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Random simple-homotopy theory

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Abstract
We implement an algorithm *RSHT (random simple-homotopy)* to study the simple-homotopy types of simplicial complexes, with a particular focus on contractible spaces and on finding substructures in higher-dimensional complexes. The algorithm combines elementary simplicial collapses with *pure elementary expansions*. For triangulated d-manifolds with $d \leq 6$, we show that RSHT reduces to (random) bistellar flips. Among the many examples on which we test RSHT, we describe an explicit 15-vertex triangulation of the Abalone, and more generally, $(14k + 1)$-vertex triangulations of a new series of Bing's houses with k rooms, $k \geq 3$, which all can be deformed to a point using only six pure elementary expansions.

Keywords Simple-homotopy theory · Simplicial collapse and anticollapse · Pure elementary expansion · Bistellar flip

Mathematics Subject Classification 57Q10 · 57Q15 · 57-04 · 57Q35 · 57M05

1 Introduction

A standard task in topology is to simplify a given presentation of a topological space. In general, this task cannot be performed algorithmically: Even the simplest homotopic property, contractibility, is undecidable in dimensions $d \geq 4$ (while open for $d = 2, 3$); cf. (Tancer 2016, Appendix). Nevertheless, here we propose a simple randomized algorithm to modify triangulations while keeping the simple-homotopy type of a space. The algorithm can be used as a heuristic to study simply-connected complexes, or, more generally, complexes whose fundamental groups have no Whitehead torsion.

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We shall see that in several contractible examples the heuristics works very well. The algorithm is also of interest when applied to manifolds or complexes of arbitrary topology, as we discuss below.

Our work builds on that of Whitehead, who in 1939 introduced a discrete version of homotopy theory, called *simple-homotopy theory* (Whitehead 1939). An *elementary collapse* is a deletion from a simplicial complex of a free face, i.e., of a non-empty face that is properly contained in only one other face, along with that face it is contained in. Elementary collapses are deformation retracts, and thus maintain the homotopy type; the same is true for their inverse moves, *elementary anticollapses*. Two simplicial complexes are of the same *simple-homotopy type* if they can be transformed into one another via some sequence of collapses and anticollapses, called a *formal deformation* (Whitehead 1939).

Equivalently, two simplicial complexes are of the same simple-homotopy type if there exists a third complex that can be reduced to both the original ones via suitable sequences of elementary collapses (Hog-Angeloni and Metzler 1993, p. 13). The size of the third complex (or, using the former definition, the length of the formal deformation) cannot be bounded a priori, because the simple-homotopy type cannot be decided algorithmically. In fact, by a famous result of Whitehead, having the simple-homotopy type of a point is equivalent to being contractible (Whitehead 1939) and thus undecidable.

In contrast, it is possible to decide algorithmically whether a given complex is *collapsible*, i.e., whether it can be reduced via collapses onto a single vertex. This decision problem was recently proved to be NP-complete by Tancer (2016). The advantage of the collapsibility notion is that all intermediate steps in the reduction are simplicial complexes of smaller and smaller size, hence very easy to encode and work with. The drawback is that collapsibility is strictly stronger than contractibility: Many "elementary" contractible complexes, like the dunce hat (Zeeman 1964) or Bing's house with two rooms (Bing 1964), are not collapsible.

In 1998, Forman introduced a second way to study contractibility combinatorially. His *discrete Morse theory* (Forman 1998, 2002) is a tool to reduce simplicial complexes using a mix of collapses and facet deletions. The advantage is that all simplicial complexes (contractible or not) can now be reduced to a vertex, possibly by using a relatively large number of facet deletions. The drawback is that even if one starts with a simplicial complex, the intermediate steps in the reduction sequence are typically non-regular CW complexes, and thus harder to handle. By only focusing on the count of facet deletions (the so-called "discrete Morse vector") it is possible to use randomness to produce fast implementations (Benedetti and Lutz 2014), but at the cost of failing to recognize many contractible complexes. See Joswig et al. (2022), Adiprasito et al. (2017), Lofano and Newman (2021), for computational and theoretical obstructions.

In this paper, we go back to Whitehead's original idea, and propose a third simplification method based on collapses in combination with certain expansions (see below and Remark 3.1). Our randomized heuristic *random simple-homotopy* (*RSHT*; see Sect. 3) has two advantages: First, all intermediate steps are indeed simplicial complexes; and second, at the moment we do not know of a single contractible complex for which our heuristics has probability zero to succeed in recognizing contractibility; cf. Remark 3.2.

Here is the idea. We perform elementary collapses until we get stuck. Then we select a top dimensional face ϱ uniformly at random, and for all d-faces ϱ' adjacent to ϱ via a $(d-1)$-dimensional face (a *ridge*), we check if the subcomplex induced on the $d+2$ vertices of $\varrho \cup \varrho'$ is a *pure* d*-dimensional ball*. This test is very fast. If for some ϱ' the answer is positive, we glue onto our complex the full $(d+1)$-simplex σ on the vertices of $\varrho \cup \varrho'$. If for several ϱ's the answer is positive, we simply choose one uniformly at random.

This gluing step is called a *pure elementary* $(d+1)$*-expansion*, and it is also classical from the topological perspective, compare (Hog-Angeloni and Metzler 1993, Chapter I). After this step, we may collapse away the newly introduced $(d+1)$-simplex σ together with any d-face τ of it. To avoid undoing the pure elementary expansion, we must select a τ that was already present in the complex we got stuck at before the pure elementary expansion. This first elementary collapse after the pure elementary expansion is called "(CC) step" below (see Sect. 3). The combination "pure elementary expansion + (CC) step", known in the topological literature as "transient move" (Hog-Angeloni and Metzler 1993), maintains both the dimension and the simple-homotopy type: In fact, any pure elementary expansion can be viewed as a composition of back-to-back elementary anticollapses.

Whitehead proved that for every contractible complex there is a formal deformation that reduces it to a single point (Whitehead 1939). It is not known if there is also a formal deformation to a point in which one performs anticollapses (or specific types of expansions) "only when stuck", i.e., only to intermediate complexes without free faces. If this is true, then indeed any contractible complex would have a positive probability to be recognized by our heuristics. Of course, we cannot in any case expect any universal upper bound on the number of elementary anticollapses needed, or else we would have found an algorithm that recognizes contractibility.

However, we shall see in Sects. 5 and 6 that in many key examples the number of pure elementary expansions needed is relatively low. As a benchmark series, we build *Bing's house with k rooms*, a one-parameter generalization of the aforementioned Bing's house with two rooms. For all $k \geq 3$, we prove that Bing's house with k rooms can be collapsed by adding only six further tetrahedra, cleverly chosen (**Theorem** 5.2). Of course, since our algorithm is randomized, there is no guarantee that precisely those tetrahedra will be selected as (successive) pure elementary expansions. But even with a quick attempt consisting of 10^4 runs, our algorithm is able to reduce Bing's house with seven rooms (which is a 2-complex on 99 vertices) to a point by adding only 41 tetrahedra; see Table 1.

Random simple-homotopy (RSHT) works with simplicial complexes of arbitrary dimension, but it is of particular interest when applied to triangulations of low-dimensional manifolds. When $d \leq 6$, we show (in **Theorem** 4.4) that on any (closed) d-manifold RSHT has basically the same effect of performing *bistellar flips*, also known as *Pachner moves*, which are the standard ergodic moves that allow to transform into one another any two PL homeomorphic triangulations of the same manifold (Pachner 1987).

In Sect. 6, we discuss how RSHT can be used to reach interesting small (or even vertex-minimal) triangulations and subcomplexes "hidden" inside triangulated manifolds. For the sake of applications, one should declare right away that RSHT is designed to focus on the (simple-)homotopy, and not the homeomorphism type. So in case we start with a collection of points in 10-space, say, which all lie "approximately" on a Möbius strip, the effect of performing RSHT on the Čech complex of the point set would be to detect an S^1, and not a Möbius strip. Yet, RSHT is capable of detecting, for example, closed surfaces or higher-dimensional closed manifolds in data, beyond just determining their homologies.

