V}_\alpha,\, t+\Delta t;\mathbf{V}_{\alpha 0},\, t)$ is the transition probability in the $\mathbf{V}_\alpha$ coordinate system satisfying the following normalization condition:

$$\int P_\alpha(\mathbf{V}_\alpha,\, t+\Delta t;\mathbf{V}_{\alpha 0},\, t)\,\mathrm{d}^3\mathbf{V}_\alpha = 1, \tag{91}$$

we can obtain from Eq. (1) that

$$F_\alpha(\mathbf{V}_\alpha,\, t+\Delta t) = \int P_\alpha(\mathbf{V}_\alpha,\, t+\Delta t;\mathbf{V}_{\alpha 0},\, t)F_\alpha(\mathbf{V}_{\alpha 0},\, t)\,\mathrm{d}^3\mathbf{V}_{\alpha 0}. \tag{92}$$

In a similar way as deriving Eq. (13) from Eq. (1), the equation governing the evolution of $F_\alpha(\mathbf{V}_\alpha,\, t)$ can be obtained from Eq. (92),

$$\frac{\partial F_\alpha}{\partial t} = -\frac{\partial}{\partial\mathbf{V}_\alpha}\cdot[\langle\Delta\mathbf{V}_\alpha\rangle F_\alpha] + \frac{1}{2}\frac{\partial^2}{\partial\mathbf{V}_\alpha\partial\mathbf{V}_\alpha} : [\langle\Delta\mathbf{V}_\alpha\Delta\mathbf{V}_\alpha\rangle F_\alpha], \tag{93}$$

where $\langle\Delta\mathbf{V}_\alpha\rangle$ and $\langle\Delta\mathbf{V}_\alpha\Delta\mathbf{V}_\alpha\rangle$ are, respectively, the first and second order magnetized FP coefficients defined by

$$\langle\Delta\mathbf{V}_\alpha\rangle = \frac{1}{\Delta t}\int\Delta\mathbf{V}_\alpha P_\alpha(\mathbf{V}_\alpha+\Delta\mathbf{V}_\alpha,\, t+\Delta t;\mathbf{V}_\alpha,\, t)\,\mathrm{d}^3\Delta\mathbf{V}_\alpha, \tag{94}$$

$$\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle = \frac{1}{\Delta t} \int \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha P_\alpha(\mathbf{V}_\alpha + \Delta \mathbf{V}_\alpha,\ t + \Delta t; \mathbf{V}_\alpha,\ t)\, \mathrm{d}^3 \Delta \mathbf{V}_\alpha. \tag{95}$$

$\langle \Delta \mathbf{V}_\alpha \rangle$ and $\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle$ can be calculated based on Eq. (88). Using the inverse transformation from $\mathbf{V}_\alpha$ to $\mathbf{v}_\alpha$ and taking the limit $t \to \tau$, the general form of the FP equation in the presence of a uniform magnetic field can be obtained and reads (Dong et al. 2016)

$$\frac{\partial f_\alpha}{\partial \tau} + \Omega_\alpha \mathbf{v}_\alpha \times \hat{\mathbf{e}}_z \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} = -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot [\langle \Delta \mathbf{V}_\alpha \rangle f_\alpha] + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v}_\alpha \partial \mathbf{v}_\alpha} : [\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle f_\alpha]. \tag{96}$$

To our surprise, the magnetized FP collision term on the RHS of the above equation has the same form as that of the usual FP collision term without magnetic field. However, it should be kept in mind that the magnetized FP coefficients have different physical implications since $\Delta \mathbf{V}_\alpha$ is not the α particle's velocity change in the laboratory frame.

2.2.2 Calculation of $\langle \Delta \mathbf{V}_\alpha \rangle$ and $\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle$ based on the wave theory

The BC theory is not a very satisfactory way to calculate $\langle \Delta \mathbf{V}_\alpha \rangle$ and $\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle$. On one hand, the flux of the BCs is difficult to determine when a magnetic field is present, as the particles' unperturbed trajectories are helical. When the thermal gyro-radii of the two types of particles involved in the BCs are both smaller than the collision scale, the relative velocity of the guiding centres along the magnetic field can be utilized to determine the flux of the BCs. However, in most cases, the magnetic field is not so strong that the guiding center approximation is appropriate for all the BCs. On the other hand, the problem of two charged particles' interaction in the presence of a uniform magnetic field cannot be solved in a closed form. To calculate the particles' velocity changes in a BC, the perturbation method (Geller and Weisheit 1997; Toepffer 2002; Nersisyan et al. 2003; Nersisyan 2003; Möllers et al. 2003; Nersisyan et al. 2007; Nersisyan and Zwicknagel 2009, 2010) is usually employed, making it necessary to introduce a lower cutoff for the impact parameter. In the work of Dong et al. (2016) using the BC theory to calculate the magnetized FP coefficients, the α particle's velocity change in a BC during the time interval Δt was first calculated by use of the perturbation theory and then summed over all the BCs with the background plasma. This kind of BC theory is different from that without magnetic field (Rosenbluth et al. 1957), but resembles the wave theory except the collective interactions are not considered. Therefore, the magnetized FP coefficients are calculated based only on the wave theory here.

The procedure is very similar to that without magnetic field. To distinguish from the no magnetic field case, the physical quantities in the case with a uniform magnetic field are marked with the subscript ($\star$) hereinafter. The distribution function fluctuation $\delta N_{\beta\star}$ can be obtained by integrating the linearized Vlasov equation

$$\frac{\partial \delta N_{\beta\star}}{\partial t} + \mathbf{v}_\beta \cdot \frac{\partial \delta N_{\beta\star}}{\partial \mathbf{r}_\beta} + \frac{q_\beta}{m_\beta}\mathbf{v}_\beta \times \mathbf{B} \cdot \frac{\partial \delta N_{\beta\star}}{\partial \mathbf{v}_\beta} = -\frac{q_\beta}{m_\beta}\delta \mathbf{E}_\star \cdot \frac{\partial f_\beta}{\partial \mathbf{v}_\beta} \tag{97}$$

along the unperturbed helical orbit:

$$\mathbf{v}^{(0)}_{\beta\star}(t) = \mathsf{T}_\beta(t-\tau)\cdot\mathbf{v}_\beta(\tau), \quad \mathbf{r}^{(0)}_{\beta\star}(t) = \mathbf{r}_\beta(\tau) + [\mathsf{H}_\beta(t-\tau) - \mathsf{H}_\beta(0)]\cdot\mathbf{v}_\beta(\tau), \tag{98}$$

where

$$\mathsf{H}_\beta(t) = \int \mathsf{T}_\beta(t)\,\mathrm{d}t = \frac{1}{\Omega_\beta}\begin{pmatrix} \sin(\Omega_\beta t) & -\cos(\Omega_\beta t) & 0 \\ \cos(\Omega_\beta t) & \sin(\Omega_\beta t) & 0 \\ 0 & 0 & \Omega_\beta t \end{pmatrix}. \tag{99}$$

The results are

$$\delta N_{\beta\star} = \delta N^S_{\beta\star} + \delta N^I_{\beta\star}, \tag{100}$$

$$\delta N^S_{\beta\star}(\mathbf{r}_\beta, \mathbf{v}_\beta, t) = \delta N_\beta(\mathbf{r}_\beta - [\mathsf{H}_\beta(0) - \mathsf{H}_\beta(\tau - t)]\cdot\mathbf{v}_\beta, \mathsf{T}^{-1}_\beta(t-\tau)\cdot\mathbf{v}_\beta, \tau), \tag{101}$$

$$\begin{aligned}\delta N^I_{\beta\star}(\mathbf{r}_\beta, \mathbf{v}_\beta, t) = -\frac{q_\beta}{m_\beta}\int_\tau^t \mathrm{d}t'\, \delta\mathbf{E}_\star(\mathbf{r}_\beta - [\mathsf{H}_\beta(0) - \mathsf{H}_\beta(t'-t)]\cdot\mathbf{v}_\beta, t') \\ \cdot \mathsf{T}^{-1}_\beta(t-t')\cdot\frac{\partial f_\beta(\mathsf{T}^{-1}_\beta(t-t')\cdot\mathbf{v}_\beta, t')}{\partial \mathbf{v}_\beta}.\end{aligned} \tag{102}$$

Assuming f_β is gyrotropic and neglecting its change due to the collisions in calculating the fluctuations, $f_\beta(\mathsf{T}^{-1}_\beta(t-t')\cdot\mathbf{v}_\beta, t') = f_\beta(\mathbf{v}_\beta, \tau)$ in Eq. (102). $\delta N^I_{\beta\star}$ thus becomes

$$\begin{aligned}\delta N^I_{\beta\star}(\mathbf{r}_\beta, \mathbf{v}_\beta, t) = -\frac{q_\beta}{m_\beta}\int_\tau^t \mathrm{d}t'\, \delta\mathbf{E}_\star(\mathbf{r}_\beta - [\mathsf{H}_\beta(0) - \mathsf{H}_\beta(t'-t)]\cdot\mathbf{v}_\beta, t') \\ \cdot \mathsf{T}^{-1}_\beta(t-t')\cdot\frac{\partial f_\beta(\mathbf{v}_\beta, \tau)}{\partial \mathbf{v}_\beta}.\end{aligned} \tag{103}$$

Substituting Eqs. (100), (101), and (103) into Eq. (40) and making the Fourier-Laplace transform, we obtain

$$\delta\tilde{\phi}_\star(\mathbf{k}, \omega) = \frac{1}{\varepsilon_0 \varepsilon_\star(\mathbf{k}, \omega) k^2}\sum_\beta q_\beta \int \delta\widetilde{N}^S_{\beta\star}(\mathbf{k}, \mathbf{v}, \omega)\,\mathrm{d}^3\mathbf{v}, \tag{104}$$

where

$$\delta\widetilde{N}^S_{\beta\star}(\mathbf{k},\,\mathbf{v},\,\omega)=\frac{1}{(2\pi)^3}\int_\tau^\infty \mathrm{d}t\int \mathrm{d}^3\mathbf{r}\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\ \times\,\delta N_\beta(\mathbf{r}-[\mathsf{H}_\beta(0)-\mathsf{H}_\beta(\tau-t)]\cdot\mathbf{v},\,\mathsf{T}^{-1}_\beta(t-\tau)\cdot\mathbf{v},\tau). \tag{105}$$

is the Fourier-Laplace transform of $\delta N^S_{\beta\star}(\mathbf{r},\,\mathbf{v},\,t)$ and

$$\varepsilon_\star(\mathbf{k},\,\omega)=1-\sum_\gamma\frac{q_\gamma^2}{\varepsilon_0 m_\gamma k^2}\mathrm{i}\int_0^\infty \mathrm{d}t\int \mathrm{d}^3\mathbf{v}\,\mathbf{k}\cdot\frac{\partial f_\gamma(\mathbf{v})}{\partial\mathbf{v}}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\gamma(t)-\mathsf{H}_\gamma(0)\right]\cdot\mathbf{v}+\mathrm{i}\omega t} \tag{106}$$

is the dielectric response function in the presence of a uniform magnetic field. $\varepsilon_\star(\mathbf{k},\,\omega)$ is defined in the upper half ω plane and can be continued analytically into the real axis and lower half plane as $\varepsilon(\mathbf{k},\,\omega)$ using the way shown in Fig. 1.

Keeping terms up to the second order in $\delta\phi_\star$ in Eq. (88) yields

$$\begin{aligned}\Delta\mathbf{V}_\alpha=&-\frac{q_\alpha}{2\pi m_\alpha}\mathrm{i}\int_\tau^{\tau+\Delta t}\mathrm{d}t\int\mathrm{d}^3\mathbf{k}\int_C\mathrm{d}\omega\,\mathsf{T}^{-1}_\alpha(t-\tau)\cdot\mathbf{k}\delta\tilde{\phi}_\star(\mathbf{k},\,\omega)\mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot\mathbf{r}^{(0)}_{\alpha\star}(t)-\omega t\right]}\\&-\frac{q_\alpha^2}{(2\pi)^2m_\alpha^2}\mathrm{i}\int_\tau^{\tau+\Delta t}\mathrm{d}t\int_\tau^t\mathrm{d}t'\int\mathrm{d}^3\mathbf{k}\int_C\mathrm{d}\omega\int\mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega'\,\mathsf{T}^{-1}_\alpha(t-\tau)\cdot\mathbf{k}\\&\times\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t-t')-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{k}'\delta\tilde{\phi}_\star(\mathbf{k},\,\omega)\delta\tilde{\phi}_\star(\mathbf{k}',\,\omega')\\&\times\mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot\mathbf{r}^{(0)}_{\alpha\star}(t)-\omega t+\mathbf{k}'\cdot\mathbf{r}^{(0)}_{\alpha\star}(t')-\omega't'\right]}.\end{aligned} \tag{107}$$

Based on $\Delta\mathbf{V}_\alpha$ given in the above equation, $\langle\Delta\mathbf{V}_\alpha\rangle$ and $\langle\Delta\mathbf{V}_\alpha\Delta\mathbf{V}_\alpha\rangle$ can be calculated. Like the no magnetic field case, $\langle\Delta\mathbf{V}_\alpha\rangle$ consists of the part $\langle\Delta\mathbf{V}_\alpha\rangle_p$ due to the polarization and the part $\langle\Delta\mathbf{V}_\alpha\rangle_c$ due to the correlation of the fluctuations, corresponding to the two terms on the RHS of the above equation. Substituting Eqs. (104) and (105) into the first term and noting that in its statistical average only the contribution from the α particle itself is not 0 as the no magnetic field case, we obtain after some variable substitutions

$$\begin{aligned}\langle\Delta\mathbf{V}_\alpha\rangle_p=&-\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\frac{\mathrm{i}}{\Delta t}\int_\tau^{\tau+\Delta t}\mathrm{d}t\int_\tau^\infty\mathrm{d}t'\int\mathrm{d}^3\mathbf{k}\int_C\mathrm{d}\omega\int\mathrm{d}^6\mathbf{X}\,\delta(\mathbf{X}-\mathbf{X}_\alpha)\\&\times\frac{\mathsf{T}^{-1}_\alpha(t-\tau)\cdot\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\left\{\mathbf{r}+[\mathsf{H}_\alpha(t'-\tau)-\mathsf{H}_\alpha(0)]\cdot\mathbf{v}\right\}+\mathrm{i}\omega t'}\mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot\mathbf{r}^{(0)}_{\alpha\star}(t)-\omega t\right]}.\end{aligned} \tag{108}$$

Carrying out the integrals over $\mathbf{X}$, substituting $\mathbf{r}^{(0)}_{\alpha\star}(t)$ from Eq. (98) into the above equation, and making the variable substitution $\mathbf{k}\to\mathsf{T}_\alpha(t'-\tau)\cdot\mathbf{k}$, Eq. (108) becomes

$$\begin{aligned}\langle\Delta\mathbf{V}_\alpha\rangle_p=&-\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\frac{\mathrm{i}}{\Delta t}\int_\tau^{\tau+\Delta t}\mathrm{d}t\int_\tau^\infty\mathrm{d}t'\int\mathrm{d}^3\mathbf{k}\int_C\mathrm{d}\omega\,\frac{\mathsf{T}^{-1}_\alpha(t-t')\cdot\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2}\\&\times\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t-t')-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\omega(t-t')}.\end{aligned} \tag{109}$$

From the Fourier-Bessel identity (Gradshteyn and Ryzhik 2007)

$$\mathrm{e}^{\mathrm{i}a\sin\zeta} = \sum_{l=-\infty}^{\infty} J_l(a)\mathrm{e}^{\mathrm{i}l\zeta}, \tag{110}$$

where $J_l(x)$ is the Bessel function of the lth order, we have

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(t-t')\cdot\mathbf{v}_\alpha} &= \mathrm{e}^{\mathrm{i}\left[k_\perp\rho_\alpha\sin\left(\Omega_\alpha(t-t')+\varphi_\mathbf{k}-\varphi_\alpha\right)+k_\parallel v_{\alpha\parallel}(t-t')\right]} \\ &= \sum_{l=-\infty}^{\infty} J_l\left(k_\perp\rho_\alpha\right)\mathrm{e}^{\mathrm{i}\left[\left(l\Omega_\alpha+k_\parallel v_{\alpha\parallel}\right)(t-t')+l\left(\varphi_\mathbf{k}-\varphi_\alpha\right)\right]},\end{aligned} \tag{111}$$

where $\rho_\alpha = v_{\alpha\perp}/\Omega_\alpha$ is the gyro-radius of the α particle; $v_{\alpha\perp}$ and $v_{\alpha\parallel}$ and φ_α are the perpendicular and parallel components and azimuthal angle of $\mathbf{v}_\alpha$, respectively; $k_\perp$ and $k_\parallel$ and $\varphi_\mathbf{k}$ are the perpendicular and parallel components and azimuthal angle of $\mathbf{k}$, respectively. Substituting Eq. (111) into Eq. (109), carrying out the integral over t', moving the contour of the ω integral into the lower half plane in a similar way as that shown in Fig. 2, and using the residue theorem, it is found that for each term in the summation, when only the contribution that does not decay with $t-\tau$ is retained, replacing $\int_\tau^\infty \mathrm{d}t' \int_C \mathrm{d}\omega$ by $\int_{-\infty}^\infty \mathrm{d}t' \int_{-\infty}^\infty \mathrm{d}\omega$ does not change its value. Therefore, $\langle\Delta\mathbf{V}_\alpha\rangle_p$ can be re-expressed as

$$\begin{aligned}\langle\Delta\mathbf{V}_\alpha\rangle_p = &-\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\frac{\mathrm{i}}{\Delta t}\int_\tau^{\tau+\Delta t}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}t'\int\mathrm{d}^3\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\mathsf{T}_\alpha^{-1}(t-t')\cdot\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2} \\ &\times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t-t')-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\omega(t-t')}.\end{aligned} \tag{112}$$

Making the variable substitution $t'\to t-t'$ and carrying out the integral over t gives

$$\begin{aligned}\langle\Delta\mathbf{V}_\alpha\rangle_p = &-\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t'\int\mathrm{d}^3\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\mathsf{T}_\alpha^{-1}(t')\cdot\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2} \\ &\times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t')-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\omega t'}.\end{aligned} \tag{113}$$

