Journal of Applied and Computational Topology (2023) 7:89–102
https://doi.org/10.1007/s41468-022-00100-x

Fiber of persistent homology on morse functions

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Received: 15 September 2021 / Revised: 4 July 2022 / Accepted: 7 September 2022 /
Published online: 10 October 2022

Abstract
Let f be a Morse function on a smooth compact manifold M with boundary. The path component $\mathrm{PH}_f^{-1}(D)$ containing f of the space of Morse functions giving rise to the same Persistent Homology $D = \mathrm{PH}(f)$ is shown to be the same as the orbit of f under pre-composition $\phi \mapsto f \circ \phi$ by diffeomorphisms of M which are isotopic to the identity. Consequently we derive topological properties of the fiber $\mathrm{PH}_f^{-1}(D)$: In particular we compute its homotopy type for many compact surfaces M. In the 1-dimensional settings where M is the unit interval or the circle we extend the analysis to continuous functions and show that the fibers are made of contractible and circular components respectively.

Keywords Persistent homology · Inverse problems · Morse theory

Mathematics Subject Classification 55N31 · 62R40 · 57S05

1 Introduction

Persistent Homology is a central and computable descriptor in Topological Data Analysis (TDA) which has been applied to a large variety of data science problems. Namely the persistence map PH associates to a real-valued function f on a topological space X a so-called barcode that captures the topological variations of its sub level-sets (Edelsbrunner and Harer 2008; Zomorodian and Carlsson 2005). It is natural to ask how

Work supported by the Mathematical Institute of Oxford & the EPSRC Grant No. EP/R018472/1. Both authors are members of the Centre for Topological Data Analysis.

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much information can be recovered from persistent homology: Given a barcode D what can we say about the fiber $\mathrm{PH}^{-1}(D)$?

For the purpose of this paper $X = M$ is an arbitrary smooth (finite-dimensional) compact manifold with boundary ∂M and $f : M \to \mathbb{R}$ is a Morse function, i.e. f has a prescribed constant value a_j on each boundary component ∂M_j and has isolated critical points none of which belong to ∂M. Denote by $D := \mathrm{PH}(f)$ the associated barcode. Acting on f by pre-composition with the group $\mathcal{D}_{\mathrm{Id}}(M)$ of diffeomorphisms which are isotopic to the identity, we get an orbit $\mathcal{O}_{\mathrm{Id}}(f)$ inside the space of Morse functions. Our core contribution (Theorem 3.1) is the equality between this orbit and the the path connected component $\mathrm{PH}_f^{-1}(D)$ containing f in the fiber of PH over D:

$$\mathrm{PH}_f^{-1}(D) = \mathcal{O}_{\mathrm{Id}}(f).$$

This result crucially relies on Mather's fibration theorem for smooth mappings (Mather 1969), which we slightly adapt to the case of Morse functions with equal critical values using results of Cerf (1970).

We then put at work the abundant literature about the homotopy type of the orbit $\mathcal{O}_{\mathrm{Id}}(f)$, especially the work of Maksymenko (2006): the mapping $\phi \mapsto f \circ \phi$ in fact defines a locally trivial fibration from $\mathcal{D}_{\mathrm{Id}}(M)$ to the orbit $\mathcal{O}_{\mathrm{Id}}(f)$, with fiber $\mathcal{S}_{\mathrm{Id}}(f) \subseteq \mathcal{D}_{\mathrm{Id}}(M)$ the diffeomorphisms stabilising f, i.e. $f \circ \phi = f$. Hence a long exact sequence links the fiber $\mathrm{PH}_f^{-1}(D) = \mathcal{O}_{\mathrm{Id}}(f)$ to well-studied diffeomorphism groups of the manifold M. In particular for any compact surface M, we compute $\pi_n(\mathrm{PH}_f^{-1}(D))$ for $n \geq 2$ (Proposition 4.6). In fact, if D has distinct interval endpoints we derive the complete homotopy type of $\mathrm{PH}_f^{-1}(D)$ for many compact surfaces (Propositions 4.8 and 4.9).

Variations of this setting have already been addressed. In the discrete setting where $X = K$ is a finite simplicial complex and f is compatible with face inclusions, the fiber $\mathrm{PH}^{-1}(D)$ is a complex of polyhedra (Leygonie and Tillmann 2021). This structure can be used to algorithmically reconstruct the fiber (Leygonie and Henselman-Petrusek 2021). In the restricted case where K is a line complex, each path connected component of $\mathrm{PH}^{-1}(D)$ is contractible (Cyranka et al. 2020), and it is homeomorphic to a circle in the case where K is a subdivision of the unit circle (Mischaikow and Weibel 2021).

In the analogous, continuous 1-dimensional setting where X is the interval or the circle and f is continuous, each component in the fiber is contractible and circular respectively as we show in Sect. 5. For the unit interval it is possible to count the number of path connected components in $\mathrm{PH}^{-1}(D)$ by means of the combinatorics of the barcode (Curry 2018). For higher dimensional X analyzing the fiber is a challenging problem: already for Morse functions on the 2-sphere $X = \mathbb{S}^2$ new tools have been designed to describe the fiber $\mathrm{PH}^{-1}(D)$, and allowed for conjectures on the number of path connected components (Catanzaro et al. 2020). However the higher dimensional homotopy groups of $\mathrm{PH}^{-1}(D)$ remain unknown, from which stems the motivation of this work.

2 Stability of morse functions

We fix a d-dimensional compact smooth manifold M with boundary ∂M, whose path connected components are denoted by ∂M_j. Given another smooth manifold N, we denote by $C^\infty(M, N)$ the space of smooth maps from M to N equipped with the C^∞ Whitney topology. We denote by $\mathcal{D}(M) \subseteq C^\infty(M, M)$ the diffeomorphisms of M, and by $\mathcal{D}_{\mathrm{Id}}(M)$ its subspace of Id-*isotopic* diffeomorphisms, i.e. the path connected component of the identity map.

Given real values a_j, a smooth map f belongs to the space $\mathrm{Morse}(M) \subseteq C^\infty(M, \mathbb{R})$ of *Morse functions* on M if:

- The Hessian of f is non-degenerate at critical points, all of which belong to $M \backslash \partial M$; and
- The restrictions $f_{|\partial M_j}$ to each boundary component ∂M_j are constant with prescribed value a_j.