It takes considerable effort to build examples of contractible complexes for which RSHT does *not* practically succeed in revealing contractibility, if interrupted after a million steps, say. Respective examples, showcased in the last Sect. 6.4 of our paper, are based on the Akbulut–Kirby 4-spheres (Akbulut and Kirby 1985) and triangulations thereof (Tsuruga and Lutz 2013). The homeomorphism type of these "tangled" triangulations of S^4 is notoriously difficult to recognize.

2 Pure elementary expansions

Any two simple-homotopy equivalent complexes are homotopy equivalent. The converse is true for complexes whose fundamental group has trivial *Whitehead group* (see Cohen (1973) or Mnev (2014) for the definition), but false in general: Counterexamples in dimension two can be obtained by triangulating the cell complexes in Lustig (1991), while counterexamples in dimension three or higher had been known to exist long before (Milnor 1966).

It is an easy consequence of the theory of Gaussian elimination for integer matrices that the Whitehead group of the trivial group is trivial. Therefore, any two homotopy-equivalent simply-connected complexes are also simple-homotopy equivalent.

More generally, it is known that the Whitehead group of a group G is trivial if G is

- $\mathbb{Z}$ (Higman 1940), $\mathbb{Z} \oplus \mathbb{Z}$ (Bass et al. 1964), and more generally, any free Abelian group (Bass et al. 1964),
- any of the cyclic groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ (Cohen 1973),
- any subgroup of the braid group B_n (Farrell and Roushon 2000), or of any Artin group of type A_n, D_n, F_4, G_2, $I_2(p)$, $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, or $G(de, e, r)$ for $d, r \geq 2$ (Roushon 2020),
- any free product of groups listed above, so in particular $\mathbb{Z} * \mathbb{Z}$ or any free group (Stallings 1965),
- and further cases (Grenier-Boley 2007); in fact, the Farrell–Jones conjecture implies that any torsion-free group should appear in the present list (Lück et al. 2017).

Any two homotopy-equivalent complexes whose fundamental group appears in the list above are of the same simple-homotopy type.

Whitehead's work allows us to be more specific on the dimension (although not on the number) of the intermediate complexes involved in the definition of simple-homotopy equivalence, as follows. An elementary collapse is called an *i-collapse* if the dimension of the two faces removed are $i - 1$ and i. Similarly, an *i-anticollapse* is one that adds a pair of faces of dimension $i - 1$ and i. The *order* of a formal deformation will be the maximum i for which i-collapses or i-anticollapses are involved in the sequence.

Theorem 2.1 *(Whitehead 1939, Theorems 20 & 21). Let K and L be two homotopy-equivalent simplicial complexes. If the Whitehead group of their fundamental group is trivial, then there is a formal deformation from K to L whose order does not exceed* $\max\{\dim K, \dim L\} + 2$. *If, in addition, L is a deformation retract of K, and* $\dim K > 2$, *then there is a formal deformation from K to L whose order does not exceed* $\dim K + 1$.

The conjecture that the last statement of the previous theorem might hold also for the case $\dim K = 2$ goes under the name of "Generalized Andrews–Curtis conjecture", and represents a major open problem in combinatorial topology. It is, however, generally believed to be false (Hog-Angeloni and Metzler 1993).

Based on Whitehead's work, we would now like to perform "random anticollapses". Yet, if we wish to add a $(d+1)$-dimensional face σ to K in an elementary anticollapse, then all d-faces of σ need to be present in K already, except for a single d-face τ. However, it is not difficult to construct contractible d-complexes K that do not allow any $(d+1)$-anticollapses; cf. (Lofano and Newman 2021). In these cases, lower-dimensional faces need to be added first. Computationally, this brings an extra difficulty to the introduction of a random model. To bypass this difficulty, we adopt a different set of moves.

Definition 2.2 Let K be a d-dimensional complex. A *pure elementary* $(d+1)$-*expansion* is the gluing of a $(d+1)$-dimensional simplex σ to K in case σ intersects K in a d-ball.

A pure elementary $(d+1)$-expansion combines together in a single move one (d+1)-anticollapse plus all the lower-dimensional anticollapses that have to be performed first. Hence a sequence of pure elementary expansions and elementary collapses can be rewritten as a formal deformation. Whenever we run out of collapsing steps, we perform exactly one pure elementary $(d+1)$-expansion, and then switch back to elementary collapses. When the complex is reduced to a point, we stop—or else after *max_step* number of steps we output the (simplified) simplicial complex that is reached.

3 Implementation of random simple-homotopy

Algorithm RSHT provides a description of the random simple-homotopy procedure in pseudocode. The actual implementation can be found on GitHub at (Lofano 2021) as a `polymake` (Gawrilow et al. 1997) extension. It is based on the two different types of basic operations: random *collapses* (C) and random *pure elementary expansions* (E) plus collapsing steps (CC) that ensure that a pure elementary expansion is not undone immediately by the next regular collapsing step (C). The step (S) allows facet *subdivisions* in case no other pure elementary expansions are available.

Random collapses (C) are discussed as part of *random discrete morse theory* in Benedetti and Lutz (2014). A fast implementation of random collapses in `polymake`

Algorithm RSHT: Random Simple-Homotopy

```
Input: connected simplicial complex K; let K' be a copy of K to be modified
Output: simplified simplicial complex K'
while dim(K') ≠ 0 and i <max_step do
    while K' has free faces do
        (C): perform a random elementary collapse
    if dim(K') = d ≠ 0 and there are induced pure d-balls on d + 2 vertices then
        (E): perform a random pure elementary (d + 1)-expansion
        (CC): perform an elementary collapse deleting the newly added
                (d + 1)-face and one of its d-faces that was already in K'
    else
        (S): perform (E) + (CC) on a d-facet with d + 1 vertices
        i++
return K'
```

is described in Joswig et al. (2022). Hence, it remains to implement random pure elementary expansions (E).

While collapses in `polymake` can be carried out fast in the Hasse diagram of K, there is no explicit implementation in `polymake` to expand the Hasse diagram of K to include the faces of a $(d+1)$-simplex σ that is added in a pure elementary expansion. Thus, for every pure elementary expansion we recompute the Hasse diagram for $K+\sigma$ and then proceed with random collapses in the new Hasse diagram of $K+\sigma$. For various input examples of non-collapsible, contractible complexes, relatively few pure elementary expansions are needed (see Sects. 5 and 6); thus the extra cost for recomputing Hasse diagrams stays low.

Remark 3.1 Pure elementary expansions are not the only operations to modify a given complex K by expanding it. Another more general possibility would be to glue additional $(d+1)$-simplices to K along induced contractible subcomplexes (of mixed dimension). This provides more options to modify K, but at the price of having to check subcomplexes for contractibility. As we experienced after running various experiments, this seems expensive without any advantage. We therefore decided to stick to pure elementary expansions. In fact, checking whether an induced subcomplex on $d+2$ vertices is a pure d-ball is very fast: It can be achieved by a lookup in the Hasse diagram.

Remark 3.2 By Whitehead's Theorem 2.1, we might be forced to first go up by two dimensions (and not just by one as we do in Algorithm RSHT) to find a formal deformation from a complex K to some homotopy equivalent complex L. This could be incorporated in the algorithm by performing not only single pure elementary $(d+1)$-expansions followed immediately by collapses, but by allowing sequences of pure elementary $(d+1)$-expansions followed by pure elementary $(d+2)$-expansions before switching back to collapses. In principle, this procedure could be set up in a simulated annealing fashion (Björner and Lutz 2000; Kirkpatrick et al. 1983) as a generalized way to what we do here; but for the examples we study in the subsequent Sects. 5 and 6, we shall restrict ourselves to the basic algorithm RSHT, as this already works well.