Inserting Eqs. (104) and (105) into the second term on the RHS of Eq. (107), we can obtain $\langle\Delta\mathbf{V}_\alpha\rangle_c$,

$$\begin{aligned}\langle\Delta\mathbf{V}_\alpha\rangle_c = &-\sum_{\beta,\gamma}\frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^8\varepsilon_0^2 m_\alpha^2}\frac{\mathrm{i}}{\Delta t}\int_\tau^{\tau+\Delta t}\mathrm{d}t\int_\tau^t\mathrm{d}t'\int_\tau^\infty\mathrm{d}t_1\int_\tau^\infty\mathrm{d}t_2\int\mathrm{d}^3\mathbf{k}\int_C\mathrm{d}\omega \\ &\times\int\mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega'\int\mathrm{d}^6\mathbf{X}\int\mathrm{d}^6\mathbf{X}'\,\mathsf{T}_\alpha^{-1}(t-\tau)\cdot\mathbf{k} \\ &\times\frac{\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t-t')-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{k}'}{\varepsilon_\star(\mathbf{k},\omega)\varepsilon_\star(\mathbf{k}',\omega')k^2k'^2}\left\langle\delta N_\beta(\mathbf{X},\,\tau)\delta N_\gamma(\mathbf{X}',\,\tau)\right\rangle \\ &\times\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\left\{\mathbf{r}+\left[\mathsf{H}_\beta(t_1-\tau)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}\right\}+\mathrm{i}\omega t_1}\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\left\{\mathbf{r}'+\left[\mathsf{H}_\gamma(t_2-\tau)-\mathsf{H}_\gamma(0)\right]\cdot\mathbf{v}'\right\}+\mathrm{i}\omega' t_2} \\ &\times\mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot\mathbf{r}_{\alpha\star}^{(0)}(t)+\mathbf{k}'\cdot\mathbf{r}_{\alpha\star}^{(0)}(t')-\omega t-\omega' t'\right]}.\end{aligned} \tag{114}$$

Substituting Eqs. (68) and (111) into the above equation, carrying out the integrals over t_1 and t_2, moving the contours $\mathcal{C}$ and $\mathcal{C}'$ of the ω and ω' integrals into the lower half planes in a similar way as that shown in Fig. 2, and using the residue theorem, it is found that when only the contributions not decaying with $t-\tau$ and $t'-\tau$ are retained, the term corresponding to $g_{\beta\gamma}$ can be ignored and for the remaining part $\int_\tau^\infty \mathrm{d}t_1 \int_\tau^\infty \mathrm{d}t_2 \int_{\mathcal{C}} \mathrm{d}\omega \int_{\mathcal{C}'} \mathrm{d}\omega'$ can be replaced by $\int_{-\infty}^\infty \mathrm{d}t_1 \int_{-\infty}^\infty \mathrm{d}t_2 \int_{-\infty}^\infty \mathrm{d}\omega \int_{-\infty}^\infty \mathrm{d}\omega'$. After some further calculations, we get

$$\begin{aligned}\langle \Delta \mathbf{V}_\alpha \rangle_c = &\sum_\beta \frac{q_\alpha^2 q_\beta^2}{(2\pi)^4 \varepsilon_0^2 m_\alpha^2} \mathrm{i} \int_0^\infty \mathrm{d}t' \int_{-\infty}^\infty \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^\infty \mathrm{d}\omega \int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v}) \\ &\times \frac{\mathsf{T}_\alpha^{-1}(t') \cdot \mathbf{k}}{|\varepsilon_\star(\mathbf{k},\omega)|^2 k^4} \mathbf{k} \cdot \left[\mathsf{H}_\alpha(t') - \mathsf{H}_\alpha(0)\right] \cdot \mathbf{k} \\ &\times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(t')-\mathsf{H}_\alpha(0)]\cdot \mathbf{v}_\alpha - \mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)]\cdot\mathbf{v} - \mathrm{i}\omega(t'-t_1)}. \end{aligned} \tag{115}$$

For $\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle$, we get from Eq. (107) by keeping terms up to the second order in $\delta\tilde{\phi}_\star$

$$\begin{aligned}\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle = &-\frac{q_\alpha^2}{(2\pi)^2 m_\alpha^2} \frac{1}{\Delta t} \int_\tau^{\tau+\Delta t} \mathrm{d}t \int_\tau^{\tau+\Delta t} \mathrm{d}t' \int \mathrm{d}^3\mathbf{k} \int_{\mathcal{C}} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{k}' \int_{\mathcal{C}'} \mathrm{d}\omega' \\ &\times \mathsf{T}_\alpha^{-1}(t'-\tau) \cdot \mathbf{k}' \mathsf{T}_\alpha^{-1}(t-\tau) \cdot \mathbf{k} \left\langle \delta\tilde{\phi}_\star(\mathbf{k},\omega) \delta\tilde{\phi}_\star(\mathbf{k}',\omega') \right\rangle \\ &\times \mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot\mathbf{r}_{\alpha\star}^{(0)}(t) + \mathbf{k}'\cdot\mathbf{r}_{\alpha\star}^{(0)}(t') - \omega t - \omega' t'\right]}. \end{aligned} \tag{116}$$

Substituting Eqs. (104) and (105) into the above equation, we obtain through a similar procedure as calculating $\langle \Delta \mathbf{V}_\alpha \rangle_c$

$$\begin{aligned}\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle = &\sum_\beta \frac{q_\alpha^2 q_\beta^2}{(2\pi)^4 \varepsilon_0^2 m_\alpha^2} \int_{-\infty}^\infty \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^\infty \mathrm{d}\omega \int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v}) \\ &\times \frac{\mathbf{k}\mathsf{T}_\alpha^{-1}(t)\cdot\mathbf{k}}{|\varepsilon_\star(\mathbf{k},\omega)|^2 k^4} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(t)-\mathsf{H}_\alpha(0)]\cdot \mathbf{v}_\alpha - \mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)]\cdot\mathbf{v} - \mathrm{i}\omega(t-t_1)}. \end{aligned} \tag{117}$$

It can be readily verified that $\langle \Delta \mathbf{V}_\alpha \rangle_c$ and $\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle$ satisfy the following relationship:

$$\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{V}_\alpha \rangle_c f_\alpha\right] = \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle f_\alpha\right], \tag{118}$$

from which it follows that

$$\begin{aligned}&-\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{V}_\alpha \rangle_c f_\alpha\right] + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v}_\alpha \partial \mathbf{v}_\alpha} : \left[\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle f_\alpha\right] \\ &= \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \cdot \langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle\right]. \end{aligned} \tag{119}$$

This part of the magnetized collision term $\mathcal{C}_{\alpha\star}$ is due to the correlation of the fluctuations as the no magnetic field case. Adding the part due to the polarization, $\mathcal{C}_{\alpha\star}$ can thus be expressed as

$$\mathcal{C}_{\alpha\star}(f_\alpha) = -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{V}_\alpha \rangle_p f_\alpha \right] + \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \cdot \langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle \right]. \tag{120}$$

Inserting Eqs. (113) and (117) into the above equation, we obtain through some simple derivation

$$\begin{aligned} \mathcal{C}_{\alpha\star}(f_\alpha) = {} & \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha^2 q_\beta^2}{32\pi^4 \varepsilon_0^2 m_\alpha} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^{\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{v}_\beta \\ & \times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t)-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha - \mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta - \mathrm{i}\omega(t-t_1)} \\ & \times \frac{\mathsf{T}_\alpha^{-1}(t)\cdot\mathbf{k}}{|\varepsilon_\star(\mathbf{k},\omega)|^2 k^4} \mathbf{k} \cdot \left(\frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}_\alpha} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}_\beta} \right) f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta) \\ = {} & \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha^2 q_\beta^2}{32\pi^4 \varepsilon_0^2 m_\alpha} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^{\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{v}_\beta \\ & \times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t)-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha + \mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta - \mathrm{i}\omega(t+t_1)} \frac{\mathbf{k}}{|\varepsilon_\star(\mathbf{k},\omega)|^2 k^4} \\ & \times \left[\mathbf{k}\cdot\mathsf{T}_\alpha(t)\cdot\frac{1}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} - \mathbf{k}\cdot\mathsf{T}_\beta(t_1)\cdot\frac{1}{m_\beta}\frac{\partial}{\partial \mathbf{v}_\beta} \right] f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta). \end{aligned} \tag{121}$$

This is the magnetized BLG collision term (Rostoker and Rosenbluth 1960; Rostoker 1960; Hassan and Watson 1977; Klimontovich 1982; Yoon 2016; Dong et al. 2017), which simultaneously takes into account the collective interactions and magnetic field effects on the collisions. When $\mathbf{B} = 0$, the usual BLG collision term without magnetic field can be easily recovered by carrying out the integrals over ω, t_1, and t in Eq. (121).

3 Derivation of the collision term based on the BBGKY approach

3.1 The case without B

The BBGKY hierarchy of equations, including all the relevant physics, can serve as a good starting point to derive the collision term. The first two equations Montgomery and Tidman (1964) of this hierarchy are written out in the following:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial}{\partial \mathbf{r}_\alpha} \right) f_\alpha(\mathbf{X}_\alpha, t) = \sum_\beta \frac{q_\alpha q_\beta}{4\pi\varepsilon_0 m_\alpha} \int \mathrm{d}^6\mathbf{X}_\beta \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\beta}} \cdot \frac{\partial f_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta, t)}{\partial \mathbf{v}_\alpha}, \tag{122}$$

$$\left[\frac{\partial}{\partial t}+\mathbf{v}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}_\alpha}+\mathbf{v}_\beta\cdot\frac{\partial}{\partial \mathbf{r}_\beta}-\frac{q_\alpha q_\beta}{4\pi\varepsilon_0 m_\alpha}\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\beta}}\cdot\frac{\partial}{\partial \mathbf{v}_\alpha}-(\alpha\leftrightarrow\beta)\right]f_{\alpha\beta}(\mathbf{X}_\alpha,\mathbf{X}_\beta,t)$$
$$=\sum_\gamma\frac{q_\gamma}{4\pi\varepsilon_0}\int \mathrm{d}^6\mathbf{X}_\gamma\left[\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\gamma}}\cdot\frac{\partial}{\partial \mathbf{v}_\alpha}+(\alpha\leftrightarrow\beta)\right]f_{\alpha\beta\gamma}(\mathbf{X}_\alpha,\mathbf{X}_\beta,\mathbf{X}_\gamma,t), \tag{123}$$

where $f_{\alpha\beta}$ and $f_{\alpha\beta\gamma}$ are, respectively, the two-particle and three-particle distribution functions, and we use the symbol $(\alpha\leftrightarrow\beta)$ to mean the immediately preceding expression with α and β interchanged. As the evolution of $f_{\alpha\beta\gamma}$ is governed by the third equation of the hierarchy, Eqs. (122) and (123) are not closed. Making use of the Mayer cluster expansion:

$$f_{\alpha\beta}(\mathbf{X}_\alpha,\mathbf{X}_\beta,t)=f_\alpha(\mathbf{X}_\alpha,t)f_\beta(\mathbf{X}_\beta,t)+g_{\alpha\beta}(\mathbf{X}_\alpha,\mathbf{X}_\beta,t), \tag{124}$$

$$\begin{aligned}f_{\alpha\beta\gamma}(\mathbf{X}_\alpha,\mathbf{X}_\beta,\mathbf{X}_\gamma,t)=&f_\alpha(\mathbf{X}_\alpha,t)f_\beta(\mathbf{X}_\beta,t)f_\gamma(\mathbf{X}_\gamma,t)+f_\alpha(\mathbf{X}_\alpha,t)g_{\beta\gamma}(\mathbf{X}_\beta,\mathbf{X}_\gamma,t)\\&+f_\beta(\mathbf{X}_\beta,t)g_{\alpha\gamma}(\mathbf{X}_\alpha,\mathbf{X}_\gamma,t)+f_\gamma(\mathbf{X}_\gamma,t)g_{\alpha\beta}(\mathbf{X}_\alpha,\mathbf{X}_\beta,t)\\&+g_{\alpha\beta\gamma}(\mathbf{X}_\alpha,\mathbf{X}_\beta,\mathbf{X}_\gamma,t),\end{aligned} \tag{125}$$

and neglecting the terms associated with the triplet correlation function $g_{\alpha\beta\gamma}$ in the binary correlation approximation, Eqs. (122) and (123) become

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}_\alpha}+\frac{q_\alpha\mathbf{E}}{m_\alpha}\cdot\frac{\partial}{\partial \mathbf{v}_\alpha}\right)f_\alpha=\sum_\beta\frac{q_\alpha q_\beta}{4\pi\varepsilon_0 m_\alpha}\int \mathrm{d}^6\mathbf{X}_\beta\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\beta}}\cdot\frac{\partial g_{\alpha\beta}}{\partial \mathbf{v}_\alpha}, \tag{126}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\mathbf{v}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}_\alpha}+\mathbf{v}_\beta\cdot\frac{\partial}{\partial \mathbf{r}_\beta}+\frac{q_\alpha\mathbf{E}}{m_\alpha}\cdot\frac{\partial}{\partial \mathbf{v}_\alpha}+\frac{q_\beta\mathbf{E}}{m_\beta}\cdot\frac{\partial}{\partial \mathbf{v}_\beta}\right)g_{\alpha\beta}\\&=\sum_\gamma\frac{q_\gamma}{4\pi\varepsilon_0}\int \mathrm{d}^6\mathbf{X}_\gamma\left[\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\gamma}}\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha}g_{\beta\gamma}+(\alpha\leftrightarrow\beta)\right]\\&\quad+\frac{q_\alpha q_\beta}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\beta}}\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}-\frac{1}{m_\beta}\frac{\partial}{\partial \mathbf{v}_\beta}\right)\left(f_\alpha f_\beta+g_{\alpha\beta}\right),\end{aligned} \tag{127}$$

where

$$\mathbf{E}(\mathbf{r},t)=-\sum_\beta\frac{q_\beta}{4\pi\varepsilon_0}\int \mathrm{d}^6\mathbf{X}_\beta\frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r}_\beta-\mathbf{r}|}f_\beta(\mathbf{X}_\beta,t). \tag{128}$$

As a result of the approximation made, Eqs. (126) and (127) constitute a closed system for the one-particle distribution functions and pair correlation functions. The RHS of Eq. (126) is the collision term. To obtain it in a closed form, Eq. (127) has to be solved for $g_{\alpha\beta}$, which is exceedingly difficult without making any approximation. The impact of $\mathbf{E}$ on the particles' trajectories is generally trivial during the collision process, so we do not carry it along in the following treatment. $g_{\alpha\beta}$ is comparable to $f_\alpha f_\beta$ for the close collisions. For the weakly coupled plasmas we are concerned with, the collision term is mainly contributed by the distant collisions. As far as the distant collisions are concerned, $g_{\alpha\beta}$ is much smaller than $f_\alpha f_\beta$ and the part involving $g_{\alpha\beta}$ in the second term on the RHS of Eq. (127) can be neglected. In this polarization approximation, Eq. (127) is simplified to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial}{\partial \mathbf{r}_\alpha} + \mathbf{v}_\beta \cdot \frac{\partial}{\partial \mathbf{r}_\beta}\right) g_{\alpha\beta} - \sum_\gamma \frac{q_\gamma}{4\pi\varepsilon_0} \int \mathrm{d}^6\mathbf{X}_\gamma \left[\frac{q_\alpha}{m_\alpha} \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\gamma}} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} g_{\beta\gamma} + (\alpha \leftrightarrow \beta)\right] = S_{\alpha\beta}, \tag{129}$$

where

$$S_{\alpha\beta} = \frac{q_\alpha q_\beta}{4\pi\varepsilon_0} \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\beta}} \cdot \left(\frac{1}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} - \frac{1}{m_\beta}\frac{\partial}{\partial \mathbf{v}_\beta}\right) f_\alpha f_\beta. \tag{130}$$

The left-hand side (LHS) of Eq. (129) is homogeneous with respect to $g_{\alpha\beta}$. Its last two terms describe the polarization, which are absent in the BC model. $S_{\alpha\beta}$ can be viewed as a driving term which acts to create the particle correlations through the Coulomb interactions.

There are many ways to solve Eq. (129). Here, we use the one based on the operator method due to Dupree (Dupree 1961; Montgomery and Tidman 1964), which is easily generalized to magnetized plasmas. We rewrite Eq. (129) in the following form:

$$\frac{\partial g_{\alpha\beta}}{\partial t} + \left(H_\alpha + H_\beta\right) g_{\alpha\beta} = S_{\alpha\beta}, \tag{131}$$

where H_α and H_β are the linear operators acting, respectively, on the variables $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$. Since they act on different variables, they commute. When they act on an arbitrary function $h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$, we have

$$H_\alpha h_{\alpha\beta} = \mathbf{v}_\alpha \cdot \frac{\partial h_{\alpha\beta}}{\partial \mathbf{r}_\alpha} - \frac{q_\alpha}{m_\alpha}\frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \cdot \sum_\gamma \frac{q_\gamma}{4\pi\varepsilon_0} \int h_{\beta\gamma} \frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\gamma}} \mathrm{d}^6\mathbf{X}_\gamma, \tag{132}$$

$$H_\beta h_{\alpha\beta} = \mathbf{v}_\beta \cdot \frac{\partial h_{\alpha\beta}}{\partial \mathbf{r}_\beta} - \frac{q_\beta}{m_\beta}\frac{\partial f_\beta}{\partial \mathbf{v}_\beta} \cdot \sum_\gamma \frac{q_\gamma}{4\pi\varepsilon_0} \int h_{\alpha\gamma} \frac{\partial}{\partial \mathbf{r}_\beta}\frac{1}{r_{\beta\gamma}} \mathrm{d}^6\mathbf{X}_\gamma. \tag{133}$$

Provided we can find an operator $\mathcal{P}_{\alpha\beta}(t)$ satisfying

$$\frac{\partial \mathcal{P}_{\alpha\beta}}{\partial t} + \left(H_\alpha + H_\beta\right)\mathcal{P}_{\alpha\beta} = 0 \tag{134}$$

with the initial condition $\mathcal{P}_{\alpha\beta}(0) = 1$ and an inverse $\mathcal{P}_{\alpha\beta}^{-1}$ defined by $\mathcal{P}_{\alpha\beta}^{-1}\mathcal{P}_{\alpha\beta} = \mathcal{P}_{\alpha\beta}\mathcal{P}_{\alpha\beta}^{-1} = 1$, the solution to Eq. (131) is readily found to be

$$g_{\alpha\beta}(t) = \mathcal{P}_{\alpha\beta}(t) \int_0^t \mathcal{P}_{\alpha\beta}^{-1}(t') S_{\alpha\beta}(t')\,\mathrm{d}t' + \mathcal{P}_{\alpha\beta}(t) g_{\alpha\beta}(t=0), \tag{135}$$

which can be verified by direct substitution.