Then $\mathcal{D}(M)$ acts on $C^\infty(M, \mathbb{R})$ by pre-composition and we denote by $\mathcal{O}(f) \subseteq \mathrm{Morse}(M)$ the orbit of f, and by $\mathcal{O}_{\mathrm{Id}}(f) \subseteq \mathcal{O}(f)$ the orbit of f under the restricted action of Id-isotopic diffeomorphisms.

To express the local triviality results of this section, we rely on the notion of local cross-sections defined below.

Definition 2.1 Let $\mathcal{G}$ be a topological group acting on a topological space X. Given $x_0 \in X$, a *local cross-section* for the action of $\mathcal{G}$ on X at x_0 is a continuous map $s : \mathcal{U} \to \mathcal{G}$ defined on an open neighborhood $\mathcal{U}$ of x_0 satisfying:

$$\forall x \in \mathcal{U},\ s(x) \cdot x_0 = x.$$

We say that the action of $\mathcal{G}$ on X *admits local cross-sections* if it does so at each point.

Up to replacing $s(x)$ by $s(x) \cdot s(x_0)^{-1}$ in the above definition, we can assume that $s(x_0)$ is the identity element.

Remark 2.2 It is well-known that, if X admits local cross-sections, then any $\mathcal{G}$-equivariant map from a $\mathcal{G}$-space to X is locally trivial, see e.g. Palais(1960, Theorem A).

The main result of this section adapts Mather's stability of smooth mappings (Mather 1969) to the case of Morse functions with equal critical points by combining results of Cerf:

Proposition 2.3 *Let* $f \in \mathrm{Morse}(M)$ *and* $\mathrm{Morse}_f(M)$ *be the subspace of Morse functions with the same critical values as* f. *Then the action of* $\mathcal{D}_{\mathrm{Id}}(M)$ *on* $\mathrm{Morse}_f(M)$ *admits local cross-sections.*

The first result of Cerf we use is essentially a version of Proposition 2.3 restricted to the space $\mathrm{Morse}_f(M; \mathrm{Crit}(f), \partial M)$ of Morse functions with the same critical points and critical values as f and the same value and derivatives of any order as f on ∂M.

Proposition 2.4 (Cerf (1970), Appendix, §1, Proposition 1) *Let $f : M \to \mathbb{R}$ be a Morse function. Let $\mathcal{G}$ be the subspace of diffeomorphisms fixing the critical points of f and ∂M, and which have the same value and derivatives of every order as the identity on ∂M. Then the action of $\mathcal{G}$ on $\mathrm{Morse}_f(M; \mathrm{Crit}(f), \partial M)$ admits local cross-sections.*

To enhance this result to Morse functions with critical points and derivatives at the boundary allowed to vary, we need the reformulation of a result of Cerf (Cerf, 1961, Theorem 5) for the C^∞ case given in Hong et al. (2012).

Proposition 2.5 *Let $N \subseteq M$ be a compact submanifold. Suppose that:*

(a) either N has no boundary and does not intersect the boundary ∂M;
(b) or N is a closed collar neighborhood of a boundary component ∂M_j.

Let L be a compact neighborhood of N. Let $\mathrm{Emb}_L(N, M)$ be the space of embeddings $j : N \hookrightarrow M$ such that $j(N) \subseteq L$, $j^{-1}(\partial M) = N \cap \partial M$, and that restricts to the identity on $N \cap \partial M$. Denote by $\mathcal{D}_L(M)$ the space of diffeomorphisms inducing the identity on $(M \setminus L)$. Then the action of $\mathcal{D}_L(M)$ on $\mathrm{Emb}_L(N, M)$ admits local cross-sections.

Proof This is a direct consequence of Theorem 3.1 in Hong et al. (2012). Note that, strictly speaking, embeddings j as defined in (Hong et al.,2012, Definition 2.5) are required to admit an extension to a diffeomorphism of M. However this assumption is unnecessary for their Theorem 3.1 and so we omit it. □

Proof of Proposition 2.3 Since $g \in \mathrm{Morse}_f(M)$ implies that $\mathrm{Morse}_g(M) = \mathrm{Morse}_f(M)$, it is enough to construct a local cross-section at f. By Proposition 2.4, there exists a local cross-section $g \in \mathcal{U}_f \mapsto \phi_g \in \mathcal{D}_{\mathrm{Id}}(M)$ defined on a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Morse}_f(M; \mathrm{Crit}(f), \partial M)$, the space of functions with the same critical points $p_1, \ldots, p_n$ and critical values as f, and with the same value and derivatives $\partial^k f$ as f on the boundary ∂M. Therefore

$$\forall g \in \mathcal{U}_f, \ \ g = f \circ \phi_g. \tag{1}$$

For the general case where critical points and derivatives on the boundary are allowed to vary we simply find a diffeomorphism sending them back to $p_1, \ldots, p_n$ and $\partial^k f$ and apply the above result.

Namely, from item (a) of Proposition 2.5, we can find disjoint compact neighborhoods $U_1, \ldots, U_n$ of $p_1, \ldots, p_n$ and continuously associate to $(p'_1, \ldots, p'_n) \in U_1 \times \cdots \times U_n$ a diffeomorphism $\psi_{(p'_1,\ldots,p'_n)} \in \mathcal{D}_{\mathrm{Id}}(M)$ such that $\psi_{(p_1,\ldots,p_n)} = \mathrm{Id}$ and:

$$\forall (p'_1, \ldots, p'_n) \in U_1 \times \cdots \times U_n, \ \ \forall 1 \leq i \leq n, \ \ \psi_{(p'_1,\ldots,p'_n)}(p_i) = p'_i.$$

Let $\mathcal{U} \subseteq \mathrm{Morse}_f(M)$ be a neighborhood of f for which any $g \in \mathcal{U}$ has critical points $\mathrm{Crit}(g)$ in $U_1, \ldots, U_n$. In particular in this case $g \circ \psi_{\mathrm{Crit}(g)}$ has the same critical points $p_1, \ldots, p_n$ as f, so it remains to deal with the boundary ∂M.

