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ORIGINAL RESEARCH PAPER

Simpson type inequalities and applications

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Abstract
A new generalized integral identity involving first order differentiable functions is obtained. Using this identity as an auxiliary result, we then obtain some new refinements of Simpson type inequalities using a new class called as strongly (s, m)-convex functions of higher order of $\sigma > 0$. We also discuss some interesting applications of the obtained results in the theory of means. In last we present applications of the obtained results in obtaining Simpson-like quadrature formula.

Keywords Convex · Strongly · Higher Order · Simpson · Inequality

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1 Introduction and preliminaries

The following inequality is known as Simpson's inequality.

Theorem 1.1 *Let* $\mathcal{F} : [a_1, a_2] \to \mathbb{R}$ *be a four times continuously differentiable function on* (a_1, a_2) *and* $\|\mathcal{F}^4\|_\infty = \sup_{x\in(a_1,a_2)} |\mathcal{F}^4(x)| < \infty$, *then*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(a_1)+\mathcal{F}(a_2)}{2}+2\mathcal{F}\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\mathcal{F}(x)\mathrm{d}x\right|$$
$$\leq \frac{1}{2880}\|\mathcal{F}^4\|_\infty (a_2-a_1)^4.$$

Simpson's inequality plays a significant role in modern analysis [4]. In past it has been extended and generalized in different directions using novel and innovative approaches. For example, Dragomir et al. [4] obtained the following version of the Simpson's inequality.

Theorem 1.2 *Let* $\mathcal{F} : [a_1, a_2] \to \mathbb{R}$ *be a differentiable function whose derivative is continuous on* (a_1, a_2) *and* $\mathcal{F}' \in L[a_1, a_2]$, *then*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(a_1)+\mathcal{F}(a_2)}{2}+2\mathcal{F}\left(\frac{a_1+a_2}{2}\right)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\mathcal{F}(x)\mathrm{d}x\right| \leq \frac{a_2-a_1}{3}\|\mathcal{F}'\|_1,$$

where $\|\mathcal{F}'\|_1 = \int_{a_1}^{a_2} |\mathcal{F}'(x)|\mathrm{d}x$.

Sarikaya et al. [13] obtained Simpson's inequality using differentiable convex functions.

Theorem 1.3 *Let* $\mathcal{F} : [a_1, a_2] \to \mathbb{R}$ *be a differentiable function on* (a_1, a_2) *such that* $\mathcal{F}' \in L[a_1, a_2]$. *If* $|\mathcal{F}'|^q$ *is convex, then*

$$\left|\frac{1}{6}\left[\mathcal{F}(a_1)+4\mathcal{F}\left(\frac{a_1+a_2}{2}\right)+\mathcal{F}(a_2)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\mathcal{F}(x)\mathrm{d}x\right|$$
$$\leq \frac{a_2-a_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|\mathcal{F}'(a_1)|^q+3|\mathcal{F}'(a_2)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3|\mathcal{F}'(a_1)|^q+|\mathcal{F}'(a_2)|^q}{4}\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Dragomir et al. [4] has written a very informative paper regarding the developments and applications of Simpson's inequality.

Fractional calculus although is very old but in recent decades has received special attention. It has emerged as an interdisciplinary field of mathematics. Due to its great utility and applications mathematicians have shown their keen interest in it.

The classical Riemann-Liouville fractional integrals are defined as:

Definition 1.1 ([7]) Let $\mathcal{F} \in L_1[a_1, a_2]$. Then the Riemann-Liouville integrals $J^{\alpha}_{a_1^+}\mathcal{F}$ and $J^{\alpha}_{a_2^-}\mathcal{F}$ of order $\alpha > 0$ with $a_1 \geq 0$ are defined by

$$J^{\alpha}_{a_1^+}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_{a_1}^{x} (x-v)^{\alpha-1}\mathcal{F}(v)\mathrm{d}v, \quad x > a_1,$$

and

$$J^{\alpha}_{a_2^-}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{a_2} (v-x)^{\alpha-1}\mathcal{F}(v)\mathrm{d}v, \quad x < a_2,$$

where

$$\Gamma(x) = \int_0^{\infty} e^{-v}v^{x-1}\mathrm{d}v, \quad x > 0$$

is the well known Gamma function.

Diaz and Pariguan [3] introduced the notion of generalized κ-gamma function. The integral form of Γ_κ is given by:

$$\Gamma_\kappa(x) = \int_0^{\infty} v^{x-1}e^{-\frac{v^\kappa}{\kappa}}\mathrm{d}v, \quad Re(x) > 0.$$

Note that

$$\Gamma_\kappa(x) = \kappa^{\frac{x}{\kappa}-1}\Gamma\left(\frac{x}{\kappa}\right).$$

κ-Beta function is defined as:

$$B_\kappa(x,y) = \frac{1}{\kappa}\int_0^1 v^{\frac{x}{\kappa}-1}(1-v)^{\frac{y}{\kappa}-1}\mathrm{d}v.$$

Obviously

$$B_\kappa(x,y) = \frac{1}{\kappa}\beta\left(\frac{x}{\kappa},\frac{y}{\kappa}\right).$$

Mubeen and Habibullah [9] were the first to extend the notion of classical Riemann-Liouville fractional integrals to κ-Riemann-Liouville fractional integrals. Sarikaya et al. [12] independently extended the notion of Riemann-Liouville fractional integrals to κ-Riemann-Liouville fractional integrals and discussed some of its interesting properties.

To be more precise let $\mathcal{F}$ be piecewise continuous on $I^* = (0, \infty)$ and integrable on any finite subinterval of $I = [0, \infty]$. Then for $v > 0$, we consider κ-Riemann-Liouville fractional integral of $\mathcal{F}$ of order α

$$_{\kappa}J^{\alpha}_{a_1}\mathcal{F}(x) = \frac{1}{\kappa\Gamma_{\kappa}(\alpha)}\int_{a_1}^{x}(x-v)^{\frac{\alpha}{\kappa}-1}\mathcal{F}(v)\mathrm{d}v, \quad x > a_1, \kappa > 0.$$

For some recent studies involving κ-fractional calculus, see [1, 2, 5, 10, 11].

The main objective of this paper is to derive some new generalized variants of Simpson's like inequalities for the functions belonging to Lebesgue L_1, L_q and L_∞ spaces. We essentially derive our results using a new class of convexity called as strongly (s, m)-convex functions of higher order of $\sigma > 0$. We also discuss some special cases of the obtained results. In order to show the significance of the obtained results, we also discuss some interesting applications of the main results. We hope that the ideas and the techniques of this paper will inspire interested readers working in this filed.

2 Main results

2.1 New definition

We now define the class of strongly (s, m)-convex functions of higher order of $\sigma > 0$.