The solution to Eq. (134) can be written as

$$\mathcal{P}_{\alpha\beta}(t) = \mathcal{P}_\alpha(t)\mathcal{P}_\beta(t), \tag{136}$$

where

$$\mathcal{P}_\alpha(t) = \mathrm{e}^{-H_\alpha t}, \tag{137}$$

$$\mathcal{P}_\beta(t) = \mathrm{e}^{-H_\beta t}. \tag{138}$$

$\mathcal{P}_\alpha(t)$ and $\mathcal{P}_\beta(t)$ satisfy the following equations:

$$\frac{\partial \mathcal{P}_\alpha}{\partial t} + H_\alpha \mathcal{P}_\alpha = 0, \tag{139}$$

$$\frac{\partial \mathcal{P}_\beta}{\partial t} + H_\beta \mathcal{P}_\beta = 0 \tag{140}$$

with the initial conditions $\mathcal{P}_\alpha(0) = \mathcal{P}_\beta(0) = 1$. Using Eqs. (136)–(138), Eq. (135) can be re-expressed as

$$\begin{aligned} g_{\alpha\beta}(t) &= \int_0^t \mathcal{P}_\alpha(t-t')\mathcal{P}_\beta(t-t') S_{\alpha\beta}(t')\,\mathrm{d}t' + \mathcal{P}_\alpha(t)\mathcal{P}_\beta(t) g_{\alpha\beta}(t=0) \\ &= \int_0^t \mathcal{P}_\alpha(t')\mathcal{P}_\beta(t') S_{\alpha\beta}(t-t')\,\mathrm{d}t' + \mathcal{P}_\alpha(t)\mathcal{P}_\beta(t) g_{\alpha\beta}(t=0). \end{aligned} \tag{141}$$

According to Bogoliubov's hypothesis, during the relaxation of $g_{\alpha\beta}$, f_α and f_β can be regarded to be time-independent. Therefore, the time dependence of $S_{\alpha\beta}$ in Eq. (141) can be ignored. $g_{\alpha\beta}(t)$ then becomes

$$g_{\alpha\beta}(t) = \int_0^t \mathcal{P}_\alpha(t')\mathcal{P}_\beta(t') S_{\alpha\beta}\,\mathrm{d}t' + \mathcal{P}_\alpha(t)\mathcal{P}_\beta(t) g_{\alpha\beta}(t=0). \tag{142}$$

The first term on the RHS of the above equation represents the component of $g_{\alpha\beta}$ produced by the Coulomb interactions and will be denoted by $g_{\alpha\beta}^I$ while the second term represents the component of $g_{\alpha\beta}$ associated with its initial value and will be denoted by $g_{\alpha\beta}^S$. From Eq. (126), the part of the collision term corresponding to $g_{\alpha\beta}^I$ is

$$C_\alpha^I(f_\alpha) = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi\varepsilon_0 m_\alpha} \int_0^t \mathrm{d}t' \int \mathrm{d}^6\mathbf{X}_\beta\, \mathcal{P}_\alpha(t')\mathcal{P}_\beta(t') S_{\alpha\beta} \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\beta}}, \tag{143}$$

and the part of the collision term corresponding to $g^S_{\alpha\beta}$ is

$$\mathcal{C}^S_\alpha(f_\alpha) = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi\varepsilon_0 m_\alpha} \int \mathrm{d}^6\mathbf{X}_\beta\, \mathcal{P}_\alpha(t)\mathcal{P}_\beta(t) g_{\alpha\beta}(t=0) \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\beta}}. \tag{144}$$

Applying $\mathcal{P}_\alpha(t_1)$ to $h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$ gives a function $h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta)$:

$$h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta) \equiv \mathcal{P}_\alpha(t_1) h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta). \tag{145}$$

Using Eq. (139), it is easy to prove that $h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta)$ satisfies the following equation:

$$\frac{\partial h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta)}{\partial t_1} + H_\alpha h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta) = 0, \tag{146}$$

which becomes by invoking Eq. (132)

$$\begin{aligned} &\frac{\partial h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta)}{\partial t_1} + \mathbf{v}_\alpha \cdot \frac{\partial h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta)}{\partial \mathbf{r}_\alpha} \\ &- \frac{q_\alpha}{m_\alpha} \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \cdot \sum_\gamma \frac{q_\gamma}{4\pi\varepsilon_0} \int h_{\gamma\beta}(\mathbf{X}_\gamma, t_1, \mathbf{X}_\beta) \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\gamma}}\, \mathrm{d}^6\mathbf{X}_\gamma = 0. \end{aligned} \tag{147}$$

The above equation has the same form as the linearized Vlasov equation as given by Eq. (43). Its solution can be readily obtained through the Fourier-Laplace transform and is

$$\begin{aligned} h_{\alpha\beta}(\mathbf{X}_\alpha, t_1, \mathbf{X}_\beta) = \frac{1}{2\pi} \int \mathrm{d}^3\mathbf{k}_1 \int_{C_1} \mathrm{d}\omega_1 \frac{\mathrm{i}}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_\alpha} \Bigg[\bar{h}_{\alpha\beta}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{X}_\beta) \\ - \frac{q_\alpha}{\varepsilon_0 m_\alpha} \frac{\mathbf{k}_1 \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha}{\varepsilon(\mathbf{k}_1, \omega_1) k_1^2} \sum_\gamma q_\gamma \int \frac{\bar{h}_{\gamma\beta}(\mathbf{k}_1, \mathbf{v}_\gamma, \mathbf{X}_\beta)}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_\gamma}\, \mathrm{d}^3\mathbf{v}_\gamma \Bigg] \mathrm{e}^{\mathrm{i}(\mathbf{k}_1 \cdot \mathbf{r}_\alpha - \omega_1 t_1)}, \end{aligned} \tag{148}$$

where $\bar{h}_{\alpha\beta}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{X}_\beta)$ is the Fourier transform of $h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$ with respect to $\mathbf{r}_\alpha$,

$$\bar{h}_{\alpha\beta}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{X}_\beta) = \frac{1}{(2\pi)^3} \int h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta) \mathrm{e}^{-\mathrm{i}\mathbf{k}_1 \cdot \mathbf{r}_\alpha}\, \mathrm{d}^3\mathbf{r}_\alpha. \tag{149}$$

Similarly, we have

$$
\begin{aligned}
&\mathcal{P}_\alpha(t_1)\mathcal{P}_\beta(t_2)h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta) \\
&\quad = -\frac{1}{(2\pi)^2}\int \mathrm{d}^3\mathbf{k}_1 \int_{\mathcal{C}_1} \mathrm{d}\omega_1 \int \mathrm{d}^3\mathbf{k}_2 \int_{\mathcal{C}_2} \mathrm{d}\omega_2 \frac{1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\alpha}\frac{1}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}_\beta} \\
&\qquad \times \mathrm{e}^{\mathrm{i}\left(\mathbf{k}_1\cdot\mathbf{r}_\alpha - \omega_1 t_1 + \mathbf{k}_2\cdot\mathbf{r}_\beta - \omega_2 t_2\right)} \Bigg\{ \bar{\bar{h}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta) \\
&\qquad - \frac{q_\beta}{\varepsilon_0 m_\beta}\frac{\mathbf{k}_2\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon(\mathbf{k}_2, \omega_2)k_2^2}\sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{\bar{\bar{h}}_{\alpha\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\gamma)}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}_\gamma} \\
&\qquad - \frac{q_\alpha}{\varepsilon_0 m_\alpha}\frac{\mathbf{k}_1\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon(\mathbf{k}_1, \omega_1)k_1^2}\sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\gamma}\Bigg[\bar{\bar{h}}_{\gamma\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\gamma, \mathbf{v}_\beta) \\
&\qquad - \frac{q_\beta}{\varepsilon_0 m_\beta}\frac{\mathbf{k}_2\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon(\mathbf{k}_2, \omega_2)k_2^2}\sum_\sigma q_\sigma \int \mathrm{d}^3\mathbf{v}_\sigma \frac{\bar{\bar{h}}_{\gamma\sigma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\gamma, \mathbf{v}_\sigma)}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}_\sigma}\Bigg]\Bigg\},
\end{aligned} \tag{150}
$$

where $\bar{\bar{h}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta)$ is the Fourier transform of $h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$ with respect to both $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$,

$$
\bar{\bar{h}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta) = \frac{1}{(2\pi)^6}\int \mathrm{d}^3\mathbf{r}_\alpha \int \mathrm{d}^3\mathbf{r}_\beta\, h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta)\mathrm{e}^{\mathrm{i}\left(\mathbf{k}_1\cdot\mathbf{r}_\alpha + \mathbf{k}_2\cdot\mathbf{r}_\beta\right)}. \tag{151}
$$

Replacing $\bar{\bar{h}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta)$ in Eq. (150) by $\bar{\bar{g}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0)$ and then substituting the equation into Eq. (144), $\mathcal{C}_\alpha^S$ can be obtained,

$$
\begin{aligned}
\mathcal{C}_\alpha^S(f_\alpha) = &-\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\sum_\beta \frac{q_\alpha q_\beta}{16\pi^3\varepsilon_0 m_\alpha}\int \mathrm{d}^3\mathbf{k}_1 \int_{\mathcal{C}_1}\mathrm{d}\omega_1 \int \mathrm{d}^3\mathbf{k}_2 \int_{\mathcal{C}_2}\mathrm{d}\omega_2 \int \mathrm{d}^6\mathbf{X}_\beta \\
&\times \frac{1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\alpha}\frac{1}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}_\beta}\mathrm{e}^{\mathrm{i}\left[\mathbf{k}_1\cdot\mathbf{r}_\alpha + \mathbf{k}_2\cdot\mathbf{r}_\beta - (\omega_1+\omega_2)t\right]}\frac{\partial}{\partial\mathbf{r}_\alpha}\frac{1}{r_{\alpha\beta}} \\
&\times \Bigg\{ \bar{\bar{g}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0) \\
&- \frac{q_\beta}{\varepsilon_0 m_\beta}\frac{\mathbf{k}_2\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon(\mathbf{k}_2, \omega_2)k_2^2}\sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{\bar{\bar{g}}_{\alpha\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\gamma, 0)}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}_\gamma} \\
&- \frac{q_\alpha}{\varepsilon_0 m_\alpha}\frac{\mathbf{k}_1\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon(\mathbf{k}_1, \omega_1)k_1^2}\sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\gamma}\Bigg[\bar{\bar{g}}_{\gamma\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\gamma, \mathbf{v}_\beta, 0) \\
&- \frac{q_\beta}{\varepsilon_0 m_\beta}\frac{\mathbf{k}_2\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon(\mathbf{k}_2, \omega_2)k_2^2}\sum_\sigma q_\sigma \int \mathrm{d}^3\mathbf{v}_\sigma \frac{\bar{\bar{g}}_{\gamma\sigma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\gamma, \mathbf{v}_\sigma, 0)}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}_\sigma}\Bigg]\Bigg\}.
\end{aligned} \tag{152}
$$

As the one-particle distribution functions are regarded to be homogeneous during the relaxation of the pair correlation functions, $g_{\alpha\beta}$ depends only on the relative position vector $\mathbf{r}_{\alpha\beta} = \mathbf{r}_\alpha - \mathbf{r}_\beta$. In this case, $\bar{\bar{g}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0)$ assumes the following form:

$$\bar{\bar{g}}_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0) = \bar{g}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0)\delta(\mathbf{k}_1 + \mathbf{k}_2), \tag{153}$$

where

$$\bar{g}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0) = \frac{1}{(2\pi)^3} \int g_{\alpha\beta}(\mathbf{r}_{\alpha\beta}, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0) \mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot\mathbf{r}_{\alpha\beta}}\, \mathrm{d}^3\mathbf{r}_{\alpha\beta}. \tag{154}$$

Inserting Eq. (153) into Eq. (152) and carrying out the integrals over $\mathbf{k}_2$ and $\mathbf{r}_\beta$ gives

$$\begin{aligned} \mathcal{C}_\alpha^S(f_\alpha) = {} & \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi^2 \varepsilon_0 m_\alpha} \mathrm{i} \int \mathrm{d}^3\mathbf{k}_1 \int_{\mathcal{C}_1} \mathrm{d}\omega_1 \int_{\mathcal{C}_2} \mathrm{d}\omega_2 \int \mathrm{d}^3\mathbf{v}_\beta \\ & \times \frac{1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\alpha} \frac{1}{\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_\beta} \frac{\mathbf{k}_1}{k_1^2} \mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t} \Bigg\{ \bar{g}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0) \\ & + \frac{q_\beta}{\varepsilon_0 m_\beta} \frac{\mathbf{k}_1\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon(-\mathbf{k}_1, \omega_2)k_1^2} \sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{\bar{g}_{\overline{\alpha\gamma}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\gamma, 0)}{\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_\gamma} \\ & - \frac{q_\alpha}{\varepsilon_0 m_\alpha} \frac{\mathbf{k}_1\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon(\mathbf{k}_1, \omega_1)k_1^2} \sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\gamma} \Bigg[\bar{g}_{\overline{\gamma\beta}}(\mathbf{k}_1, \mathbf{v}_\gamma, \mathbf{v}_\beta, 0) \\ & + \frac{q_\beta}{\varepsilon_0 m_\beta} \frac{\mathbf{k}_1\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon(-\mathbf{k}_1, \omega_2)k_1^2} \sum_\sigma q_\sigma \int \mathrm{d}^3\mathbf{v}_\sigma \frac{\bar{g}_{\overline{\gamma\sigma}}(\mathbf{k}_1, \mathbf{v}_\gamma, \mathbf{v}_\sigma, 0)}{\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_\sigma} \Bigg] \Bigg\}. \end{aligned} \tag{155}$$

Interchanging the order of the summations over β and γ and using the definition of $\varepsilon(\mathbf{k}, \omega)$ in Eq. (50), the above equation can be simplified to

$$\begin{aligned} \mathcal{C}_\alpha^S(f_\alpha) = {} & \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi^2 \varepsilon_0 m_\alpha} \mathrm{i} \int \mathrm{d}^3\mathbf{k}_1 \int_{\mathcal{C}_1} \mathrm{d}\omega_1 \int_{\mathcal{C}_2} \mathrm{d}\omega_2 \int \mathrm{d}^3\mathbf{v}_\beta \frac{1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\alpha} \\ & \times \frac{1}{\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_\beta} \frac{\mathbf{k}_1}{\varepsilon(-\mathbf{k}_1, \omega_2)k_1^2} \mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t} \Bigg\{ \bar{g}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0) \\ & - \frac{q_\alpha}{\varepsilon_0 m_\alpha} \frac{\mathbf{k}_1\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon(\mathbf{k}_1, \omega_1)k_1^2} \sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{\bar{g}_{\overline{\gamma\beta}}(\mathbf{k}_1, \mathbf{v}_\gamma, \mathbf{v}_\beta, 0)}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_\gamma} \Bigg\}. \end{aligned} \tag{156}$$

Since $\bar{g}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta, 0)$ is smooth, by deforming the contour of the ω_2 integral to the lower half plane in a similar way as that shown in Fig. 2 and using the residue theorem, $\mathcal{C}_\alpha^S$ is readily found to decay exponentially with t when the integrals over ω_2 and $\mathbf{v}_\beta$ are carried out and can thus be neglected for large t.

Proceeding in a similar way as calculating $\mathcal{C}_\alpha^S$, $\mathcal{C}_\alpha^I$ can be obtained,

$$
\begin{aligned}
\mathcal{C}_\alpha^I(f_\alpha) = {} & \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi^2 \varepsilon_0 m_\alpha} \mathrm{i} \int_0^t \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k}_1 \int_{C_1} \mathrm{d}\omega_1 \int_{C_2} \mathrm{d}\omega_2 \int \mathrm{d}^3\mathbf{v}_\beta \frac{1}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_\alpha} \\
& \times \frac{1}{\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_\beta} \frac{\mathbf{k}_1}{\varepsilon(-\mathbf{k}_1, \omega_2) k_1^2} \mathrm{e}^{-\mathrm{i}(\omega_1 + \omega_2) t_1} \Bigg\{ \bar{S}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta) \\
& - \frac{q_\alpha}{\varepsilon_0 m_\alpha} \frac{\mathbf{k}_1 \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha}{\varepsilon(\mathbf{k}_1, \omega_1) k_1^2} \sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{\bar{S}_{\overline{\gamma\beta}}(\mathbf{k}_1, \mathbf{v}_\gamma, \mathbf{v}_\beta)}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_\gamma} \Bigg\},
\end{aligned}
\tag{157}
$$

where

$$
\begin{aligned}
\bar{S}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta) &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}_{\alpha\beta} \frac{q_\alpha q_\beta}{4\pi\varepsilon_0} \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\beta}} \cdot \left(\frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}_\alpha} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}_\beta} \right) f_\alpha f_\beta \\
& \quad \times \mathrm{e}^{-\mathrm{i}\mathbf{k}_1 \cdot \mathbf{r}_{\alpha\beta}} \\
&= \mathrm{i} \frac{q_\alpha q_\beta}{(2\pi)^3 \varepsilon_0} \frac{\mathbf{k}_1}{k_1^2} \cdot \left(\frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}_\alpha} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}_\beta} \right) f_\alpha f_\beta.
\end{aligned}
\tag{158}
$$

Carrying out the integral over t_1, Eq. (157) becomes

$$
\begin{aligned}
\mathcal{C}_\alpha^I(f_\alpha) = {} & -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi^2 \varepsilon_0 m_\alpha} \int \mathrm{d}^3\mathbf{k}_1 \int_{C_1} \mathrm{d}\omega_1 \int_{C_2} \mathrm{d}\omega_2 \int \mathrm{d}^3\mathbf{v}_\beta \frac{1}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_\alpha} \\
& \times \frac{1}{\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_\beta} \frac{\mathrm{e}^{-\mathrm{i}(\omega_1 + \omega_2) t} - 1}{\omega_1 + \omega_2} \frac{\mathbf{k}_1}{\varepsilon(-\mathbf{k}_1, \omega_2) k_1^2} \Bigg[\bar{S}_{\overline{\alpha\beta}}(\mathbf{k}_1, \mathbf{v}_\alpha, \mathbf{v}_\beta) \\
& - \frac{q_\alpha}{\varepsilon_0 m_\alpha} \frac{\mathbf{k}_1 \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha}{\varepsilon(\mathbf{k}_1, \omega_1) k_1^2} \sum_\gamma q_\gamma \int \mathrm{d}^3\mathbf{v}_\gamma \frac{\bar{S}_{\overline{\gamma\beta}}(\mathbf{k}_1, \mathbf{v}_\gamma, \mathbf{v}_\beta)}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_\gamma} \Bigg].
\end{aligned}
\tag{159}
$$

