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# Homology and Euler characteristic of generalized anchored configuration spaces of graphs

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**Abstract**
In this paper we consider the generalized anchored configuration spaces on $n$ labeled points on a graph. These are the spaces of all configurations of $n$ points on a fixed graph $G$, subject to the condition that at least $q$ vertices in some pre-determined set $K$ of vertices of $G$ are included in each configuration. We give a non-alternating formula for the Euler characteristic of such spaces for arbitrary connected graphs, which are not trees. Furthermore, we completely determine the homology groups of the generalized anchored configuration spaces of $n$ points on a circle graph.

**Keywords** Graphs · Configuration spaces · Topological combinatorics · Applied topology · Chain complexes

**Mathematics Subject Classification** Primary 55U05 · Secondary 57Z25 · 05E45

## 1 Introduction

The study of the anchored configuration spaces was initiated in Kozlov (2021) and continued in Hoekstra-Mendoza (2022), Kozlov (2022). These spaces are motivated by certain considerations in logistics and differ from classical configuration spaces in a crucial way. The formal definition is as follows.

**Definition 1.1** Let $X$ be a non-empty topological space, let $K$ be a set of distinct points in $X$, and let $n$ be an arbitrary positive integer. An **anchored configuration space**,

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denoted $\Sigma(X, K, n)$, is defined as the subspace of the direct product $X^n$, consisting of all tuples $(x_1, \dots, x_n)$, such that $K \subseteq \{x_1, \dots, x_n\}$.

In this paper we continue with this line of research and generalize Definition 1.1, by relaxing the conditions on the allowed $n$-tuples $(x_1, \dots, x_n)$.

**Definition 1.2** As above, let $X$ be a non-empty topological space, let $K$ be a set of $k$ distinct points in $X$, $k \geq 0$, and let $n$ be an arbitrary positive integer. Furthermore, let $q$ be an arbitrary integer, such that $k \geq q \geq 0$. A **generalized anchored configuration space**, denoted $\Sigma(X, K, n, q)$, is defined as the subspace of the direct product $X^n$, consisting of all tuples $(x_1, \dots, x_n)$, such that $|K \cap \{x_1, \dots, x_n\}| \geq q$.

Clearly, the case $k = q$ in Definition 1.2 corresponds to Definition 1.1.

So far, the anchored configuration spaces have been studied in the situation when $X$ is a geometric realization of a graph $G$, and $K$ is a subset of the set of the vertices of $G$. These spaces are of particular interest for the logistics, since they formalize networks with moving goods, with the extra condition that at each point in time a certain set of nodes is securely supplied with the goods. Accordingly, the generalized anchored configuration spaces relax this condition and only require that at each point in time at least $q$ nodes out of some previously fixed set $K$ are supplied.

The case of the anchored spaces, where the graph $G$ is a tree has been settled in Kozlov (2021), where the homotopy type of $\Sigma(X, K, n)$ has been completely determined. Accordingly, in this paper, we consider the case when $G$ is not a tree.

As a first result we provide a non-recursive formula for the Euler characteristic of $\Sigma(G, K, n, q)$, expressed as a sum of positive terms (rather than a sign-alternating sum). After that we turn to study the topology of these spaces.

We consider the case when $G$ is a circle graph, which appears to be the most natural next step, after the case of $G$ being a tree. This is the same as to consider the case when the topological space $X$ is homeomorphic to a circle, since changing the positions of the points in $K$ will produce homeomorphic anchored configuration spaces. Since all we need to record is the cardinality of $K$, we let $\Omega(k, n)$ denote $\Sigma(G, K, n)$, where $G$ is a cycle graph with $k$ vertices, and $K$ is the set of vertices of $G$. Passing on to the generalized anchored configuration spaces, we let $\Omega(k, n, q)$ denote $\Sigma(G, K, n, q)$ in this case.

The spaces $\Omega(2, n)$ were the focus of investigations in Kozlov (2022) and in [HM]. More specifically, the homology of these spaces was calculated in Kozlov (2022) using discrete Morse theory. This work was continued in [HM], where the cup product structure was completely described, and connection to the topological complexity was established.

In this paper we study the spaces $\Omega(k, n)$ of an arbitrary $k$, and more generally $\Omega(k, n, q)$, for an arbitrary $q \leq k$, and calculate their homology groups in all dimensions. Rather than using discrete Morse theory, our method is to consider classical long sequences for the corresponding combinatorially given chain complexes. For the standard concepts of Algebraic Topology we refer to (Fulton 1995; Greenberg and Harper 1981; Hatcher 2002; Munkres 1984). Our study lies within the field of Applied Topology, see (Carlsson 2009; Edelsbrunner and Harer 2010; Kozlov 2008, 2020) for more information.

## 2 The Euler characteristic of the generalized anchored configuration spaces

Let $G$ be a non-empty connected graph,[1] which is not a tree, and let $V$ and $E$ denote its sets of vertices and edges, respectively. Let $K$ be an arbitrary subset of $V$, and let $q \leq |K|$. In this section we give a theorem which provides a complete non-recursive and non-alternating formula for the Euler characteristic of the spaces $\Sigma(G, K, n, q)$. Before we proceed with its formulation and its proof, let recall the following concepts.

First, for arbitrary positive integers $a \geq b$ the *Stirling numbers of the second kind*, denoted $\left\{ a \atop b \right\}$, count the number of ways to partition a set of $a$ labelled objects into $b$ nonempty unlabelled subsets. Clearly, then $b!\left\{ a \atop b \right\}$ is the number of ways to partition a set of $a$ labelled objects into $b$ nonempty labelled subsets.

Second, if we have a set $U$, a subset $S \subseteq U$ and an element $x \in U$, we let $S\,\mathrm{XOR}\,x$ denote the subset of $U$ obtained from $S$ by the *exclusive or* operation with respect to $x$. Formally, we set

$$S\,\mathrm{XOR}\,x := \begin{cases} S \setminus x, & \text{if } x \in S; \\ S \cup x, & \text{if } x \notin S. \end{cases}$$

We can now formulate the main result of this section.