Definition 2.1 A function $\mathcal{F} : [0, \infty) \to \mathbb{R}$ is said to be strongly (s, m)-convex functions of higher order of $\sigma > 0$, for certain fixed $(s, m) \in (0, 1]^2$, if

$$\begin{aligned}\mathcal{F}(\tau a_1 + m(1-\tau)a_2) \leq & \tau^s\mathcal{F}(a_1) + m(1-\tau)^s\mathcal{F}(a_2) \\ & - \mu[\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma]\|a_2 - a_1\|^\sigma,\end{aligned}$$

where $\mu, \sigma > 0$ and $\tau \in [0, 1]$.

It is worth to point out here that, if $\sigma = 0$, then we have the class of (s, m)-convex functions introduced and studied by Eftkhari [6]. And if we take $\sigma = 2$, them the class of strongly (s, m)-convex functions of higher order of $\sigma > 0$ reduces to the class of strongly (s, m)-convex functions of order 2. To the best of our knowledge this is also new in the literature.

Definition 2.2 A function $\mathcal{F} : [0, \infty) \to \mathbb{R}$ is said to be strongly (s, m)-convex functions of order 2, for certain fixed $(s, m) \in (0, 1]^2$, if

$$\mathcal{F}(\tau a_1 + m(1-\tau)a_2) \leq \tau^s\mathcal{F}(a_1) + m(1-\tau)^s\mathcal{F}(a_2) - \mu\tau(1-\tau)\|a_2 - a_1\|^2,$$

where $\mu > 0$ and $\tau \in [0, 1]$.

2.2 A key lemma

In this section, we derive a new integral identity which will be used as an auxiliary result in obtaining results of the subsequent sections.

Lemma 2.1 *Let* $0 < m \leq 1$, $\alpha > 0$, $n \in \mathbb{N}$, *and* $\mathcal{F} : J = [ma_1, \frac{a_2}{m}] \to \mathbb{R}$ *be a differentiable function with* $a_1 < a_2$. *If* $\mathcal{F} \in L_1[J]$, *then*

$$\begin{aligned}
&\Omega(a_1, a_2; m, \alpha, w) \\
&\quad = \frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)} \int_0^1 \left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4} \right) \mathcal{F}'\left(\frac{1-\tau}{n+1} ma_1 + \frac{n+\tau}{n+1} w \right) \mathrm{d}\tau \\
&\qquad + \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)} \int_0^1 \left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} \right) \mathcal{F}'\left(\frac{n+\tau}{n+1} w + \frac{1-\tau}{n+1} a_2 \right) \mathrm{d}\tau,
\end{aligned}$$

where

$$\begin{aligned}
&\Omega(a_1, a_2; m, \alpha, w) \\
&= \frac{3}{(n+1)(n+4)(a_2 - ma_1)} \mathcal{F}(w) \\
&\quad + \frac{1}{(n+4)(a_2 - ma_1)} \left[(w - ma_1) \mathcal{F}\left(\frac{(nw + ma_1)}{(n+1)} \right) \right. \\
&\quad \left. + (a_2 - w) \mathcal{F}\left(\frac{(nw + a_2)}{(n+1)} \right) \right] \\
&\quad - \frac{(n+1)^{\frac{\alpha}{\kappa}-1} \Gamma_\kappa(\alpha + \kappa)}{(a_2 - ma_1)(a_2 - w)^{\frac{\alpha}{\kappa}-1}} {}_kJ^{\alpha}_{w^+} \mathcal{F}\left(\frac{(nw + a_2)}{n+1} \right) \\
&\quad - \frac{(n+1)^{\frac{\alpha}{\kappa}-1} \Gamma_\kappa(\alpha + \kappa)}{(a_2 - ma_1)(w - ma_1)^{\frac{\alpha}{\kappa}-1}} {}_kJ^{\alpha}_{w^-} \mathcal{F}\left(\frac{(nw + ma_1)}{n+1} \right),
\end{aligned}$$

and $w = (1 - \beta)ma_1 + \beta a_2, \beta \in [0, 1]$.

Proof Consider

$$\begin{aligned}
I &= \frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)} \int_0^1 \left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4} \right) \mathcal{F}'\left(\frac{1-\tau}{n+1} ma_1 + \frac{n+\tau}{n+1} w \right) \mathrm{d}\tau \\
&\quad + \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)} \int_0^1 \left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} \right) \mathcal{F}'\left(\frac{n+\tau}{n+1} w + \frac{1-\tau}{n+1} a_2 \right) \mathrm{d}\tau \qquad (2.1) \\
&= \frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)} I_1 + \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)} I_2.
\end{aligned}$$

Now

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right)\mathcal{F}'\left(\frac{1-\tau}{n+1}ma_1 + \frac{n+\tau}{n+1}w\right)\mathrm{d}\tau \\ &= \frac{3}{(n+4)(w-ma_1)}\mathcal{F}(w) + \frac{n+1}{(n+4)(w-ma_1)}\mathcal{F}\left(\frac{(nw+ma_1)}{n+1}\right) \\ &\quad - \frac{(n+1)^{\frac{\alpha}{\kappa}}\Gamma_\kappa(\alpha+\kappa)}{(w-ma_1)^{\frac{\alpha}{\kappa}+1}}\,{}_kJ^{\alpha}_{w^-}\mathcal{F}\left(\frac{(nw+ma_1)}{n+1}\right). \end{aligned} \tag{2.2}$$

Similarly

$$\begin{aligned} I_2 &= \int_0^1 \left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}\right)\mathcal{F}'\left(\frac{n+\tau}{n+1}w + \frac{1-\tau}{n+1}a_2\right)\mathrm{d}\tau \\ &= \frac{3}{(n+4)(a_2-w)}\mathcal{F}(w) + \frac{n+1}{(n+4)(a_2-w)}\mathcal{F}\left(\frac{(nw+a_2)}{n+1}\right) \\ &\quad - \frac{(n+1)^{\frac{\alpha}{\kappa}}\Gamma_\kappa(\alpha+\kappa)}{(a_2-w)^{\frac{\alpha}{\kappa}+1}}\,{}_kJ^{\alpha}_{w^+}\mathcal{F}\left(\frac{(nw+a_2)}{n+1}\right). \end{aligned} \tag{2.3}$$

Using the values of I_1 and I_2 in (2.1), we get the required result. □

2.3 Simpson's like inequalities

We now derive some new Simpson's like inequalities using the class of strongly (s, m) convex functions of higher order $\sigma > o$.