The integrand corresponding to -1 in the factor $\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t} - 1$ in the above equation is analytic in the upper half $\omega_{1(2)}$ plane and proportional to $1/|\omega_{1(2)}|^2$ there as $|\omega_{1(2)}| \to \infty$. Hence, the contour of the $\omega_{1(2)}$ integral for this part of the integrand can be closed around the upper half plane, i.e., adding the semi-circle at $\mathrm{Im}\omega_{1(2)} = \infty$ which contributes nothing, and the integral gives 0. Inserting Eq. (158) into the remaining term proportional to $\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t}$ in Eq. (159), $\mathcal{C}_\alpha^I$ can be separated into two parts: $\mathcal{C}_\alpha^I = \mathcal{C}_\alpha^{IP} + \mathcal{C}_\alpha^{IC}$. $\mathcal{C}_\alpha^{IP}$ corresponds to the part proportional to f_α and $\mathcal{C}_\alpha^{IC}$ to the part proportional to $\partial f_\alpha / \partial \mathbf{v}_\alpha$. $\mathcal{C}_\alpha^{IP}$ is given by

$$
\begin{aligned}
\mathcal{C}_\alpha^{IP}(f_\alpha) = {} & \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^5 \varepsilon_0 m_\alpha} \mathrm{i} \int \mathrm{d}^3\mathbf{k} \int_{C_1} \mathrm{d}\omega_1 \int_{C_2} \mathrm{d}\omega_2 \frac{\mathbf{k}}{k^2} \left[\frac{1}{\varepsilon(-\mathbf{k}, \omega_2)} - 1 \right] \\
& \times \frac{1}{\omega_1 - \mathbf{k} \cdot \mathbf{v}_\alpha} \frac{\mathrm{e}^{-\mathrm{i}(\omega_1 + \omega_2) t}}{\omega_1 + \omega_2} f_\alpha,
\end{aligned}
\tag{160}
$$

where Eq. (50) for $\varepsilon(\mathbf{k}, \omega)$ has been used. The ω_1 integral in the above equation is equal to $-2\pi\mathrm{i}$ times the sum of the residues of the poles at $\omega_1 = -\omega_2$ and $\omega_1 = \mathbf{k} \cdot \mathbf{v}_\alpha$

by closing the integration contour around the lower half plane, i.e., adding the semi-circle at $\mathrm{Im}\omega_1 = -\infty$ which contributes nothing, and using the residue theorem. Equation (160) thus becomes

$$\begin{aligned} \mathcal{C}_\alpha^{IP}(f_\alpha) = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^4 \varepsilon_0 m_\alpha} \int \mathrm{d}^3\mathbf{k} \int_{C_2} \mathrm{d}\omega_2 \, \frac{\mathbf{k}}{k^2} \left[\frac{1}{\varepsilon(-\mathbf{k}, \omega_2)} - 1 \right] \\ \times \frac{\mathrm{e}^{-\mathrm{i}(\omega_2 + \mathbf{k}\cdot\mathbf{v}_\alpha)t} - 1}{\omega_2 + \mathbf{k}\cdot\mathbf{v}_\alpha} f_\alpha. \end{aligned} \tag{161}$$

The integrand corresponding to -1 in the factor $\mathrm{e}^{-\mathrm{i}(\omega_2+\mathbf{k}\cdot\mathbf{v}_\alpha)t} - 1$ in the above equation is analytic in the upper half ω_2 plane and proportional to $1/|\omega_2|^2$ there as $|\omega_2| \to \infty$, so its integral over ω_2 gives 0 by closing the integration contour around the upper half plane.For the integrandcorresponding to $\mathrm{e}^{-\mathrm{i}(\omega_2+\mathbf{k}\cdot\mathbf{v}_\alpha)t}$, deforming the contour of the ω_2 integral to the lower half plane in a similar way as that shown in Fig. 2 and retaining only the contribution from the residue of the pole at $\omega_2 = -\mathbf{k}\cdot\mathbf{v}_\alpha$ which does not decay with t, we obtain

$$\begin{aligned} \mathcal{C}_\alpha^{IP}(f_\alpha) &= -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^3 \varepsilon_0 m_\alpha} \mathrm{i} \int \mathrm{d}^3\mathbf{k} \, \frac{\mathbf{k}}{k^2} \left[\frac{1}{\varepsilon(-\mathbf{k}, -\mathbf{k}\cdot\mathbf{v}_\alpha)} - 1 \right] f_\alpha \\ &= \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^3 \varepsilon_0 m_\alpha} \int \mathrm{d}^3\mathbf{k} \, \frac{\mathbf{k}\varepsilon_i(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_\alpha)}{k^2 |\varepsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_\alpha)|^2} f_\alpha \\ &= -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta \mathbf{v}_\alpha \rangle_p f_\alpha \right]. \end{aligned} \tag{162}$$

Compared with the results obtained based on the FP approach, it can be seen that $\mathcal{C}_\alpha^{IP}$ is due to the polarization.

For $\mathcal{C}_\alpha^{IC}$, interchanging the order of the summations over β and γ in Eq. (159) and using Eq. (50), we have

$$\begin{aligned} \mathcal{C}_\alpha^{IC}(f_\alpha) = -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^5 \varepsilon_0^2 m_\alpha^2} \mathrm{i} \int \mathrm{d}^3\mathbf{k} \int_{C_1} \mathrm{d}\omega_1 \int_{C_2} \mathrm{d}\omega_2 \, \frac{\mathbf{k}\mathbf{k}\cdot\partial f_\alpha/\partial \mathbf{v}_\alpha}{k^4} \frac{1}{\omega_1 - \mathbf{k}\cdot\mathbf{v}_\alpha} \\ \times \frac{\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t}}{\omega_1 + \omega_2} \left[-\frac{u(\mathbf{k}, \omega_1) + u(-\mathbf{k}, \omega_2)}{\varepsilon(\mathbf{k}, \omega_1)\varepsilon(-\mathbf{k}, \omega_2)} + \frac{u(\mathbf{k}, \omega_1)}{\varepsilon(\mathbf{k}, \omega_1)} \right], \end{aligned} \tag{163}$$

where

$$u(\mathbf{k}, \omega) \equiv \sum_\beta q_\beta^2 \int \frac{f_\beta(\mathbf{v})}{\mathbf{k}\cdot\mathbf{v} - \omega} \mathrm{d}^3\mathbf{v}. \tag{164}$$

$u(\mathbf{k}, \omega)$ is defined in the upper half ω plane and can be continued analytically into the real axis and lower half plane as $\varepsilon(\mathbf{k}, \omega)$ by using the way shown in Fig. 1. Leaving $\mathrm{Im}\omega_1$ fixed and moving the contour of the ω_2 integral in Eq. (163) to the lower half plane below the pole at $\omega_2 = -\omega_1$ in a similar way as that shown in Fig. 2, we obtain by retaining only the contribution from the residue of the pole at $\omega_2 = -\omega_1$ which does not decay with t

$$\begin{aligned} \mathcal{C}_\alpha^{IC}(f_\alpha) = &-\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^4 \varepsilon_0^2 m_\alpha^2} \int \mathrm{d}^3\mathbf{k} \int_{C_1} \mathrm{d}\omega_1 \frac{\mathbf{k}\mathbf{k} \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha}{k^4} \frac{1}{\omega_1 - \mathbf{k} \cdot \mathbf{v}_\alpha} \\ &\times \left[-\frac{u(\mathbf{k}, \omega_1) + u(-\mathbf{k}, -\omega_1)}{\varepsilon(\mathbf{k}, \omega_1)\varepsilon(-\mathbf{k}, -\omega_1)} + \frac{u(\mathbf{k}, \omega_1)}{\varepsilon(\mathbf{k}, \omega_1)} \right]. \end{aligned} \tag{165}$$

The integrand corresponding to the second term in the square brackets on the RHS of the above equation is analytic in the upper half ω_1 plane and proportional to $1/|\omega_1|^2$ there as $|\omega_1| \to \infty$, so its integral over ω_1 gives 0 by closing the integration contour around the upper half plane. For the remaining term, moving the contour of the ω_1 integral to the real axis and using the following relation for this case:

$$u(\mathbf{k}, \omega_1) + u(-\mathbf{k}, -\omega_1) = \mathrm{i}2\pi \sum_\beta q_\beta^2 \int f_\beta(\mathbf{v}) \delta(\omega_1 - \mathbf{k} \cdot \mathbf{v}) \, \mathrm{d}^3\mathbf{v}, \tag{166}$$

we obtain

$$\begin{aligned} \mathcal{C}_\alpha^{IC}(f_\alpha) = &\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha^2 q_\beta^2}{(2\pi)^3 \varepsilon_0^2 m_\alpha^2} \mathrm{i} \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^{\infty} \mathrm{d}\omega_1 \int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v}) \\ &\times \frac{\mathbf{k}\mathbf{k} \cdot \partial f_\alpha(\mathbf{v}_\alpha) / \partial \mathbf{v}_\alpha}{|\varepsilon(\mathbf{k}, \omega_1)|^2 k^4} \frac{\delta(\omega_1 - \mathbf{k} \cdot \mathbf{v})}{\omega_1 - \mathbf{k} \cdot \mathbf{v}_\alpha + \mathrm{i}0^+}. \end{aligned} \tag{167}$$

Substituting the Plemelj formula

$$\frac{1}{\omega_1 - \mathbf{k} \cdot \mathbf{v}_\alpha + \mathrm{i}0^+} = \frac{P}{\omega_1 - \mathbf{k} \cdot \mathbf{v}_\alpha} - \mathrm{i}\pi\delta(\omega_1 - \mathbf{k} \cdot \mathbf{v}_\alpha), \tag{168}$$

into the above equation where P denotes the principal value, noting that the term corresponding to $P/(\omega_1 - \mathbf{k} \cdot \mathbf{v}_\alpha)$ is 0 as the integrand becomes to its opposite when ω_1 and $\mathbf{k}$ are, respectively, changed to $-\omega_1$ and $-\mathbf{k}$, and carrying out the integral over ω_1 yields

$$\begin{aligned} \mathcal{C}_\alpha^{IC}(f_\alpha) &= \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha^2 q_\beta^2}{8\pi^2 \varepsilon_0^2 m_\alpha^2} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v}) \frac{\mathbf{k}\mathbf{k} \cdot \partial f_\alpha(\mathbf{v}_\alpha) / \partial \mathbf{v}_\alpha}{|\varepsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 k^4} \\ &\quad \times \delta(\mathbf{k} \cdot \mathbf{v} - \mathbf{k} \cdot \mathbf{v}_\alpha) \\ &= \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left[\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\alpha \rangle \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \right]. \end{aligned} \tag{169}$$

Compared with the results obtained based on the FP approach, it is clear that $\mathcal{C}_\alpha^{IC}$ is due to the correlation of the fluctuations. Combining Eqs. (162) and (169) gives the BLG collision tern.

3.2 The case with a uniform B

In the presence of a uniform magnetic field, the collision term $\mathcal{C}_{\alpha\star}$ is still given by the RHS of Eq. (126),

$$\mathcal{C}_{\alpha\star}(f_\alpha) = \sum_\beta \frac{q_\alpha q_\beta}{4\pi\epsilon_0 m_\alpha} \int \mathrm{d}^6\mathbf{X}_\beta \frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta\star}}{\partial \mathbf{v}_\alpha}, \tag{170}$$

where $g_{\alpha\beta\star}$ satisfies the following evolution equation:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial}{\partial \mathbf{r}_\alpha} + \mathbf{v}_\beta \cdot \frac{\partial}{\partial \mathbf{r}_\beta} + \frac{q_\alpha}{m_\alpha}\mathbf{v}_\alpha \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{v}_\alpha} + \frac{q_\beta}{m_\beta}\mathbf{v}_\beta \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{v}_\beta}\right) g_{\alpha\beta\star} \\ &\quad - \sum_\gamma \frac{q_\gamma}{4\pi\varepsilon_0}\int \mathrm{d}^6\mathbf{X}_\gamma \left[\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\gamma}} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} g_{\beta\gamma\star} + (\alpha \leftrightarrow \beta)\right] = S_{\alpha\beta}.\end{aligned} \tag{171}$$

As in the no magnetic field case, $\mathcal{C}_{\alpha\star}$ can be expressed as the sum of two terms:

$$\mathcal{C}_{\alpha\star}(f_\alpha) = \mathcal{C}^S_{\alpha\star}(f_\alpha) + \mathcal{C}^I_{\alpha\star}(f_\alpha), \tag{172}$$

where

$$\mathcal{C}^I_{\alpha\star}(f_\alpha) = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi\varepsilon_0 m_\alpha}\int_0^t \mathrm{d}t' \int \mathrm{d}^6\mathbf{X}_\beta\, \mathcal{P}_{\alpha\star}(t')\mathcal{P}_{\beta\star}(t') S_{\alpha\beta}\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\beta}}, \tag{173}$$

$$\mathcal{C}^S_{\alpha\star}(f_\alpha) = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{4\pi\varepsilon_0 m_\alpha}\int \mathrm{d}^6\mathbf{X}_\beta\, \mathcal{P}_{\alpha\star}(t)\mathcal{P}_{\beta\star}(t) g_{\alpha\beta}(t=0)\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\beta}} \tag{174}$$

represent the part produced by the Coulomb interactions and that associated with the initial pair correlation, respectively. For simplicity, $g_{\alpha\beta}(t=0) = 0$ is assumed here, so $\mathcal{C}^S_{\alpha\star} = 0$. As a matter of fact, it can be easily proven that $\mathcal{C}^S_{\alpha\star}$ decays exponentially with t when $g_{\alpha\beta}(t=0) \neq 0$ like the no magnetic field case and can thus be neglected for large t. To calculate $\mathcal{C}^I_{\alpha\star}$, $\mathcal{P}_{\alpha\star}(t')\mathcal{P}_{\beta\star}(t')S_{\alpha\beta}$ has to be determined first.

Acting $\mathcal{P}_{\alpha\star}(t)$ on $h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$ gives a function $h_{\alpha\beta\star}(\mathbf{X}_\alpha, t, \mathbf{X}_\beta) \equiv \mathcal{P}_{\alpha\star}(t)h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$ which satisfies the following equation:

$$\begin{aligned}&\frac{\partial h_{\alpha\beta\star}(\mathbf{X}_\alpha, t, \mathbf{X}_\beta)}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial h_{\alpha\beta\star}(\mathbf{X}_\alpha, t, \mathbf{X}_\beta)}{\partial \mathbf{r}_\alpha} + \frac{q_\alpha}{m_\alpha}\mathbf{v}_\alpha \times \mathbf{B} \cdot \frac{\partial h_{\alpha\beta\star}(\mathbf{X}_\alpha, t, \mathbf{X}_\beta)}{\partial \mathbf{v}_\alpha} \\ &\quad - \frac{q_\alpha}{m_\alpha}\frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} \cdot \sum_\gamma \frac{q_\gamma}{4\pi\varepsilon_0}\int h_{\gamma\beta\star}(\mathbf{X}_\gamma, t, \mathbf{X}_\beta)\frac{\partial}{\partial \mathbf{r}_\alpha}\frac{1}{r_{\alpha\gamma}}\,\mathrm{d}^6\mathbf{X}_\gamma = 0.\end{aligned} \tag{175}$$

By the Fourier-Laplace transform, the solution of the above equation is found to be

$$
\begin{aligned}
&h_{\alpha\beta\star}(\mathbf{X}_\alpha, \tau_1, \mathbf{X}_\beta) \\
&\quad = \frac{1}{2\pi}\int_0^\infty \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{\mathcal{C}} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left\{\mathbf{r}_\alpha - \left[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)\right]\cdot\mathbf{v}_\alpha\right\}+\mathrm{i}\omega(t_1-\tau_1)} \\
&\qquad \times\left\{\bar{h}_{\alpha\beta}\left(\mathbf{k}, \mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{v}_\alpha, \mathbf{X}_\beta\right) + \mathrm{i}\frac{q_\alpha}{\varepsilon_0 m_\alpha}\frac{\mathbf{k}\cdot\mathsf{T}_\alpha^{-1}(t_1)\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon_\star(\mathbf{k}, \omega)k^2}\right. \\
&\qquad \left.\times\sum_\gamma q_\gamma \int_0^\infty \mathrm{d}t_2 \int \mathrm{d}^3\mathbf{v}_\gamma\, \bar{h}_{\gamma\beta}(\mathbf{k}, \mathbf{v}_\gamma, \mathbf{X}_\beta)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\gamma(t_2)-\mathsf{H}_\gamma(0)\right]\cdot\mathbf{v}_\gamma+\mathrm{i}\omega t_2}\right\}.
\end{aligned}
\tag{176}
$$

