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ORIGINAL RESEARCH PAPER

Geometric characterization of level set families of harmonic functions on Riemannian manifolds

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Abstract

In (Bivens in Mathematics Magazine 65: 226–235, 1992), it is shown that the appearance of the curves completely determines whether a family of curves in the Euclidean plane is a family of level curves of some harmonic function free of critical points. In this paper, we extend the result of (Bivens in Mathematics Magazine 65: 226–235, 1992) to higher dimensional Riemannian manifolds and give a geometric characterization of the level set family of the solutions of the differential equation $|\text{grad}\, u|^{-1}\Delta u = \psi$, where ψ is a smooth function on the manifold.

Keywords Harmonic function · Level set · Riemannian manifold · Mean curvature

Mathematics Subject Classification 31C12

1 Introduction

Harmonic functions defined on Euclidean space have a good geometric characterization. Specifically, a function is harmonic if and only if its mean value in any spherical neighborhood is equal to the function value at the center of the ball (cf. [2, pp: 25–26]). A natural question is whether the level sets of harmonic functions also have a good geometric characterization. Many studies have shown that the level sets of harmonic functions, or even more generally, the level sets of solutions of elliptic differential equations, have many beautiful geometric properties, especially with respect to curvature estimation and convexity tests (see [3, 4]). In general, to clearly distinguish the level sets of harmonic functions from other hypersurfaces in Euclidean space, we often expand the object under consideration to the entire hypersurface family instead of considering only a single member of it.

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What kind of conditions make a family of hypersurfaces the level set family of a harmonic function? In the case that the hypersurface family can be regarded as the level set family of a certain C^2 function free of critical points, the level set family of a harmonic function can be characterized by the differential properties of the function, cf. [5, 6, pp: 111], [7, pp: 227–228], and the characterization conditions given by them are essentially the deformation of the following proposition.

Proposition 1 *Let us suppose* $f \in C^2(\mathbb{R}^n)$ *is free of critical points; then, a necessary and sufficient condition for a family of hypersurfaces* $f = \mathrm{const}$ *to be the level set family of some harmonic function is that* $|\mathrm{grad}\, f|^{-2} \Delta f$ *be a function of* f *alone.*

For the general case, a necessary and sufficient geometric condition that a family of hypersurfaces be the level set family of some harmonic function has been proposed only in the two-dimensional case, see [1, 8, pp: 520–523], where a simple expression of the result of [1] is shown in the following proposition.

Proposition 2 *A family of curves in an open subset of the Euclidean plane is the level set family of a harmonic function free of critical points if and only if at every point the sum of the derivatives of the respective (signed) curvatures along the respective tangent directions of the curves and the orthogonal trajectories equals zero.*

The primary goal of this paper is to generalize the above proposition to the case of Riemannian manifolds with general dimensions which will improve the results of [5] and thus provide a geometric characterization of the level set families of harmonic functions free of critical points on Riemannian manifolds. The main result is now described as follows.

Let (M, g) be a Riemannian manifold of dimension m and $\{N_t\}_{t\in\mathcal{T}}$ is the set of leaves of an $(m-1)$-dimensional foliation of M.

Theorem 1 *For any* $p \in M$, *the collection of plaques near* p *is the level set family of some smooth solution of the differential equation*

$$|\mathrm{grad}\, u|^{-1} \Delta u = \psi, \tag{1}$$

where u is unknown and ψ is an arbitrary smooth function on M, if and only if

$$\mathrm{grad}_{N_t}(H + \psi) = (H + \psi)\dot{e}_m + \mathrm{D}_{e_m}\dot{e}_m + \mathrm{D}_{\dot{e}_m} e_m + |\dot{e}_m|^2 e_m \tag{2}$$

holds on M, in which

$$\begin{aligned} e_m &= \text{a local unit vector field normal to the foliation,} \\ H &= \text{mean curvature (sum of principal curvatures) of } \{N_t\} \\ &\quad\ \text{with respect to } e_m, \\ \mathrm{D} &= \text{the Christoffel-Levi-Civita connection on } M, \\ \dot{e}_m &= \mathrm{D}_{e_m} e_m. \end{aligned}$$

If we set $\psi = 0$ in Theorem 1, then a corollary follows.