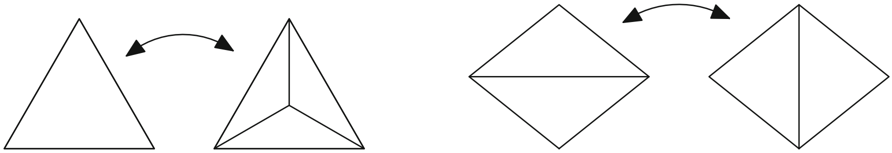


Fig. 1 Pure elementary expansions as bistellar flips

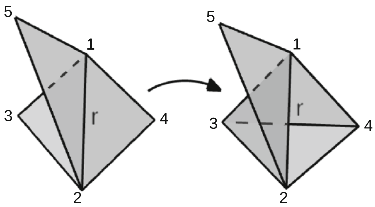


Fig. 2 Reduction of the degree of the edge r

4 Bistellar flips and artifacts

Pure elementary $(d+1)$-expansions have (at least for d-manifolds of low dimension $d \leq 6$) a clear interpretation in terms of bistellar flips. In fact, let K be a d-complex. In a pure elementary $(d+1)$-expansion, some $(d+1)$-simplex σ is glued to K along a d-ball consisting of $1 \leq k \leq d+1$ of the d-faces of σ; let r be the intersection of these k faces. If r is contained in no further d-face of K, then after adding σ, collapsing it away with one of the k d-faces, and collapsing further lower-dimensional faces, we are left with a complex K' that is obtained from K via a bistellar move; cf. (Björner and Lutz 2000). If instead r is contained in more than k d-faces of K, then in passing from K to K' the facet degree of r is decreased by one.

Example 4.1 If we glue a tetrahedron σ to a 2-complex K along a 2-disk in $\partial\sigma$, the disk can either consist of 1, 2, or 3 triangles. In the first case, the complex K' resulting after the collapses is a subdivision of K. (The triangle τ of K is subdivided using the unique vertex of σ not in K; see Fig. 1, left.) In the second case, if r is the edge common to the two triangles of $\partial\sigma$ in which σ intersects K and r is contained in exactly these two triangles of K, then r is flipped to yield K'; see Fig. 1, right. In the third case, the transition from K to K' "undoes" a subdivision.

Example 4.2 Let K be 2-dimensional and let σ be a tetrahedron glued to K along two triangles whose intersection is r, and suppose that this r is contained in exactly three triangles of K. Then after the addition of σ and its removal, r will be contained in two of the triangles of K'.

In Fig. 2, the tetrahedron 1234 is glued from below to the three triangles 123, 124, and 125 (as part of K). Then the free triangle 124 is used to collapse away the tetrahedron 1234, yielding the four triangles 123, 125, 134, and 234. This can be interpreted as a generalized bistellar flip on the triangles 123 and 124 (of K), yielding the triangles 134 and 234—in the presence of the triangle 125 and by keeping the triangle 123.

If, in a pure elementary 3-expansion, some tetrahedron is glued on top of two adjacent triangles ϱ_1, ϱ_2 of a triangulated 2-manifold, then, after collapsing away the tetrahedron together with ϱ_1, the resulting triangulation will still contain ϱ_2 and (as a free face) the edge $e = \varrho_1 \cap \varrho_2$. This edge e is thus the only free (1-)face; hence, it will be selected in the incoming (C) step of RSHT. As a result, the combination (E) + (CC) + (C) is a proper bistellar flip—and the diagonal of the two initial triangles gets flipped. In the case of a subdivision, the combination (E) + (CC) is a proper bistellar flip as well. Thus, it remains to inspect the case when a subdivision is undone. After the addition of a tetrahedron (E) and the deletion of one of the initial three triangles along with the tetrahedron in the (CC) step, the other two initial triangles remain, and we have (two) free edges for two further (C) steps. In contrast to the previous cases, after the two (C) steps, the resulting triangulation is not a surface yet—as we still have the intersection vertex of the three initial triangles as a free vertex that is connected to the modified triangulated surface by an edge. That is, the result of (E) + (CC) + (C) + (C) is a triangulated surface with an additional edge sticking out. This edge is then collapsed away in another (C) step.

This situation generalizes as follows:

Lemma 4.3 *Let K be a triangulation (without free faces) of a d-manifold M and suppose that the $(d+1)$-simplex $\sigma = [0, 1, \ldots, d+1]$ intersects K in a pure d-ball B with $1 \leq k \leq d+1$ d-facets on the $d+2$ vertices $0, 1, \ldots, d+1$ of σ so that (w.l.o.g.) $B = [0, 1, \ldots, d-k+1] * \partial[d-k+2, d-k+1, \ldots, d+1]$. We add σ (and its faces) to K and, by step (CC) of RSHT, ban those facets of σ as free faces that do not contain $[0, 1, \ldots, d-k+1]$.*

- *If $k \leq 7$, then running RSHT on $K \cup \sigma$ until no further free faces are available yields a triangulation $K' = K - B + B'$ of M with*

$$B' = \partial[0, 1, \ldots, d-k+1] * [d-k+2, d-k+1, \ldots, d+1],$$

  i.e., K' is obtained from K by a bistellar flip.
- *If $k > 7$ (which can occur for $d > 6$ only), then running RSHT on $K \cup \sigma$ until no further free faces are available might terminate in a non-pure simplicial complex K'' that is the union of a triangulation of M with a contractible, but non-collapsible lower-dimensional complex on the vertices $d-k+2, d-k+1, \ldots, d+1$.*

Proof Step (CC) of RSHT implies that our first collapsing move will remove a facet of B along with the added $(d+1)$-simplex σ. At any consecutive collapsing step (C), the faces involved in the collapses will be of the form $[0, 1, \ldots, d-k+1] * \tau$, where $\tau \in \partial[d-k+2, d-k+1, \ldots, d+1]$ (because our starting complex K had no free faces). The restriction of the collapsing sequence to $\partial[d-k+2, d-k+1, \ldots, d+1]$ gives us a valid collapsing sequence of the simplex $[d-k+2, d-k+1, \ldots, d+1]$, where the first collapsing move is induced by the initial step (CC). Now:

- If $k \leq 7$, the simplex $[d-k+2, d-k+1, \ldots, d+1]$ has at most seven vertices; and by (Bagchi and Datta 2005) every contractible simplicial complex with $k \leq 7$ vertices is collapsible, i.e., the collapsing sequence induced by RSHT on $[d-k+2, d-k+1, \ldots, d+1]$ will terminate at a single point. It follows that $K' = K - B + B'$.

- If $k > 7$, then the collapsing sequence on $[d-k+2, d-k+1, \ldots, d+1]$ might get stuck on a contractible, but non-collapsible subcomplex of dimension at least two (Crowley et al. 2003; Lofano and Newman 2021), and thus the resulting complex K' need not be pure. □

In the special case when $d = 7$ and $k = 8$ we might get stuck on a modified manifold triangulation union, say, a complex $[0] * D \subseteq [0] * \partial[1, \ldots, 8]$, with D an 8-vertex triangulation of the dunce hat; cf. (Benedetti and Lutz 2013). The resulting complex $K' = K - B + B' + [0] * D$ then deviates from the modification via a bistellar flip, $K - B + B'$, by the additional cone $[0] * D$ with apex $[0]$ over the (contractible) dunce hat D in the 2-skeleton of $\partial[1, \ldots, 8]$. The complex $K' = K - B + B' + [0] * D$ deformation retracts to $K - B + B'$, but has no free faces. We regard the contractible complex $[0] * D$ as an *artifact* of the modification.