Theorem 2.2 *Under all the assumption of Lemma 2.1, if $|\mathcal{F}'|$ is bounded, that is $\|\mathcal{F}'\|_\infty = \sup_{x\in[ma_1,a_2]}|\mathcal{F}'(x)| < \infty$, then*

$$|\Omega(a_1, a_2; m, \alpha, w)| \le \phi(\tau)\left[\frac{(w-ma_1)^2 + (a_2-w)^2}{(n+1)(a_2-ma_1)}\right]\|\mathcal{F}'\|_\infty.$$

Proof Using Lemma 2.1, boundedness $|\mathcal{F}'|$ and properties of absolute value, we have

$$\begin{aligned} &|\Omega(a_1, a_2; m, \alpha, w)| \\ &\le \frac{(w-ma_1)^2}{(n+1)(a_2-ma_1)}\int_0^1 \left|\left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right)\right|\left|\mathcal{F}'\left(\frac{1-\tau}{n+1}ma_1 + \frac{n+\tau}{n+1}w\right)\right|\mathrm{d}\tau \\ &\quad + \frac{(a_2-w)^2}{(n+1)(a_2-ma_1)}\int_0^1 \left|\left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}\right)\right|\left|\mathcal{F}'\left(\frac{n+\tau}{n+1}w + \frac{1-\tau}{n+1}a_2\right)\right|\mathrm{d}\tau \\ &\le \left[\frac{(w-ma_1)^2 + (a_2-w)^2}{(n+1)(a_2-ma_1)}\right]\|\mathcal{F}'\|_\infty \int_0^1 \left|\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right|\mathrm{d}\tau \\ &= \phi(\tau)\left[\frac{(w-ma_1)^2 + (a_2-w)^2}{(n+1)(a_2-ma_1)}\right]\|\mathcal{F}'\|_\infty, \end{aligned}$$

where

$$\int_0^1 \left| \left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} \right) \right|$$
$$= \frac{\alpha}{(n+1)(\alpha+\kappa)} \left(\frac{n+1}{n+4} \right)^{\frac{\kappa}{\alpha}} + \frac{\kappa}{(n+1)(\alpha+\kappa)} \left(1 - \left(\frac{n+1}{n+4} \right)^{\frac{\kappa}{\alpha}+1} \right)$$
$$- \frac{1}{n+4} \left(1 - \left(\frac{n+1}{n+4} \right)^{\frac{\kappa}{\alpha}} \right) := \phi(\tau)$$

This completes the proof. □

Theorem 2.3 *Under all the assumption of Lemma 2.1, if $\mathcal{F}' \in L_1[ma_1, a_2]$, then*

$$|\Omega(a_1, a_2; m, \alpha, w)| \leq \frac{3}{(n+1)(n+4)} \|\mathcal{F}'\|_1,$$

where $\|\mathcal{F}'\|_1 = \int_{ma_1}^{a_2} |\mathcal{F}'(x)| \mathrm{d}x$.

Proof Using Lemma 2.1, boundedness $|\mathcal{F}'|$ and properties of absolute value, we have

$$\begin{aligned}
&|\Omega(a_1, a_2; m, \alpha, w)| \\
&\leq \frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)} \int_0^1 \left| \left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4} \right) \right| \left| \mathcal{F}'\left(\frac{1-\tau}{n+1} ma_1 + \frac{n+\tau}{n+1} w \right) \right| \mathrm{d}\tau \\
&\quad + \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)} \int_0^1 \left| \left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} \right) \right| \left| \mathcal{F}'\left(\frac{n+\tau}{n+1} w + \frac{1-\tau}{n+1} a_2 \right) \right| \mathrm{d}\tau \\
&\leq \frac{3}{(n+1)(n+4)} \left[\frac{w - ma_1}{a_2 - ma_1} \int_{\frac{(ma_1+nw)}{n+1}}^{w} |\mathcal{F}'(x)| \mathrm{d}x + \right. \\
&\left. \frac{a_2 - w}{a_2 - ma_1} \int_w^{\frac{(a_2+nw)}{n+1}} |\mathcal{F}'(x)| / \mathrm{d}x \right] \\
&\leq \frac{3}{(n+1)(n+4)} \left[\int_{\frac{(ma_1+nw)}{n+1}}^{w} |\mathcal{F}'(x)| \mathrm{d}x + \int_w^{\frac{(nw+a_2)}{n+1}} |\mathcal{F}'(x)| \mathrm{d}x \right] \\
&\leq \frac{3}{(n+1)(n+4)} \int_{\frac{(ma_1+nw)}{n+1}}^{\frac{(nw+a_2)}{n+1}} |\mathcal{F}'(x)| \mathrm{d}x \\
&\leq \frac{3}{(n+1)(n+4)} \int_{ma_1}^{a_2} |\mathcal{F}'(x)| \mathrm{d}x \\
&= \frac{3}{(n+1)(n+4)} \|\mathcal{F}'\|_1,
\end{aligned}$$

where we have used the fact that $\left| \frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} \right| \leq \frac{3}{(n+1)(n+4)}$, for $\tau \in [0, 1]$ and $\alpha > 0$. This completes the proof. □

Theorem 2.4 *Under all the assumption of Lemma 2.1, if* $\mathcal{F}' \in L_q[ma_1, a_2]$ *with* $1 < q < \infty$, *then*

$$|\Omega(a_1, a_2; m, \alpha, w)| \leq N^{\frac{1}{p}}\left((n+1)^{\frac{1}{q}}(a_2 - ma_1)^{\frac{1}{p}}\right)\|\mathcal{F}'\|_p,$$

where

$$T = \frac{\kappa(n+1)^{\frac{\kappa}{\alpha}}\left((n+1)^{\frac{-\kappa}{\alpha}} - (n+4)^{\frac{-\kappa}{\alpha}}\right)^{\frac{\alpha p}{\kappa}+1}}{\alpha p + \kappa} + \frac{\kappa(n+1)^{\frac{\kappa}{\alpha}}(n+4)^{-p-\frac{\kappa}{\alpha}}}{\alpha p + \kappa},$$

and $\|\mathcal{F}'\|_q = \left(\int_{ma_1}^{a_2} |\mathcal{F}'(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 2.1, boundedness $|\mathcal{F}'|$ and properties of absolute value, we have

