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ORIGINAL PAPER

Multicriteria decision-making based on distance measures and knowledge measures of Fermatean fuzzy sets

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Abstract

Fermatean fuzzy sets are more powerful than fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets in handling various problems involving uncertainty. The distance measures in the fuzzy and non-standard fuzzy frameworks have got their applicability in various areas such as pattern analysis, clustering, medical diagnosis, etc. Also, the fuzzy and non-standard fuzzy knowledge measures have played a vital role in computing the criteria weights in the multicriteria decision-making problems. As there is no study concerning the distance and knowledge measures of Fermatean fuzzy sets, so in this paper, we propose some novel distance measures for Fermatean fuzzy sets using t-conorms. We also discuss their various desirable properties. With the help of suggested distance measures, we introduce some knowledge measures for Fermatean fuzzy sets. Through numerical comparison and linguistic hedges, we establish the effectiveness of the suggested distance measures and knowledge measures, respectively, over the existing measures in the Pythagorean/Fermatean fuzzy setting. At last, we demonstrate the application of the suggested measures in pattern analyis and multicriteria decision-making.

Keywords Pythagorean fuzzy set · Fermatean fuzzy set · t-conorm · Knowledge measure · Pattern recognition · Multicriteria decision-making

1 Introduction

The concept of the fuzzy set (FS) theory was put forward by Zadeh (1965) for handling imprecise and vague information. In an FS, each element is assigned a membership value lying between 0 and 1, indicating its degree of belongingness to the set. Fuzzy sets (FSs) have been applied in many fields such as pattern recognition, medical diagnosis, clustering, etc. Since in an FS, the non-membership value of an element cannot be chosen independently, so Atanassov (1986) introduced the concept of intuitionistic fuzzy sets (IFSs). In an intuitionistic fuzzy set (IFS), each element has a membership value and a non-membership value lying in the interval [0, 1] with their sum less or equal to one. This restriction on the sum of membership values limits the scope of IFSs and so the concept of Pythagorean fuzzy set (PFS) was proposed by Yager (2013) as an extension of IFSs (Atanassov 1986) (IFSs) and FSs (Zadeh 1965) for solving the problems involving uncertainty more precisely. Each element of a PFS has a membership grade (μ) and a non-membership grade (ϑ) with their square sum at most one ($\mu^2 + \vartheta^2 \leq 1$). The technique for order of preference by similarity to ideal solution (TOPSIS) in the Pythagorean fuzzy (PF) setting and the concept of the Pythagorean fuzzy number were suggested by Zhang and Xu (2014). Various PF aggregation functions with their utility in decision-making were given by Yager (2014). Wei and Lu (2018) introduced some power aggregation functions for PFSs. Using Einstein operations, Garg (2016) proposed some new aggregation functions in the PF environment. Wei (2017) suggested some PF interaction aggregation functions with their utility in multicriteria decision-making (MCDM). Many studies (Garg 2017; Lu et al. 2017; Wei et al. 2017; Wei and Lu 2017) concerning the PF aggregation functions with their various applications are available in the literature. The TODIM (an acronym in Portuguese for Interactive and Multicriteria Decision Making) method for PFSs was introduced by Ren et al (2016). Peng et al. (2017) proposed