Corollary 1 *For any $p \in M$, the collection of plaques near p is a family of level sets of some harmonic function free of critical points, if and only if*

$$\mathrm{grad}_{N_t} H = H\dot{e}_m + \mathrm{D}_{\dot{e}_m} e_m + \mathrm{D}_{e_m} \dot{e}_m + |\dot{e}_m|^2 e_m. \tag{3}$$

2 The level set family of a smooth function

Let us fix any point $p \in M$; there exists a local coordinate system $(U; u^A)$ such that each plaques in U has the description

$$u^1 = \text{const.}$$

We can obtain an orthogonal frame field $e_1, \ldots, e_m$ from $\partial/\partial u^1, \ldots, \partial/\partial u^m$ on U, with its dual frame field denoted by $\omega_1, \ldots, \omega_m$, in which e_m is in the same direction as $\partial/\partial u^1$. For simplicity, we set $\mathcal{F} = \{\text{the plaques in } U\}$.

In the following exposition, the ranges of the indices $A, B, C \cdots$ and $i, j, k \cdots$ are as follows:

$$\begin{cases} 1 \le A, B, C, \cdots \le m, \\ 1 \le i, j, k, \cdots \le m-1. \end{cases}$$

We denote the connection matrix with respect to the local frame field $e_1, \cdots, e_m$ by (ω_{AB}); thus,

$$\mathrm{D}e_A = \sum_B \omega_{AB} \otimes e_B,$$

$$\mathrm{d}\omega_A = \sum_B \omega_{AB} \wedge \omega_B,$$

$$\omega_{AB} + \omega_{BA} = 0,$$

and the corresponding Christoffel symbols are defined by

$$\Gamma_{ABC} = \omega_{AB}(e_C) = (\mathrm{D}_{e_C} e_A) \cdot e_B,$$

satisfying $\Gamma_{ABC} + \Gamma_{BAC} = 0$. Then, the mean curvature of $\{N_t\}$ with respect to e_m is

$$H = \sum_k \Gamma_{kmk}.$$

Using the notation above, we give the following lemma.

Lemma 1 *Let us suppose e_A, Γ_{ABC} has been locally defined; then,*

1. $\Gamma_{mij} - \Gamma_{mji} = \Gamma_{jmi} - \Gamma_{imj} = 0$;
2. $e_i \Gamma_{mjm} - e_j \Gamma_{mim} - \sum_k \Gamma_{mkm}(\Gamma_{kij} - \Gamma_{kji}) = 0$;
3. $\sum_i \left(e_m \Gamma_{mim} + \sum_k \Gamma_{mkm}(\Gamma_{kim} - \Gamma_{kmi})\right) e_i = \mathrm{D}_{e_m} \dot{e}_m + \mathrm{D}_{\dot{e}_m} e_m + |\dot{e}_m|^2 e_m$.

Proof For 1, we only need to note that on the submanifold $i_t : N_t \hookrightarrow M$, we have

$$0 = \mathrm{d}(i_t^* \omega_m) = i_t^* \mathrm{d}\omega_m = \sum_{i<j} (\Gamma_{mij} - \Gamma_{mji}) \circ i_t i_t^* \omega_j \wedge i_t^* \omega_i,$$

then the conclusion can be derived from the arbitrariness of t.

For 2, by the definition of the Riemannian curvature tensor of M,

$$-\frac{1}{2} \sum_{D,E} R_{ABDE} \omega_D \wedge \omega_E = \Omega_{AB} = \mathrm{d}\omega_{AB} - \sum_C \omega_{AC} \wedge \omega_{CB}$$
$$= \frac{1}{2} \sum_{D,E} \Bigg\{ e_D \Gamma_{ABE} - e_E \Gamma_{ABD} + \sum_C \Gamma_{ABC} (\Gamma_{CED} - \Gamma_{CDE})$$
$$+ \sum_C (\Gamma_{CAD} \Gamma_{CBE} - \Gamma_{CAE} \Gamma_{CBD}) \Bigg\} \omega_D \wedge \omega_E,$$

and so

$$-R_{mjim} = e_i \Gamma_{mjm} - e_m \Gamma_{mji} - \Gamma_{mjm} \Gamma_{mim}$$
$$- \sum_k \left(\Gamma_{mjk} \Gamma_{mki} + \Gamma_{mjk} \Gamma_{kim} + \Gamma_{mki} \Gamma_{kjm} - \Gamma_{mkm} \Gamma_{kji} \right).$$

According to the Bianchi identity, we have

$$0 = R_{mijm} - R_{mjim}$$
$$= e_i \Gamma_{mjm} - e_j \Gamma_{mim} - \sum_k \Gamma_{mkm} \left(\Gamma_{kij} - \Gamma_{kji} \right).$$