**Theorem 2.1** *Let $G$ be an arbitrary non-empty connected graph, whose set of vertices is $V$, and whose set of edges is $E$. Let $K$ be an arbitrary non-empty subset of $V$, and let $q$ be a positive integer, such that $q \leq |K|$. Finally, let $n$ be a natural number, such that $n \geq q$.*

*Assume the graph $G$ is not a tree. Then, the Euler characteristic of the cell complex $\Sigma(G, K, n, q)$ is given by the formula*[2]

$$\frac{(-1)^{n-q}}{q!}\chi(\Sigma(G, K, n, q)) = \sum_{\lambda=\varepsilon}^{\varepsilon+k-q} \sum_{t=0}^{n-q} \binom{\lambda - \varepsilon + q - 1}{q-1} \binom{n}{t} \left\{ n-t \atop q \right\} \lambda^t, \quad (2.1)$$

*where $k := |K|$ and $\varepsilon := |E| - |V|$.*

Before proceeding with the proof of Theorem 2.1, we would like to make a couple of remarks. First of all note that, since $G$ is a non-empty connected and not a tree, it must have some vertices and some edges, and $\varepsilon = |E| - |V| \geq 0$.

Consider the simplest case where $G$ has one vertex and one loop edge. We then have $\varepsilon = 0$ and $k = q = 1$. The right hand side of (2.1) then reads $\sum_{t=0}^{n-1} \binom{n}{t} 0^t = 1$, where we use the convention for $0^t$ from Footnote 2. On the other hand, the cell complex $\Sigma(G, K, n, q)$ in this case is obtained from an $n$-torus by removing an $n$-cell, so the Euler characteristic of it is equal to $(-1)^{n-1}$.

[1] Note, that we do not assume that the graph $G$ is simple. In other words, it may have multiple edges, as well as loops.

[2] Note, that in the formula (2.1) we use the convention $0^0 = 1$, while of course $0^t = 0$, for $t > 0$.

For a slightly more involved situation, assume $G$ has a single vertex and $d$ loops, $d \geq 2$. In that case, the right hand side of (2.1) becomes

$$\sum_{t=0}^{n-1} \binom{n}{t} (d-1)^t = d^n - (d-1)^n.$$

The cell complex $\Sigma(G, K, n, q)$ is the $(n-1)$-skeleton of the cubical complex $G^n$. The Euler characteristic is multiplicative, so $\chi(G^n) = \chi(G)^n = (-1)^n (d-1)^n$. The cell complex $\Sigma(G, K, n, q)$ is obtained from $G^n$ by removing $d^n$ n-cells, so we get

$$\chi(\Sigma(G, K, n, q)) = \chi(G^n) - (-1)^n d^n = (-1)^{n-1}(d^n - (d-1)^n),$$

which agrees with the right hand side of (2.1).

In general, the cubical complex $\Sigma(G, K, n, q)$ has dimension $n - q$. So, it makes sense to mention the lowest dimensional case, which is $n = q$. Clearly, the complex $\Sigma(G, K, q, q)$ is a disjoint union of $q!\binom{k}{q}$ points, because we just need to choose which $q$ of the $k$ points in $K$ will be covered by the $n$ points, and after that permute arbitrarily these $n$ points. On the other hand, in the right hand side of (2.1) we must have $t = 0$, so it becomes

$$\sum_{\lambda=\varepsilon}^{\varepsilon+k-q} \binom{\lambda - \varepsilon + q - 1}{q-1} \left\{ {q \atop q} \right\} = \binom{q-1}{q-1} + \binom{q}{q-1} + \cdots + \binom{k-1}{q-1} = \binom{k}{q},$$

which agrees with our computation.

***Proof of Theorem 2.1*** Set $\alpha := |V|$, $\beta := |E|$, and let us write $V = \{v_1, \ldots, v_\alpha\}$, $E = \{e_1, \ldots, e_\beta\}$. Since $G$ is a connected graph which is not a tree, we have $\beta \geq \alpha$, or, using our notation $\varepsilon = \beta - \alpha \geq 0$. Without the loss of generality we can assume that $K = \{v_1, \ldots, v_k\}$.

We can think about the Euler characteristic $\chi(\Sigma(G, K, n, q))$ as a sum of $\pm 1$, more precisely $(-1)^{\dim \sigma}$, ranging over the set of all cells $\sigma$ in $\Sigma(G, K, n, q)$. By definition of $\Sigma(G, K, n, q)$, each such cell is indexed by a function $\varphi : V \cup E \to 2^{[n]}$, which satisfies two conditions:

(1) The number of vertices $v \in K$, for which $\varphi(v) \neq \emptyset$ is at least $q$;
(2) The set of images $\{\varphi(x) \mid x \in V \cup E\}$ is a partition of $[n] = \{1, \ldots, n\}$ into disjoint sets.

Let us now introduce some further notation. Consider the following collection of sets:

$$A_i := \varphi(v_i) \cup \varphi(e_i), \text{ for } 1 \leq i \leq \alpha,$$
$$U := \bigcup_{i=\alpha+1}^{\beta} \varphi(e_i).$$

Set $P_\sigma := (A_1, \dots, A_\alpha, U)$. Clearly, the tuple $P_\sigma$ is an ordered set partition of $[n]$, in which we allow empty sets. We shall now group all the cells $\sigma \in \Sigma(G, K, n, q)$ according to their tuple $P_\sigma$, and calculate the contribution to the Euler characteristic separately in each group.