Similarly, we obtain

$$
\begin{aligned}
&\mathcal{P}_{\alpha\star}(\tau_1)\mathcal{P}_{\beta\star}(\tau_2)h_{\alpha\beta}(\mathbf{X}_\alpha, \mathbf{X}_\beta) \\
&\quad = \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_3 \int \mathrm{d}^3\mathbf{k}_1 \int_{\mathcal{C}_1}\mathrm{d}\omega_1 \int \mathrm{d}^3\mathbf{k}_2 \int_{\mathcal{C}_2}\mathrm{d}\omega_2 \\
&\qquad \times \mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\left\{\mathbf{r}_\alpha-\left[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)\right]\cdot\mathbf{v}_\alpha\right\}+\mathrm{i}\omega_1(t_1-\tau_1)}\mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\left\{\mathbf{r}_\beta-\left[\mathsf{H}_\beta(0)-\mathsf{H}_\beta(-t_3)\right]\cdot\mathbf{v}_\beta\right\}+\mathrm{i}\omega_2(t_3-\tau_2)} \\
&\qquad \times\left\{\bar{\bar{h}}_{\alpha\beta}\left(\mathbf{k}_1, \mathbf{k}_2, \mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{v}_\alpha, \mathsf{T}_\beta^{-1}(t_3)\cdot\mathbf{v}_\beta\right) + \mathrm{i}\frac{q_\beta}{\varepsilon_0 m_\beta}\frac{\mathbf{k}_2\cdot\mathsf{T}_\beta^{-1}(t_3)\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon_\star(\mathbf{k}_2, \omega_2)k_2^2}\right. \\
&\qquad \times\sum_\gamma q_\gamma\int_0^\infty \mathrm{d}t_4 \int \mathrm{d}^3\mathbf{v}_\gamma\, \bar{\bar{h}}_{\alpha\gamma}\left(\mathbf{k}_1, \mathbf{k}_2, \mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{v}_\alpha, \mathbf{v}_\gamma\right)\mathrm{e}^{-\mathrm{i}\mathbf{k}_2\cdot\left[\mathsf{H}_\gamma(t_4)-\mathsf{H}_\gamma(0)\right]\cdot\mathbf{v}_\gamma+\mathrm{i}\omega_2 t_4} \\
&\qquad +\mathrm{i}\frac{q_\alpha}{\varepsilon_0 m_\alpha}\frac{\mathbf{k}_1\cdot\mathsf{T}_\alpha^{-1}(t_1)\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon_\star(\mathbf{k}_1, \omega_1)k_1^2}\sum_\gamma q_\gamma\int_0^\infty \mathrm{d}t_2\int\mathrm{d}^3\mathbf{v}_\gamma\, \mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot\left[\mathsf{H}_\gamma(t_2)-\mathsf{H}_\gamma(0)\right]\cdot\mathbf{v}_\gamma+\mathrm{i}\omega_1 t_2} \\
&\qquad \times\left[\bar{\bar{h}}_{\gamma\beta}\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\gamma, \mathsf{T}_\beta^{-1}(t_3)\cdot\mathbf{v}_\beta\right) + \mathrm{i}\frac{q_\beta}{\varepsilon_0 m_\beta}\frac{\mathbf{k}_2\cdot\mathsf{T}_\beta^{-1}(t_3)\cdot\partial f_\beta/\partial\mathbf{v}_\beta}{\varepsilon_\star(\mathbf{k}_2, \omega_2)k_2^2}\right. \\
&\qquad \left.\left.\times\sum_\sigma q_\sigma\int_0^\infty \mathrm{d}t_4\int\mathrm{d}^3\mathbf{v}_\sigma\, \bar{\bar{h}}_{\gamma\sigma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{v}_\gamma, \mathbf{v}_\sigma)\mathrm{e}^{-\mathrm{i}\mathbf{k}_2\cdot\left[\mathsf{H}_\sigma(t_4)-\mathsf{H}_\sigma(0)\right]\cdot\mathbf{v}_\sigma+\mathrm{i}\omega_2 t_4}\right]\right\}.
\end{aligned}
\tag{177}
$$

Replacing $\bar{\bar{h}}_{\alpha\beta}$ in the above equation by $\bar{\bar{S}}_{\alpha\beta} = \bar{S}_{\overline{\alpha\beta}}\delta(\mathbf{k}_1 + \mathbf{k}_2)$ and then substituting the equation into Eq. (173), we obtain through some similar manipulations as those in the no magnetic field case

$$
\begin{aligned}
\mathcal{C}^{I}_{\alpha\star}(f_\alpha) = &-\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^2\varepsilon_0 m_\alpha}\mathrm{i}\int_0^t \mathrm{d}t' \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 \int \mathrm{d}^3\mathbf{k}_1 \int_{\mathcal{C}_1}\mathrm{d}\omega_1 \\
&\times \int_{\mathcal{C}_2}\mathrm{d}\omega_2 \int \mathrm{d}^3\mathbf{v}_\beta \,\frac{\mathbf{k}_1}{\varepsilon_\star(-\mathbf{k}_1,\,\omega_2)k_1^2}\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t'} \\
&\times \mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)]\cdot\mathbf{v}_\alpha+\mathrm{i}\omega_1 t_1}\mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot[\mathsf{H}_\beta(t_2)-\mathsf{H}_\beta(0)]\cdot\mathbf{v}_\beta+\mathrm{i}\omega_2 t_2} \\
&\times \left\{ \bar{S}_{\overline{\alpha\beta}}\big(\mathbf{k}_1,\,\mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{v}_\alpha,\,\mathbf{v}_\beta\big) + \mathrm{i}\frac{q_\alpha}{\varepsilon_0 m_\alpha}\frac{\mathbf{k}_1\cdot\mathsf{T}_\alpha^{-1}(t_1)\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon_\star(\mathbf{k}_1,\,\omega_1)k_1^2} \right. \\
&\left. \times \sum_\gamma q_\gamma \int_0^\infty \mathrm{d}t_3 \int \mathrm{d}^3\mathbf{v}_\gamma\, \bar{S}_{\overline{\gamma\beta}}(\mathbf{k}_1,\,\mathbf{v}_\gamma,\,\mathbf{v}_\beta)\mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot[\mathsf{H}_\gamma(t_3)-\mathsf{H}_\gamma(0)]\cdot\mathbf{v}_\gamma+\mathrm{i}\omega_1 t_3} \right\}.
\end{aligned}
\tag{178}
$$

The integral over t' of $\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t'}$ in the above equation gives $\mathrm{i}\left[\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t}-1\right]/(\omega_1+\omega_2)$. Carrying out the integrals over t_1 and t_2, the integrand corresponding to the factor $-\mathrm{i}/(\omega_1+\omega_2)$ in Eq. (178) is found to be analytic in the upper half $\omega_{1(2)}$ plane and approaches $1/|\omega_{1(2)}|^2$ there as $|\omega_{1(2)}|\to\infty$. Therefore, its integral over $\omega_{1(2)}$ is 0 by closing the integration contour around the upper half plane. Substituting $\bar{S}_{\overline{\alpha\beta}}$ in Eq. (158), the remaining termproportional to $\mathrm{i}\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t}/(\omega_1+\omega_2)$ in Eq. (178) is separated into two parts $\mathcal{C}^{IP}_{\alpha\star}$ and $\mathcal{C}^{IC}_{\alpha\star}$ like the no magnetic field case, corresponding to the parts involving f_α and $\partial f_\alpha/\partial\mathbf{v}_\alpha$, respectively. Using $\varepsilon_\star(\mathbf{k},\,\omega)$ in Eq. (106), $\mathcal{C}^{IP}_{\alpha\star}$ can be expressed as

$$
\begin{aligned}
\mathcal{C}^{IP}_{\alpha\star}(f_\alpha) = &\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^5\varepsilon_0 m_\alpha}\int_0^\infty \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k}\int_{\mathcal{C}_1}\mathrm{d}\omega_1\int_{\mathcal{C}_2}\mathrm{d}\omega_2\,\frac{\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t}}{\omega_1+\omega_2}\frac{\mathbf{k}}{k^2} \\
&\times\left[\frac{1}{\varepsilon_\star(-\mathbf{k},\,\omega_2)}-1\right]\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)]\cdot\mathbf{v}_\alpha+\mathrm{i}\omega_1 t_1}f_\alpha.
\end{aligned}
\tag{179}
$$

Expanding $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(-t_1)\cdot\mathbf{v}_\alpha}$ in the above equation into a series of Bessel functions using Eq. (111) and carrying out the integral over t_1 yields

$$
\begin{aligned}
\mathcal{C}^{IP}_{\alpha\star}(f_\alpha) = &\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^5\varepsilon_0 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\int_{\mathcal{C}_1}\mathrm{d}\omega_1\int_{\mathcal{C}_2}\mathrm{d}\omega_2\sum_{m=-\infty}^{\infty}\frac{J_m(k_\perp\rho_\alpha)\mathrm{e}^{\mathrm{i}m(\varphi_\mathbf{k}-\varphi_\alpha)}}{\omega_1-m\Omega_\alpha-k_\parallel v_{\alpha\parallel}} \\
&\times\frac{\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t}}{\omega_1+\omega_2}\frac{\mathbf{k}}{k^2}\left[\frac{1}{\varepsilon_\star(-\mathbf{k},\,\omega_2)}-1\right]\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(0)\cdot\mathbf{v}_\alpha}f_\alpha.
\end{aligned}
\tag{180}
$$

Carrying out the integral over ω_1 by closing the integration contour around the lower half plane, the result is equal to $-2\pi\mathrm{i}$ times the sum of the residues of the poles at $\omega_1=-\omega_2$ and $\omega_1=m\Omega_\alpha+k_\parallel v_{\alpha\parallel}$. Therefore, Eq. (180) becomes

$$
\begin{aligned}
\mathcal{C}_{\alpha\star}^{IP}(f_\alpha) = \frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\int \mathrm{d}^3\mathbf{k}\int_{\mathcal{C}_2}\mathrm{d}\omega_2\sum_{m=-\infty}^{\infty}\frac{J_m(k_\perp\rho_\alpha)\mathrm{e}^{\mathrm{i}m(\varphi_\mathbf{k}-\varphi_\alpha)}}{\omega_2+m\Omega_\alpha+k_\parallel v_{\alpha\parallel}} \\
\times\left[\mathrm{e}^{-\mathrm{i}(\omega_2+m\Omega_\alpha+k_\parallel v_{\alpha\parallel})t}-1\right]\left[\frac{1}{\varepsilon_\star(-\mathbf{k},\,\omega_2)}-1\right]\frac{\mathbf{k}}{k^2}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(0)\cdot\mathbf{v}_\alpha}f_\alpha.
\end{aligned}
\tag{181}
$$

The integrand corresponding to -1 in the factor $\mathrm{e}^{-\mathrm{i}(\omega_2+m\Omega_\alpha+k_\parallel v_{\alpha\parallel})t}-1$ in the above equation is analytic in the upper half ω_2 plane and approaches $1/|\omega_2|^2$ there as $|\omega_2|\to\infty$, so its integral over ω_2 is 0 by closing the integration contour around the upper half plane. For the remaining term proportional to $\mathrm{e}^{-\mathrm{i}(\omega_2+m\Omega_\alpha+k_\parallel v_{\alpha\parallel})t}$, moving the contour of the ω_2 integral into the lower half plane in a similar way as that shown in Fig. 2 and retaining only the contributions from the residues of the poles at $\omega_2=-m\Omega_\alpha-k_\parallel v_{\alpha\parallel}$ not decaying with t, we get

$$
\begin{aligned}
\mathcal{C}_{\alpha\star}^{IP}(f_\alpha) = -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^3\varepsilon_0 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\sum_{m=-\infty}^{\infty}J_m(k_\perp\rho_\alpha)\mathrm{e}^{\mathrm{i}m(\varphi_\mathbf{k}-\varphi_\alpha)} \\
\times\left[\frac{1}{\varepsilon_\star(-\mathbf{k},\,-m\Omega_\alpha-k_\parallel v_{\alpha\parallel})}-1\right]\frac{\mathbf{k}}{k^2}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(0)\cdot\mathbf{v}_\alpha}f_\alpha.
\end{aligned}
\tag{182}
$$

The integrand corresponding to the term -1 in the square brackets on the RHS of the above equation is an odd function of $\mathbf{k}$ by noting that

$$
\sum_{m=-\infty}^{\infty}J_m(k_\perp\rho_\alpha)\mathrm{e}^{\mathrm{i}m(\varphi_\mathbf{k}-\varphi_\alpha)}=\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(0)\cdot\mathbf{v}_\alpha},
\tag{183}
$$

so its integral over $\mathbf{k}$ gives 0. Then $\mathcal{C}_{\alpha\star}^{IP}$ becomes

$$
\begin{aligned}
\mathcal{C}_{\alpha\star}^{IP}(f_\alpha) = -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^3\varepsilon_0 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\sum_{m=-\infty}^{\infty}\frac{J_m(k_\perp\rho_\alpha)\mathrm{e}^{\mathrm{i}m(\varphi_\mathbf{k}-\varphi_\alpha)}}{\varepsilon_\star(-\mathbf{k},\,-m\Omega_\alpha-k_\parallel v_{\alpha\parallel})}\frac{\mathbf{k}}{k^2} \\
\times\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(0)\cdot\mathbf{v}_\alpha}f_\alpha.
\end{aligned}
\tag{184}
$$

Using Eq. (111), $\mathcal{C}_{\alpha\star}^{IP}$ can also be expressed as

$$
\begin{aligned}
\mathcal{C}_{\alpha\star}^{IP}(f_\alpha) = -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t_1\int \mathrm{d}^3\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega_2\frac{\mathbf{k}}{\varepsilon_\star(-\mathbf{k},\,\omega_2)k^2} \\
\times\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(-t_1)-\mathsf{H}_\alpha(0)]\cdot\mathbf{v}_\alpha-\mathrm{i}\omega_2 t_1}f_\alpha.
\end{aligned}
\tag{185}
$$

Making the variable substitution $\mathbf{k}\to-\mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{k}$, the above equation becomes

$$
\begin{aligned}
\mathcal{C}_{\alpha\star}^{IP}(f_\alpha) &= \frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t_1\int \mathrm{d}^3\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega_2\frac{\mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega_2)k^2} \\
&\quad\times\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(t_1)-\mathsf{H}_\alpha(0)]\cdot\mathbf{v}_\alpha-\mathrm{i}\omega_2 t_1}f_\alpha \\
&= -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left[\langle\Delta\mathbf{V}_\alpha\rangle_p f_\alpha\right].
\end{aligned}
\tag{186}
$$

Compared with the results obtained based on the FP approach, it is found that $\mathcal{C}^{IP}_{\alpha\star}$ is due to the polarization.

Interchanging the order of the summations over β and γ in Eq. (178) and using Eq. (106), we obtain the expression for $\mathcal{C}^{IC}_{\alpha\star}$

$$\mathcal{C}^{IC}_{\alpha\star}(f_\alpha) = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^5 \varepsilon_0^2 m_\alpha^2} \int_0^\infty \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{\mathcal{C}_1} \mathrm{d}\omega_1 \int_{\mathcal{C}_2} \mathrm{d}\omega_2 \\ \times \frac{\mathbf{kk} \cdot \mathsf{T}_\alpha^{-1}(t_1) \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha}{k^4} \frac{\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t}}{\omega_1 + \omega_2} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)]\cdot\mathbf{v}_\alpha + \mathrm{i}\omega_1 t_1} \\ \times \left[\frac{u_\star(-\mathbf{k}, \omega_2) + u_\star(\mathbf{k}, \omega_1)}{\varepsilon_\star(-\mathbf{k}, \omega_2)\varepsilon_\star(\mathbf{k}, \omega_1)} - \frac{u_\star(\mathbf{k}, \omega_1)}{\varepsilon_\star(\mathbf{k}, \omega_1)} \right], \tag{187}$$

where

$$u_\star(\mathbf{k}, \omega) \equiv \sum_\beta q_\beta^2 \mathrm{i} \int_0^\infty \mathrm{d}t \int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\beta(t)-\mathsf{H}_\beta(0)]\cdot\mathbf{v}_\beta + \mathrm{i}\omega t}. \tag{188}$$

$u_\star(\mathbf{k}, \omega)$ is defined in the upper half ω plane and can be continued analytically into the real axis and lower half plane as $\varepsilon(\mathbf{k}, \omega)$ using the way shown in Fig. 1. Carrying out the ω_2 integral by moving the integration contour into the lower half plane below the pole at $\omega_2 = -\omega_1$ in a similar way as that shown in Fig. 2 and retaining only the contribution from the residue of the pole at $\omega_2 = -\omega_1$ which does not decay with t, we have

$$\mathcal{C}^{IC}_{\alpha\star}(f_\alpha) = -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \frac{q_\alpha^2}{(2\pi)^4 \varepsilon_0^2 m_\alpha^2} \mathrm{i} \int_0^\infty \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{\mathcal{C}_1} \mathrm{d}\omega_1 \\ \times \frac{\mathbf{kk} \cdot \mathsf{T}_\alpha^{-1}(t_1) \cdot \partial f_\alpha / \partial \mathbf{v}_\alpha}{k^4} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)]\cdot\mathbf{v}_\alpha + \mathrm{i}\omega_1 t_1} \\ \times \left[\frac{u_\star(-\mathbf{k}, -\omega_1) + u_\star(\mathbf{k}, \omega_1)}{\varepsilon_\star(-\mathbf{k}, -\omega_1)\varepsilon_\star(\mathbf{k}, \omega_1)} - \frac{u_\star(\mathbf{k}, \omega_1)}{\varepsilon_\star(\mathbf{k}, \omega_1)} \right]. \tag{189}$$

The integrand corresponding to the second term in the square brackets on the RHS of the above equation is analytic in the upper half ω_1 plane and found to approach $1/|\omega_1|^2$ there as $|\omega_1| \to \infty$ by carrying out the integral over t_1, so its integral over ω_1 is 0 by closing the integration contour around the upper half plane. For the remaining term, pushing the contour of the ω_1 integral to the real axis, and noting that in this case

$$u_\star(-\mathbf{k}, -\omega_1) + u_\star(\mathbf{k}, \omega_1) = \sum_\beta q_\beta^2 \mathrm{i} \int_{-\infty}^\infty \mathrm{d}t \int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta) \\ \times \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\beta(t)-\mathsf{H}_\beta(0)]\cdot\mathbf{v}_\beta + \mathrm{i}\omega_1 t}, \tag{190}$$

we have

$$\mathcal{C}_{\alpha\star}^{IC}(f_\alpha)=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta\frac{q_\alpha^2 q_\beta^2}{(2\pi)^4\varepsilon_0^2 m_\alpha^2}\int_0^\infty \mathrm{d}t_1\int_{-\infty}^\infty \mathrm{d}t_2\int \mathrm{d}^3\mathbf{k}\int_{\mathcal{C}_1}\mathrm{d}\omega_1$$
$$\times\int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta)\frac{\mathbf{k}\mathbf{k}\cdot\mathsf{T}_\alpha^{-1}(t_1)\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{|\varepsilon_\star(\mathbf{k},\,\omega_1)|^2k^4} \tag{191}$$
$$\times\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_2)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta+\mathrm{i}\omega_1\left(t_1+t_2\right)}.$$

Making the variable substitutions $\omega_1\rightarrow-\omega_1$, $t_2\rightarrow-t_2$, and $\mathbf{k}\rightarrow-\mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{k}$, $\mathcal{C}_{\alpha\star}^{IC}$ becomes

$$\mathcal{C}_{\alpha\star}^{IC}(f_\alpha)=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta\frac{q_\alpha^2 q_\beta^2}{(2\pi)^4\varepsilon_0^2 m_\alpha^2}\int_0^\infty \mathrm{d}t_1\int_{-\infty}^\infty \mathrm{d}t_2\int \mathrm{d}^3\mathbf{k}\int_{\mathcal{C}_1}\mathrm{d}\omega_1$$
$$\times\int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta)\frac{\mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{k}\mathbf{k}\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{|\varepsilon_\star(\mathbf{k},\,\omega_1)|^2k^4} \tag{192}$$
$$\times\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t_1)-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_2)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta-\mathrm{i}\omega_1\left(t_1-t_2\right)}$$
$$=\frac{1}{2}\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\left[\frac{\partial f_\alpha}{\partial\mathbf{v}_\alpha}\cdot\langle\Delta\mathbf{V}_\alpha\Delta\mathbf{V}_\alpha\rangle\right].$$

Compared with the results obtained based on the FP approach, it is clear that $\mathcal{C}_{\alpha\star}^{IC}$ is due to the correlation of the fluctuations. Combining Eqs. (186) and (192) yields the magnetized BLG collision term.