$$\begin{aligned}
&|\Omega(a_1, a_2; m, \alpha, w)| \\
&\leq \frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)}\left(\int_0^1 \left|\left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right)\right|^p \mathrm{d}\tau\right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1 \left|\mathcal{F}'\left(\frac{1-\tau}{n+1}ma_1 + \frac{n+\tau}{n+1}w\right)\right|^q \mathrm{d}\tau\right)^{\frac{1}{q}} \\
&\quad + \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)}\left(\int_0^1 \left|\left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}\right)\right|^p \mathrm{d}\tau\right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1 \left|\mathcal{F}'\left(\frac{n+\tau}{n+1}w + \frac{1-\tau}{n+1}a_2\right)\right|^q \mathrm{d}\tau\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left|\left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right)\right|^p \mathrm{d}\tau\right)^{\frac{1}{p}} \\
&\quad \left[\frac{(w - ma_1)^{2-\frac{1}{q}}}{(n+1)^{1-\frac{1}{q}}(a_2 - ma_1)}\left(\int_{\frac{(ma_1+nw)}{n+1}}^{w} |\mathcal{F}'(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \frac{(a_2 - w)^{2-\frac{1}{q}}}{(n+1)^{1-\frac{1}{q}}(a_2 - ma_1)}\left(\int_w^{\frac{(a_2+nw)}{n+1}} |\mathcal{F}'(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}\right] \\
&\leq \left(\int_0^1 \left|\left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right)\right|^p \mathrm{d}\tau\right)^{\frac{1}{p}}[(n+1)^{\frac{1}{q}-1}(a_2 - ma_1)^{\frac{1}{p}}] \\
&\quad \times \left[\left(\int_{\frac{(ma_1+nw)}{n+1}}^{w} |\mathcal{F}'(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} + \left(\int_w^{\frac{(a_2+nw)}{n+1}} |\mathcal{F}'(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}\right] \\
&\leq \left(\int_0^1 \left|\left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right)\right|^p \mathrm{d}\tau\right)^{\frac{1}{p}}[(n+1)^{\frac{1}{q}}(a_2 - ma_1)^{\frac{1}{p}}] \\
&\quad \times \left[\left(\int_{\frac{(ma_1+nw)}{n+1}}^{w} |\mathcal{F}'(x)|^q \mathrm{d}x + \int_w^{\frac{(a_2+nw)}{n+1}} |\mathcal{F}'(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}\right] \\
&\leq \left(\int_0^1 \left|\left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4}\right)\right|^p \mathrm{d}\tau\right)^{\frac{1}{p}}[(n+1)^{\frac{1}{q}}(a_2 - ma_1)^{\frac{1}{p}}]\left(\int_{ma_1}^{a_2} |\mathcal{F}'(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} \\
&= T^{\frac{1}{p}}\left((n+1)^{\frac{1}{q}}(a_2 - ma_1)^{\frac{1}{p}}\right)\|\mathcal{F}'\|_p,
\end{aligned}$$

where

$$\int_0^1 \left| \left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4} \right) \right|^p \mathrm{d}\tau \leq \int_0^1 \left| \left(\frac{\tau}{(n+4)^{\frac{\kappa}{\alpha}}} - \frac{1}{(n+4)^{\frac{\kappa}{\alpha}}} \right) \right|^{\frac{\alpha p}{\kappa}} \mathrm{d}\tau$$
$$= \frac{\kappa (n+1)^{\frac{\kappa}{\alpha}} \left((n+1)^{\frac{-\kappa}{\alpha}} - (n+4)^{\frac{-\kappa}{\alpha}} \right)^{\frac{\alpha p}{\kappa}+1}}{\alpha p + \kappa} + \frac{\kappa (n+1)^{\frac{\kappa}{\alpha}} (n+4)^{-p-\frac{\kappa}{\alpha}}}{\alpha p + \kappa} := N$$

which follows from $\left|A^\theta - B^\theta\right| \leq |A - B|^\theta$ for any $A, B \geq 1$ and $\theta \in (0,1]$. This completes the proof. □

Theorem 2.5 *Let $\alpha > 0$, $\mu > 0$, $n \in \mathbb{N}$ and $\mathcal{F} : [ma_1, \frac{a_2}{m}] \to \mathbb{R}$ be a differentiable function with $a_1 < a_2$ such that $\mathcal{F} \in L_1[ma_1, \frac{a_2}{m}]$. If $|\mathcal{F}'|^q$ is strongly (s, m)-convex function of higher order of $\sigma > 0$ with $(s, m) \in (0,1]^2$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $w \in [ma_1, a_2]$ and $\alpha \in (0,1]$, we have*

$$\begin{aligned}
&\Omega(a_1, a_2; m, \alpha, w) \\
&\leq T^{\frac{1}{p}} \left[\frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)} \left(\frac{m}{(n+1)^s (s+1)} |\mathcal{F}'(a_1)|^q \right.\right. \\
&\left. + \frac{(n+1)^{s+1} - n^{s+1}}{(n+1)^s (s+1)} |\mathcal{F}'(w)|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)} \|w - a_1\|^\sigma \right)^{\frac{1}{q}} \\
&+ \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)} \left(\frac{(n+1)^{s+1} - n^{s+1}}{(n+1)^s (s+1)} |\mathcal{F}'(w)|^q \right. \\
&\left.\left. + \frac{m}{(n+1)^s (s+1)} \left| \mathcal{F}'(\frac{a_2}{m}) \right|^q - \frac{2\mu}{(\sigma+1)(\sigma+2)} \| \frac{a_2}{m} - w \|^\sigma \right)^{\frac{1}{q}} \right],
\end{aligned}$$

where T is defined in Theorem 2.4.

Proof Using Lemma 2.1 and the fact that $|\mathcal{F}'|^q$ is strongly (s, m)-convex function of higher order of $\sigma > 0$, we have

$$\begin{aligned}
&|\Omega(a_1, a_2; m, \alpha, w)| \\
&\leq \frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)} \left(\int_0^1 \left| \left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4} \right) \right|^p \mathrm{d}\tau \right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1 \left| \mathcal{F}' \left(\frac{1-\tau}{n+1} ma_1 + \frac{n+\tau}{n+1} w \right) \right|^q \mathrm{d}\tau \right)^{\frac{1}{q}} \\
&+ \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)} \left(\int_0^1 \left| \left(\frac{1}{n+4} - \frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} \right) \right|^p \mathrm{d}\tau \right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1 \left| \mathcal{F}' \left(\frac{n+\tau}{n+1} w + \frac{1-\tau}{n+1} a_2 \right) \right|^q \mathrm{d}\tau \right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left| \left(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1} - \frac{1}{n+4} \right) \right|^p \mathrm{d}\tau \right)^{\frac{1}{p}} \\
&\quad \left[\frac{(w - ma_1)^2}{(n+1)(a_2 - ma_1)} \left(\int_0^1 \left(m \left(\frac{1-\tau}{n+1} \right)^s |\mathcal{F}'(a_1)|^q + \left(\frac{n+\tau}{n+1} \right)^s |\mathcal{F}'(w)|^q \right) \mathrm{d}\tau \right.\right. \\
&\quad \left. - \mu(\tau^\sigma (1-\tau) + \tau(1-\tau)^\sigma) \|w - a_1\|^\sigma \right)^{\frac{1}{q}} \\
&\quad + \frac{(a_2 - w)^2}{(n+1)(a_2 - ma_1)} \left(\int_0^1 \left(\left(\frac{n+\tau}{n+1} \right)^s |\mathcal{F}'(w)|^q + m \left(\frac{1-\tau}{n+1} \right)^s \left| \mathcal{F}'(\frac{a_2}{m}) \right|^q \right.\right. \\
&\quad \left.\left.\left. - \mu(\tau^\sigma (1-\tau) + \tau(1-\tau)^\sigma) \| \frac{a_2}{m} - w \|^\sigma \right) \mathrm{d}\tau \right)^{\frac{1}{q}} \right],
\end{aligned}$$