Theorem 4.4 *(Reduction of pure elementary $(d+1)$-expansions to bistellar flips). Let K be a triangulation of a d-manifold M with $d \leq 6$. Any pure elementary $(d+1)$-expansion followed by collapses (as long as free faces are available) induces a bistellar flip on K.*

Proof The statement follows from Lemma 4.3 and the fact that the maximum number of facets of a pure d-ball on $d+2$ vertices is $d+1$. □

Corollary 4.5 *(Manifold stability). Let K be a (not necessarily pure) simplicial complex. If we run RSHT on K and at some point reach a simplicial complex K' that triangulates a d-manifold with $d \leq 6$, then from then on, whenever there are no free faces in the further run of RSHT, the respective temporary complex $\tilde{K}$ is a d-manifold as well, and $\tilde{K}$ is bistellarly equivalent to K'.*

To avoid lower-dimensional artifacts $[0, 1, \ldots, d-k+1] * N \subseteq [0, 1, \ldots, d-k+1] * \partial[d-k+2, d-k+1, \ldots, d+1]$ in the modification $K' = K - B + B' + [0, 1, \ldots, d-k+1] * N$ of a triangulated manifold K, involving a contractible, non-collapsible complex N for $d \geq 7$ and $k \geq 8$, we should switch to bistellar flips $K' = K - B + B'$ once we know that K is a manifold. Quite often, this is not clear a priori—in fact, testing whether K is a manifold is an undecidable problem for $d \geq 6$; cf. (Joswig et al. 2022).

In practice (Joswig et al. 2022), on a 7-simplex it is nearly impossible to get stuck with random collapses. On the 8-simplex, only about 0.0000012% of the runs of random collapses get stuck. But in higher dimensions, the situation changes dramatically: For example, for the 25-simplex, contractible but non-collapsible substructures are encountered in 92% of the runs.

Another option to deal with the artifacts would be to run RSHT on lower-dimensional parts to "melt away" the artifacts. However, in our experiments in Sects. 5 and 6 we only focus on top-dimensional pure elementary expansions, since the terminal triangulations of the examples we consider are all of dimension $d \leq 6$.

In case a general complex K has no free faces and is not a manifold, then a sequence (E) + (CC) + (C) + …+ (C) until no further collapses are possible might reduce K in dimension or can reduce (or increase) the degree of a face in K, as we have seen in Example 4.2 and Fig. 2. In the latter case, we can regard the sequence as a *generalized bistellar flip*. These generalized operations give flexibility in the modification of a given complex K.

4.1 Selection of expansions and simplification of complexes

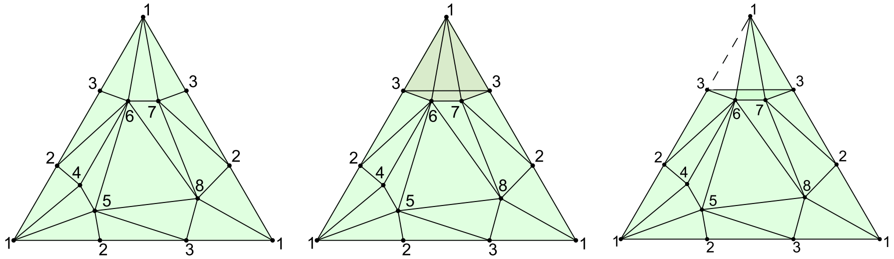
We next discuss in more detail how the pure elementary expansions are selected and why Algorithm RSHT has a tendency to simplify simplicial complexes to yield small or even vertex-minimal triangulations. First, we note that RSHT, apart from temporarily adding (d+1)-faces in the pure elementary expansion steps (E), never increases the dimension of the complex.

As outlined in the introduction, for any d-facet of a d-dimensional complex K, chosen uniformly at random, we can check for all neighboring d-facets whether the induced subcomplexes on the combined $d+2$ vertices are pure d-dimensional. From the collection of all available such pure induced d-balls on $d+2$ vertices, we pick one uniformly at random for a pure elementary d-expansion step (E). However, in general, such pure induced d-balls on $d+2$ vertices need not exist. For example, in the case of neighborly triangulations of surfaces, the induced subcomplexes on the four vertices of two adjacent triangles are the two triangles *plus* the opposite diagonal edge; such subcomplexes are not contractible. In such a case, the only possible pure elementary expansion is by picking a facet (uniformly at random) as a pure d-ball and initiating a subdivision (S). An example of a triangulated 3-sphere on 16 vertices that allows no bistellar flips (apart from subdivisions of tetrahedra) is given in (Dougherty et al. 2004).

Let K be a triangulated circle S^1 with $n > 3$ vertices. Then K is reduced by Algorithm RSHT to the boundary of a triangle in $n-3$ pure elementary expansion steps (E), each followed by two collapsing steps (CC) + (C).

In the case of triangulations of S^2 with $n > 4$ vertices, there always are admissible edge flips, and thus Algorithm RSHT never adds a vertex in a subdivision step (S). A vertex can get removed in the reversal of a subdivision once the current triangulation has a vertex of degree 3. However, the boundary of the octahedron has all of its vertices of degree 4; in fact, there are infinitely many triangulations of S^2 with all vertex degrees at least four. In any such example, the removal of a vertex is not immediately possible. But after a suitably long sequence of random edge flips, eventually vertices of degree 3 show up, and the three incident triangles to such a vertex have the chance to get chosen for an induced pure 2-ball to remove the vertex of degree 3.

Similarly, general complexes K are simplified and reduced in size by collapsing away collapsible parts and by reversing subdivisions to reduce the number of vertices—but without a universal guarantee for success (as contractibility is undecidable).



(a) An 8-vertex triangulation of the Dunce Hat

(b) Anticollapsing the tetrahedron 1367

(c) Collapsing away the tetrahedron 1367

Fig. 3 A formal deformation of the dunce hat

5 Classical examples

In this section, we test how the Algorithm RSHT performs on the Dunce Hat, on Bing's house with two rooms, and on similar, "classical" examples of contractible complexes. It turns out that the number of pure elementary expansions needed to reduce these complexes to a single vertex is conveniently low: one pure elementary expansion suffices for an 8-vertex triangulation of the dunce hat; five pure elementary expansions suffice for a simplicial version of Bing's house with two rooms; and in general, six tetrahedra are sufficient to collapse Bing's house with k rooms (**Theorem** 5.2). Triangulations of these examples can be found online at the "Library of Triangulations" (Benedetti and Lutz 2013).

5.1 The dunce hat

The dunce hat (Zeeman 1964) is the most famous example of a contractible, but non-collapsible complex; cf. (Benedetti and Lutz 2013). It is obtained by gluing together the three edges of a single triangle in a non-coherent way. The dunce hat can be triangulated as a simplicial complex with eight vertices (see Fig. 3a); and eight vertices is fewest possible, as every contractible simplicial complex on seven vertices is collapsible (Bagchi and Datta 2005). No triangulation of the dunce hat is collapsible, since there are no free edges to start with.

The dunce hat of Fig. 3a admits two (proper) anticollapsing moves, the addition of the tetrahedron 1245 or alternatively the addition of the tetrahedron 1367. In Fig. 3b we added 1367. All of the triangles in 1367 are free, since this is now the only tetrahedron present. If we collapse away the triangle 367, we recover the initial complex of Fig. 3a. If instead we choose to delete the free triangle 136, we obtain the triangulation displayed in Fig. 3c. This triangulation has a free edge, 16, that allows us to get rid of the triangle 167. After this elementary collapse, the edge 17 becomes free, allowing us to remove the triangle 137. But now the edge 13 is free, and it can easily be seen that the deletion of the triangle 138 paves the way to a full collapse down to a single vertex.