Consider first an arbitrary tuple $P_\sigma$, such that $\cup_{i=k+1}^{\alpha} A_i \neq \emptyset$. Let $M$ be the set of all cells with this tuple. Let $l$ be the minimal element of $\cup_{i=k+1}^{\alpha} A_i$, and let $t$ denote the index $k+1 \leq t \leq \alpha$, for which $l \in A_t = \varphi(v_t) \cup \varphi(e_t)$. We can then define an involution $\mu : M \to M$, by moving the element $l$ from $\varphi(v_t)$ to $\varphi(e_t)$, and vice versa. Formally, we set

$$\mu(\varphi)(u) := \begin{cases} \varphi(u) \,\mathrm{XOR}\, l, & \text{if } u = v_t, \text{ or } u = e_t; \\ \varphi(u), & \text{otherwise.} \end{cases}$$

Since $k+1 \leq t \leq \alpha$, there are no conditions on $\varphi(v_t)$, so the involution $\mu$ is well-defined. It produces a perfect matching on the set $M$. The difference of dimensions of any two matched cells is 1, so their contributions to the Euler characteristic of $\Sigma(G, K, n, q)$ have opposite signs. It follows that the contribution of each matched pair is 0, and hence also the total contribution of all the cells in $M$ is 0.

This means, that when computing the Euler characteristic of $\Sigma(G, K, n, q)$ we can limit ourselves to considering the tuples $P_\sigma$, for which $A_{k+1} = \dots = A_\alpha = \emptyset$, which we do for the rest of the argument.

Assume now $\sigma$ is one of the remaining cells. Set

$$\{i_1, \dots, i_m\} := \{1 \leq i \leq k \mid \varphi(v_i) \neq \emptyset\}, \text{ where } i_1 < \dots < i_m,$$

and set $r(\sigma) := i_q$. This is well-defined since by condition (1) above, we know that $m \geq q$.

Let us now fix the following data $\Omega$:

- The index set $\{i_1, \dots, i_q\}$, where $i_1 < \dots < i_q$,
- The non-empty sets $A_{i_1}, \dots, A_{i_q}$.

Let $M$ denote the set of all cells with this data $\Omega$ (and with $A_{k+1} = \dots = A_\alpha = \emptyset$). Note, that for each cell $\sigma \in M$, we have $r(\sigma) = i_q$. Let us calculate the total contribution of the cells in $M$ to the Euler characteristic of $\Sigma(G, K, n, q)$.

Let $\widetilde{M}$ denote the subset of $M$ consisting of all cells $\sigma$ for which the union $\bigcup_{j=r(\sigma)+1}^{s} A_j$ is not empty. For $\sigma \in \widetilde{M}$, set $\rho(\sigma) := \min \bigcup_{j=r(\sigma)+1}^{s} A_j$. Let $s \geq t > r(\sigma)$ be the index, for which $\rho(\sigma) \in A_t$. In a complete analogy to the above, we define a matching $\mu : \widetilde{M} \to \widetilde{M}$ by moving the element $\rho(\sigma)$ from $\varphi(v_t)$ to $\varphi(e_t)$ and vice versa. This is a perfect matching on $\widetilde{M}$, since $t \geq i_q + 1$, so there is no restriction $\varphi(v_t)$ being non-empty. The difference of the dimensions of the matched cells is equal to 1. This implies that the total contribution to the Euler characteristic by the cells from $\widetilde{M}$ is 0. We can therefore from now on concentrate on the cells from $M \setminus \widetilde{M}$.

For $1 \leq j \leq q$, we set $l_j$ to be the minimum of $A_{i_j}$. Since $A_{i_j} \neq \emptyset$, the element $l_j$ is well-defined.

We now partition the set $M \setminus \widetilde{M}$ into the sets $M_1, M_2, \dots, M_{q+1}$ as follows. For each cell $\sigma \in M \setminus \widetilde{M}$ we define $h(\sigma)$ to be the index $z$, uniquely determined by the following condition:

$$\varphi(v_{i_z}) \neq l_z, \text{ and } \varphi(v_{i_j}) = l_j, \text{ for all } j < z.$$

Here, if $\varphi(v_{i_j}) = l_j$, for all $1 \leq j \leq k$, we set $h(\sigma) = q+1$. Clearly $1 \leq h(\sigma) \leq q+1$, and we define the above partition of $M \setminus \widetilde{M}$ by saying that $\sigma \in M_i$ if and only if $h(\sigma) = i$.

Next, fix an index $1 \leq d \leq q$, and calculate the contribution of the cells in $M_d$. Same way as earlier in the proof, we can define an involution $\mu : M_d \to M_d$. This time it is shifting $l_d$ from $\varphi(v_{i_d})$ to $\varphi(e_{i_d})$ and back. Formally,

$$\mu(\varphi)(u) := \begin{cases} \varphi(u) \,\mathrm{XOR}\, l_d, & \text{if } u = v_{i_d}, \text{ or } u = e_{i_d}; \\ \varphi(u), & \text{otherwise.} \end{cases}$$

Since $\varphi(v_{i_d}) \neq l_d$, the involution $\mu$ is well-defined. As before, it matches cells with dimension difference 1, so the contribution of these two cells, and hence also the contribution of the total set $M_d$ to the Euler characteristic of $\Sigma(G, K, n, q)$ is 0.

The only interesting contribution occurs in $M_{q+1}$. Note, that all cells in $M_{q+1}$ have dimension $n - q$. Indeed, if $\sigma \in M_{q+1}$, we have $\varphi(v_{i_j}) = l_j$, for all $1 \leq j \leq q$, and $\varphi(v_t) = \emptyset$, for $t \notin \{i_1, \dots, i_q\}$. It follows that $\sum_{v \in V} |\varphi(v)| = q$, hence $\sum_{e \in E} |\varphi(e)| = n - q$.

This means that each $\sigma \in M_{q+1}$ gives the contribution $(-1)^{n-q}$, and we need to compute the cardinality $|M_{q+1}|$. Set $W := [n] \setminus \bigcup_{j=1}^{q} A_{i_j}$. The cells $\sigma \in M_{q+1}$ are obtained by arbitrarily distributing the elements of $W$ among the sets $\varphi(e_j)$, for

- Either $j \in \{1, \dots, r(\sigma)\} \setminus \{i_1, \dots, i_q\}$,
- Or $\alpha + 1 \leq j \leq \beta$.