Let ∂M_j be a boundary component. By flowing along the normalized gradient of f (or its inverse) from the boundary ∂M_j we get a compact collar $V_j \cong \partial M_j \times [0, \alpha]$ that is adapted to f in the sense that $f(x,t) = a_j \pm t$, w.l.o.g. $f(x,t) = a_j + t$. For g in a small neighborhood $\mathcal{V} \subseteq \mathrm{Morse}_f(M)$ of f, we have $g(\partial M_j \times [0, \frac{\alpha}{2}]) \subseteq [a_j, a_j + \alpha]$. Therefore, after potentially shrinking $\mathcal{V}$, we can continuously associate to $g \in \mathcal{V}$ the embedding $\iota_g : (x,t) \in \partial M_j \times [0, \frac{\alpha}{2}] \mapsto (x, g(x,t) - a_j) \in V_j$ that preserves ∂M_j and satisfies $g = f \circ \iota_g$ on $\partial M_j \times [0, \frac{\alpha}{2}]$. Hence by using item (b) of Proposition 2.5, up to shrinking $\mathcal{V}$, we can extend ι_g to a diffeomorphism χ_g that induces the identity outside V_j. By repeating this process for each boundary component ∂M_j, we can continuously associate to $g \in \mathcal{V}$ a diffeomorphism χ_g such that g and $f \circ \chi_g$ agrees on a closed collar neighborhood of the boundary.

By reducing the neighorhoods U_i and V_j to avoid overlaps, we have that for any g in $\mathcal{U} \cap \mathcal{V}$ the Morse function $g \circ \psi_{\mathrm{Crit}(g)} \circ \chi^{-1}_{g \circ \psi_{\mathrm{Crit}(g)}}$ has the same critical points as f and agrees with f on a neighborhood of the boundary ∂M, in particular it belongs to $\mathrm{Morse}_f(M; \mathrm{Crit}(f), \partial M)$. Hence by Eq. (1), up to shrinking $\mathcal{U} \cap \mathcal{V}$, we have

$$\begin{aligned}\forall g \in \mathcal{U} \cap \mathcal{V}, g &= (g \circ \psi_{\mathrm{Crit}(g)} \circ \chi^{-1}_{g \circ \psi_{\mathrm{Crit}(g)}}) \circ \chi_{g \circ \psi_{\mathrm{Crit}(g)}} \circ \psi^{-1}_{\mathrm{Crit}(g)} \\ &= f \circ \phi_{g \circ \psi_{\mathrm{Crit}(g)} \circ \chi^{-1}_{g \circ \psi_{\mathrm{Crit}(g)}}} \circ \chi_{g \circ \psi_{\mathrm{Crit}(g)}} \circ \psi^{-1}_{\mathrm{Crit}(g)},\end{aligned}$$

hence the local cross-section:

$$s : g \in \mathcal{U} \cap \mathcal{V} \longmapsto \phi_{g \circ \psi_{\mathrm{Crit}(g)} \circ \chi^{-1}_{g \circ \psi_{\mathrm{Crit}(g)}}} \circ \chi_{g \circ \psi_{\mathrm{Crit}(g)}} \circ \psi^{-1}_{\mathrm{Crit}(g)} \in \mathcal{D}_{\mathrm{Id}}(M).$$

□

Corollary 2.6 *Let $(f_t)_{t \in [0,1]} \subseteq \mathrm{Morse}(M)$ be a path of Morse functions with the same critical values. Then there exists $\phi \in \mathcal{D}_{\mathrm{Id}}(M)$ an* Id*-isotopic diffeomorphism such that $f_1 = f_0 \circ \phi$.*

Proof By Proposition 2.3, each $t \in [0,1]$ has a neighborhood $\mathrm{I}_t \subseteq [0,1]$ such that f_h can be written $f_h = f_t \circ \phi_h$ whenever $h \in \mathrm{I}_t$. By compactness $[0,1]$ is covered by finitely many such intervals. Therefore f_1 equals $f_0 \circ \phi$, where ϕ is a finite composition of diffeomorphisms in $\mathcal{D}_{\mathrm{Id}}(M)$ hence is itself in $\mathcal{D}_{\mathrm{Id}}(M)$. □

3 Covering the fiber with diffeomorphisms isotopic to the identity

Given $f \in \mathrm{Morse}(M)$, we get a nested sequence of sub level-sets $f^{-1}((-\infty, x])$. In turn, by applying homology in degree $0 \leq k \leq d$ with coefficients in an arbitrary field, we get the *persistent homology module of* f: the sequence of vector spaces $\mathrm{H}_k(f^{-1}((-\infty, x]))$ with linear maps between them induced by inclusions, in other words a functor from the poset $(\mathbb{R}, \leq)$ to finite dimensional vector spaces. The *barcode* of f in degree k is the isomorphism class of this functor up to natural isomorphism. From Crawley-Boevey (2015) any such functor uniquely decomposes as a direct sum of functors $\bigoplus_{(b,d) \in D} \mathbb{I}_{[b,d)}$, with $[b,d) \subseteq \mathbb{R}$ an interval closed on the left and open on the right (hence possibly $d = +\infty$): each $\mathbb{I}_{[b,d)}$ consists of 1-dimensional vector spaces linked with identity maps on $[b,d)$, and it is the zero vector space everywhere outside of $[b, d)$. Therefore the barcode of f, denoted by $\mathrm{PH}_k(f)$, can be equivalently described as the multi-set D of pairs (b, d) indexing this decomposition, and will be described in this way in the rest of this document. By abuse of terminology we refer to pairs (b, d) as *intervals* or *bars* of the barcode $D = \mathrm{PH}_k(f)$. Intuitively (b, d) corresponds to the appeareance of a k-cycle in $f^{-1}((-\infty, b])$ that is further cancelled in $f^{-1}((-\infty, d])$ (or persists forever if $d = \infty$). We refer to Edelsbrunner and Harer (2008); Zomorodian and Carlsson (2005) for extensive treatments of the theory of Persistence.

In this work the persistence map is defined on Morse functions and returns the $d+1$ barcodes of interest:

$$\mathrm{PH} : f \in \mathrm{Morse}(M) \longmapsto [\mathrm{PH}_0(f), \ldots, \mathrm{PH}_d(f)] \in \mathrm{Bar}^{d+1}.$$

We assume that Bar is equipped with its natural *bottleneck metric* which turns PH into a continuous map by the Stability Theorem (Cohen-Steiner et al. 2007). Given a barcode D and a Morse function $f \in \mathrm{Morse}(M)$ such that $\mathrm{PH}(f) = D$, we denote by $\mathrm{PH}_f^{-1}(D)$ the path connected component of the fiber $\mathrm{PH}^{-1}(D) \subseteq \mathrm{Morse}(M)$ containing f.