To obtain our required result, we use the following:

$$\int_0^1 \left(\frac{n+\tau}{n+1}\right)^s \mathrm{d}\tau = \frac{(n+1)^{s+1} - n^{s+1}}{(n+1)^s(s+1)},$$
$$\int_0^1 \left(\frac{1-\tau}{n+1}\right)^s \mathrm{d}\tau = \frac{1}{(n+1)^s(s+1)}$$

and

$$\int_0^1 \tau^\sigma(1-\tau)\mathrm{d}\tau = \frac{1}{(\sigma+1)(\sigma+2)}.$$

□

We now discuss some special cases.

Corollary 2.1 *If* $w = ma_1$ *and* $\mu = 0$ *in Theorem* 2.5, *we get*

$$\begin{aligned} &|\Omega(a_1, a_2; m, \alpha, ma_1)| \\ &\leq T^{\frac{1}{p}} \frac{(a_2 - ma_1)}{(n+1)} \left[\left(\frac{(n+1)^{s+1} - n^{s+1}}{(n+1)^s(s+1)} |\mathcal{F}'(ma_1)|^q \right.\right. \\ &\left.\left. + \frac{m}{(n+1)^s(s+1)} \left| \mathcal{F}'(\frac{a_2}{m}) \right|^q \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{2.4}$$

Corollary 2.2 *If* $w = a_2$ *and* $\mu = 0$ *in Theorem* 2.5, *we get*

$$\begin{aligned} &|\Omega(a_1, a_2; m, \alpha, a_2)| \\ &\leq T^{\frac{1}{p}} \frac{(a_2 - ma_1)}{(n+1)} \left[\left(\frac{m}{(n+1)^s(s+1)} |\mathcal{F}'(a_1)|^q. \right.\right. \\ &\left.\left. + \frac{(n+1)^{s+1} - n^{s+1}}{(n+1)^s(s+1)} |\mathcal{F}'(a_2)|^q \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{2.5}$$

Corollary 2.3 *If* $w = \frac{ma_1+a_2}{n+1}$ *and* $\mu = 0$ *in Theorem* 2.5, *we get*

$$\begin{aligned} &\left|\Omega(a_1, a_2; m, \alpha, \frac{ma_1 + a_2}{n+1})\right| \\ &\leq T^{\frac{1}{p}} \left[\left(\frac{(a_2 - mna)^2}{(n+1)^3(a_2 - ma_1)} \right) \left(\frac{m}{(n+1)^s(s+1)} \right)^{\frac{1}{q}} |\mathcal{F}'(a_1)| \right. \\ &\left(\frac{(n+1)^{s+1} - n^{s+1}}{(n+1)^s(s+1)} \right)^{\frac{1}{q}} \left\{ \left(\frac{(a_2 - mna)^2}{(n+1)^3(a_2 - ma_1)} \right) + \left(\frac{(nb - ma_1)^2}{(n+1)^3(a_2 - ma_1)} \right) \right\} \\ &\left. \left| \mathcal{F}'\left(\frac{ma_1 + a_2}{n+1} \right) \right| + \left(\frac{m}{(n+1)^s(s+1)} \right)^{\frac{1}{q}} \left| \mathcal{F}'\left(\frac{a_2}{m} \right) \right| \right]. \end{aligned}$$

Corollary 2.4 *Combining* (2.4) *and* (2.5), *we have*

$$\begin{aligned}
&\Bigg|\frac{3}{(n+1)(n+4)}[\mathcal{F}(a_2)+\mathcal{F}(ma_1)] \\
&\quad+\frac{1}{(n+4)}\left[\mathcal{F}\left(\frac{(nb+ma_1)}{(n+1)}\right)+\mathcal{F}\left(\frac{(nma+a_2)}{(n+1)}\right)\right] \\
&\quad-\frac{(n+1)^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(a_2-ma_1)^{\frac{\alpha}{\kappa}}}\,{}_kJ^{\alpha}_{(ma_1)^+}\mathcal{F}\left(\frac{(nma+a_2)}{n+1}\right) \\
&\quad-\frac{(n+1)^{\frac{\alpha}{\kappa}-1}\Gamma_\kappa(\alpha+\kappa)}{(a_2-ma_1)^{\frac{\alpha}{\kappa}}}\,{}_kJ^{\alpha}_{a_2^-}\mathcal{F}\left(\frac{(nb+ma_1)}{n+1}\right)\Bigg| \\
&\leq\frac{(a_2-ma_1)^2}{(n+1)}T^{\frac{1}{p}}\Bigg[\left(\frac{m}{(n+1)^s(s+1)}|\mathcal{F}'(a_1)|^q.\right. \\
&\quad\left.+\frac{(n+1)^{s+1}-n^{s+1}}{(n+1)^s(s+1)}|\mathcal{F}'(a_2)|^q\right)^{\frac{1}{q}} \\
&\left(\frac{(n+1)^{s+1}-n^{s+1}}{(n+1)^s(s+1)}|\mathcal{F}'(ma_1)|^q+\frac{m}{(n+1)^s(s+1)}\left|\mathcal{F}'(\frac{a_2}{m})\right|^q\right)^{\frac{1}{q}}\Bigg]
\end{aligned} \tag{2.6}$$

Corollary 2.5 *By taking* $\alpha=\kappa=m=1$ *in* (2.6), *we have*

$$\begin{aligned}
&\Bigg|\frac{3(a_2-a_1)}{(n+1)(n+4)}[\mathcal{F}(a_1)+\mathcal{F}(a_2)] \\
&\quad+\frac{a_2-a_1}{(n+4)}\left[\mathcal{F}\left(\frac{(nb+a_1)}{(n+1)}\right)+\mathcal{F}\left(\frac{(na+a_2)}{(n+1)}\right)\right] \\
&-\left[\int_{a_1}^{\frac{na+a_2}{n+1}}\mathcal{F}(x)\mathrm{d}x+\int_{\frac{a_1+nb}{n+1}}^{a_2}\mathcal{F}(x)\mathrm{d}x\right]\Bigg| \\
&\quad\leq\frac{(a_2-a_1)^2}{(n+1)}\left[\frac{(n+1)[3^{p+1}-(n+1)^{p+1}]}{(n+1)^{p+1}(n+4)^{p+1}(p+1)}\right]^{\frac{1}{p}} \\
&\times\left[\left(\frac{m}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(n+1)^{s+1}-n^{s+1}}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}\right][|\mathcal{F}'(a_1)|+|\mathcal{F}'(a_2)|].
\end{aligned} \tag{2.7}$$