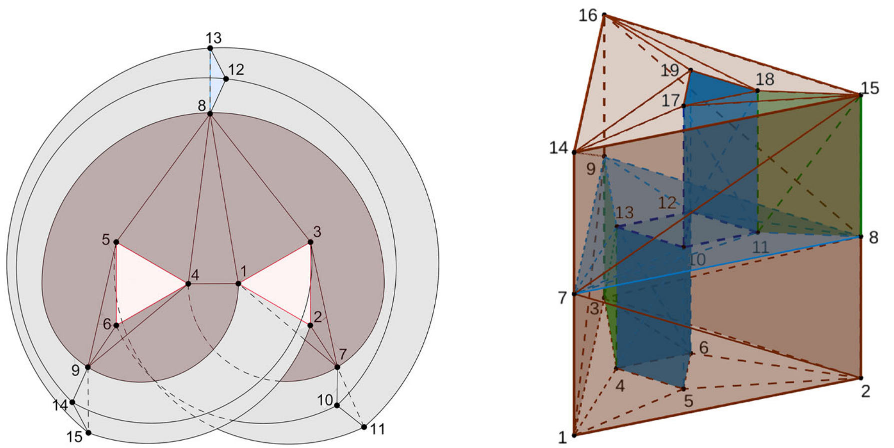


Fig. 4 Triangulations `Abalone` of the Abalone (left) and `BH` of Bing's house with two rooms (right)

Lemma 5.1 *One pure elementary* 3*-expansion suffices to reduce to a vertex the* 8*-vertex triangulation of the dunce hat from Fig. 3b(a).*

In 10^4 runs, RSHT used on average 2.4145 pure elementary 3-expansions to reduce the 8-vertex dunce hat to a point; see Sect. 6.1 and Table 1.

5.2 The Abalone

The Abalone (Hog-Angeloni and Metzler 1993), sometimes called *Bing's house with one room*, is another example of a contractible but non-collapsible complex. We are not aware of any triangulation of this space in the literature, so we present one, `Abalone`, with 15 vertices:

1 2 7	1 2 9	1 3 8	1 3 9	1 4 7	1 4 8	1 4 9	2 3 7
2 3 15	2 9 15	3 7 8	3 9 14	3 14 15	4 5 7	4 5 8	4 6 7
4 6 9	5 6 9	5 6 10	5 7 10	5 8 9	6 7 11	6 10 11	7 8 10
7 8 11	8 9 12	8 9 13	8 10 12	8 11 13	8 12 13	9 12 14	9 13 15
10 11 12	11 12 13	12 13 14	13 14 15				

Figure 4 displays this triangulation, although some diagonals have been omitted for reasons of pictorial clarity. Essentially, the triangulation consists of a membrane (in dark) from which two prismatic tunnels (in light) originate at the two empty triangles 1 2 3 and 4 5 6; and the tunnels are separated by the highlighted triangle 8 12 13. The Abalone is contractible as can be seen by filling in the two tunnels.

RSHT can reduce the Abalone to a point using only three pure elementary expansions. One way to do so is to free the edge 8 9 of Fig. 4 by first adding the three tetrahedra 8 9 12 13, 9 12 13 14, and 9 13 14 15, in this order, as anticollapsing moves. The resulting complex is then collapsible. This can either be verified by hand, or via the random_discrete_morse algorithm (Benedetti and Lutz 2014) implemented in polymake (Joswig et al. 2022): The three tetrahedra fill in the prism between the triangle 8 12 13 and the (formerly empty) triangle 9 14 15. By collapsing away this prism, the edge 8 9 becomes free so that the (dark) membrane around the empty triangle 4 5 6 can be collapsed away, which frees the tunnel originating at this empty triangle. Its removal then allows to collapse the remaining disk.

We can interpret the anticollapsing moves followed by collapses as operations that move the walls of the tunnel so that eventually the obstruction to collapsibility vanishes.

5.3 Bing's house with two rooms

Bing's house with two rooms (Bing 1964) is an early example of a contractible space no triangulation of which is collapsible. For our purposes, we triangulate Bing's house as a triangular prism with two floors, two tunnels to reach the floors, and all rectangular walls subdivided into two triangles each. Figure 4 displays the following (small) triangulation BH with $f = (19, 65, 47)$ (with the list of facets also available online as example BH at (Benedetti and Lutz 2013)):

1 2 5	1 2 7	1 3 4	1 3 9	1 4 5	1 7 9	2 3 6	2 3 8
2 5 6	2 7 8	3 4 6	3 4 13	3 8 9	3 9 13	4 5 10	4 6 13
4 10 13	4 12 13	5 6 10	6 10 12	6 12 13	7 8 11	7 8 15	7 9 13
7 9 14	7 10 11	7 10 13	7 14 15	8 9 12	8 9 16	8 11 12	8 11 15
8 15 16	9 12 13	9 14 16	10 11 17	10 12 17	11 12 18	11 15 18	11 17 18
12 17 19	12 18 19	14 15 17	14 16 19	14 17 19	15 16 18	15 17 18	16 18 19

RSHT is able to reduce Bing's house to a point by means of five (successive) pure elementary expansions (in the upper room, each followed by collapses so that the outer walls of Bing's house are moved towards the upper tunnel). Here is a possible strategy. By successively adding five tetrahedra in the upper room of our Bing's house triangulation, we fill in a cubical prism between the horizontal square 7–8–11–10 of the medium floor and the square 14–15–18–17 of the ceiling. The first two tetrahedra 7 8 11 15 and 11 15 17 18 can be added independently, and their addition are proper anticollapsing steps. The third tetrahedron 7 11 15 17 is a pure elementary expansion, and the addition of the two final tetrahedra 7 10 11 17 and 7 14 15 17 are again anticollapsing steps. The newly introduced cubical prism connects the outer vertical square 7–8–15–14 with the vertical square 10–11–18–17 of the upper tunnel. The resulting complex is collapsible; an explicit collapsing sequence proving this claim is detailed below.

We start from the outside, by perforating the back square 7–8–15–14. Then we entirely remove the interior of the cubical prism along with the two triangles 7 8 15 and 7 14 15 of the back square and the two triangles 14 15 17 and 15 17 18 of the top square. The result is an indented Bing's house triangulation with two new side triangles 7 10 17 and 7 14 17. But now the edge 17 18 has been freed, and we can use it to collapse away the subdivided squares of the triangulation one by one. First the

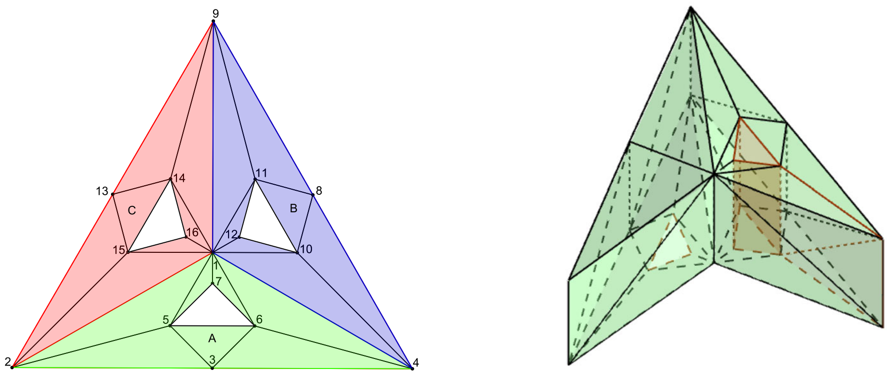


Fig. 5 Ground floor (left) and room R_2 (right) of Bing's house $BH(3)$ with three rooms

square 10–11–18–17 is collapsed away, which frees the edge 10 11. This edge in turn can be used to remove the horizontal square 7–8–11–10, thus freeing the edge 78. Next, we remove the squares 1–2–8–7, 2–3–9–8, 1–3–9–7, the vertical wall 3–4–13–9, then all triangles of the lower floor, then the lower tunnel, to end up with the indented upper room with empty triangle 10 12 13. This remaining complex is a triangulated disc and thus collapsible.