4 Derivation of the collision term based on the QL approach

4.1 The case without B

Besides the FP and BBGKY approaches described in the preceding two sections, the QL approach is also an elegant way to derive the collision term. It is as systematic as the BBGKY approach but much simpler in the derivation process. The starting point is the Klimontovich (1982) equation:

$$\frac{\partial N_\alpha}{\partial t}+\mathbf{v}_\alpha\cdot\frac{\partial N_\alpha}{\partial \mathbf{r}_\alpha}+\frac{q_\alpha}{m_\alpha}\mathbf{E}^M\cdot\frac{\partial N_\alpha}{\partial\mathbf{v}_\alpha}=0. \tag{193}$$

Statistical average of the above equation yields the kinetic equation:

$$\frac{\partial f_\alpha}{\partial t}+\mathbf{v}_\alpha\cdot\frac{\partial f_\alpha}{\partial \mathbf{r}_\alpha}+\frac{q_\alpha}{m_\alpha}\mathbf{E}\cdot\frac{\partial f_\alpha}{\partial\mathbf{v}_\alpha}=-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\langle\delta\mathbf{E}\delta N_\alpha\rangle. \tag{194}$$

It is clear that the RHS of the above equation represents the collision term $\mathcal{C}_\alpha$ with which we are concerned. In the QL approximation, $\delta\mathbf{E}$ and δN_α are determined by

Eqs. (40) and (43) (Chavanis 2012). Recalling that $\delta N_\alpha = \delta N_\alpha^S + \delta N_\alpha^I$, $\mathcal{C}_\alpha$ can be formally separated into two parts:

$$\mathcal{C}_\alpha(f_\alpha) = -\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left\langle \delta \mathbf{E} \delta N_\alpha^S \right\rangle - \frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left\langle \delta \mathbf{E} \delta N_\alpha^I \right\rangle. \tag{195}$$

These two parts will be considered separately in the following.

The relevant results in Sec. II can be used directly. Using $\delta N_\alpha^S(\mathbf{r}_\alpha, \mathbf{v}_\alpha, t) = \delta N_\alpha(\mathbf{r}_\alpha - \mathbf{v}_\alpha t, \mathbf{v}_\alpha, 0)$, expressing $\delta \mathbf{E} = -\nabla \delta\phi$ in terms of $\delta\tilde{\phi}$ through the inverse Fourier-Laplace transform, and substituting Eq. (48) for $\delta\tilde{\phi}$ and Eq. (49) for $\delta\tilde{N}_\alpha^S$, we have

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left\langle \delta \mathbf{E} \delta N_\alpha^S \right\rangle \\
&\quad = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^4 \varepsilon_0 m_\alpha} \mathrm{i} \int_0^\infty \mathrm{d}t_1 \int \mathrm{d}^3\mathbf{k} \int_{\mathcal{C}} \mathrm{d}\omega \int \mathrm{d}^6\mathbf{X}\, \frac{\mathbf{k}}{\varepsilon(\mathbf{k}, \omega) k^2} \\
&\qquad \times \langle \delta N_\alpha(\mathbf{r}_\alpha - \mathbf{v}_\alpha t, \mathbf{v}_\alpha, 0) \delta N_\beta(\mathbf{r} - \mathbf{v} t_1, \mathbf{v}, 0) \rangle \mathrm{e}^{\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}_\alpha - \mathbf{r}) - \omega(t - t_1)]}.
\end{aligned} \tag{196}$$

Making the variable substitution $\mathbf{r} \to \mathbf{r} + \mathbf{v} t_1$ and carrying out the integral over t_1, the above equation becomes

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left\langle \delta \mathbf{E} \delta N_\alpha^S \right\rangle \\
&\quad = -\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^4 \varepsilon_0 m_\alpha} \int \mathrm{d}^3\mathbf{k} \int_{\mathcal{C}} \mathrm{d}\omega \int \mathrm{d}^6\mathbf{X}\, \frac{\mathbf{k}}{\varepsilon(\mathbf{k}, \omega) k^2} \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}} \\
&\qquad \times \langle \delta N_\alpha(\mathbf{r}_\alpha - \mathbf{v}_\alpha t, \mathbf{v}_\alpha, 0) \delta N_\beta(\mathbf{r}, \mathbf{v}, 0) \rangle \mathrm{e}^{\mathrm{i}[\mathbf{k}\cdot(\mathbf{r}_\alpha - \mathbf{r}) - \omega t]}.
\end{aligned} \tag{197}$$

Moving the contour of the ω integral in the above equation into the lower half plane in a similar way as that shown in Fig. 2 and noting that the contribution not decaying with t comes only from the residue of the pole at $\omega = \mathbf{k}\cdot\mathbf{v}$ as all the zeros of $\varepsilon(\mathbf{k}, \omega)$ are assumed to lie in the lower half ω plane, we have for large t

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left\langle \delta \mathbf{E} \delta N_\alpha^S \right\rangle \\
&\quad = \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^3 \varepsilon_0 m_\alpha} \mathrm{i} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^6\mathbf{X}\, \frac{\mathbf{k}}{\varepsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}) k^2} \\
&\qquad \times \langle \delta N_\alpha(\mathbf{r}_\alpha - \mathbf{v}_\alpha t, \mathbf{v}_\alpha, 0) \delta N_\beta(\mathbf{r}, \mathbf{v}, 0) \rangle \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_\alpha - \mathbf{r} - \mathbf{v}t)}.
\end{aligned} \tag{198}$$

Substituting Eq. (68) into the above equation, neglecting the term corresponding to $g_{\alpha\beta}$ as it decays exponentially with t when the integral over $\mathbf{v}$ is performed, and carrying out the integral over $\mathbf{X}$ for the remaining term, we obtain

$$-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}\delta N_\alpha^S\right\rangle = \frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^3\varepsilon_0 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\,\frac{\mathbf{k}}{\varepsilon(\mathbf{k},\,\mathbf{k}\cdot\mathbf{v}_\alpha)k^2}f_\alpha \\ = -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left[\langle\Delta\mathbf{v}_\alpha\rangle_p f_\alpha\right]. \tag{199}$$

Compared with the results obtained based on the FP approach, this part of $\mathcal{C}_\alpha$ can be considered to result from the polarization.

Expressing $\delta\mathbf{E}$ and δN_α^I through the inverse Fourier-Laplace transform, the second part of the collision term can be written as

$$-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}\delta N_\alpha^I\right\rangle \\ = \frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha}{(2\pi)^2 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega'\,\mathbf{k}\left\langle\delta\tilde{\phi}(\mathbf{k},\,\omega)\delta\tilde{N}_\alpha^I(\mathbf{k}',\,\mathbf{v}_\alpha,\,\omega')\right\rangle \\ \times \mathrm{e}^{\mathrm{i}\left[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_\alpha-(\omega+\omega')t\right]}. \tag{200}$$

$\delta\tilde{N}_\alpha^I$ can be obtained by directly taking the Fourier-Laplace transform on the second term on the RHS of Eq. (45),

$$\delta\tilde{N}_\alpha^I(\mathbf{k}',\,\mathbf{v}_\alpha,\,\omega') = -\frac{q_\alpha}{m_\alpha}\frac{\mathbf{k}'\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\omega'-\mathbf{k}'\cdot\mathbf{v}_\alpha}\delta\tilde{\phi}(\mathbf{k}',\,\omega'). \tag{201}$$

Inserting the above equation and Eqs. (48) and (49) into Eq. (200), we obtain

$$-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}\delta N_\alpha^I\right\rangle \\ = -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_{\beta,\gamma}\frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^8\varepsilon_0^2 m_\alpha^2}\mathrm{i}\int_0^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t_2\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega' \\ \times\int \mathrm{d}^6\mathbf{X}\int \mathrm{d}^6\mathbf{X}'\,\frac{\mathbf{k}\mathbf{k}'\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon(\mathbf{k},\,\omega)\varepsilon(\mathbf{k}',\,\omega')k^2k'^2}\frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}_\alpha} \\ \times\left\langle\delta N_\beta(\mathbf{r}-\mathbf{v}t_1,\,\mathbf{v},\,0)\delta N_\gamma(\mathbf{r}'-\mathbf{v}'t_2,\,\mathbf{v}',\,0)\right\rangle \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t_1+\mathbf{k}'\cdot\mathbf{r}'-\omega' t_2)} \\ \times \mathrm{e}^{\mathrm{i}\left[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_\alpha-(\omega+\omega')t\right]}. \tag{202}$$

Making the variable substitutions $\mathbf{r}\to\mathbf{r}+\mathbf{v}t_1$ and $\mathbf{r}'\to\mathbf{r}'+\mathbf{v}'t_2$ and carrying out the integrals over t_1 and t_2 in the above equation yields

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}\delta N_\alpha^I\right\rangle \\
&=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_{\beta,\gamma}\frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^8\varepsilon_0^2 m_\alpha^2}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega'\int \mathrm{d}^6\mathbf{X}\int \mathrm{d}^6\mathbf{X}' \\
&\quad\times\frac{\mathbf{k}\mathbf{k}'\cdot\partial f_\alpha/\partial \mathbf{v}_\alpha}{\varepsilon(\mathbf{k},\,\omega)\varepsilon(\mathbf{k}',\,\omega')k^2k'^2}\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}'}\frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}_\alpha} \\
&\quad\times\left\langle \delta N_\beta(\mathbf{X},\,0)\delta N_\gamma(\mathbf{X}',\,0)\right\rangle \mathrm{e}^{-\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r}+\mathbf{k}'\cdot\mathbf{r}'\right)}\mathrm{e}^{\mathrm{i}\left[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_\alpha-(\omega+\omega')t\right]}.
\end{aligned}
\tag{203}
$$

Moving the contours of the ω and ω' integrals in the above equation into the lower half planes in a similar way as that shown in Fig. 2 and retaining only the contributions from the residues of the poles at $\omega = \mathbf{k}\cdot\mathbf{v}$, $\omega' = \mathbf{k}'\cdot\mathbf{v}'$, and $\omega' = \mathbf{k}'\cdot\mathbf{v}_\alpha$ which do not decay with t, we obtain

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}\delta N_\alpha^I\right\rangle \\
&=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_{\beta,\gamma}\frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^6\varepsilon_0^2 m_\alpha^2}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{k}'\int \mathrm{d}^6\mathbf{X}\int \mathrm{d}^6\mathbf{X}'\frac{\mathbf{k}\mathbf{k}'\cdot\partial f_\alpha/\partial \mathbf{v}_\alpha}{\varepsilon(\mathbf{k},\,\mathbf{k}\cdot\mathbf{v})k^2k'^2} \\
&\quad\times\frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{v}t}}{\mathbf{k}'\cdot\mathbf{v}_\alpha-\mathbf{k}'\cdot\mathbf{v}'}\left[\frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{v}'t}}{\varepsilon(\mathbf{k}',\,\mathbf{k}'\cdot\mathbf{v}')}-\frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{v}_\alpha t}}{\varepsilon(\mathbf{k}',\,\mathbf{k}'\cdot\mathbf{v}_\alpha)}\right] \\
&\quad\times\left\langle \delta N_\beta(\mathbf{X},\,0)\delta N_\gamma(\mathbf{X}',\,0)\right\rangle \mathrm{e}^{-\mathrm{i}\left(\mathbf{k}\cdot\mathbf{r}+\mathbf{k}'\cdot\mathbf{r}'\right)}\mathrm{e}^{\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_\alpha}.
\end{aligned}
\tag{204}
$$

Substituting Eq. (68) into the above equation, neglecting the term corresponding to $g_{\beta\gamma}$ as it decays exponentially with t when the integral over $\mathbf{v}$ is performed, and carrying out the integrals over $\mathbf{X}'$, $\mathbf{r}$, and $\mathbf{k}'$ for the remaining term, we get

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}\delta N_\alpha^I\right\rangle \\
&=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_{\beta}\frac{q_\alpha^2 q_\beta^2}{(2\pi)^3\varepsilon_0^2 m_\alpha^2}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v})\frac{\mathbf{k}\mathbf{k}\cdot\partial f_\alpha(\mathbf{v}_\alpha)/\partial \mathbf{v}_\alpha}{\varepsilon(\mathbf{k},\,\mathbf{k}\cdot\mathbf{v})k^4} \\
&\quad\times\left[\frac{1}{\varepsilon(-\mathbf{k},\,-\mathbf{k}\cdot\mathbf{v}_\alpha)}\frac{1}{\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha}\mathrm{e}^{-\mathrm{i}\left(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha\right)t}\right. \\
&\quad\left.-\frac{1}{\varepsilon(-\mathbf{k},\,-\mathbf{k}\cdot\mathbf{v})}\frac{1}{\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha}\right].
\end{aligned}
\tag{205}
$$

As can be seen, $\mathbf{k}\cdot\mathbf{v} = \mathbf{k}\cdot\mathbf{v}_\alpha$ is the pole of both the two terms in the square brackets on the RHS of the above equation, but is not the pole of their sum. Therefore, we can change $1/(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha)$ to $1/(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha+\mathrm{i}0^+)$ in Eq. (205) without changing its value, i.e., bypassing the point $\mathbf{k}\cdot\mathbf{v}=\mathbf{k}\cdot\mathbf{v}_\alpha$ from above. Retaining only the contribution from the residue of the pole at $\mathbf{k}\cdot\mathbf{v}=\mathbf{k}\cdot\mathbf{v}_\alpha$ to the $\mathbf{v}$ integral for the first term in the square brackets on the RHS of Eq. (205), which does not decay with t, and using the Plemelj formula for the second term,

$$\frac{1}{\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha+\mathrm{i}0^+}=\frac{P}{\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha}-\mathrm{i}\pi\delta(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha), \tag{206}$$

we obtain

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\left\langle\delta\mathbf{E}\delta N_\alpha^I\right\rangle\\
&=-\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\sum_\beta\frac{q_\alpha^2q_\beta^2}{(2\pi)^3\varepsilon_0^2m_\alpha^2}\mathrm{i}\int\mathrm{d}^3\mathbf{k}\int\mathrm{d}^3\mathbf{v}f_\beta(\mathbf{v})\frac{\mathbf{kk}\cdot\partial f_\alpha(\mathbf{v}_\alpha)/\partial\mathbf{v}_\alpha}{|\varepsilon(\mathbf{k},\,\mathbf{k}\cdot\mathbf{v})|^2k^4}\\
&\quad\times\left[\mathrm{i}\pi\delta(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha)+\frac{P}{\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha}\right].
\end{aligned}\tag{207}$$

The integrand corresponding to $P/(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha)$ in the above equation is an odd function of $\mathbf{k}$, so its integral over $\mathbf{k}$ is 0. We then have

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\left\langle\delta\mathbf{E}\delta N_\alpha^I\right\rangle\\
&=\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\sum_\beta\frac{q_\alpha^2q_\beta^2}{8\pi^2\varepsilon_0^2m_\alpha^2}\int\mathrm{d}^3\mathbf{k}\int\mathrm{d}^3\mathbf{v}f_\beta(\mathbf{v})\frac{\mathbf{kk}\cdot\partial f_\alpha(\mathbf{v}_\alpha)/\partial\mathbf{v}_\alpha}{|\varepsilon(\mathbf{k},\,\mathbf{k}\cdot\mathbf{v})|^2k^4}\delta(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}_\alpha)\\
&=\frac{1}{2}\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\left[\langle\Delta\mathbf{v}_\alpha\Delta\mathbf{v}_\alpha\rangle\cdot\frac{\partial f_\alpha}{\partial\mathbf{v}_\alpha}\right].
\end{aligned}\tag{208}$$

Compared with the results obtained based on the FP approach, this part of $\mathcal{C}_\alpha$ can be thought to come from the correlation of the electric field fluctuations. Combining Eqs. (199) and (208) produces the BLG collision term.