Theorem 2.6 *Let* $\alpha>0$, $n\in\mathbb{N}$ *and* $\mathcal{F}:[ma_1,\frac{a_2}{m}]\to\mathbb{R}$ *be a differentiable function with* $a_1<a_2$ *such that* $\mathcal{F}\in L_1[ma_1,\frac{a_2}{m}]$. *If* $|\mathcal{F}'|^q$ *is strongly* (s,m)*-convex function of higher order of* $\sigma>0$ *with* $(s,m)\in(0,1]^2$ *for* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then for* $w\in[ma_1,a_2]$ *and* $\alpha\in(0,1]$, *we have*

$$\begin{aligned}
&|\Omega(a_1,a_2;m,\alpha,w)|\\
&\leq T^{\frac{1}{p}}\Bigg[\frac{(w-ma_1)^2}{(n+1)(a_2-ma_1)}\Bigg(m\bigg(\frac{1}{(n+1)^{qs}(qs+1)}\bigg)^{\frac{1}{q}}|\mathcal{F}'(a_1)|\\
&\quad+\bigg(\frac{(n+1)^{qs+1}-n^{qs+1}}{(n+1)^{qs}(qs+1)}\bigg)^{\frac{1}{q}}|\mathcal{F}'(w)|\\
&\quad-2\mu\|w-a_1\|^{\sigma}(kB_{\kappa}(\kappa(q\sigma+1),\kappa(q+1)))^{\frac{1}{q}}\Bigg)\\
&+\frac{(a_2-w)^2}{(n+1)(a_2-ma_1)}\Bigg(\bigg(\frac{(n+1)^{qs+1}-n^{qs+1}}{(n+1)^{qs}(qs+1)}\bigg)^{\frac{1}{q}}|\mathcal{F}'(w)|\\
&\quad+m\bigg(\frac{1}{(n+1)^{qs}(qs+1)}\bigg)^{\frac{1}{q}}\Big|\mathcal{F}'(\frac{a_2}{m})\Big|\\
&\quad-2\mu\|\frac{a_2}{m}-w\|^{\sigma}(kB_{\kappa}(\kappa(q\sigma+1,\kappa(q+1))))^{\frac{1}{q}}\Bigg)\Bigg],
\end{aligned}\tag{2.8}$$

where T is defined in Theorem 2.4.

Proof Using Lemma 2.1 and convexity of $|\mathcal{F}'|^q$ with $p,q>1$, we have

$$\begin{aligned}
&|\Omega(a_1,a_2;m,\alpha,w)|\\
&\quad\leq\frac{(w-ma_1)^2}{(n+1)(a_2-ma_1)}\bigg(\int_0^1\bigg|\bigg(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}-\frac{1}{n+4}\bigg)\bigg|\bigg|\mathcal{F}'\bigg(\frac{1-\tau}{n+1}ma_1+\frac{n+\tau}{n+1}w\bigg)\bigg|\mathrm{d}\tau\bigg)\\
&\quad+\frac{(a_2-w)^2}{(n+1)(a_2-ma_1)}\bigg(\int_0^1\bigg|\bigg(\frac{1}{n+4}-\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}\bigg)\bigg|\bigg|\mathcal{F}'\bigg(\frac{n+\tau}{n+1}w+\frac{1-\tau}{n+1}a_2\bigg)\bigg|\mathrm{d}\tau\bigg)\\
&\quad\leq\frac{(w-ma_1)^2}{(n+1)(a_2-ma_1)}\bigg(\int_0^1\bigg|\bigg(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}-\frac{1}{n+4}\bigg)\bigg|\\
&\qquad\bigg(m\bigg(\frac{1-\tau}{n+1}\bigg)^s|\mathcal{F}'(a_1)|+\bigg(\frac{n+\tau}{n+1}\bigg)^s|\mathcal{F}'(w)|\\
&\quad-\mu(\tau^{\sigma}(1-\tau)+\tau(1-\tau)^{\sigma})\|w-a_1\|^{\sigma})\mathrm{d}\tau)\\
&\quad+\frac{(a_2-w)^2}{(n+1)(a_2-ma_1)}\bigg(\int_0^1\bigg|\bigg(\frac{1}{n+4}-\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}\bigg)\bigg|\\
&\qquad\bigg(\bigg(\frac{n+\tau}{n+1}\bigg)^s|\mathcal{F}'(w)|+m\bigg(\frac{1-\tau}{n+q}\bigg)^s\Big|\mathcal{F}'(\frac{a_2}{m})\Big|\\
&\quad-\mu(\tau^{\sigma}(1-\tau)+\tau(1-\tau)^{\sigma})\|\frac{a_2}{m}-w\|^{\sigma}\bigg)\mathrm{d}\tau\bigg)\\
&\quad\leq\bigg(\int_0^1\bigg|\bigg(\frac{\tau^{\frac{\alpha}{\kappa}}}{n+1}-\frac{1}{n+4}\bigg)\bigg|^p\mathrm{d}\tau\bigg)^{\frac{1}{p}}\\
&\quad\Bigg[\frac{(w-ma_1)^2}{(n+1)(a_2-ma_1)}\Bigg(\bigg(\bigg(\int_0^1\bigg(m\bigg(\frac{1-\tau}{n+1}\bigg)^{qs}\mathrm{d}\tau\bigg)^{\frac{1}{q}}|\mathcal{F}'(a_1)|\\
&\qquad+\bigg(\int_0^1\bigg(\frac{n+\tau}{n+1}\bigg)^{qs}\mathrm{d}\tau\bigg)^{\frac{1}{q}}|\mathcal{F}'(w)|\\
&\quad-\mu\|w-a_1\|^{\sigma}\bigg(\bigg(\int_0^1\tau^{q\sigma}(1-\tau)^q\mathrm{d}\tau\bigg)^{\frac{1}{q}}+\bigg(\int_0^1\tau^q(1-\tau)^{q\sigma}\mathrm{d}\tau\bigg)^{\frac{1}{q}}\bigg)\bigg)\Bigg)\\
&\quad+\frac{(a_2-w)^2}{(n+1)(a_2-ma_1)}\Bigg(\bigg(\bigg(\int_0^1\bigg(\frac{n+\tau}{n+1}\bigg)^{qs}\mathrm{d}\tau\bigg)^{\frac{1}{q}}|\mathcal{F}'(w)|\\
&\qquad+m\bigg(\int_0^1\bigg(\frac{1-\tau}{n+1}\bigg)^{qs}\mathrm{d}\tau\bigg)^{\frac{1}{q}}\Big|\mathcal{F}'(\frac{a_2}{m})\Big|\\
&\quad-\mu\|\frac{a_2}{m}-w\|^{\sigma}\bigg(\bigg(\int_0^1\tau^{q\sigma}(1-\tau)^q\mathrm{d}\tau\bigg)^{\frac{1}{q}}+\bigg(\int_0^1\tau^q(1-\tau)^{q\sigma}\mathrm{d}\tau\bigg)^{\frac{1}{q}}\bigg)\bigg)\Bigg)\Bigg].
\end{aligned}$$