In total, there are $\beta - \alpha + r(\sigma) - q = \varepsilon + r(\sigma) - q$ such sets, so we have $|M_{q+1}| = (\varepsilon + r(\sigma) - q)^{|W|}$.

At this point, let us specifically consider what happens when $\varepsilon + r(\sigma) - q = 0$, which of course is equivalent to saying that $\varepsilon = 0$ and $r(\sigma) = q$. In this case, there are no sets to distribute the elements of $W$ to. Therefore, the number of ways to distribute the elements of $W$, and hence also the cardinality of $M_{q+1}$, is equal to 0, unless, of course, the set $W$ itself is empty, in which case the cardinality of $M_{q+1}$ is equal to 1. Note, how this is compatible with our convention for $0^t$, cf. the footnote on page 3.

Summing over all choices of $\Omega$, we have

$$\chi(\Sigma(G, K, n, q)) = \sum_{\Omega} (-1)^{n-q} (\varepsilon + i_q - q)^{|W|}. \tag{2.2}$$

To further evaluate (2.2) we can choose the data $\Omega$ in the following order. First, pick $r$, such that $q \leq r \leq k$. Set $i_q := r$ and choose the remaining elements $i_1, \dots, i_{q-1}$, such that $i_1 < \cdots < i_{q-1} < i_q$, in $\binom{r-1}{q-1}$ ways. After that, choose the cardinality

$t := |W|$, we have $0 \leq t \leq n-q$. Proceed by choosing $W$ itself, there are $\binom{n}{t}$ possibilities. Finally, distribute the elements of $[n] \setminus W$ into the sets $A_{i_1}, \dots, A_{i_q}$, so that they are non-empty. The number of ways to do that is $q!\left\{ {n-t \atop q} \right\}$.

Summarizing, we obtain

$$\chi(\Sigma(G, K, n, q)) = (-1)^{n-q} \sum_{r=q}^{k} \binom{r-1}{q-1} \sum_{t=0}^{n-q} \binom{n}{t} q! \left\{ {n-t \atop q} \right\} (\varepsilon + r - q)^t.$$

Now, set $\lambda := \varepsilon + r - q$. Then $r = q, \dots, k$ translates to $\lambda = \varepsilon, \dots, \varepsilon + k - q$, and $r - 1 = \lambda - \varepsilon + q - 1$, so we obtain (2.1). $\square$

We can now specialize Theorem 2.1 to the case of the regular anchored configuration spaces.

**Corollary 2.2** *The Euler characteristic of* $\Sigma(G, K, n)$ *is given by the formula*

$$\begin{aligned} \frac{(-1)^{n-k}}{k!} \chi(\Sigma(G, K, n)) &= \sum_{t=0}^{n-k} \binom{n}{t} \left\{ {n-t \atop k} \right\} \varepsilon^t \\ &= \left\{ {n \atop k} \right\} + \binom{n}{1} \left\{ {n-1 \atop k} \right\} \varepsilon + \binom{n}{2} \left\{ {n-2 \atop k} \right\} \varepsilon^2 + \dots + \binom{n}{n-k} \left\{ {k \atop k} \right\} \varepsilon^{n-k}. \end{aligned} \tag{2.3}$$

***Proof*** Substitute $q := k$ into (2.1). We then have $\lambda = \varepsilon$, so the first summation is trivial, and the $\binom{\lambda-\varepsilon+q-1}{q-1} = \binom{q-1}{q-1} = 1$. This yields (2.3). $\square$

## 3 The chain complexes for the generalized anchored configuration spaces on circle graphs

Let us fix positive integers $k$ and $n$, such that $n \geq k \geq 2$. Let $C_k$ be a cycle graph with $k$ vertices and $k$ edges. Let $E$ denote its set of edges, and let $V$ denote its set of vertices. We can choose the index set to be $\mathbb{Z}_k$, and write $E = \{e_1, \dots, e_k\}$ and $V = \{v_1, \dots, v_k\}$, in such a way that the adjacency map $\partial : E \to 2^V$ is given by $\partial(e_i) = \{v_i, v_{i+1}\}$.[3]

**Definition 3.1** Given $n$, a **vertex-edge $n$-tuple** is an $n$-tuple $\sigma = (\sigma_1, \dots, \sigma_n)$, such that $\sigma_i \in V \cup E$, for all $i$.

For a vertex-edge $n$-tuple $\sigma = (\sigma_1, \dots, \sigma_n)$ we define two subsets of $\mathbb{Z}_k$, which we call **vertex** and **edge support sets**, and which we denote $\text{supp}_V(\sigma)$ and $\text{supp}_E(\sigma)$, as follows:

$$\text{supp}_V(\sigma) := \{i \in \mathbb{Z}_k \mid v_i \in \{\sigma_1, \dots, \sigma_n\}\},$$

and

$$\text{supp}_E(\sigma) := \{j \in \mathbb{Z}_k \mid e_j \in \{\sigma_1, \dots, \sigma_n\}\}.$$

[3] Of course, $k + 1 = 1$ in $\mathbb{Z}_k$.

Finally, the **dimension** of $\sigma$ is defined to be $\dim \sigma := |\{i \mid \sigma_i \in E\}|$. So, in particular, we have $0 \leq \dim \sigma \leq n$. □

Clearly, for any vertex-set $n$-tuple $\sigma = (\sigma_1, \dots, \sigma_n)$, the set $\{\sigma_1, \dots, \sigma_n\}$ is a disjoint union of the sets $\{v_i \mid i \in \operatorname{supp}_V(\sigma)\}$ and $\{e_j \mid j \in \operatorname{supp}_E(\sigma)\}$.