4.2 The case with a uniform B

When a uniform magnetic field is present, the magnetized collision term can still be separated into the following two parts as the no magnetic field case:

$$\mathcal{C}_{\alpha\star}(f_\alpha)=-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\left\langle\delta\mathbf{E}_\star\delta N_{\alpha\star}^S\right\rangle-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial\mathbf{v}_\alpha}\cdot\left\langle\delta\mathbf{E}_\star\delta N_{\alpha\star}^I\right\rangle. \tag{209}$$

Expressing $\delta\mathbf{E}_\star=-\nabla\delta\phi_\star$ in terms of $\delta\tilde{\phi}_\star$ through the inverse Fourier-Laplace transform and substituting Eqs. (101), (104) and (105), we obtain

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^S_{\alpha\star}\right\rangle \\
&=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^4\varepsilon_0 m_\alpha}\mathrm{i}\int_0^\infty \mathrm{d}t_1\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^6\mathbf{X}_\beta\,\frac{\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2}\\
&\quad\times \mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot(\mathbf{r}_\alpha-\mathbf{r}_\beta)-\omega(t-t_1)\right]}\Big\langle \delta N_\alpha(\mathbf{r}_\alpha-[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t)]\cdot\mathbf{v}_\alpha,\,\mathsf{T}_\alpha^{-1}(t)\cdot\mathbf{v}_\alpha,\,0)\\
&\quad\times \delta N_\beta(\mathbf{r}_\beta-[\mathsf{H}_\beta(0)-\mathsf{H}_\beta(-t_1)]\cdot\mathbf{v}_\beta,\,\mathsf{T}_\beta^{-1}(t_1)\cdot\mathbf{v}_\beta,\,0)\Big\rangle.
\end{aligned}
\tag{210}
$$

Making the variable substitutions $\mathbf{r}_\beta \to \mathbf{r}_\beta + [\mathsf{H}_\beta(0)-\mathsf{H}_\beta(-t_1)]\cdot\mathbf{v}_\beta$ and $\mathbf{v}_\beta \to \mathsf{T}_\beta(t_1)\cdot\mathbf{v}_\beta$, the above equation becomes

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^S_{\alpha\star}\right\rangle \\
&=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^4\varepsilon_0 m_\alpha}\mathrm{i}\int_0^\infty \mathrm{d}t_1\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^6\mathbf{X}_\beta\,\frac{\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2}\\
&\quad\times \mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot(\mathbf{r}_\alpha-\mathbf{r}_\beta)-\omega(t-t_1)\right]}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta}\\
&\quad\times \left\langle \delta N_\alpha(\mathbf{r}_\alpha-[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t)]\cdot\mathbf{v}_\alpha,\,\mathsf{T}_\alpha^{-1}(t)\cdot\mathbf{v}_\alpha,\,0)\delta N_\beta(\mathbf{r}_\beta,\,\mathbf{v}_\beta,\,0)\right\rangle.
\end{aligned}
\tag{211}
$$

Expanding $\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\beta(t_1)\cdot\mathbf{v}_\beta}$ in the above equation into a series of Bessel functions using Eq. (111) and carrying out the integral over t_1 yields

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^S_{\alpha\star}\right\rangle \\
&=-\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^4\varepsilon_0 m_\alpha}\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^6\mathbf{X}_\beta\sum_{m=-\infty}^{\infty}\frac{J_m(k_\perp\rho_\beta)\mathrm{e}^{-\mathrm{i}m(\varphi_\mathbf{k}-\varphi_\beta)}}{\omega-m\Omega_\beta-k_\parallel v_{\beta\parallel}}\\
&\quad\times \frac{\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2}\mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot(\mathbf{r}_\alpha-\mathbf{r}_\beta)-\omega t\right]}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\beta(0)\cdot\mathbf{v}_\beta}\\
&\quad\times \left\langle \delta N_\alpha(\mathbf{r}_\alpha-[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t)]\cdot\mathbf{v}_\alpha,\,\mathsf{T}_\alpha^{-1}(t)\cdot\mathbf{v}_\alpha,\,0)\delta N_\beta(\mathbf{r}_\beta,\,\mathbf{v}_\beta,\,0)\right\rangle.
\end{aligned}
\tag{212}
$$

Moving the contour of the ω integral in the above equation into the lower half plane in a similar way as that shown in Fig. 2 and retaining only the contributions from the residues of the poles at $\omega = m\Omega_\beta + k_\parallel v_{\beta\parallel}$ which do not decay with t, we obtain

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^S_{\alpha\star}\right\rangle \\
&=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta \frac{q_\alpha q_\beta}{(2\pi)^3\varepsilon_0 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^6\mathbf{X}_\beta\sum_{m=-\infty}^{\infty}\frac{J_m(k_\perp\rho_\beta)\mathrm{e}^{-\mathrm{i}m(\Omega_\beta t+\varphi_\mathbf{k}-\varphi_\beta)}}{\varepsilon_\star(\mathbf{k},\,m\Omega_\beta+k_\parallel v_{\beta\parallel})}\\
&\quad\times \frac{\mathbf{k}}{k^2}\mathrm{e}^{\mathrm{i}\left[\mathbf{k}\cdot(\mathbf{r}_\alpha-\mathbf{r}_\beta)-k_\parallel v_{\beta\parallel}t\right]}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\beta(0)\cdot\mathbf{v}_\beta}\\
&\quad\times \left\langle \delta N_\alpha(\mathbf{r}_\alpha-[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t)]\cdot\mathbf{v}_\alpha,\,\mathsf{T}_\alpha^{-1}(t)\cdot\mathbf{v}_\alpha,\,0)\delta N_\beta(\mathbf{r}_\beta,\,\mathbf{v}_\beta,\,0)\right\rangle.
\end{aligned}
\tag{213}
$$

Inserting Eq. (68) into the above equation, neglecting the term corresponding to $g_{\alpha\beta}$ as it decays exponentially with t when the integral over $v_{\beta\|}$ is performed, and carrying out the integral over $\mathbf{X}_\beta$ for the remaining term gives

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^S_{\alpha\star}\right\rangle \\
&\quad=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^3\varepsilon_0 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\sum_{m=-\infty}^{\infty}\frac{J_m(k_\perp\rho_\alpha)\mathrm{e}^{-\mathrm{i}m(\varphi_\mathbf{k}-\varphi_\alpha)}}{\varepsilon_\star(\mathbf{k},\, m\Omega_\alpha+k_\| v_{\alpha\|})}\frac{\mathbf{k}}{k^2}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathsf{H}_\alpha(0)\cdot\mathbf{v}_\alpha}f_\alpha.
\end{aligned} \tag{214}$$

Using Eq. (111), the above equation can be rewritten as

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^S_{\alpha\star}\right\rangle \\
&\quad=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t_1\int \mathrm{d}^3\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2} \\
&\qquad\times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t_1)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\omega t_1}f_\alpha.
\end{aligned} \tag{215}$$

Making the variable substitution $\mathbf{k}\rightarrow\mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{k}$, the above equation becomes

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^S_{\alpha\star}\right\rangle \\
&\quad=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha^2}{(2\pi)^4\varepsilon_0 m_\alpha}\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t_1\int \mathrm{d}^3\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\mathsf{T}_\alpha^{-1}(t_1)\cdot\mathbf{k}}{\varepsilon_\star(\mathbf{k},\,\omega)k^2} \\
&\qquad\times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t_1)-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\omega t_1}f_\alpha \\
&\quad=-\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left[\langle\Delta\mathbf{V}_\alpha\rangle_p f_\alpha\right].
\end{aligned} \tag{216}$$

Compared with the results obtained based on the FP approach, this part of $\mathcal{C}_{\alpha\star}$ is found to stem from the polarization.

By use of the inverse Fourier-Laplace transform, the second term on the RHS of Eq. (209) can be expressed as

$$\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^I_{\alpha\star}\right\rangle \\
&\quad=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\frac{q_\alpha}{(2\pi)^2 m_\alpha}\mathrm{i}\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega'\,\mathbf{k}\left\langle\delta\tilde{\phi}_\star(\mathbf{k},\,\omega)\delta\tilde{N}^I_{\alpha\star}(\mathbf{k}',\,\mathbf{v}_\alpha,\,\omega')\right\rangle \\
&\qquad\times \mathrm{e}^{\mathrm{i}\left[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_\alpha-(\omega+\omega')t\right]}.
\end{aligned} \tag{217}$$

$\delta\tilde{N}^I_{\alpha\star}(\mathbf{k}',\,\mathbf{v}_\alpha,\,\omega')$ can be got by taking the Fourier-Laplace transform of Eq. (103),

$$\begin{aligned}
\delta\tilde{N}^I_{\alpha\star}(\mathbf{k}',\,\mathbf{v}_\alpha,\,\omega') = \mathrm{i}\frac{q_\alpha}{m_\alpha}\int_0^\infty \mathrm{d}t'\,\mathbf{k}'\cdot\mathsf{T}_\alpha^{-1}(t')\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha}\delta\tilde{\phi}_\star(\mathbf{k}',\,\omega') \\
\times \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\left[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t')\right]\cdot\mathbf{v}_\alpha+\mathrm{i}\omega' t'}.
\end{aligned} \tag{218}$$

Inserting the above equation and Eqs. (104) and (105) into Eq. (217) yields

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^I_{\alpha\star}\right\rangle \\
&= -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_{\beta,\gamma}\frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^8\varepsilon_0^2 m_\alpha^2}\int_0^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t_2\int_0^\infty \mathrm{d}t'\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega' \\
&\quad\times\int \mathrm{d}^6\mathbf{X}\int \mathrm{d}^6\mathbf{X}'\,\frac{\mathbf{k}\mathbf{k}'\cdot\mathsf{T}_\alpha^{-1}(t')\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon_\star(\mathbf{k},\omega)\varepsilon_\star(\mathbf{k}',\omega')k^2k'^2}\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t_1)}\mathrm{e}^{-\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}'-\omega' t_2)} \\
&\quad\times\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t')]\cdot\mathbf{v}_\alpha+\mathrm{i}\omega' t'}\mathrm{e}^{\mathrm{i}[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_\alpha-(\omega+\omega')t]} \\
&\quad\times\Big\langle \delta N_\beta(\mathbf{r}-[\mathsf{H}_\beta(0)-\mathsf{H}_\beta(-t_1)]\cdot\mathbf{v},\,\mathsf{T}_\beta^{-1}(t_1)\cdot\mathbf{v},\,0) \\
&\quad\times\delta N_\gamma(\mathbf{r}'-[\mathsf{H}_\gamma(0)-\mathsf{H}_\gamma(-t_2)]\cdot\mathbf{v}',\,\mathsf{T}_\gamma^{-1}(t_2)\cdot\mathbf{v}',\,0)\Big\rangle.
\end{aligned}
\tag{219}
$$

Making the variable substitutions $\mathbf{r}\to\mathbf{r}+[\mathsf{H}_\beta(0)-\mathsf{H}_\beta(-t_1)]\cdot\mathbf{v}$, $\mathbf{r}'\to\mathbf{r}'+[\mathsf{H}_\gamma(0)-\mathsf{H}_\gamma(-t_2)]\cdot\mathbf{v}'$, $\mathbf{v}\to\mathsf{T}_\beta(t_1)\cdot\mathbf{v}$, and $\mathbf{v}'\to\mathsf{T}_\gamma(t_2)\cdot\mathbf{v}'$, the above equation becomes

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^I_{\alpha\star}\right\rangle \\
&= -\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_{\beta,\gamma}\frac{q_\alpha^2 q_\beta q_\gamma}{(2\pi)^8\varepsilon_0^2 m_\alpha^2}\int_0^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t_2\int_0^\infty \mathrm{d}t'\int \mathrm{d}^3\mathbf{k}\int_C \mathrm{d}\omega\int \mathrm{d}^3\mathbf{k}'\int_{C'}\mathrm{d}\omega' \\
&\quad\times\int \mathrm{d}^6\mathbf{X}\int \mathrm{d}^6\mathbf{X}'\,\frac{\mathbf{k}\mathbf{k}'\cdot\mathsf{T}_\alpha^{-1}(t')\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{\varepsilon_\star(\mathbf{k},\omega)\varepsilon_\star(\mathbf{k}',\omega')k^2k'^2}\left\langle\delta N_\beta(\mathbf{X},0)\delta N_\gamma(\mathbf{X}',0)\right\rangle \\
&\quad\times\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\{\mathbf{r}+[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)]\cdot\mathbf{v}\}+\mathrm{i}\omega t_1}\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\{\mathbf{r}'+[\mathsf{H}_\gamma(t_2)-\mathsf{H}_\gamma(0)]\cdot\mathbf{v}'\}+\mathrm{i}\omega' t_2} \\
&\quad\times\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t')]\cdot\mathbf{v}_\alpha+\mathrm{i}\omega' t'}\mathrm{e}^{\mathrm{i}[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_\alpha-(\omega+\omega')t]}.
\end{aligned}
\tag{220}
$$

Substituting Eq. (68) into the above equation and taking similar procedures as deriving Eq. (215) from Eq. (211), it is found that the term corresponding to $g_{\beta\gamma}$ is negligible as it decays exponentially with t and $\int_0^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t_2\int_C \mathrm{d}\omega\int_{C'}\mathrm{d}\omega'$ can be replaced by $\int_{-\infty}^\infty \mathrm{d}t_1\int_{-\infty}^\infty \mathrm{d}t_2\int_{-\infty}^\infty \mathrm{d}\omega\int_{-\infty}^\infty \mathrm{d}\omega'$ for the remaining term. Under this condition, carrying out the integrals over $\mathbf{X}'$, $\mathbf{r}$, $\mathbf{k}'$, t_2, and ω', we obtain

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^I_{\alpha\star}\right\rangle \\
&= \frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_{\beta}\frac{q_\alpha^2 q_\beta^2}{(2\pi)^4\varepsilon_0^2 m_\alpha^2}\int_{-\infty}^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t'\int \mathrm{d}^3\mathbf{k}\int_{-\infty}^\infty \mathrm{d}\omega\int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v}) \\
&\quad\times\frac{\mathbf{k}\mathbf{k}\cdot\mathsf{T}_\alpha^{-1}(t')\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{|\varepsilon_\star(\mathbf{k},\omega)|^2k^4}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\alpha(0)-\mathsf{H}_\alpha(-t')]\cdot\mathbf{v}_\alpha-\mathrm{i}\mathbf{k}\cdot[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)]\cdot\mathbf{v}-\mathrm{i}\omega(t'-t_1)}.
\end{aligned}
\tag{221}
$$

Making the variable substitution $\mathbf{k}\to\mathsf{T}_\alpha^{-1}(t')\cdot\mathbf{k}$, the above equation becomes

$$
\begin{aligned}
&-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left\langle \delta\mathbf{E}_\star \delta N^I_{\alpha\star}\right\rangle \\
&=\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\sum_\beta \frac{q_\alpha^2 q_\beta^2}{32\pi^4\varepsilon_0^2 m_\alpha^2}\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t'\int \mathrm{d}^3\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega\int \mathrm{d}^3\mathbf{v} f_\beta(\mathbf{v}) \\
&\quad\times\frac{\mathsf{T}_\alpha^{-1}(t')\cdot\mathbf{k}\mathbf{k}\cdot\partial f_\alpha/\partial\mathbf{v}_\alpha}{|\varepsilon_\star(\mathbf{k},\omega)|^2 k^4}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t')-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathrm{i}\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_1)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}-\mathrm{i}\omega(t'-t_1)} \\
&=\frac{1}{2}\frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left[\frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha}\cdot\langle\Delta\mathbf{V}_\alpha\Delta\mathbf{V}_\alpha\rangle\right].
\end{aligned}
\tag{222}
$$

Compared with the results obtained based on the FP approach, this part of $\mathcal{C}_{\alpha\star}$ can be considered to result from the correlation of the electric field fluctuations. Combining Eqs. (216) and (222) produces the magnetized BLG collision term.

5 Properties of the collision term

In this section, conservation of particles, momentum, and energy by the collision term will be proven. We will also show that the collision term satisfies the H theorem and ensures $f_\alpha \geq 0$ for all time if $f_\alpha \geq 0$ initially. Since the proof process is similar for the magnetized and unmagnetized cases, only the magnetized case is considered here.

Equation (121) shows that $\mathcal{C}_{\alpha\star}$ can be expressed as the divergence of a vector which approaches 0 for $|\mathbf{v}_\alpha| \to \infty$, upon using the divergence theorem we can immediately obtain

$$
\int \mathcal{C}_{\alpha\star}(f_\alpha)\,\mathrm{d}^3\mathbf{v}_\alpha = 0, \tag{223}
$$

implying that $\mathcal{C}_{\alpha\star}$ conserves the particles.

The time rate of change of the momentum $\mathbf{P} = \sum_\alpha \int m_\alpha \mathbf{v}_\alpha f_\alpha\,\mathrm{d}^3\mathbf{v}_\alpha$ in a unit volume of the system due to the collisions is

$$
\left(\frac{\partial \mathbf{P}}{\partial t}\right)_c = \sum_\alpha \int m_\alpha \mathbf{v}_\alpha \mathcal{C}_{\alpha\star}(f_\alpha)\,\mathrm{d}^3\mathbf{v}_\alpha. \tag{224}
$$

Substituting Eq. (121) for $\mathcal{C}_{\alpha\star}$ into the above equation and integrating by parts over $\mathbf{v}_\alpha$ gives

$$\left(\frac{\partial \mathbf{P}}{\partial t}\right)_c = -\sum_{\alpha,\beta} \frac{q_\alpha^2 q_\beta^2}{32\pi^4 \varepsilon_0^2} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^{\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{v}_\alpha \int \mathrm{d}^3\mathbf{v}_\beta$$
$$\times\, \mathrm{e}^{\mathrm{i}\left\{\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t_1)-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha+\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_2)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta-\omega(t_1+t_2)\right\}} \frac{\mathbf{k}}{|\varepsilon_\star(\mathbf{k},\,\omega)|^2 k^4}$$
$$\times \left[\mathbf{k}\cdot \mathsf{T}_\alpha(t_1)\cdot \frac{1}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} - \mathbf{k}\cdot \mathsf{T}_\beta(t_2)\cdot \frac{1}{m_\beta}\frac{\partial}{\partial \mathbf{v}_\beta}\right] f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta), \tag{225}$$

which is obviously equal to its own negative upon the interchanges $\alpha \leftrightarrow \beta$ and $t_1 \leftrightarrow t_2$. Therefore,

$$\left(\frac{\partial \mathbf{P}}{\partial t}\right)_c = 0, \tag{226}$$

indicating that $\mathcal{C}_{\alpha\star}$ conserves the total momentum.

Similarly, the time rate of change of the kinetic energy $E = \sum_\alpha \int m_\alpha v_\alpha^2/2 f_\alpha \,\mathrm{d}^3\mathbf{v}_\alpha$ in a unit volume of the system due to the collisions is

$$\begin{aligned}\left(\frac{\partial E}{\partial t}\right)_c &= \sum_\alpha \int \frac{1}{2} m_\alpha v_\alpha^2 \mathcal{C}_{\alpha\star}(f_\alpha)\,\mathrm{d}^3\mathbf{v}_\alpha \\ &= -\sum_{\alpha,\beta} \frac{q_\alpha^2 q_\beta^2}{32\pi^4 \varepsilon_0^2} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^{\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{v}_\beta \\ &\quad \times\, \mathrm{e}^{\mathrm{i}\left\{\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t_1)-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_2)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta-\omega(t_1-t_2)\right\}} \\ &\quad \times \frac{\mathbf{k}\cdot\mathsf{T}_\alpha(t_1)\cdot\mathbf{v}_\alpha}{|\varepsilon_\star(\mathbf{k},\,\omega)|^2 k^4}\mathbf{k}\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} - \frac{1}{m_\beta}\frac{\partial}{\partial \mathbf{v}_\beta}\right) f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta).\end{aligned} \tag{227}$$

Making the interchanges $\alpha \leftrightarrow \beta$ and $t_1 \leftrightarrow t_2$ and variable substitutions $\omega \to -\omega$ and $\mathbf{k} \to -\mathbf{k}$, adding the resulting equivalent expressions, and dividing by 2 yields

$$\begin{aligned}\left(\frac{\partial E}{\partial t}\right)_c &= -\sum_{\alpha,\beta} \frac{q_\alpha^2 q_\beta^2}{64\pi^4 \varepsilon_0^2} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int \mathrm{d}^3\mathbf{k} \int_{-\infty}^{\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{v}_\beta \\ &\quad \times\, \mathrm{e}^{\mathrm{i}\left\{\mathbf{k}\cdot\left[\mathsf{H}_\alpha(t_1)-\mathsf{H}_\alpha(0)\right]\cdot\mathbf{v}_\alpha-\mathbf{k}\cdot\left[\mathsf{H}_\beta(t_2)-\mathsf{H}_\beta(0)\right]\cdot\mathbf{v}_\beta-\omega(t_1-t_2)\right\}} \\ &\quad \times \frac{\mathbf{k}\cdot\mathsf{T}_\alpha(t_1)\cdot\mathbf{v}_\alpha - \mathbf{k}\cdot\mathsf{T}_\beta(t_2)\cdot\mathbf{v}_\beta}{|\varepsilon_\star(\mathbf{k},\,\omega)|^2 k^4} \\ &\quad \times \mathbf{k}\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha} - \frac{1}{m_\beta}\frac{\partial}{\partial \mathbf{v}_\beta}\right) f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta).\end{aligned} \tag{228}$$

Integrating by parts over t_1 for the term involving $\mathbf{k}\cdot\mathsf{T}_\alpha(t_1)\cdot\mathbf{v}_\alpha$ and over t_2 for the term involving $\mathbf{k}\cdot\mathsf{T}_\beta(t_2)\cdot\mathbf{v}_\beta$ in the above equation, we find that the two terms yield the opposite results with the sum being 0. We thus have

$$\left(\frac{\partial E}{\partial t}\right)_c = 0, \tag{229}$$

indicating that $\mathcal{C}_{\alpha\star}$ conserves the total kinetic energy.