By using these integrals in above expression, we obtain our required result

$$\int_0^1 \left(\frac{n+\tau}{n+1}\right)^{qs} d\tau = \frac{(n+1)^{qs+1} - n^{qs+1}}{(n+1)^{qs}(qs+1)},$$

$$\int_0^1 \left(\frac{1-\tau}{n+1}\right)^{qs} d\tau = \frac{1}{(n+1)^{qs}(qs+1)}$$

and

$$\int_0^1 \tau^q (1-\tau)^{q\sigma} d\tau = \kappa B_\kappa(\kappa(q\sigma+1, \kappa(q+1))).$$

This completes the proof. □

Corollary 2.6 *If we put* $w = \frac{ma_1+a_2}{n+1}$ *in Theorem 2.6, then*

$$\begin{aligned}
&\left|\Omega(a_1, a_2; m, \alpha, \frac{ma_1+a_2}{n+1})\right| \\
&\quad \le T^{\frac{1}{p}}\Bigg[\left(\frac{(a_2-mna)^2}{(n+1)^3(a_2-ma_1)}\right)\left(\frac{m}{(n+1)^{qs}(qs+1)}\right)^{\frac{1}{q}}|\mathcal{F}'(a_1)| \\
&\qquad + \left(\frac{(n+1)^{qs+1}-n^{qs+1}}{(n+1)^{qs}(qs+1)}\right)^{\frac{1}{q}}\left\{\left(\frac{(a_2-mna)^2}{(n+1)^3(a_2-ma_1)}\right)+\left(\frac{(nb-ma_1)^2}{(n+1)^3(a_2-ma_1)}\right)\right\} \\
&\qquad \left|\mathcal{F}'\left(\frac{ma_1+a_2}{n+1}\right)\right| \\
&\qquad + \left(\frac{(nb-ma_1)^2}{(n+1)^3(a_2-ma_1)}\right)\left(\frac{m}{(n+1)^{qs}(qs+1)}\right)^{\frac{1}{q}}\left|\mathcal{F}'\left(\frac{a_2}{m}\right)\right|\Bigg].
\end{aligned} \tag{2.9}$$

3 Applications

3.1 Applications to means

In this section, we give some applications to means. First of all we recall the definitions of some special means.

For $0 < a_1 < a_2$ and $n \in \mathbb{N}$, we have

1. The arithmetic mean is defined as: $A(a_1, a_2) = \frac{a_1+a_2}{n+1}$,
2. The β-logarithmic mean is defined as: $L_\beta(a_1, a_2) = \left[\frac{{a_2}^{\beta+1} - {a_1}^{\beta+1}}{(\beta+1)(a_2-a_1)}\right]^{\frac{1}{\beta}}$ where $\beta \in \mathbb{Z} - \{0, 1\}$.

We now discuss main results of this section.

Proposition 3.1 *Let* $0 < a_1 < a_2$, $0 < s < 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $q > 1$, *then*

$$\begin{aligned}
&\Bigg|\frac{3q}{(n+1)(n+4)(a_2-a_1)(s+q)}A^{\frac{s}{q}+1}(a_1,a_2)\frac{q}{(n+4)(a_2-a_1)(s+q)}\\
&\left[\left(\frac{a_2-na}{n+1}\right)A^{\frac{s}{q}+1}\left(\frac{(2n+1)a_1}{n+1},\frac{nb}{n+1}\right)\right.\\
&\left.+\left(\frac{nb-a_1}{n+1}\right)A^{\frac{s}{q}+1}\left(\frac{na}{n+1},\frac{(2n+1)a_2}{n+1}\right)\right]\\
&-\frac{q}{(s+q)}\left[L^{\frac{s}{q}+1}_{\frac{s}{q}+1}\left(\frac{(2n+1)a_1+nb}{(n+1)^2},\frac{na+(2n+1)a_2}{(n+1)^2}\right)\right]\Bigg|\\
&\le\left[\frac{(n+1)(3^{p+1}+(n+1)^{p+1})}{(n+1)^{p+1}(n+4)^{p+1}(p+1)}\right]^{\frac{1}{p}}\left[\left(\frac{(a_2-na)^2}{(n+1)^3(a_2-a_1)}\right)\right.\\
&\left(\frac{1}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}a_1{}^{\frac{s}{q}}\left(\frac{(n+1)^{s+1}-n^{s+1}}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}\cdot\\
&\left\{\left(\frac{(a_2-na)^2}{(n+1)^3(a_2-a_1)}\right)+\left(\frac{(nb-a_1)^2}{(n+1)^3(a_2-a_1)}\right)\right\}A^{\frac{s}{q}}(a_1,a_2)\\
&\left.+\left(\frac{(nb-a_1)^2}{(n+1)^3(a_2-a_1)}\right)\left(\frac{1}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}a_2{}^{\frac{s}{q}}\right].
\end{aligned}$$

Proof Using $\alpha=\kappa=m=1$ and $\mathcal{F}(x)=\frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$ in Corollary 2.3 completes the proof. □