5.4 Bing's house with *k* rooms

A recent example of a non-collapsible, contractible complex is *Bing's house with three rooms (and thin walls)* by Tancer (2016). He introduced the example as a gadget to prove that the problem of recognizing collapsible complexes is NP-complete. The basic layout of the example can be found in (Tancer 2016). Here, we give an explicit triangulation $BH(3)$; and extend this construction to k rooms, $BH(k)$, $k \geq 3$.

The starting point for the construction of $BH(3)$ is to have a ground floor with three triangular holes as depicted in Fig. 5. The floor has the following triangles:

1 2 5	1 2 15	1 4 6	1 4 10	1 5 7	1 6 7	1 9 11	1 9 14
1 10 12	1 11 12	1 14 16	1 15 16	2 3 5	2 13 15	3 4 6	3 5 6
4 8 10	8 9 11	8 10 11	9 13 14	13 14 15			

Onto the ground floor, we glue three rooms in a coherent way. Room R_1 is glued onto the two regions A and B and uses nine additional vertices, from 17 to 25. Room R_2, depicted in Fig. 5, is glued onto the regions B and C and uses the nine vertices from 26 to 34. Finally, room R_3 is glued onto the regions C and A with further nine vertices ranging from 35 to 43. The rooms R_2 and R_3 are cyclic copies of the room R_1, where 9 and 18 are added to the vertex-labels 17 to 25 of room R_1, respectively.

Concretely, the triangles of room R_1 are

1 2 17	1 9 17	2 3 18	2 5 18	2 17 18	3 4 19	3 18 19	4 8 20
4 19 20	5 6 21	5 7 21	5 18 21	6 7 22	6 21 22	7 21 23	7 22 23
8 9 24	8 20 24	9 17 25	9 24 25	17 18 21	17 20 22	17 20 24	17 21 23
17 21 23	17 24 25	18 19 21	19 20 22	19 21 22			

Those of room R_2 are

1 2 26	1 4 26	2 13 33	2 26 34	2 33 34	4 8 27	4 10 27	4 26 27
8 9 28	8 27 28	9 13 29	9 28 29	10 11 30	10 12 30	10 27 30	11 12 31
11 30 31	12 30 32	12 31 32	13 29 33	26 27 30	26 29 31	26 29 33	26 30 32
26 31 32	26 33 34	27 28 30	28 29 31	28 30 31			

and those of room R_3 are

1 4 35	1 9 35	2 3 38	2 13 37	2 37 38	3 4 42	3 38 42	4 35 43
4 42 43	9 13 36	9 14 36	9 35 36	13 36 37	14 15 39	14 16 39	14 36 39
15 16 40	15 39 40	16 39 41	16 40 41	35 36 39	35 38 40	35 38 42	35 39 41
35 40 41	35 42 43	36 37 39	37 38 40	37 39 40			

The three rooms R_1, R_2, and R_3 are then all glued to the upper side of the ground floor. Since the vertices of the upper layer of a room are distinct from the vertices of the upper layers of the other two rooms, there is no conflict for the chosen gluing to the same side. To enter the interior of a room, one has to first pass through the tunnel from above of the room to the left, before the room itself can be entered from below through the lower left empty triangle.

The previous triangulation $BH(3)$ of Tancer's Bing's house with three rooms can be generalized to create a new series of triangulated Bing's houses $BH(k)$ with k rooms for all $k \geq 3$. Instead of just three regions, start with k regions that have a triangular hole each, cyclically arranged around a central vertex 1 on the ground floor, and attach to it k rooms, $R_1, \ldots, R_k$, in a coherent way, as before. The resulting triangulation has face vector

$$f = (14k + 1, 50k, 36k).$$

A C++-implementation `BH_k.cc` by Lofano to generate the examples $BH(k)$ along with explicit triangulations `BH_3`, `BH_4`, and `BH_5` can be found online at (Benedetti and Lutz 2013).

Our next result highlights that in terms of simple-homotopy theory, $BH(k)$ is easy to understand.

Theorem 5.2 *For any $k \geq 3$, Bing's house with k rooms, $BH(k)$, can be formally deformed to a point using only six pure elementary expansions.*

Proof Since the rooms $R_1, \ldots, R_k$ are all identical, we extend to $BH(k)$ the labelling scheme that we used for the ground floor and the rooms of $BH(3)$. First we do all the expansions in room R_1. By adding the following six tetrahedra

$$2\,3\,5\,18,\quad 3\,5\,18\,19,\quad 5\,18\,19\,21,\quad 3\,5\,6\,19,\quad 5\,6\,19\,21,\quad 6\,19\,21\,22$$

we fill in the cubical prism between the horizontal square on the vertices 2–3–6–5 of the main floor and the horizontal square on the vertices 18–19–22–21 of room R_1's ceiling. We may now start the collapsing sequence from the outside. We perforate the back square 2–3–19–18 and then remove the whole interior of the prism, along with the back square 2–3–19–18 and the horizontal square 18–19–22–21 of the ceiling. Now the edge 21 22 is free. Thus, we can proceed exactly as for Bing's house with two rooms: We collapse away the squares 5–6–22–21 and 2–3–5–6, in this order. But now the edge 23 is free; so we can use it to collapse away room R_k. By induction, we can thus collapse all the rooms one by one. □

How does this compare with the experimental results? In 10^4 runs, RSHT was always able to reduce Bing's house with three rooms, $BH(3)$, to a point, using on average about 148 additional tetrahedra. In the "best run", only 12 additional tetrahedra were used. For Bing's house with k rooms, $BH(k)$, $4 \leq k \leq 7$, in 10^4 runs, even in the best case, RSHT tends to perform a growing number of expansions; see Table 1. This growing number of used tetrahedra is not surprising, due to the probabilistic model that we used: When selecting from more rooms, the number of options for possible expansions gets larger. So if we keep the number of rounds fixed, the chances to pick the cleverest sequence of pure elementary expansions will get thinner.

6 Experiments on various topologies and substructures

In this section, we explore how our algorithm RSHT performs for further interesting simplicial complexes, whether contractible or not. All timings were taken on an Intel(R) Core(TM) i7-4720HQ CPU with 2.60 GHz and 16 GB RAM.

6.1 Contractible, non-collapsible complexes

Table 1 lists the number of pure elementary expansions used for the dunce hat and Bing's houses described in the previous section, as well as for the contractible complex `two_optima` of (Adiprasito 2017) and for some knotted balls (Lutz 2004; Benedetti and Lutz 2013). Furch's knotted 3-ball is the only example in this set for which the runtime is not negligible. In fact, due to the large number of expansions required (to overcome the knottedness), it took an average of 85 s to complete one round of the algorithm for this 3-ball.

The explanation of Table 2 is as follows. If one starts with a single d-simplex, with $8 \leq d \leq 15$, and one tries to collapse it down to a point, sometimes one gets

Table 1 RSHT run for a selection of contractible, non-collapsible complexes from (Benedetti and Lutz 2013)

Complex	f-vector	Rounds	# Expansions (smallest found)	# Expansions (mean)
Dunce hat	(8, 24, 17)	10^4	1	2.4145
Abalone	(15, 50, 36)	10^4	3	32.4156
Bing's house	(19, 65, 47)	10^4	5	58.0964
Bing's house with three rooms, $BH(3)$	(43, 150, 108)	10^4	12	147.9727
$BH(4)$	(57, 200, 144)	10^4	15	167.7727
$BH(5)$	(71, 250, 180)	10^4	27	195.8890
$BH(6)$	(85, 300, 216)	10^4	34	221.2596
$BH(7)$	(99, 350, 252)	10^4	41	244.5763
`two_optima`	(106, 596, 1064, 573)	10^3	1	7.050
Knotted balls				
Furch's knotted ball	(380, 1929, 2722, 1172)	10^3	1459	1949.950
`double_trefoil_ball`	(15, 93, 145, 66)	10^3	1	29.600
`triple_trefoil_arc`	(17, 127, 208, 97)	10^3	6	94.678

Table 2 RSHT run for contractible, non-collapsible complexes obtained when trying to collapse a d-simplex

d	# Examples × # Rounds	# Expansions (smallest found)	# Expansions (mean)
8	10×10^3	1.0	1.9310
9	10×10^3	1.0	3.6845
10	10×10^3	1.0	8.4502
11	10×10^3	1.1	7.6552
12	10×10^3	2.5	29.4564
13	10×10^3	1.2	38.3988
14	10×10^3	7.9	174.7835
15	10×10^3	36.3	205.1362

stuck in contractible, non-collapsible complexes of intermediate dimension (Lofano and Newman 2021). For each initial d-simplex we recorded 10 such examples, and on each one of these 10 examples we let RSHT run for 10^3 rounds. In each of the rounds, RSHT was able to reduce the respective examples to a point: In columns three and four of Table 2, we recorded the smallest found and average numbers of pure elementary expansions used. With the increase of the dimension, the runtime started to become an issue. For the largest examples, with $d = 15$, it took on average around 25 s to complete one round.