Theorem 3.1 *Let D be a barcode and $f \in \mathrm{PH}^{-1}(D)$. Then $\mathrm{PH}_f^{-1}(D) = \mathcal{O}_{\mathrm{Id}}(f)$.*

Proof Let $(\phi_t)_{0 \le t \le 1}$ be a path in $\mathcal{D}_{\mathrm{Id}}(M)$. Each ϕ_t restricts to a homeomorphism between the sub level-sets of $f \circ \phi_t$ and f, hence it induces an isomorphism between the associated persistent homology modules. In turn $\mathrm{PH}(f \circ \phi_t) = \mathrm{PH}(f)$, so that $(f \circ \phi_t)_{0 \le t \le 1}$ is a path in the fiber $\mathrm{PH}_f^{-1}(D)$, which implies $\mathcal{O}_{\mathrm{Id}}(f) \subseteq \mathrm{PH}_f^{-1}(D)$.

Conversely let $g \in \mathrm{PH}_f^{-1}(D)$ and let $(f_t)_{0 \le t \le 1}$ be a path in the fiber $\mathrm{PH}_f^{-1}(D)$ joining f to g, thus $\mathrm{PH}(f_t) = D$ for each t. As is well-known, when M has no boundary there is a one-to-one correspondence between the set $\mathcal{D}$ of (bounded) interval endpoints in the barcode and the set $\mathcal{C}$ of critical values (counted with multiplicity) for Morse functions because the associated persistent homology module and Morse-Smale complex are isomorphic (Barannikov 1994), see also (Leygonie et al. 2021, Proposition 2.14) for a self-contained proof.

When M has a boundary the correspondence adapts by adding in $\mathcal{C}$ the value a_j with multiplicity $\sum_i \beta_i(\partial M_j)$ for each boundary component ∂M_j that is a local minimum. Note that a Morse function is constant on ∂M_j and has no critical points there, so either it has ∂M_j as a local minimum or as a local maximum, and this choice is fixed inside a path connected component of $\mathrm{Morse}(M)$.

Therefore each f_t has the same critical values as f, because the barcode $\mathrm{PH}(f_t) = D$ is constant. By corollary 2.6 there exists an Id-isotopic diffeomorphism ϕ such that $g = f \circ \phi$. Consequently $\mathrm{PH}_f^{-1}(D) \subseteq \mathcal{O}_{\mathrm{Id}}(f)$. □

4 Topological properties of the fiber

We derive direct consequences of Theorem 3.1 combined with the extensive study of $\mathcal{O}_{\mathrm{Id}}(f)$ by Maksymenko (2006). Strictly speaking, it is $\mathcal{O}_f(f)$, the path component of $\mathcal{O}(f)$ containing f, whose properties are studied in Maksymenko (2006). However, there is an obvious inclusion $\mathcal{O}_{\mathrm{Id}}(f) \subseteq \mathcal{O}_f(f)$, and the reverse inclusion holds as well by corollary 2.6. Therefore $\mathcal{O}_f(f) = \mathcal{O}_{\mathrm{Id}}(f)$.

Denote by $\mathcal{S}_{\mathrm{Id}}(f)$ the subspace of Id-isotopic diffeomorphisms ϕ preserving a Morse function f, i.e. $f \circ \phi = f$.[1]

Proposition 4.1 *Assume that M is connected. Let D be a barcode and $f \in \mathrm{PH}^{-1}(D)$. Then the action of $\mathcal{D}_{\mathrm{Id}}(M)$ on $\mathrm{PH}_f^{-1}(D)$ defines a locally trivial principal $\mathcal{S}_{\mathrm{Id}}(f)$-fibration.*

Proof From Maksymenko(2006, Theorem 2.1, (2)) the action of diffeomorphisms $\mathcal{D}(M)$ on $\mathcal{O}(f)$ defines a locally trivial principal fibration with fiber the diffeomorphisms ϕ satisfying $f \circ \phi = f$. Restricting to the action of $\mathcal{D}_{\mathrm{Id}}(M)$ on $\mathcal{O}_{\mathrm{Id}}(f)$ defines a locally trivial principal $\mathcal{S}_{\mathrm{Id}}(f)$-fibration, and $\mathrm{PH}_f^{-1}(D)$ equals $\mathcal{O}_{\mathrm{Id}}(f)$ by Theorem 3.1. □

Remark 4.2 The principal bundle $\mathcal{S}_{\mathrm{Id}}(f) \to \mathcal{D}_{\mathrm{Id}}(M) \to \mathrm{PH}_f^{-1}(D)$ has computationally useful and direct implications. First, it is a locally trivial fibration hence it induces a homotopy long exact sequence:

$$\begin{aligned} \cdots &\to \pi_n(\mathcal{S}_{\mathrm{Id}}(f)) \to \pi_n(\mathcal{D}_{\mathrm{Id}}(M)) \\ &\to \pi_n(\mathrm{PH}_f^{-1}(D)) \to \pi_{n-1}(\mathcal{S}_{\mathrm{Id}}(f)) \to \cdots \to \pi_0(\mathcal{D}_{\mathrm{Id}}(M)). \end{aligned}$$

Second, we have the homeomorphism:

$$\mathrm{PH}_f^{-1}(D) \cong \mathcal{D}_{\mathrm{Id}}(M)/\mathcal{S}_{\mathrm{Id}}(f).$$

We apply this result to compute the path components of the fiber $\mathrm{PH}^{-1}(D)$ when M is a circle:

Proposition 4.3 *Assume $M = \mathbb{S}^1$. Let D be a barcode and $f \in \mathrm{PH}^{-1}(D)$. Then $\mathrm{PH}_f^{-1}(D)$ is homotopy equivalent to $\mathbb{S}^1$.*

Proof From Proposition 4.1 $\mathrm{PH}_f^{-1}(D)$ is homeomorphic to $\mathcal{D}_{\mathrm{Id}}(\mathbb{S}^1)/\mathcal{S}_{\mathrm{Id}}(f)$. Let n be the number of minima of f, which is then also the number of maxima of f because $\chi(\mathbb{S}^1) = 0$. Without loss of generality we assume that the associated $2n$ critical points of f are evenly spaced on $\mathbb{S}^1$. The space $\mathcal{D}_{\mathrm{Id}}(\mathbb{S}^1)$ of Id-isotopic diffeomorphisms of the circle deformation retracts to $\mathbb{S}^1$, i.e. the rotations of the circle. The subgroup $\mathcal{S}_{\mathrm{Id}}(f)$ of Id-isotopic diffeomorphisms ϕ preserving f, that is $f \circ \phi = f$, is then (isomorphic to) the subgroup of rotations consisting of the $2n$-th roots of unity that preserve the sequence of extremal values of f. The result follows since the quotient of $\mathbb{S}^1$ by a finite subgroup is again $\mathbb{S}^1$. □

When $M = [0, 1]$ recall that Morse functions have prescribed values a_0 and a_1 on the boundary points 0 and 1.