The direct product $\underbrace{C_k \times \cdots \times C_k}_{n}$ has a natural structure of the cubical complex, whose geometric realization is an $n$-torus. Its cells are indexed by the vertex-edge $n$-tuples, whose dimensions, as described in Definition 3.1, coincide with the geometric dimension of the corresponding cells. Therefore, the chain complex whose chain groups are generated by the vertex-edge $n$-tuples, with appropriately defined boundary operators, will calculate the homology of an $n$-torus.

We shall now consider the chain complexes whose chain groups are generated by the vertex-edge $n$-tuples, satisfying additional conditions on the vertex support set $\operatorname{supp}_V(\sigma)$.

**Definition 3.2** Assume we are given an arbitrary subset $P \subseteq \mathbb{Z}_k$, and a nonnegative integer $q$, such that $q \leq |P|$. We define a chain complex $\mathcal{C}^{P,q} = (C_*^{P,q}, \partial_*)$, where $C_*^{P,q}$ are free abelian groups, as follows.

(1) For each $d$, the free abelian group $C_d^{P,q}$ is generated by the vertex-edge $n$-tuples $\sigma = (\sigma_1, \dots, \sigma_n)$, with $\dim \sigma = d$, satisfying the following two conditions:

- $\operatorname{supp}_V(\sigma) \subseteq P$;
- $|\operatorname{supp}_V(\sigma)| \geq q$.

(2) The boundary operator takes the vertex-edge $n$-tuple $\sigma$, and replaces, with an appropriate sign, any of the edges $\sigma_i \in E$ by any of its boundary vertices, subject to the condition that the index of that vertex lies in $P$. Formally we have

$$\partial \sigma = \sum_{i \in \operatorname{supp}_E(\sigma)} \sum_{\tilde{\sigma}_i \in \partial \sigma_i \cap V_P} (-1)^{\rho(\sigma,i)} (\sigma_1, \dots, \sigma_{i-1}, \tilde{\sigma}_i, \sigma_{i+1}, \dots, \sigma_n), \tag{3.1}$$

where $V_P := \{v_j \mid j \in P\}$, and $\rho(\sigma, i) := |\operatorname{supp}_E(\sigma) \cap \{1, \dots, i-1\}|$.

Note the special case when $|P| = q$, when the chain groups $C_d^{P,q}$ are generated by all $\sigma$, satisfying $\dim \sigma = d$ and $\operatorname{supp}_V(\sigma) = P$.

For convenience, we introduce additional notation for the complement set $H := \mathbb{Z}_k \setminus P$, and $h := |H| = n - |P|$.

***Remark 3.3*** Obviously, $C_i^{P,q} = 0$, for $i < 0$. Furthermore, if a vertex-edge $n$-tuple $\sigma = (\sigma_1, \dots, \sigma_n)$ satisfies $|\operatorname{supp}_V(\sigma)| \geq q$, then $\dim \sigma \leq n - q$, so $C_i^{P,q} = 0$ also for all $i > n - q$.

In what follows, we shall compute the homology groups of the chain complexes $\mathcal{C}^{P,q}$. When $P$ is a proper subset of $\mathbb{Z}_k$, the complexes $\mathcal{C}^{P,q}$ do not correspond to topological spaces, and play here an auxilliary role. Accordingly, the case which interests us most is when $P = \mathbb{Z}_k$, since it gives us the homology of the generalized anchored configuration spaces $\Omega(k, n, q)$. We stress this observation for a later reference.

**Fact 3.4** *The chain complex $\mathcal{C}^{\mathbb{Z}_k,q}$ is isomorphic to the cubical chain complex of the generalized anchored configuration space $\Omega(k,n,q)$. In particular, the chain complex $\mathcal{C}^{\mathbb{Z}_k,k}$ is isomorphic to the cubical chain complex of the anchored configuration space $\Omega(k,n)$.*

Our calculation will proceed by induction, and we shall compute the homology groups for all values of $P$ and $q$.

# 4 Calculation of the homology groups of $\mathcal{C}^{P,q}$

## 4.1 The case $q = 0$

Let us start with the case $q = 0$. When $q = 0$ the condition $|\mathrm{supp}_V(\sigma)| \geq q$ is void, which radically simplifies the situation. The homology is then given by the following proposition.

**Proposition 4.1** *(1) The chain complex $\mathcal{C}^{\mathbb{Z}_k,0}$ calculates the homology of an $n$-torus. In fact, it is a chain complex of the cubical complex obtained as an $n$-fold direct product of the $k$-cycle.*

*(2) When $P$ is a proper subset of $\mathbb{Z}_k$, we have $H_n(\mathcal{C}^{P,0}) \approx \mathbb{Z}^{h^n}$, and all other homology groups are trivial.*

***Proof*** Statement (1) is trivial and simply formalizes our earlier observation, so we proceed to proving the statement (2).

Let $\widetilde{C}_k$ be the graph which is in a sense dual to $C_k$. It is also a cycle graph with $k$ vertices and $k$ edges, but with a different indexing. Let $\widetilde{E}$ denote its set of edges, and let $\widetilde{V}$ denote its set of vertices. Both again are indexed by $\mathbb{Z}_k$, $\widetilde{E} = \{\tilde{e}_1, \dots, \tilde{e}_k\}$ and $\widetilde{V} = \{\tilde{v}_1, \dots, \tilde{v}_k\}$, but now in such a way, that the boundary map $\partial : \widetilde{E} \to 2^{\widetilde{V}}$ is given by $\partial(\tilde{e}_i) = \{\tilde{v}_{i-1}, \tilde{v}_i\}$. So, compared to $C_k$, the relative indexing is shifted by 1.