6.2 Submanifolds and non-manifold substructures in manifolds

If we remove a facet from a triangulation of the d-dimensional sphere S^d, the resulting simplicial complex is a triangulated d-ball, and thus has the simple-homotopy type of a point by Whitehead's Theorem 2.1. In case the initial d-manifold M^d is not a sphere, the removal of a simplex from a triangulation yields a simplicial complex that, depending on M^d, may deform to a submanifold or to a non-manifold substructure in M^d. Table 3 provides results for some classical examples:

Starting with the vertex-minimal triangulation of $\mathbb{R}P^3$ with 11 vertices, and removing a facet, in 10^4 runs of RSHT it took on average 25.2510 expansions to reach the 6-vertex triangulation of $\mathbb{R}P^2$. From $\mathbb{R}P^4$ to $\mathbb{R}P^3$ it took 885.5957 expansions. From $\mathbb{C}P^2$ to S^2 no expansions were used around half of the times; the average number of expansions needed was 2.3543. Finally, it took 30.0784 expansions to reach S^4 from $\mathbb{H}P^2$. For the Poincaré homology 3-sphere (Björner and Lutz 2000), the RSHT algorithm found a 2-dimensional $\mathbb{Z}$-acyclic 2-complex on 10 vertices (the boundary of the identified dodecahedron) using 2031.732 expansions in less than two minutes per run.

The 3-dimensional lens spaces $L(p, q)$, introduced by (Tietze 1908), are well-known topological spaces with torsion in first homology. Starting from triangulations of the 3-manifolds $L(p, 1)$ (Brehm and Świątkowski 1993; Lutz 2003) for $p \geq 3$, we aimed for small triangulations of 2-dimensional simplicial complexes that still

Table 3 RSHT run for manifold triangulations minus a facet

Initial complex	Initial f-vector	Resulting complex	Resulting f-vector
$\mathbb{R}P^3$ (Walkup 1970)	(11, 51, 80, 40)	$\mathbb{R}P^2$	(6, 15, 10)
$\mathbb{R}P^4$ (Lutz 1999, Ch. 3)	(16, 120, 330, 375, 150)	$\mathbb{R}P^3$	(11, 51, 80, 40)
$\mathbb{C}P^2$ (Kühnel and Banchoff 1983)	(9, 36, 84, 90, 36)	S^2	(4, 6, 4)
$\mathbb{H}P^2$ (Brehm and Kühnel 1992)	(15, 105, 455, 1365, 3003, 4515, 4230, 2205, 490)	S^4	(6, 15, 20, 15, 6)
Poincaré 3-sphere (Björner and Lutz 2000)	(16, 106, 180, 90)	$\mathbb{Z}$-acyclic 2-complex	(10, 40, 31)

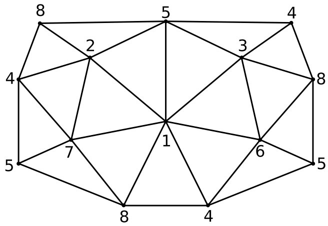
Table 4 Face vectors of small substructures with p-torsion of the lens spaces $L(p, 1)$

Torsion	f-vector
$\mathbb{Z}_3$	(8, 24, 17)
$\mathbb{Z}_4$	(8, 26, 19)
$\mathbb{Z}_5$	(9, 32, 24)
$\mathbb{Z}_6$	(9, 33, 25)
$\mathbb{Z}_7$	(9, 34, 26)
$\mathbb{Z}_8$	(9, 35, 27)
$\mathbb{Z}_9$	(9, 36, 28)
$\mathbb{Z}_{10}$	(9, 36, 28)
$\mathbb{Z}_{11}$	(10, 42, 33)
$\mathbb{Z}_{12}$	(10, 42, 33)
$\mathbb{Z}_{13}$	(10, 43, 34)
$\mathbb{Z}_{14}$	(11, 50, 40)
$\mathbb{Z}_{15}$	(11, 50, 40)

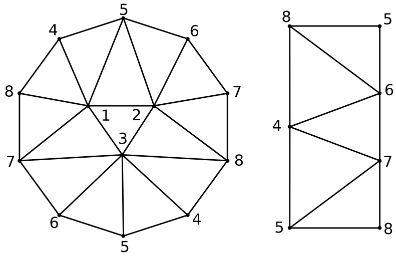
have p-torsion. (The case $p = 2$ has been already considered, since $L(2, 1) = \mathbb{R}P^3$.) Table 4 gives the f-vectors of these smaller complexes; Fig. 6a–c shows resulting small triangulations `d2_n8_3torsion`, `d2_n8_4torsion`, and `d2_n8_5torsion` (with facets lists available at (Benedetti and Lutz 2013)) with torsion $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$, respectively. The example `d2_n8_3torsion` has the combinatorial symmetry (2, 3)(4, 8)(6, 7); the example `d2_n8_4torsion` has symmetry (1, 2)(4, 6)(7, 8). In (b), the obtained complex is the union of an 8-vertex triangulation of the projective plane and a Möbius band. The complex `d2_n8_5torsion` origins from a triangulated disk with identifications highlighted in blue (the "path" 6–5–3–6–5–3–6–5–3–6) and red (the "path" 6–2–4–7–1–6–2–4–7–1–6).

The following natural problem is open for $p \geq 3$:

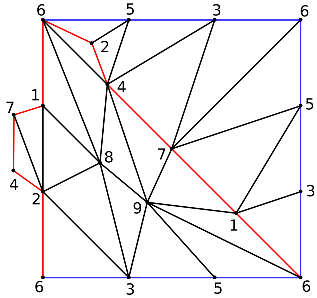
Question 1 What is the minimal number of vertices $n_{\min}(p)$ for a simplicial 2-complex with p-torsion?



(a) Complex `d2_n8_3torsion` with 3-torsion.



(b) Complex `d2_n8_4torsion` with 4-torsion.



(c) Complex `d2_n8_5torsion` with 5-torsion.

Fig. 6 Small substructures with p-torsion of the lens spaces $L(p, 1)$

An earlier construction of a 2-dimensional simplicial complex with 3-torsion as a *sum complex* on eight vertices is by Linial, Meshulam, and Rosenthal (2010). Their example is based on the following collection of subsets of $\mathbb{Z}_8$:

$$X_{\{0,1,3\}} = \{\sigma \subset \mathbb{Z}_8 : |\sigma| = 3, \sum_{x \in \sigma} x \equiv 0, 1 \text{ or } 3 \pmod 8\}.$$

This complex has complete 1-skeleton and face vector $f = (8, 28, 21)$. Three edges of the complex are free, and after collapsing the respective triangles we reach a 2-complex with $f = (8, 25, 18)$, which still has one triangle and one edge more than the example `d2_n8_3torsion`. By runninng RSHT on the triangulation with 18 triangles repeatedly, we again reach `d2_n8_3torsion`—or a second non-isomorphic triangulation with the same f-vector that is obtained from `d2_n8_3torsion` by flipping the edge 1–5.