[1] In Maksymenko (2006) the notation $\mathcal{S}_{\mathrm{Id}}(f)$ rather stands for the space of diffeomorphisms ϕ preserving f that are isotopic to Id_M *though* maps preserving f, thus it is the path connected component of Id_M in our $\mathcal{S}_{\mathrm{Id}}(f)$.

Proposition 4.4 *Assume $M = [0,1]$. Let D be a barcode and $f \in \mathrm{PH}^{-1}(D)$. Then $\mathrm{PH}_f^{-1}(D)$ is contractible.*

Proof From Proposition 4.1 $\mathrm{PH}_f^{-1}(D)$ is homeomorphic to $\mathcal{D}_{\mathrm{Id}}([0,1])/\mathcal{S}_{\mathrm{Id}}(f)$. However $\mathcal{D}_{\mathrm{Id}}([0,1])$ deformation retracts on the identity diffeomorphism $\mathrm{Id}_{[0,1]}$ by straight-line interpolations, and $\mathcal{S}_{\mathrm{Id}}(f) = \{\mathrm{Id}_{[0,1]}\}$. □

Note that we could easily derive a similar statement for Morse functions on [0, 1] without boundary conditions. In Sect. 5 we prove the analogues of Propositions 4.3 and 4.4 for continuous functions. The analogues for lower-star filtrations on the subdivided interval and circle have been proved by Cyranka et al. (2020) and Mischaikow and Weibel (2021) respectively.

Remark 4.5 When $M = M_1 \sqcup M_2$ has more than one connected component, the path component $\mathrm{PH}_f^{-1}(D)$ in the fiber over $D = \mathrm{PH}(f)$ can be retrieved as the product of the path components of the fibers over $D_1 := \mathrm{PH}(f_{|M_1})$ and $D_2 := \mathrm{PH}(f_{|M_2})$ containing the restrictions $f_{|M_1}$ and $f_{|M_2}$ respectively:

$$\mathrm{PH}_f^{-1}(D) = \mathrm{PH}_{f_{|M_1}}^{-1}(D_1) \times \mathrm{PH}_{f_{|M_2}}^{-1}(D_2).$$

For this reason we focus our analysis to the interesting case where M is connected.

For the rest of the section we fix a compact connected surface M and a function f with barcode D, whose number of critical points of index 1 is denoted by c_1. We make use of the analysis of the orbit $\mathcal{O}_{\mathrm{Id}}(f)$ by Maksymenko (2006).

Proposition 4.6 *Assume that $c_1 > 0$. Then $\pi_2(\mathrm{PH}_f^{-1}(D)) = 0$ and $\pi_n(\mathrm{PH}_f^{-1}(D)) = \pi_n(M)$ for $n \geq 3$.*

Proof $\mathrm{PH}_f^{-1}(D) = \mathcal{O}_{\mathrm{Id}}(f)$ by Theorem 3.1, and by Maksymenko(2006, (2), Theorem 1.5) we have $\pi_2(\mathcal{O}_{\mathrm{Id}}(f)) = 0$ and $\pi_n(\mathcal{O}_{\mathrm{Id}}(f)) = \pi_n(M)$ for $n \geq 3$. □

Remark 4.7 From Maksymenko(2006, (2), Theorem 1.5) we can also derive a short exact sequence $0 \to \pi_1(\mathcal{D}_{\mathrm{Id}}(M)) \oplus \mathbb{Z}^{k_f} \to \pi_1(\mathrm{PH}_f^{-1}(D)) \to G \to 0$ where G is a finite group and the integer $k_f \geq 0$ depends on the component $\mathrm{PH}_f^{-1}(D)$ in the fiber, on the number c_1 of saddles and the surface M.

Proposition 4.8 *Assume that $c_1 = 0$. Then the homotopy type of the fiber $\mathrm{PH}_f^{-1}(D)$ is classified as follows:*

Surface M	$\mathbb{S}^2$	$\mathbb{S}^1 \times \mathrm{I}$	$\mathbb{D}^2$
Fiber $\mathrm{PH}_f^{-1}(D)$	$\mathbb{S}^2$	$\{*\}$	$\{*\}$

Proof $\mathrm{PH}_f^{-1}(D) = \mathcal{O}_{\mathrm{Id}}(f)$ by Theorem 3.1, and the homotopy type of $\mathcal{O}_{\mathrm{Id}}(f)$ is computed by Maksymenko(citeyearmaksymenko2006homotopy, Theorem 1.9). □

For instance the case where $f : \mathbb{S}^2 \to \mathbb{R}$ has no saddle ($c_1 = 0$) can be interpreted as follows: The fiber sequence $\mathcal{S}_{\mathrm{Id}}(f) \to \mathcal{D}_{\mathrm{Id}}(\mathbb{S}^2) \to \mathrm{PH}_f^{-1}(D)$ of Proposition 4.1 can be identified up to homotopy with the standard fiber sequence $\mathbb{S}^1 \to \mathrm{SO}(3) \to \mathbb{S}^2$. This is because $\mathcal{D}_{\mathrm{Id}}(\mathbb{S}^2)$ deformation retracts to SO(3) by the 2-dimensional Smale conjecture (see Smale (1959)), and if without loss of generality we assume that f is the standard height function, then $\mathcal{S}_{\mathrm{Id}}(f)$ consists of those rotations fixing the poles, so $\mathcal{S}_{\mathrm{Id}}(f)$ is fixed by the retraction and $\mathcal{S}_{\mathrm{Id}}(f) \sim \mathbb{S}^1$.

Proposition 4.9 *Assume that D has pairwise distinct bounded interval endpoints, and that $c_1 > 0$. Then we have the following homotopy types for the fiber $\mathrm{PH}_f^{-1}(D)$:*

Surface M	$\mathbb{S}^2$	Projective Plane	Torus	$\mathbb{S}^1 \times \mathrm{I}$	$\mathbb{D}^2$
Fiber $\mathrm{PH}_f^{-1}(D)$	$\mathrm{SO}(3) \times (\mathbb{S}^1)^{c_1-1}$	$\mathrm{SO}(3) \times (\mathbb{S}^1)^{c_1-1}$	$(\mathbb{S}^1)^{c_1+1}$	$(\mathbb{S}^1)^{c_1}$	$(\mathbb{S}^1)^{c_1}$

When M is obtained from the surfaces in the above tables by removing finitely many 2-disks, then $\mathrm{PH}_f^{-1}(D) \sim (\mathbb{S}^1)^{c_1-1}$. If M is the Möbius strip, then $\mathrm{PH}_f^{-1}(D) \sim (\mathbb{S}^1)^{c_1}$. For other orientable surfaces M, we have $\mathrm{PH}_f^{-1}(D) \sim (\mathbb{S}^1)^{c_1+\chi(M)}$. For the remaining non-orientable surfaces, we have $\mathrm{PH}_f^{-1}(D) \sim (\mathbb{S}^1)^{k_f}$ for some integer $k_f \leq c_1 + \chi(M)$, unless M is the Klein bottle in which case $k_f \leq c_1 + 1$.