Let $G$ denote the subgraph of the cycle graph $\widetilde{C}_k$, obtained by deleting all edges indexed by $H$. Consider the cubical complex $G^n = \underbrace{G \times \cdots \times G}_{n}$, and consider the cochain complex of $G^n$, let us call it $\tilde{\mathcal{C}}^*$. It is easy to see that $\mathcal{C}^{P,0}$ is isomorphic to this cochain complex, with the isomorphism $\varphi$ given by $\varphi(v_i) := \tilde{e}_i$, and $\varphi(e_i) := \tilde{v}_i$, for all $i \in \mathbb{Z}_k$. In particular, we have $H_i(\mathcal{C}^{P,0}) \approx H^{n-i}(\tilde{\mathcal{C}}^*)$, for all $i$.

On the other hand, we assumed that $h \geq 1$, so topologically, the graph $G$ consists of $h$ disjoint intervals. In particular, the direct product $G^n$ is homotopy equivalent to the discrete space with $h^n$ points. Therefore, we have

$$H^i(\tilde{\mathcal{C}}^*) = \begin{cases} \mathbb{Z}^{h^n}, & \text{if } i = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and it follows that

$$H_i(\mathcal{C}^{P,0}) = \begin{cases} \mathbb{Z}^{h^n}, & \text{if } i = n; \\ 0, & \text{otherwise.} \end{cases}$$

□

### 4.2 Structure of the relative chain complexes

Assume now $q \geq 1$, and consider the chain complex $\mathcal{C}^{P,q-1}$. The condition as to which vertex-edge $n$-tuples are allowed to be taken as generators of the chain groups is weaker for $\mathcal{C}^{P,q-1}$, than it is for $\mathcal{C}^{P,q}$, so the latter is its chain subcomplex. The following lemma states that their quotient can be decomposed as a direct sum of chain complexes of the same type.

**Lemma 4.2** *For any $P \subseteq \mathbb{Z}_k$, and any $q \geq 1$, we have the following chain complex isomorphism:*

$$\mathcal{C}^{P,q-1}/\mathcal{C}^{P,q} \approx \bigoplus_S \mathcal{C}^{S,q-1}, \tag{4.1}$$

*where the sum is taken over all subsets of $P$ of cardinality $q-1$.*

***Proof*** For each $d$, the relative chain group $C_d(\mathcal{C}^{P,q-1}/\mathcal{C}^{P,q}) = C_d^{P,q-1}/C_d^{P,q}$ is generated by the cosets of $C_d^{P,q}$, whose representatives are the vertex-edge $n$-tuples $\sigma = (\sigma_1, \dots, \sigma_n)$, satisfying $|\text{supp}_V(\sigma)| = q-1$.

Call such a coset $\bar{\sigma}$. The relative boundary operator in $\mathcal{C}^{P,q-1}/\mathcal{C}^{P,q}$ is then given by the following formula, cf. (3.1),

$$\partial\bar{\sigma} = \sum_{i \in \text{supp}_E(\sigma)} \sum_{\tilde{\sigma}_i \in \partial\sigma_i \cap \bar{V}} (-1)^{\rho(\sigma,i)} (\sigma_1, \dots, \sigma_{i-1}, \tilde{\sigma}_i, \sigma_{i+1}, \dots, \sigma_n), \tag{4.2}$$

where $\bar{V} = \{v_j \mid j \in \text{supp}_V(\sigma)\}$, and $\rho(\sigma, i)$ is the same as in (3.1).

In other words, when taking the boundary, we are allowed to replace an edge with any of its boundary vertices, subject to the condition, that this does not change the vertex support set.

Since the boundary operator preserves the vertex support set, the chain complex $\mathcal{C}^{P,q-1}/\mathcal{C}^{P,q}$ decomposes as a direct sum, with direct summands indexed by all possible choices of $\text{supp}_V(\sigma)$, which is the same as to say all possible choices of $(q-1)$-subsets of $P$. This proves (4.1). □

### 4.3 The case $P \neq \mathbb{Z}_k$

When $P$ is a proper subset of $\mathbb{Z}_k$, it turns out that all the homology of the chain complex $\mathcal{C}^{P,q}$ is concentrated in its top dimension.

**Theorem 4.3** *Assume $P$ is a proper subset of $\mathbb{Z}_k$. Then, the homology of $\mathcal{C}^{P,q}$ is concentrated in dimension $n-q$, in other words, $H_i(\mathcal{C}^{P,q})=0$, for $i \neq n-q$.*

***Proof*** The proof proceeds by induction on $q$. For the base case $q=0$, this has been proved in Proposition 4.1(2).

Assume now $q \geq 1$. Since the chain complex $\mathcal{C}^{P,q}$ is a subcomplex of $\mathcal{C}^{P,q-1}$, we have the following long exact sequence:

$$\ldots \to H_*(\mathcal{C}^{P,q}) \to H_*(\mathcal{C}^{P,q-1}) \to H_*(\mathcal{C}^{P,q-1}/\mathcal{C}^{P,q}) \xrightarrow{\partial} H_{*-1}(\mathcal{C}^{P,q}) \to \ldots \tag{4.3}$$

Note, that by induction assumption, the homology of the complex $\mathcal{C}^{P,q-1}$ is concentrated in dimension $n-(q-1)=n-q+1$. Furthermore, due to dimensional reasons, see Remark 3.3, the homology of $\mathcal{C}^{P,q}$ must be 0 in dimension $n-q+1$ and above.

By Lemma 4.2 we have $\mathcal{C}^{P,q-1}/\mathcal{C}^{P,q} \approx \oplus_S \mathcal{C}^{S,q-1}$, where the sum is taken over all subsets of $P$ of cardinality $q-1$. Since each $S$ is a proper subset of $\mathbb{Z}_k$, by induction assumption, the homology of $\mathcal{C}^{S,q-1}$ is also concentrated in dimension $n-q+1$. It follows that the only nontrivial part of the long exact sequence (4.3) is

$$0 \to H_{n-q+1}(\mathcal{C}^{P,q-1}) \to H_{n-q+1}(\mathcal{C}^{P,q-1}/\mathcal{C}^{P,q}) \to H_{n-q}(\mathcal{C}^{P,q}) \to 0,$$

so it follows that $H_i(\mathcal{C}^{P,q})=0$, for $i \neq n-q$. □

### 4.4 The case $P = \mathbb{Z}_k$

We are now ready to deal with the main case.