For 3, we note that

$$\dot{e}_m = \mathrm{D}_{e_m} e_m = \sum_i \Gamma_{mim} e_i,$$

$$\mathrm{D}_{e_m} \dot{e}_m = \sum_i \left(e_m \Gamma_{mim} + \sum_k \Gamma_{mkm} \Gamma_{kim} \right) e_i - |\dot{e}_m|^2 e_m,$$

$$\mathrm{D}_{\dot{e}_m} e_m = \sum_{A,k} \Gamma_{mkm} \Gamma_{mAk} e_A = - \sum_{i,k} \Gamma_{mkm} \Gamma_{kmi} e_i.$$

Then, the conclusion can be directly verified. □

The next lemma is a generalization of (the differential version of) Theorem 2 in [5].

Lemma 2 *If $\mathcal{F}$ is the level set family of a smooth function f on U free of critical points, then we have*

$$\operatorname{grad} \ln |\operatorname{grad} f| = \dot{e}_m + \left(H + |\operatorname{grad} f|^{-1} \Delta f \right) e_m. \tag{4}$$

Proof Let $\varphi = \ln|\mathrm{grad}f|$, then φ satisfies

$$0 = \mathrm{d}(e^{\varphi}\omega_m) = e^{\varphi}\sum_{A<B}(\varphi_A\delta_{mB} - \varphi_B\delta_{mA} + \Gamma_{AmB} - \Gamma_{BmA})\omega_A \wedge \omega_B,$$

Combined with the first equation of Lemma 1, the above equation is equivalent to

$$\varphi_i = \Gamma_{mim} \quad (\text{or simply } \mathrm{grad}_{N_t}\varphi = \dot{e}_m).$$

However, we note that

$$\begin{aligned}\Delta f &= \sum_A f_{AA} = \sum_A \left\langle e_A, \mathrm{d}f_A + \sum_B f_B\omega_{BA} \right\rangle \\ &= |\mathrm{grad}f|(e_m \ln|\mathrm{grad}f| - H),\end{aligned}$$

and thus, (4) is verified. □

Lemma 3 *If $\mathcal{F}$ is the level set family of a smooth function f on U free of critical points, then we have*

$$\begin{aligned}&\mathrm{grad}_{N_t}\left(H + |\mathrm{grad}f|^{-1}\Delta f\right) \\ &\quad = \left(H + |\mathrm{grad}f|^{-1}\Delta f\right)\dot{e}_m + \mathrm{D}_{e_m}\dot{e}_m + \mathrm{D}_{\dot{e}_m}e_m + |\dot{e}_m|^2 e_m.\end{aligned} \tag{5}$$

Proof Let $u = e_m \ln|\mathrm{grad}f| = H + |\mathrm{grad}f|^{-1}\Delta f$, then from equation (4), it can be seen that

$$\begin{aligned}0 &= \mathrm{d}\left(u\omega_m + \sum_i \Gamma_{mim}\omega_i\right) \\ &= \sum_{A<B}\Bigg\{u_A\delta_{mB} - u_B\delta_{mA} + u(\Gamma_{mBA} - \Gamma_{mAB}) + e_A\Gamma_{mBm} - e_B\Gamma_{mAm} \\ &\quad + \sum_k \Gamma_{mkm}(\Gamma_{kBA} - \Gamma_{kAB})\Bigg\}\omega_A \wedge \omega_B.\end{aligned}$$

According to the first two equations of Lemma 1, the equation above is equivalent to

$$u_i = \Gamma_{mim}u + e_m\Gamma_{mim} + \sum_k \Gamma_{mkm}(\Gamma_{kim} - \Gamma_{kmi}).$$

Then, according to the third equation of Lemma 1, we can rearrange things as

$$\mathrm{grad}_{N_t}u = u\dot{e}_m + \mathrm{D}_{e_m}\dot{e}_m + \mathrm{D}_{\dot{e}_m}e_m + |\dot{e}_m|^2 e_m.$$

□

3 Proof of main results and corollaries

Finally, we are in a position to prove Theorem 1.

Proof of Theorem 1 If f is a smooth solution of equation (1) with $\mathcal{F}$ as the level set family, then (2) is a direct consequence of (5).