Proof $\mathrm{PH}_f^{-1}(D) = \mathcal{O}_{\mathrm{Id}}(f)$ by Theorem 3.1. Since D has distinct bounded interval endpoints, f has distinct critical points, and then the homotopy type of $\mathcal{O}_{\mathrm{Id}}(f)$ is computed by Maksymenko (2006, (2) & (3), Theorem 1.5). □

Remark 4.10 When M has no boundary, $\partial M = \emptyset$, the number c_1 of saddles of Morse functions f in the fiber $\mathrm{PH}^{-1}(D)$ can be directly inferred from the barcode D. Namely, if we denote by k_D the number of intervals in D, then the quantity $\mathrm{k}_D - \beta_0 - \beta_2$ counts (i) all the intervals (b, d) of D in degree 1, which correspond by their birth value b to saddle points of f whose attaching handle increases the 1-dimensional homology of the sub level-set $f^{-1}((-\infty, b])$, and (ii) all the bounded intervals (b, d) of D in degree 0, which correspond by their death value $d < \infty$ to saddle points of f whose attaching handle decreases the 0-dimensional homology of the sub level-set $f^{-1}((-\infty, d])$. Hence $c_1 = \mathrm{k}_D - \beta_0 - \beta_2$. When $\partial M = \bigsqcup_j \partial M_j \neq \emptyset$, we can partition the boundary components ∂M_j into the sets $\partial M^{\min}$ (resp. $\partial M^{\max}$) of components ∂M_j that are local minimum (resp. maximum) of one (hence any) function f in the component of $\mathrm{PH}^{-1}(D)$ at stake. Since M is a surface each ∂M_j is a circle, therefore if $\partial M_j \subseteq \partial M^{\min}$, then it corresponds in the barcode D to the births of one interval in degree 0 and one interval in degree 1. Otherwise $\partial M_j \subseteq \partial M^{\max}$ induces no topological change when entering the sub level-sets of f. Consequently the correspondence between critical points and interval endpoints adapts and yields $c_1 = \mathrm{k}_D - \beta_0 - \beta_2 - \#\partial M^{\min}$.

Remark 4.11 For manifolds M of dimension 3 for which the Smale conjecture $\mathcal{D}(M) \cong \mathrm{Isom}(M)$ holds, e.g. the 3-sphere, lens spaces, prism and quaternionic manifolds (see Hong et al. (2012)), the homotopy type of $\mathcal{D}_{\mathrm{Id}}(M)$ is quite well-understood. For instance we have $\mathcal{D}_{\mathrm{Id}}(\mathbb{S}^3) \cong \mathrm{SO}(4)$. However, to deduce the homotopy groups of $\mathrm{PH}_f^{-1}(D)$, we lack the understanding of less-studied topological properties of $\mathcal{S}_{\mathrm{Id}}(f)$.

5 Fiber of persistent homology for continuous maps on the circle and on the interval

In this section the domain of the persistence map consists of continuous maps on the circle:

$$\mathrm{PH} : \mathcal{C}^0(\mathbb{S}^1, \mathbb{R}) \longrightarrow \mathrm{Bar}^2.$$

Note that in the codomain we record the two barcodes with non-trivial homology, those in degree 0 and 1. In fact the second barcode contains a unique unbounded interval starting at the maximum of the function on the circle.

We fix a barcode D with finitely many intervals. When $f = \underline{\mathrm{cst}}$ is constant it forms the fiber by itself over the trivial barcode $D = \mathrm{PH}(f)$ with only two infinite bars $(\underline{\mathrm{cst}}, +\infty)$, one in each degree 0 and 1. Other barcodes such that $\mathrm{PH}^{-1}(D) \neq \emptyset$ have one infinite interval $(b_0, +\infty)$ in degree 0, one infinite interval $(b_1, +\infty)$ with $b_0 < b_1$ in degree 1, finitely many bounded intervals in degree 0 with endpoints in $[b_0, b_1]$, and no other intervals. In the rest of this section we assume that D is non-trivial and denote by $(n-1)$, for some $n \geq 1$, its number of bounded intervals in degree 0.

Let $\mathrm{Aut}_{\leq}(\mathbb{S}^1)$ be the space of orientation-preserving homeomorphisms of the circle, and $\mathrm{End}_{\leq}(\mathbb{S}^1) = \overline{\mathrm{Aut}_{\leq}(\mathbb{S}^1)}$ be its closure in $\mathcal{C}^0(\mathbb{S}^1, \mathbb{R})$ in the compact-open topology. Given $f \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{R})$ we have the pre-composition map $\phi \in \mathrm{End}_{\leq}(\mathbb{S}^1) \mapsto f \circ \phi \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{R})$; we denote by $\mathcal{S}_{\mathrm{Id}}(f)$ the stabiliser of f and by $\mathcal{O}_{\mathrm{Id}}(f)$ its orbit.

Proposition 5.1 *The fiber* $\mathrm{PH}^{-1}(D)$ *has finitely many path connected components. In each such component* $\Omega(D)$ *there exists some* $f_{\Omega(D)} : \mathbb{S}^1 \to \mathbb{R}$ *such that:*

$$\Omega(D) = \mathcal{O}_{\mathrm{Id}}(f_{\Omega(D)}),$$

and then $\Omega(D)$ *is homeomorphic to the quotient* $\mathrm{End}_{\leq}(\mathbb{S}^1)/\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$*, and in particular is homotopy equivalent to* $\mathbb{S}^1$.

Unlike the smooth case the component $\Omega(D) \subseteq \mathrm{PH}^{-1}(D)$ in the fiber equals the orbit of a function only for a careful choice of function $f_{\Omega(D)}$: the requirement will be that $f_{\Omega(D)}$ is injective between its consecutive extrema. Nevertheless the fact that the pre-composition map induces a homeomorphism from $\mathrm{End}_{\leq}(\mathbb{S}^1)/\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$ to the orbit $\mathcal{O}_{\mathrm{Id}}(f_{\Omega(D)})$ is reminiscent of the smooth case, and in fact with slightly more work it can be shown that it defines a $\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$-principal bundle. We state without proof the analogous and simpler result for the unit interval [0, 1], which works with or without fixed values on the boundary points 0 and 1.