**Theorem 4.4** *The homology of the chain complex $\mathcal{C}^{\mathbb{Z}_k,q}$ is given by the following formula:*

$$H_i(\mathcal{C}^{\mathbb{Z}_k,q}) \cong \begin{cases} \mathbb{Z}^{\binom{n}{i}}, & \text{if } 0 \leq i \leq n-q-1, \\ 0, & \text{if } i<0 \text{ or } i>n-q. \end{cases} \tag{4.4}$$

***Proof*** Once again, we proceed by induction on $q$. When $q=0$, we simply have the homology of the $n$-torus, see Proposition 4.1(1).

Assume now $q \geq 1$. Consider again the long exact sequence:

$$\ldots \to H_*(\mathcal{C}^{\mathbb{Z}_k,q}) \to H_*(\mathcal{C}^{\mathbb{Z}_k,q-1}) \to H_*(\mathcal{C}^{\mathbb{Z}_k,q-1}/\mathcal{C}^{\mathbb{Z}_k,q}) \xrightarrow{\partial} H_{*-1}(\mathcal{C}^{\mathbb{Z}_k,q}) \to \ldots \tag{4.5}$$

Lemma 4.2 together with Theorem 4.3 imply that $H_i(\mathcal{C}^{\mathbb{Z}_k,q-1}/\mathcal{C}^{\mathbb{Z}_k,q})=0$, for all $i \neq n-q+1$. Furthermore, for dimensional reasons, we have $C_i^{\mathbb{Z}_k,q}=0$, whenever $i<0$, or $i>n-q$, see Remark 3.3, so we know that we must have $H_i(\mathcal{C}^{\mathbb{Z}_k,q})=0$, unless $0 \leq i \leq n-q$.

It follows that the long exact sequence (4.5) falls into several short pieces $H_i(\mathcal{C}^{\mathbb{Z}_k,q}) \approx H_i(\mathcal{C}^{\mathbb{Z}_k,q-1})$, for $0 \leq i \leq n-q-1$, and one longer piece

$$\begin{aligned} 0 \to H_{n-q+1}(\mathcal{C}^{\mathbb{Z}_k,q-1}) \to H_{n-q+1}(\mathcal{C}^{\mathbb{Z}_k,q-1}/\mathcal{C}^{\mathbb{Z}_k,q}) \to \\ H_{n-q}(\mathcal{C}^{\mathbb{Z}_k,q}) \to H_{n-q}(\mathcal{C}^{\mathbb{Z}_k,q-1}) \to 0. \end{aligned} \tag{4.6}$$

This implies the statement of the theorem. □

Note, that for dimensional reasons, see Remark 3.3, the top-dimensional homology group $H_{n-q}(\mathcal{C}^{\mathbb{Z}_k,q})$ must be free. The Betti number $\beta_{n-q}(\mathcal{C}^{\mathbb{Z}_k,q})$ can then be computed using the Euler-Poincaré formula.

When $G$ is a cycle, we have $|V| = |E|$. This means that $\varepsilon = 0$, so in (2.3) all the terms except for the first one vanish, and we have the following formula.

**Corollary 4.5** *We have* $\chi(\Omega(n,k)) = (-1)^{n-k} k! \left\{ {n \atop k} \right\}$.

Together with Theorem 4.4 this finishes the calculation of the Betti numbers of $\Omega(n,k)$.

For the generalized anchored configuration spaces, we need to substitute $\varepsilon = 0$ in (2.1), and obtain the following corollary.

**Corollary 4.6** *Assume* $1 \leq q \leq k$, *we have*

$$\chi(\Omega(n,k,q)) = (-1)^{n-q} q! \sum_{\lambda=0}^{k-q} \sum_{t=0}^{n-q} \binom{\lambda+q-1}{q-1} \binom{n}{t} \left\{ {n-t \atop q} \right\} \lambda^t.$$

Again, together with Theorem 4.4 this gives us the Betti numbers of $\Omega(n,k,q)$.

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## Appendix

An alternative duality argument by Florian Frick[4] and Martin Raussen[5]

The purpose of this appendix is to give an alternative proof of Theorem 4.4. using duality arguments. For $P = \mathbb{Z}_k$, as noted in the paper, the complex $\mathcal{C}^{\mathbb{Z}_k,q}$ is the chain complex of a cubical subcomplex $\Omega(k, q, n)$ of the $n$-torus $T^n$, i.e., the (closed) configuration space of all $n$-tuples in the circle graph $S^1 = C_k$ containing at least $q$ elements of the $k$ anchor points. Remark that $\Omega(k, q, n)$ is contained in the $(n - q)$-skeleton of the $n$-torus $T^n = (C_k)^n$ and that $\dim \Omega(k, q, n) = n - q$.

The *complement* $T^n \setminus \Omega(k, q, n)$ within $T^n$ will be denoted by $A(k, q, n)$.

**Theorem 4.7** *(1) $A(k, q, n)$ is homotopy equivalent to a simplicial complex of dimension $q - 1$.*

*(2) Inclusion $\Omega(k, q, n) \hookrightarrow T^n$ induces isomorphisms in homology up to dimension $n - q - 1$.*

*(3) $H_{n-q}(\Omega(k, q, n)) \cong \mathbb{Z}^{\binom{n}{q}+d(k,q,n)}$ with $d(k, q, n)$ denoting the dimension of the kernel of the map $i_*\colon H_{q-1}(A(k, q, n)) \to H_{q-1}(T^n)$ induced by inclusion.*

It follows immediately from the definitions that $A(k, q, n) = \bigcup_{J\subset\mathbb{Z}_k,|J|=k-q+1}(C_k\setminus J)^n$. Each part $(C_k\setminus J)^n$ is a union of $|J|^n$ disjoint contractible open $n$-boxes since $C_k \setminus J$ is a union of $|J|$ disjoint open intervals.