To prove the remaining two properties of $\mathcal{C}_{\alpha\star}$, we rewrite $\mathcal{C}_{\alpha\star}$ in Eq. (121) by expanding the exponential functions into series of the Bessel functions using Eq. (111). Proceeding to carry out the integrals over t and t_1 and retaining the terms surviving the $\varphi_{\mathbf{k}}$ and φ_β integration, $\mathcal{C}_{\alpha\star}$ becomes

$$\begin{aligned}\mathcal{C}_{\alpha\star}(f_\alpha) = {} & \sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\sum_\beta \frac{q_\alpha^2 q_\beta^2}{8\pi^2\varepsilon_0^2 m_\alpha}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{v}_\beta \left(\frac{n}{\rho_\alpha}\frac{\partial}{\partial v_{\alpha\perp}} + k_\parallel \frac{\partial}{\partial v_{\alpha\parallel}}\right)\\ & \times \frac{\delta\big(k_\parallel v_{\alpha\parallel} + n\Omega_\alpha - k_\parallel v_{\beta\parallel} - n'\Omega_\beta\big)}{|\varepsilon_\star(\mathbf{k},\, k_\parallel v_{\alpha\parallel} + n\Omega_\alpha)|^2 k^4} J_n^2(k_\perp\rho_\alpha) J_{n'}^2(k_\perp\rho_\beta)\\ & \times \left[\frac{1}{m_\alpha}\left(\frac{n}{\rho_\alpha}\frac{\partial}{\partial v_{\alpha\perp}} + k_\parallel\frac{\partial}{\partial v_{\alpha\parallel}}\right) - \frac{1}{m_\beta}\left(\frac{n'}{\rho_\beta}\frac{\partial}{\partial v_{\beta\perp}} + k_\parallel\frac{\partial}{\partial v_{\beta\parallel}}\right)\right]\\ & \times f_\alpha(\mathbf{v}_\alpha) f_\beta(\mathbf{v}_\beta).\end{aligned} \tag{230}$$

We employ proof by contradiction to prove that $\mathcal{C}_{\alpha\star}$ ensures $f_\alpha \geq 0$ for all time if $f_\alpha \geq 0$ initially. Assuming $f_\alpha \geq 0$ at $t = 0$ and becomes negative at some time later, there must be an instant at which its minimum value first becomes negative. At such a point, the following four conditions must be satisfied: (i) $f_\alpha = 0$, (ii) $\partial f_\alpha/\partial \mathbf{v}_\alpha = 0$, (iii) $\partial^2 f_\alpha/\partial\mathbf{v}_\alpha\partial\mathbf{v}_\alpha$ is a non-negative tensor, and (iv) $\partial f_\alpha/\partial t < 0$. When conditions (i) and (ii) apply, $\mathcal{C}_{\alpha\star}$ in Eq. (230) becomes

$$\begin{aligned}\mathcal{C}_{\alpha\star}(f_\alpha) = {} & \sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\sum_\beta \frac{q_\alpha^2 q_\beta^2}{8\pi^2\varepsilon_0^2 m_\alpha^2}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta)\\ & \times \frac{\delta\big(k_\parallel v_{\alpha\parallel} + n\Omega_\alpha - k_\parallel v_{\beta\parallel} - n'\Omega_\beta\big)}{|\varepsilon_\star(\mathbf{k},\, k_\parallel v_{\alpha\parallel} + n\Omega_\alpha)|^2 k^4} J_n^2(k_\perp\rho_\alpha) J_{n'}^2(k_\perp\rho_\beta)\\ & \times \left(\frac{n}{\rho_\alpha}\frac{\partial}{\partial v_{\alpha\perp}} + k_\parallel\frac{\partial}{\partial v_{\alpha\parallel}}\right)^2 f_\alpha(\mathbf{v}_\alpha)\\ = {} & \sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\sum_\beta \frac{q_\alpha^2 q_\beta^2}{8\pi^2\varepsilon_0^2 m_\alpha^2}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{v}_\beta f_\beta(\mathbf{v}_\beta)\\ & \times \frac{\delta\big(k_\parallel v_{\alpha\parallel} + n\Omega_\alpha - k_\parallel v_{\beta\parallel} - n'\Omega_\beta\big)}{|\varepsilon_\star(\mathbf{k},\, k_\parallel v_{\alpha\parallel} + n\Omega_\alpha)|^2 k^4} J_n^2(k_\perp\rho_\alpha) J_{n'}^2(k_\perp\rho_\beta)\\ & \times \left(\frac{n}{\rho_\alpha}\frac{\mathbf{v}_{\alpha\perp}}{v_{\alpha\perp}} + k_\parallel\hat{\mathbf{b}}\right)\cdot\frac{\partial^2 f_\alpha(\mathbf{v}_\alpha)}{\partial\mathbf{v}_\alpha\partial\mathbf{v}_\alpha}\cdot\left(\frac{n}{\rho_\alpha}\frac{\mathbf{v}_{\alpha\perp}}{v_{\alpha\perp}} + k_\parallel\hat{\mathbf{b}}\right).\end{aligned} \tag{231}$$

Using condition (iii), it is easy to find that $\mathcal{C}_{\alpha\star}(f_\alpha) \geq 0$. This is incompatible with condition (iv). Therefore, conditions (i)-(iv) cannot be met at the same time. In other words, $\mathcal{C}_{\alpha\star}$ will ensure $f_\alpha \geq 0$ for all t if $f_\alpha \geq 0$ at $t = 0$.

Consider the quantity H, defined by

$$H \equiv \sum_{\alpha} \int f_{\alpha}(\mathbf{v}_{\alpha}) \ln f_{\alpha}(\mathbf{v}_{\alpha}) \, \mathrm{d}^3\mathbf{v}_{\alpha}. \tag{232}$$

Its time rate of change due to the collisions is

$$\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_c = \sum_{\alpha} \int \left(1 + \ln f_{\alpha}\right) \mathcal{C}_{\alpha\star}(f_{\alpha}) \, \mathrm{d}^3\mathbf{v}_{\alpha}. \tag{233}$$

Substituting Eq. (230) for $\mathcal{C}_{\alpha\star}(f_{\alpha})$ into the above equation, integrating by parts over $v_{\alpha\perp}$ and $v_{\alpha\parallel}$, making the interchanges $\alpha \leftrightarrow \beta$ and $n \leftrightarrow n'$, adding the resulting equivalent expressions, and dividing by 2, we obtain

$$\begin{aligned}
\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_c = &- \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \sum_{\alpha,\beta} \frac{q_{\alpha}^2 q_{\beta}^2}{16\pi^2 \varepsilon_0^2} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{v}_{\alpha} \int \mathrm{d}^3\mathbf{v}_{\beta} f_{\alpha} f_{\beta} \\
&\times \frac{\delta\left(k_{\parallel} v_{\alpha\parallel} + n\Omega_{\alpha} - k_{\parallel} v_{\beta\parallel} - n'\Omega_{\beta}\right)}{|\varepsilon_{\star}(\mathbf{k}, k_{\parallel} v_{\alpha\parallel} + n\Omega_{\alpha})|^2 k^4} J_n^2\left(k_{\perp}\rho_{\alpha}\right) J_{n'}^2\left(k_{\perp}\rho_{\beta}\right) \\
&\times \left\{ n\Omega_{\alpha} \left[\frac{1}{m_{\alpha} v_{\alpha\perp}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\perp}} - \frac{1}{v_{\alpha\parallel} - v_{\beta\parallel}} \left(\frac{1}{m_{\alpha}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\parallel}} - \frac{1}{m_{\beta}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\parallel}} \right) \right] \right. \\
&\left. - n'\Omega_{\beta} \left[\frac{1}{m_{\beta} v_{\beta\perp}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\perp}} - \frac{1}{v_{\alpha\parallel} - v_{\beta\parallel}} \left(\frac{1}{m_{\alpha}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\parallel}} - \frac{1}{m_{\beta}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\parallel}} \right) \right] \right\}^2 \\
&\leqq 0.
\end{aligned} \tag{234}$$

The equality holds if and only if

$$\begin{aligned}
&n\Omega_{\alpha} \left[\frac{1}{m_{\alpha} v_{\alpha\perp}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\perp}} - \frac{1}{v_{\alpha\parallel} - v_{\beta\parallel}} \left(\frac{1}{m_{\alpha}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\parallel}} - \frac{1}{m_{\beta}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\parallel}} \right) \right] \\
&\quad - n'\Omega_{\beta} \left[\frac{1}{m_{\beta} v_{\beta\perp}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\perp}} - \frac{1}{v_{\alpha\parallel} - v_{\beta\parallel}} \left(\frac{1}{m_{\alpha}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\parallel}} - \frac{1}{m_{\beta}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\parallel}} \right) \right] = 0.
\end{aligned} \tag{235}$$

As n and n' are arbitrary, the above equation is equivalent to

$$\frac{1}{m_{\alpha} v_{\alpha\perp}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\perp}} - \frac{1}{v_{\alpha\parallel} - v_{\beta\parallel}} \left(\frac{1}{m_{\alpha}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\parallel}} - \frac{1}{m_{\beta}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\parallel}} \right) = 0, \tag{236}$$

$$\frac{1}{m_{\beta} v_{\beta\perp}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\perp}} - \frac{1}{v_{\alpha\parallel} - v_{\beta\parallel}} \left(\frac{1}{m_{\alpha}} \frac{\partial \ln f_{\alpha}}{\partial v_{\alpha\parallel}} - \frac{1}{m_{\beta}} \frac{\partial \ln f_{\beta}}{\partial v_{\beta\parallel}} \right) = 0. \tag{237}$$

The subtraction of the above two equations yields

$$\frac{1}{m_\alpha v_{\alpha\perp}}\frac{\partial \ln f_\alpha}{\partial v_{\alpha\perp}} - \frac{1}{m_\beta v_{\beta\perp}}\frac{\partial \ln f_\beta}{\partial v_{\beta\perp}} = 0. \tag{238}$$

Since $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$ are independent, the solution of the above equation is

$$\frac{1}{m_\alpha v_{\alpha\perp}}\frac{\partial \ln f_\alpha}{\partial v_{\alpha\perp}} = \frac{1}{m_\beta v_{\beta\perp}}\frac{\partial \ln f_\beta}{\partial v_{\beta\perp}} = \text{const}, \tag{239}$$

which implies

$$f_\gamma(\mathbf{v}_\gamma) = C_\gamma(v_{\gamma\|})\mathrm{e}^{-\frac{m_\gamma v_{\gamma\perp}^2}{2k_B T}}, \tag{240}$$

where T is the common temperature. Substituting the above equation into Eq. (236), we can obtain in a similar way

$$C_\gamma(v_{\gamma\|}) = \frac{n_\gamma}{(2\pi k_B T/m_\gamma)^{3/2}}\mathrm{e}^{-\frac{m_\gamma\left(v_{\gamma\|}-U_\|\right)^2}{2k_B T}}, \tag{241}$$

where $U_\|$ is the common parallel fluid velocity. We thus have

$$f_\gamma(\mathbf{v}_\gamma) = \frac{n_\gamma}{(2\pi k_B T/m_\gamma)^{3/2}}\mathrm{e}^{-\frac{m_\gamma\left[v_{\gamma\perp}^2+\left(v_{\gamma\|}-U_\|\right)^2\right]}{2k_B T}}. \tag{242}$$

It is easy to verify that the distribution function of the above form satisfies $\mathcal{C}_{\alpha\star}(f_\alpha) = 0$. Therefore, the magnetized collision term will relax the system to a Maxwellian state with different species of particles having the same temperature and parallel velocity. Due to the assumption of gyrotropic distributions we have made in deriving the magnetized collision term, the system finally relaxes to a state with no perpendicular fluid velocity.

6 Conclusion

The FP, BBGKY, and QL approaches to deriving the collision terms in the electrostatic approximation for plasmas without and with a uniform magnetic field are reviewed in this paper. All the derivations involved are based on the perturbation theory except the calculation of the FP coefficients within the BC model in the no magnetic field case. It is shown that the three approaches are equivalent in deriving the (magnetized) BLG collision term which has all the desired features connected with conservation laws and entropy production. For completeness and the convenience of readers interested in only one of the unmagnetized and magnetized collision terms, some similar derivation processes are repeated. Due

Table 1 Comparison of the characteristics of the three approaches to deriving the collision term

	Initial form of $\mathcal{C}_{\alpha(\star)}$	Advantages	Disadvantages
FP	$-\frac{\partial}{\partial \mathbf{v}_\alpha} \cdot [\langle \Delta \mathbf{v}_\alpha \rangle f_\alpha] + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v}_\alpha \partial \mathbf{v}_\alpha} : [\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle f_\alpha]^\dagger$	Straightforward, separating $\mathcal{C}_{\alpha(\star)}$ into the part from polarization and the part from correlation of the fluctuations naturally	Somewhat phenomenological
BBGKY	$\sum_\beta \frac{q_\alpha q_\beta}{4\pi\epsilon_0 m_\alpha} \int \mathrm{d}^6 \mathbf{X}_\beta \times \frac{\partial}{\partial \mathbf{r}_\alpha} \frac{1}{r_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta(\star)}}{\partial \mathbf{v}_\alpha}$	Systematic, easy to identify the physical basis of various approximations	Complex derivation
QL	$-\frac{q_\alpha}{m_\alpha} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \left\langle \delta \mathbf{E}_{(\star)} \delta N_{\alpha(\star)} \right\rangle$	Systematic, relatively simple derivation	Unable to handle close collisions

† Replacing $\langle \Delta \mathbf{v}_\alpha \rangle$ by $\langle \Delta \mathbf{V}_\alpha \rangle$ and $\langle \Delta \mathbf{v}_\alpha \Delta \mathbf{v}_\alpha \rangle$ by $\langle \Delta \mathbf{V}_\alpha \Delta \mathbf{V}_\alpha \rangle$ gives the initial form of $\mathcal{C}_{\alpha\star}$

to the use of the perturbation theory in the derivation process, the (magnetized) BLG collision term is valid in the weak coupling approximation. When collective effects are neglected it reduces to the (magnetized) Landau collision term. Besides the plasmas interacting through the inverse square Coulomb forces, the BLG collision term can be readily transposed to other systems with long-range interactions.

Relatively speaking, the BBGKY and QL approaches are more systematic and it is generally accepted that the derivation process of the QL approach is simpler. The BBGKY approach is based on the BBGKY hierarchy of equations which include all the relevant physics. One can make approximations as needed and know exactly which physics is kept and which physics is ignored. The FP approach is rather straightforward in the no magnetic field case. In addition, it has the advantage of directly separating the collision term into the part from the polarization and the part from the correlation of the electric field fluctuations. The main features of the three approaches are summarized in Table 1.

The topic of the paper is very focused and most of the results presented are very old. As more and more plasmas (Fajans and Surko 2020; Anderegg et al. 1997; Hollmann et al. 1999; Affolter et al. 2016, 2018; Zhang et al. 2008; Gorman et al. 2021; Greenwald et al. 2014; Creely et al. 2020; Harding and Lai 2006; Valyavin et al. 2014; Wilks et al. 1992; Mason and Tabak 1998; Kennedy and Helander 2021a, b) are found to satisfy the condition of strong magnetization, the strong magnetic field effects on the collision process and associated transport have aroused some interest recently. Considering this, the magnetized collision term sporadically discussed in the literature is reviewed in a great detail in this paper for the spatially homogeneous magnetized plasmas. For the unmagnetized and weakly magnetized plasmas, the collisions can be viewed to occur locally, implying that the collision term derived for the uniform case applies directly to the nonuniform case. For the strongly magnetized nonuniform plasmas, however, the nonlocality of the collision process cannot be ignored since the collisions with impact parameters larger than the particles' gyro-radii occur on scales larger than the step length of the perpendicular classical transport. Research in this respect is relatively rare (Ichimaru and Tange 1974; Øien 1995) and more relevant studies are need to be performed. For the nonuniformly magnetized plasmas, Mynick (1988) has employed the FP approach to generalize the standard BLG collision term to its electromagnetic counterpart in the action-angle formalism initially developed by Kaufman (1972) to study the quasilinear diffusion of tokamak plasmas. The UCSD group found that the reflection and "collisional caging" due to velocity diffusion in guiding center collisions could greatly enhance the transport of particles, momentum, and energy in the presence of a strong magnetic field. The theoretical results they obtained (Dubin 1997; Dubin and O'Neil 1997; Dubin 2014) can successfully explain the experimental findings (Anderegg et al. 1997; Hollmann et al. 1999; Affolter et al. 2016, 2018). However, these relevant works are beyond the scope of the perturbation theory and were not described by a collision term. In our opinion, they still need to be improved and further verified by experiments and are thus not included in the present paper. There is no doubt that it is an important research direction.

This paper is essentially pedagogical and aim to help everyone to adopt the appropriate collision term in the study of magnetized plasmas. We hope the results, ideas, and methods presented in this paper will be useful to readers working in this field.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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