Proposition 3.2 *Let* $0<a_1<a_2$, $0<s<1$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *with* $q>1$, *then*

$$\begin{aligned}
&\Bigg|\frac{3q}{(n+1)(n+4)(a_2-a_1)(s+q)}A^{\frac{s}{q}+1}(a_1,a_2)\frac{q}{(n+4)(a_2-a_1)(s+q)}\\
&\left[\left(\frac{a_2-na}{n+1}\right)A^{\frac{s}{q}+1}\left(\frac{(2n+1)a_1}{n+1},\frac{nb}{n+1}\right)\right.\\
&\left.+\left(\frac{nb-a_1}{n+1}\right)A^{\frac{s}{q}+1}\left(\frac{na}{n+1},\frac{(2n+1)a_2}{n+1}\right)\right]\\
&-\frac{q}{(s+q)}\left[L^{\frac{s}{q}+1}_{\frac{s}{q}+1}\left(\frac{(2n+1)a_1+nb}{(n+1)^2},\frac{na+(2n+1)a_2}{(n+1)^2}\right)\right]\Bigg|\\
&\le\left[\frac{(n+1)(3^{p+1}+(n+1)^{p+1})}{(n+1)^{p+1}(n+4)^{p+1}(p+1)}\right]^{\frac{1}{p}}\\
&\left[\left(\frac{(a_2-na)^2}{(n+1)^3(a_2-a_1)}\right)\left(\frac{1}{(n+1)^{qs}(qs+1)}\right)^{\frac{1}{q}}a_1{}^{\frac{s}{q}}\right.\\
&\left(\frac{(n+1)^{qs+1}-n^{qs+1}}{(n+1)^{qs}(qs+1)}\right)^{\frac{1}{q}}\left\{\left(\frac{(a_2-na)^2}{(n+1)^3(a_2-a_1)}\right)\right.\\
&\left.+\left(\frac{(nb-a_1)^2}{(n+1)^3(a_2-a_1)}\right)\right\}A^{\frac{s}{q}}(a_1,a_2)\\
&\left.+\left(\frac{(nb-a_1)^2}{(n+1)^3(a_2-a_1)}\right)\left(\frac{1}{(n+1)^{qs}(qs+1)}\right)^{\frac{1}{q}}a_2{}^{\frac{s}{q}}\right].
\end{aligned}$$

Proof Using $\alpha = \kappa = m = 1$ and $\mathcal{F}(x) = \frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$ in Corollary 2.6 completes the proof. □

3.2 Simpson-like quadrature formula

Let P be the partition of the interval $[a_1, a_2]$, that is $P : a_1 = x_o < x_1 < x_2 < \ldots < x_{n-1} < x_n = a_2$. Consider the quadrature formula

$$\int_{a_1}^{a_2} \mathcal{F}(x)\mathrm{d}x = {}_sA(\mathcal{F}, P) + {}_sR(\mathcal{F}, P),$$

where

$$\begin{aligned} {}_sA(\mathcal{F}, P) &= \frac{3}{10}\sum_{i=1}^{n-1}[\mathcal{F}(x_i) + \mathcal{F}(x_{i+1})](x_{i+1} - x_i) \\ &+ \frac{2}{5}\sum_{i=0}^{n-1}\mathcal{F}\left(\frac{(x_i + x_{i+1})^2}{2}\right)(x_{i+1} - x_i), \end{aligned}$$

for Simpson-Like formula and ${}_sR(\mathcal{F}, P)$ denotes the relative approximation error of the integral $\int_{a_1}^{b} f(x)\mathrm{d}x$.

Proposition 3.3 *Suppose that all the assumptions of Theorem 2.5 are satisfied. then the Simpson-like error estimate exists*

$$\begin{aligned} &|{}_sR(\mathcal{F}, P)| \\ &\le \left[\frac{(n+1)[3^{p+1} + (n+1)^{p+1}]}{(n+1)^{p+1}(n+4)^{p+1}(p+1)}\right]^{\frac{1}{p}} \\ &\times \left[\left(\frac{1}{(n+1)^s(s+1)}\right)^{\frac{1}{q}} + \left(\frac{(n+1)^{s+1} - n^{s+1}}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}\right] \\ &\sum_{i=0}^{n-1}\frac{(x_{i+1} - x_i)^2}{(n+1)}[|\mathcal{F}'(x_i)| + |\mathcal{F}'(x_{i+1})|]. \end{aligned} \tag{3.1}$$

Proof Considering the subintervals of $[a_1, a_2]$, that is $[x_i, x_{i+1}] \subset [a_1, a_2]$ and using in Corollary 2.5, we obtain

$$\begin{aligned}&\Bigg|\frac{3(x_{i+1}-x_i)}{(n+1)(n+4)}[\mathcal{F}(x_i)+\mathcal{F}(x_{i+1})]\\&\quad+\frac{x_{i+1}-x_i}{(n+4)}\left[\mathcal{F}\left(\frac{(nx_{i+1}+x_i)}{(n+1)}\right)+\mathcal{F}\left(\frac{(nx_i+x_{i+1})}{(n+1)}\right)\right]\\&\quad-\left[\int_{x_i}^{\frac{nx_i+x_{i+1}}{n+1}}\mathcal{F}(x)\mathrm{d}x+\int_{\frac{x_i+nx_{i+1}}{n+1}}^{x_{i+1}}\mathcal{F}(x)\mathrm{d}x\right]\Bigg|\\&\leq\left[\frac{(n+1)[3^{p+1}+(n+1)^{p+1}]}{(n+1)^{p+1}(n+4)^{p+1}(p+1)}\right]^{\frac{1}{p}}\\&\quad\times\left[\left(\frac{1}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(n+1)^{s+1}-n^{s+1}}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}\right]\\&\quad\frac{(x_{i+1}-x_i)^2}{(n+1)}[|\mathcal{F}'(x_i)|+|\mathcal{F}'(x_{i+1})|].\end{aligned}$$

By taking sum over i from 0 to $n-1$, we get

$$\begin{aligned}|_sR(\mathcal{F},P)|&=\sum_{i=0}^{n-1}\Bigg|\frac{3(x_{i+1}-x_i)}{(n+1)(n+4)}[\mathcal{F}(x_i)+\mathcal{F}(x_{i+1})]\\&\quad+\frac{x_{i+1}-x_i}{(n+4)}\left[\mathcal{F}\left(\frac{(nx_{i+1}+x_i)}{(n+1)}\right)+\mathcal{F}\left(\frac{(nx_i+x_{i+1})}{(n+1)}\right)\right]\\&\quad-\left[\int_{x_i}^{\frac{nx_i+x_{i+1}}{n+1}}\mathcal{F}(x)\mathrm{d}x+\int_{\frac{x_i+nx_{i+1}}{n+1}}^{x_{i+1}}\mathcal{F}(x)\mathrm{d}x\right]\Bigg|\\&\leq\left[\frac{(n+1)[3^{p+1}+(n+1)^{p+1}]}{(n+1)^{p+1}(n+4)^{p+1}(p+1)}\right]^{\frac{1}{p}}\\&\quad\times\left[\left(\frac{1}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(n+1)^{s+1}-n^{s+1}}{(n+1)^s(s+1)}\right)^{\frac{1}{q}}\right]\\&\quad\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{(n+1)}[|\mathcal{F}'(x_i)|+|\mathcal{F}'(x_{i+1})|].\end{aligned}$$

This completes the proof. □

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Declarations

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Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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