We can understand $A(k, q, n)$ as a colimit of contractible spaces over a poset category enumerating the *connected components of intersections* of a number of spaces $(C_k \setminus J)^n$ for various subsets $J$. More precisely,

- Associate to two vertices $i, j \in \mathbb{Z}_k$ the symbol $(i, j) \in S_k := \mathbb{Z}_k \times \mathbb{Z}_k$ and the open interval $I(i, j)$ *from* $i$ *to* $j$; in particular, $I(j, j) = C_k \setminus \{j\}$.
- Consider *admissible* maps $i : [1 : n] \to S_k$ with the property that corresponding intervals $I(i(l))$, $1 \leq l \leq n$, are *either disjoint or equal*.
- For an admissible map $i$, let $V_i := \{v \in \mathbb{Z}_k |\ v \in \bigcup_{l=1}^n I(i(l))\}$. Call $s(i) := |V_i|$ the *size* of $i$. Remark that $0 \leq s(i) \leq k$; $s(i) = 0$ iff all intervals $I(i(l))$ have length one and thus do not contain a vertex; $s(i) = k$ iff there exists $j \in \mathbb{Z}_k$ such that $i(l) = (j, j)$ for all $1 \leq l \leq n$.
- The *realization* of an admissible map $i$ is the open (and contractible) box $B(i) := \prod_{l=1}^n I(i(l)) \subset T^n$.

The spaces $B(i)$ are the connected components of spaces $(C_k \setminus K)^n$ with $K \subseteq \mathbb{Z}_k$. The poset category $\mathcal{P}(k, q, n)$ has as

**Objects** all admissible maps $i : [1 : n] \to S_k$ of size $s(i) \leq q - 1$;
**(Unique) morphisms** $\gamma : i \to i'$ if and only if $i(l) \subseteq i'(l)$ for all $1 \leq l \leq n$.

***Remark 4.8*** If there exists a morphism $\gamma : i \to i'$, then either $i = i'$ or $s(i) < s(i')$. The smallest objects in $\mathcal{P}(k, q, n)$ have size 0, the largest have size $q - 1$. Hence, the longest chain of (non-identity) morphisms increases size by one at every step and consists therefore of $q - 1$ morphisms.

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The boxes $B(i)$ and inclusions $B(i) \hookrightarrow B(i')$ define a functor on $\mathcal{P}(k, q, n)$ into a convenient category of topological spaces.

***Proof of Theorem 4.7*** (1) Note that $A(k, q, n) = \operatorname{colim}_{\mathcal{P}(k,q,n)} B$. Since all spaces $B(i)$ are contractible, the nerve lemma (Borsuk 1948) asserts that $A(k, q, n)$ is homotopy equivalent to the nerve $\mathcal{N}\mathcal{P}(k, q, n)$ of the poset category $\mathcal{P}(k, q, n)$. In particular, it is homotopy equivalent to a simplicial complex of dimension $q - 1$, and all its homology and cohomology groups of dimensions greater or equal than $q$ vanish. Moreover, top homology $H_{q-1}(A(k, q, n))$ is free.

(2) By Poincaré-Lefschetz duality, see e.g. (Bredon 1993, Cor. VI.8.4), the (co)homology groups $\tilde{H}^s(T^n, \Omega(k, q, n)) \cong H_{n-s}(A(k, q, n)) = 0$ for $s \leq n - q$, and $H^{n-q+1}(T^n, \Omega(k, q, n)) \cong H_{q-1}(A(k, q, n))$ is free. As a consequence of the universal coefficient theorem, see e.g. (Bredon 1993, Thm. V.7.1), $H_s(T^n, \Omega(k, q, n)) = 0$ for $s \leq n - q$. Hence the inclusion of $\Omega(k, q, n)$ into $T^n$ induces an isomorphism in homology in dimensions $s \leq n - q - 1$.

(3) Since $\dim \Omega(k, q, n) = n - q$, the homology group $H_{n-q}(\Omega(k, q, n))$ is free. The universal coefficient theorem allows to conclude that $H_{n-q}(\Omega(k, q, n)) \cong H^{n-q}(\Omega(k, q, n))$. Combining Poincaré-Lefschetz duality and exact sequences of the pairs $(T^n, \Omega(k, q, n))$, resp. $(T^n, A(k, q, n))$ yields the diagram

$$\begin{tikzcd}
0 = H^{n-q}(T^n, \Omega(k,q,n)) \arrow[r] & H^{n-q}(T^n) \arrow[r] & H^{n-q}(\Omega(k,q,n)) \arrow[r] & H^{n-q+1}(T^n, \Omega(k,q,n)) \arrow[r] & H^{n-q+1}(T^n) \arrow[r] & 0 \\
0 = H_q(A(k,q,n)) = 0 \arrow[r] \arrow[u, "\cong"'] & H_q(T^n) \arrow[r] \arrow[u, "\cong"'] & H_q(T^n, A(k,q,n)) \arrow[r] \arrow[u, "\cong"'] & H_{q-1}(A(k,q,n)) \arrow[r, "i_*"] \arrow[u, "\cong"'] & H_{q-1}(T^n) \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

Let $d(k, q, n)$ denote the dimension of the kernel of the map $i_*$ induced by inclusion in $H_{q-1}$. Then $\beta_{n-q}(\Omega(k, q, n)) = \beta_q(T^n) + d(k, q, n) = \binom{n}{q} + d(k, q, n)$ and hence $H_{n-q}(\Omega(k, q, n)) \cong \mathbb{Z}^{\binom{n}{q}+d(k,q,n)}$.

$\square$

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