Proposition 5.2 *For any finite barcode D the fiber* $\mathrm{PH}^{-1}(D) \subseteq \mathcal{C}^0([0,1],\mathbb{R})$ *has finitely many path connected components, each of which is contractible.*

Using a fixed orientation on $\mathbb{S}^1$ and going around starting from the north pole we can order the n minima and n maxima of a non-constant $f \in \mathcal{C}^0(\mathbb{S}^1,\mathbb{R})$ into a sequence $\mathrm{Val}(f)$ which we view as an element in $\mathbb{R}^{2n}$:

$$\mathrm{Val}(f) := m_1(f) < M_1(f) > \cdots > m_n(f) < M_n(f).$$

Associated to this sequence we have the sequence of critical sets of f:

$$\mathrm{Seq}(f) : c_1(f),\ d_1(f), \ldots,\ c_n(f),\ d_n(f).$$

Explicitly, each $c_i(f)$ (resp. $d_i(f)$) is a connected component of $f^{-1}(m_i(f))$ (resp. of $f^{-1}(M_i(f))$).

Proposition 5.3 *Let $f \in \mathrm{PH}^{-1}(D)$. Then f has $2n$ extrema, i.e.* $\mathrm{Val}(f) \in \mathbb{R}^{2n}$. *In addition, let Γ_n be the group of cyclic permutations on n elements, which acts on $\mathbb{R}^{2n}$ by cyclically permuting the n pairs of entries. Then the connected component $\Omega(D)$ in the fiber containing f is made of functions g whose sequence of extrema is the same as that of f up to a different ordering, that is:*

$$\Omega(D) = \left\{g \in \mathcal{C}^0(\mathbb{S}^1,\mathbb{R}) \mid \mathrm{Val}(g) \in \Gamma_n.\mathrm{Val}(f)\right\} \tag{2}$$

We omit the proof of this elementary statement. So if $\Omega(D)$ is a component in the fiber, we can pick the following simple function $f_{\Omega(D)}$ in $\Omega(D)$, whose critical sets and extrema are denoted by c_i, d_i, m_i, M_i for simplicity: the critical sets c_i and d_i are singletons arranged on the regular $2n$-gon in $\mathbb{S}^1$ and on each circular arc $[c_i, d_i]$, $f_{\Omega(D)}$ restricts to the linear homeomorphism to $[m_i, M_i]$.

Proposition 5.4 *Let $\Omega(D) \subseteq \mathrm{PH}^{-1}(D)$ be a path component in the fiber. Then the pre-composition map $\phi \mapsto f_{\Omega(D)} \circ \phi$ induces a homeomorphism from* $\mathrm{End}_{\leq}(\mathbb{S}^1)/\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$ *to* $\Omega(D)$.

Proof The map $\phi \in \mathrm{End}_{\leq}(\mathbb{S}^1) \mapsto f_{\Omega(D)} \circ \phi \in \Omega(D)$ is well-defined, i.e. $\mathrm{PH}(f \circ \phi) = \mathrm{PH}(f) = D$. This is because a homeomorphism $\phi \in \mathrm{Aut}_{\leq}(\mathbb{S}^1)$ restricts to a homeomorphism between the sub level-sets of $f_{\Omega(D)} \circ \phi$ and those of $f_{\Omega(D)}$, hence it induces an isomorphism of persistent homology modules and the equality of barcodes $\mathrm{PH}(f_{\Omega(D)} \circ \phi) = \mathrm{PH}(f_{\Omega(D)})$, which holds as well for any $\phi \in \mathrm{End}_{\leq}(\mathbb{S}^1) = \overline{\mathrm{Aut}_{\leq}(\mathbb{S}^1)}$ by continuity of PH.

Let $f \in \Omega(D)$. From Proposition 5.3 there are cyclic permutations $\pi \in \Gamma_n$ such that $\mathrm{Val}(f) = \pi.\mathrm{Val}(f_{\Omega(D)})$. For each such permutation π there is a unique map $\phi^{f,\pi}$ satisfying both $f_{\Omega(D)} \circ \phi^{f,\pi} = f$ and $\phi^{f,\pi}(c_i(f)) = c_{\pi(i)}$ (and $\phi^{f,\pi}(d_i(f)) = d_{\pi(i)}$): It is defined on each circular arc $[c_i(f), d_i(f)]$ by

$$\phi^{f,\pi}_{|[c_i(f),d_i(f)]} := [(f_{\Omega(D)})_{|[c_{\pi(i)},d_{\pi(i)}]}]^{-1} \circ f_{|[c_i(f),d_i(f)]}, \tag{3}$$

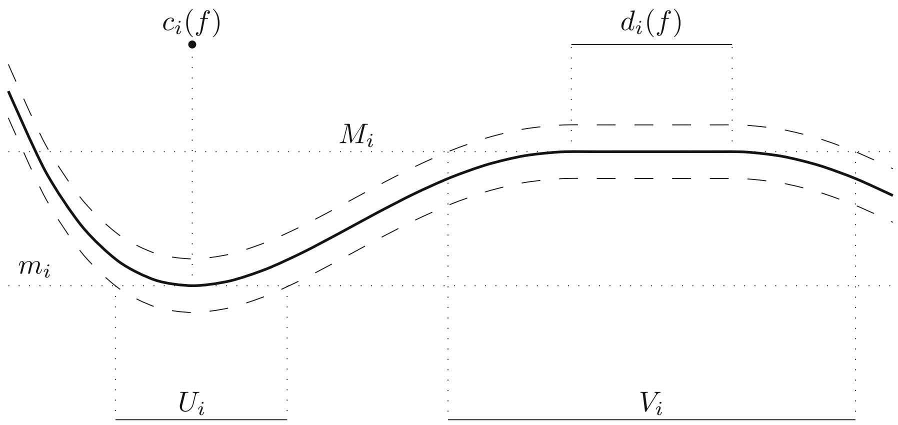

Fig. 1 A piece of a continuous function $f : \mathbb{S}^1 \to \mathbb{R}$ and a small neighborhood $\mathcal{U}$ indicated by dashed curves. Any function in $\mathrm{PH}^{-1}(D)$ between the dashed curves must have a critical value in each U_i and V_i, provided the band between the dashed curves is thin enough to separate critical values. If g is such a function then these must be the only critical values. Then the critical value of g in U_i must be m_i, in V_i must be M_i, and so on

and similarly on circular arcs $[d_{i-1}(f), c_i(f)]$. In particular for $f = f_{\Omega(D)}$ the set of such $\phi^{f,\pi}$ equals the group $\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$ of stabilisers. Therefore $\phi \mapsto f_{\Omega(D)} \circ \phi$ descends to a continuous bijection from $\mathrm{End}_{\leq}(\mathbb{S}^1)/\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$ to $\Omega(D)$.