Conversely, if (2) holds on M, according to the fact that $\mathcal{F}$ is the level set family of some smooth function $f \in C^\infty(U)$ free of critical points (e.g. $f = u^1$), f satisfies (4) and (5). Combining (5) and (2) we obtain

$$\mathrm{grad}_{N_t} \ln\left||\mathrm{grad}\, f|^{-1} \Delta f - \psi\right| = \dot{e}_m,$$

and then, incorporating (4), we have

$$\mathrm{grad}_{N_t} \ln\left||\mathrm{grad}\, f|^{-2} \Delta f - \psi |\mathrm{grad}\, f|^{-1}\right| = 0.$$

Therefore, $|\mathrm{grad}\, f|^{-2} \Delta f - \psi |\mathrm{grad}\, f|^{-1}$ is a function of f alone, so there exists a monotone function $h \in C^\infty(\mathbb{R})$ such that

$$-\frac{h'' \circ f}{h' \circ f} = -\frac{\mathrm{d}}{\mathrm{d}f} \ln|h'(f)| = |\mathrm{grad}\, f|^{-2} \Delta f - \psi |\mathrm{grad} f|^{-1};$$

Now, we have

$$\psi = \frac{h'' \circ f}{h' \circ f} |\mathrm{grad} f| + |\mathrm{grad}\, f|^{-1} \Delta f = |\mathrm{grad}(h \circ f)|^{-1} \Delta(h \circ f).$$

Consequently, $\mathcal{F}$ is exactly the level set family of the smooth solution $h \circ f$ of equation (1). □

Corollary 2 ([9]) *For any $p \in M$, the collection of plaques near p is a family of $(m-1)$-dimensional isoparametric submanifolds of M, if and only if*

$$\mathrm{grad}_{N_t} H = \dot{e}_m = 0,$$

i.e., each plaque is a submanifold with constant mean curvature, and the integral curves of e_m are geodesics of M.

Proof Fix any $p \in M$, define $\mathcal{F}$ as above. If $\mathcal{F}$ is a family of isoparametric submanifolds, that is, there is an isoparametric function f with $\mathcal{F}$ as the level set family, which satisfies both Δf and $|\mathrm{grad}\, f|$ as functions of f, then according to (4), we obtain $\dot{e}_m = 0$. On the other hand, because $\mathcal{F}$ is locally a family of level sets of some harmonic function free of critical points, we conclude that $\mathrm{grad}_{N_t} H = 0$ from (3).

Conversely, if $\mathrm{grad}_{N_t} H = \dot{e}_m = 0$, there is a smooth function f free of critical points with $\mathcal{F}$ as the level set family. From (4), $\mathrm{grad}_{N_t} \ln|\mathrm{grad} f| = 0$, i.e., $|\mathrm{grad}\, f|$ is a function of f. In addition, we note that (3) is established, so $|\mathrm{grad}\, f|^{-2} \Delta f$ is a function of f; thus, f is an isoparametric function. □

Finally we use Corollary 1 to give a proof of the result in [1].

Proof of Proposition 2 Let us suppose that there is a family of oriented curves A in the open subset U of the Euclidean plane whose orthogonal trajectory family is denoted by B. Their normal directions are defined as the counterclockwise rotation of their own tangent directions by 90 degrees. Without loss of generality, we can think that the tangent vectors of A and B together form a positive oriented (smooth) orthogonal frame field e_1, e_2 of U. We note that k_1 and k_2 are the (signed) curvatures of A and B, respectively, then $\dot{e}_2 = \mathrm{D}_{e_2} e_2$ is the curvature vector of B, and so $e_1 = -k_2^{-1} \dot{e}_2$. Finally, according to Corollary 1, A is the level set family of some harmonic function free of critical points on U if and only if

$$\begin{aligned} e_1 k_1 e_1 &= k_1 \dot{e}_2 + \mathrm{D}_{e_2} \dot{e}_2 + \mathrm{D}_{\dot{e}_2} e_2 + |\dot{e}_2|^2 e_2 \\ &= -k_1 k_2 e_1 + \mathrm{D}_{e_2}(-k_2 e_1) + \mathrm{D}_{-k_2 e_1} e_2 + k_2^2 e_2 \\ &= -k_1 k_2 e_1 - e_2 k_2 e_1 - k_2 \mathrm{D}_{e_2} e_1 - k_2 \mathrm{D}_{e_1} e_2 + k_2^2 e_2 \\ &= -k_1 k_2 e_1 - e_2 k_2 e_1 - k_2^2 e_2 + k_2 k_1 e_1 + k_2^2 e_2 \\ &= -e_2 k_2 e_1, \end{aligned}$$

that is,

$$e_1 k_1 + e_2 k_2 = 0.$$

□

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Declarations

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Ethical approval This article does not contain any studies with human participants or animals.

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