Conjecture 1 *The examples `d2_n8_3torsion` and `d2_n8_4torsion` have component-wise minimal f-vectors for complexes with* 3- *and* 4-*torsion, respectively.*

In the description of the torus $S^1 \times S^1$ as a square with opposite edges identified, the removal of the interior of the identified square yields the wedge product $S^1 \vee S^1$ of two circles S^1 that are glued together at a point. In general, if we remove a facet from a triangulation of a sphere product, the resulting complex is (simple-)homotopy equivalent to the wedge product of the constituting spheres. In the case of $S^2 \times S^1$, the wedge product $S^2 \vee S^1$ is of mixed dimension. Since in the implementation of RSHT our

Table 5 RSHT run for triangulations of sphere products minus a facet

Initial complex	Initial f-vector	Resulting complex	Size of the resulting complex
$S^2 \times S^1$	(12, 48, 72, 36)	$S^2 \vee S^1$	$\partial\Delta_3 \cup K^1(3.5382)$
$S^3 \times S^1$	(15, 75, 150, 150, 60)	$S^3 \vee S^1$	$\partial\Delta_4 \cup K^1(7.7617)$
$S^2 \times S^2$	(16, 84, 216, 240, 96)	$S^2 \vee S^2$	$\partial\Delta_3 \cup \partial\Delta_3$
$S^3 \times S^2$	(20, 130, 420, 710, 600, 200)	$S^3 \vee S^2$	$\partial\Delta_4 \cup K^2(11.5460)$

focus is on the top-dimensional faces, RSHT is not further touching lower-dimensional parts once these are reached via collapses. Thus, the resulting triangulations of $S^2 \vee S^1$ are of the form $\partial\Delta_3 \cup K^1$, consisting of the vertex-minimal triangulation of S^2 as the boundary complex $\partial\Delta_3$ of a 3-simplex Δ_3 union a 1-dimensional complex K^1.

Depending on the intersection of K^1 with $\partial\Delta_3$, K^1 either is a path (a 1-dimensional ball) or a loop (a 1-sphere S^1). For a unified description in Table 5, we write $K^1(4.5382)$ to point out that K^1 has (in 10^4 runs of RSHT) on average 4.5382 edges. Table 5 gives results for further sphere products, where for the lower-dimensional parts the average number of facets are listed. The initial triangulations of the sphere products in Table 5 are produced via product triangulations of boundaries of simplices (Lutz 2003).

In a separate experiment, we started with a triangulation of S^1 with 10 vertices and with a triangulation of S^2 with 100 vertices as the boundary complex of a random simplicial 3-polytope, for which 100 points on the round 2-dimensional sphere were chosen randomly via the `rand_sphere` client of the software system `polymake` (Gawrilow et al. 1997)). The initial triangulation of $S^2 \times S^1$ has face-vector $f =$ (1000, 6880, 11760, 5880). It took RSHT an average of 1108.23 expansions, in 10^2 runs, to reduce the triangulation (minus a facet) to a triangulation $\partial\Delta_3 \cup K^1(21.76)$ of the wedge product $S^2 \vee S^1$. We repeated the same experiment, but this time applying 200, 000 preliminary random bistellar edge flips to the 100-vertex triangulation of S^2, before taking the sphere product. The results of this experiment are similar to the one before (though with a slightly higher average number of expansions). This suggests that RSHT may be reliable even for larger complexes.

6.3 Dimensionality reduction

"Finding meaningful low-dimensional structures hidden in their high-dimensional observations" (Tenenbaum et al. 2000) is a major theme in analyzing higher-dimensional data of various origins. Usually, the data is given as a finite set of points in some Euclidean or metric space and is then often transformed to (higher-dimensional) simplicial complexes via taking Čech complexes or Vietoris–Rips complexes. Here, we did not start with explicit data sets, but instead "hid" a (closed) surface in a higher-dimensional product as another model to test RSHT on.

Starting with the standard 7-vertex triangulation T of the torus, we first took connected sums of T to create surfaces of higher genus g_k, $k \geq 2$. Then we took the cross product $g_k \times I$ of g_k with an interval (subdivided into 10 edges on 11 vertices), and reduced the resulting triangulation of the cross product with RSHT. In every single

Table 6 RSHT run for triangulations of products of surfaces with subdivided intervals

Initial complex	Initial f-vector	Resulting complex	Resulting smallest f-vector
$T \times I$	(77, 511, 854, 420)	T	(7, 21, 14)
$g_2 \times I$	(121, 929, 1586, 780)	g_2	(10, 36, 24)
$g_5 \times I$	(253, 2183, 3782, 1860)	g_5	(12, 60, 40)
$g_6 \times I$	(297, 2601, 4514, 2220)	g_6	(13, 69, 46)
$g_{10} \times I$	(473, 4273, 7442, 3660)	g_{10}	(18, 108, 72)
$g_{50} \times I$	(2233, 20993, 36722, 18060)	g_{50}	(51, 683, 534)

one out of 10^2 runs, the product $g_k \times I$ gets reduced back to a small or even vertex-minimal triangulation of the original surface of genus g_k, as displayed in Table 6. In a second experiment, we performed 200, 000 random edge flips to "randomize" the surfaces g_k; then, we took cross products with the 10-edge interval I. Again, in 10^2 runs of RSHT, we always achieved the respective f-vectors of Table 6. While at least 10 vertices are needed to triangulate g_2, Borghini and Minian describe in (Borghini and Minian 2019) a 2-complex with only 9 vertices that is homotopy equivalent to g_2 (we are grateful to Dejan Govc and Petar Pavešić for pointing this example out to us); but in our runs we never found this 9-vertex example.

In a final experiment, we started with the triangulation of the surface g_{50} from before, but this time we added 100 vertices in subdivision steps before performing the 200, 000 random edge flips. We then took again the cross product with the interval I to get a randomized triangulation of $g_{50} \times I$ with $f = (2728, 24278, 42212, 20760)$. We then took another cross product of this 3-manifold with boundary with the 4-simplex Δ_4. The resulting complex is 7-dimensional with around 34 million faces and face vector

$$f = (13420, 386630, 2446620, 6910210, 10432052, 8786210, 3909060, 718200).$$

In less than an hour and by using a few thousand expansions, in each out of 10^2 runs of RSHT, we were able to reduce this complex back to a triangulation of the 2-dimensional orientable surface of genus 50 with fewer than 60 vertices. In some cases we were even able to reach the same f-vector with 51 vertices as in Table 6. Due to memory constraints that come from the computation of the Hasse diagram of the starting complex (requiring around 10 GB of RAM for this example), this was the largest complex that we were able to study.

6.4 Akbulut–Kirby 4-spheres

As stated early on, contractibility is, in general, undecidable. However, it takes considerable effort to pose challenges to RSHT. A notoriously hard series of complexes is given by the triangulations of the Akbulut–Kirby 4-dimensional spheres (Tsuruga and Lutz 2013). These PL-triangulated standard 4-dimensional spheres (Akbulut and Kirby 1985; Akbulut 2010) are built in an intricate way via non-trivial presentations of the trivial group as their fundamental group (Akbulut and Kirby 1985). By Pachner's theorem, these examples are bistellarly equivalent to the boundary of the 5-simplex, and by Whitehead's theorem, the examples minus a facet are simple-homotopy equivalent to a single vertex. However, establishing connecting sequences of bistellar flips failed in (Tsuruga and Lutz 2013), beyond the first easy examples of the series. Indeed, here RSHT made no progress either, even when we set *max_ step* = 1,000,000 and waited for a total runtime of 60 h. On the level of 2-complexes involved here, their reduction to a single vertex would prove Andrews–Curtis conjecture (cf. (Hog-Angeloni and Metzler 1993, Ch. XII)) for these complexes, which is open.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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