Finally we show that the inverse is continuous. Let $f \in \Omega(D)$ and $\phi^{f,\pi}$ as in (3). Up to pre-composing f by a suitable homeomorphism the north pole does not belong to any extremal set $c_i(f), d_i(f)$. Consequently, for g in a small neighborhood $\mathcal{U} \subseteq \Omega(D)$ around f, we also have $\mathrm{Val}(g) = \pi.\mathrm{Val}(f_{\Omega(D)})$, hence we can define $\phi^{g,\pi} \in \mathrm{End}_{\leq}(\mathbb{S}^1)$ like in Eq. (3) and then $f_{\Omega(D)} \circ \phi^{g,\pi} = g$. Hence the map $g \in \mathcal{U} \longmapsto \phi^{g,\pi} \in \mathrm{End}_{\leq}(\mathbb{S}^1)$ is a local section, whose continuity is a consequence of the fact that on each circular arc $[c_i, d_i]$ the linear restriction $(f_{\Omega(D)})_{|[c_i,d_i]}$ and its inverse are Lipschitz, and of the fact that the maximal distance from points in the critical sets $c_i(g), d_i(g)$ to the critical sets $c_i(f), d_i(f)$ of f can be continuously tracked in a sufficiently small neighborhood $\mathcal{U}$ of f, see Fig. 1. The technical details are omitted. □

Proof of Proposition 5.1 From Proposition 5.4 the pre-composition map $\phi \mapsto f_{\Omega(D)} \circ \phi$ induces a homeomorphism from $\mathrm{End}_{\leq}(\mathbb{S}^1)/\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$ to the path connected component $\Omega(D)$. Besides it is well-known that $\mathrm{End}_{\leq}(\mathbb{S}^1)$ deformation retracts to the group $\mathrm{SO}(2) \cong \mathbb{S}^1$ of orientation preserving rotations.[2] Recall that $f_{\Omega(D)}$ is a piecewise linear interpolation between extremal values arranged on a regular $2n$-gon, therefore its stabiliser $\mathcal{S}_{\mathrm{Id}}(f_{\Omega(D)})$ is a finite subgroup of SO(2) which is preserved under the deformation retraction. Hence $\Omega(D)$ is homotopy equivalent to the quotient of $\mathrm{SO}(2) \cong \mathbb{S}^1$ by a finite subgroup, so it is in fact homotopy equivalent to $\mathbb{S}^1$. □

[2] For instance the deformation retract of $\mathrm{Aut}_{\leq}(\mathbb{S}^1)$ of Hamstrom et al.(1974, Theorem 1.1.2) extends to $\mathrm{End}_{\leq}(\mathbb{S}^1)$.

Acknowledgements We wish to thank Ulrike Tillmann for her various insights on this project. Funding was provided by Engineering and Physical Sciences Research Council (Grant No. EP/R018472/1).

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

Barannikov, S.: The framed Morse complex and its invariants. American mathematical society translations, Series 2, (1994)

Catanzaro, M.J., Curry, J.M., Fasy, B.T., Lazovskis, J., Malen, G., Riess, H., Wang, B., Zabka, M.: Moduli spaces of Morse functions for persistence. J Appl Comput Topol **4**(3), 353–385 (2020)

Cerf, Jean: Topologie de certains espaces de plongements. Bulletin de la Société Mathématique de France **89**, 227–380 (1961)

Cerf, Jean: La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. Inst. Hautes Études Sci. Publ. Math. **39**, 5–173 (1970). (**ISSN 0073-8301**)

Cohen-Steiner, D., Edelsbrunner, Herbert, Harer, J.: Stability of persistence diagrams. Discret. Comput. Geom. **37**(1), 103–120 (2007)

Crawley-Boevey, William: Decomposition of pointwise finite-dimensional persistence modules. J. Algeb. Appl. **14**(05), 1550066 (2015)

Curry, Justin: The fiber of the persistence map for functions on the interval. J. Appl. Comput. Topol. **2**(3–4), 301–321 (2018)

Cyranka, J., Mischaikow, K., Weibel, C.: Contractibility of a persistence map preimage. J. Appl. Comput. Topol. **4**(4), 509–523 (2020)

Edelsbrunner, H., Harer, J.: Persistent homology-a survey. Contemp. Math. **453**, 257–282 (2008)

Hamstrom, M.E., et al.: Homotopy in homeomorphism spaces, TOP and PL. Bullet. Am. Math. Soci. **80**(2), 207–230 (1974)

Hong, S., Kalliongis, J., McCullough, D., Rubinstein, J.H.: Diffeomorphisms of elliptic 3-manifolds. Springer, Berlin (2012)

Leygonie, J., Henselman-Petrusek, G.: Algorithmic reconstruction of the fiber of persistent homology on cell complexes. arXiv preprint arXiv:2110.14676, (2021)

Leygonie, J., Tillmann, U.: The fiber of persistent homology for simplicial complexes. J. Pure Appl. Algebra **226**(12), 107099 (2022)

Jacob Leygonie, J., Oudot, S., Tillmann, U.: A framework for differential calculus on persistence barcodes. Foundat. Comput. Math. **22**(4), 1069–1131 (2021)

Maksymenko, S.: Homotopy types of stabilizers and orbits of Morse functions on surfaces. Ann. Glob. Anal. Geom. **29**(3), 241–285 (2006)

Mather, J.N.: Stability of C^∞ mappings II Infinitesimal stability implies stability. Ann. Math. (1969). https://doi.org/10.2307/1970668

Mischaikow, K., Weibel, C.: Persistent homology with non-contractible preimages. arXiv preprint arXiv:2105.08130, (2021)
Palais, R.S.: Local triviality of the restriction map for embeddings. Commentarii Mathematici Helvetici **34**(1), 305–312 (1960)
Smale, S.: Diffeomorphisms of the 2-sphere. Proceed. Am. Math. Soc. **10**(4), 621–626 (1959)
Zomorodian, A., Carlsson, G.: Computing persistent homology. Discret. Computat. Geomet. **33**(2), 249–274 (2005)