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some information measures for PFSs. A novel PF distance measure was proposed by Peng and Dai (2017). Some PF measures of correlation with their utility were proposed by Singh and Ganie (2020). Various researchers (Garg 2019a, b; Rahman and Abdullah 2019; Khan et al. 2019a, b; Akram and Ali 2020; Ejegwa 2020a, b; Rahman et al. 2020, 2021; Talukdar and Dutta 2021; Rahman 2021; Akram and Shahzadi 2021; Biswas and Deb 2021; Verma and Agarwal 2021; Touqeer et al. 2021) have studied PFSs and applied them in distinct uncertain situations. Though PFSs have a lot of applications in various fields but they are unable to handle situations, where $\mu^2 + \vartheta^2 \geq 1$, e.g., if $\mu = 0.8$ and $\vartheta = 0.7$, then $\mu^2 + \vartheta^2 = 1.13 > 1$. So, Senapati and Yager (2020) proposed the concept of fermatean fuzzy sets (FFSs). In a fermatean fuzzy set (FFS), we have $\mu^3 + \vartheta^3 \leq 1$. This means that FFSs are more powerful than FSs, IFSs, and PFSs because they all are contained in the space of FFSs. Some FFS aggregation operators with their applicability in decision-making were given by Senapati and Yager (2019). The weighted aggregated sum product assessment (WASPAS) method in the fermatean fuzzy (FF) environment was suggested by Mishra and Rani (2021). A novel FF decision-making approach was given by Ghorabaee et al. (2020). The application of FF aggregation functions in the COVID-19 testing facility was shown by Garg et al. (2020). Yang et al. (2020) studied the continuities and derivatives of FF functions. Some FF capital budgeting techniques were proposed by Sergi and Sari (2021). Sahoo (2021a, b) proposed some score functions for FFSs and their utility in transportation problems and decision-making. The TOPSIS (technique for order of preference by similarity to ideal solution) method for FFSs was proposed by Aydemir and Yilmaz (2020). Akram et al. (2020) suggested some of Einstein's norm operations-based aggregation functions for FFSs. The concept of fermatean fuzzy soft sets (FFSSs) with their applicability in the selection of an antivirus mask was given by Shahzadi and Akram (2021). The TOPSIS technique for fermatean fuzzy soft sets (FFSSs) was suggested by Salsabeela and John (2021). To solve the sustainable third-party reverse logistics providers (S3PRLPs) selection problem a hybrid methodology based on FFSs was proposed by Mishra et al. (2021). Shahzadi et al. (2021a) introduced some FF Hamacher aggregation functions with their utility in multicriteria group decision-making. Aydin (2021) proposed some cosine FF similarity measures with their use in decision-making. The applicability of the FFSs for occupational risk assesment in manufacturing was shown by Gul et al. (2021). Some FF Hamacher arithematic and geometric aggregation operators and their application in MCDM was established by Hadi et al. (2021). Shit and Ghorai (2021) suggested some Dombi aggregation functions based on FFSs and applied them in decision-making. A novel divergence measure and multi-objective optimization based on the ratio analysis with the full multiplicative form (MULTIMOORA) method in FF setting were proposed by Rani and Mishra (2021). Some Hamacher interactive geometric FF aggregation functions and their usability in medical field were given by Shahzadi et al. (2021b). The concept of fermatean fuzzy bipolar soft sets with their utility in MCDM was given by Ali and Ansari (2021). The current study is related to the development of some novel FF distance measures and knowledge measures.

Distance measures are very powerful in comparing two objects based on their inequality content. Application of some PF measures of distance and similarity in MADM was shown by Zeng et al (2018). Hussain and Yang (2019) proposed some Hausdorff metric-based PF measures of distance and similarity with their applicability in PF TOPSIS. Some generalized measures of distance and their continuous versions for PFSs were given by Li and Lu (2019). They also proposed set-theoretic-based, matching function-based, and complement-based PF similarity measures. Based on the membership grades, Ejegwa (2020a) proposed some distance and similarity measures for PFSs. Some cosine function-based PF similarity measures were suggested by Wei and Wei (2018). Twelve PF measures of distance and similarity with their applicability were given by Peng et al (2017). For PFSs Zhang (2016) introduced a measure of similarity and its utility in decision-making. Some novel measures of similarity and distance for PFSs based on L_p norm and level of uncertainty were given by Peng (2019). By combining the Euclidean distance measure and cosine similarity measures, Mohd and Abdullah (2018) developed some novel PF similarity measures. Zhang et al. (2019) proposed some exponential PF similarity measures and demonstrated their application in MADM, pattern analysis, and medical diagnosis. Some PF Dice similarity measures with application in decision-making were given by Wang et al (2019). Verma and Merigo (2019) developed some trigonometric function-based PF measures of similarity. The application of some multiparametric PF measures of similarity in classification problems was demonstrated by Peng and Garg (2019). Some novel PF similarity measures based on exponential function with their application in classification problems were given by Nguyen et al (2019).

The entropy of a PFS is the ambiguity content present in it. Entropy measure is very essential for computing the weight of attributes in an MADM problem involving PF data. Xue et al. (2018) introduced the axiomatic definition of PF entropy measure and used the PF entropy measure in decision-making. Some probabilistic and non-probabilistic

PF entropy measures were given by Yang and Hussain (2018). With the help of a new PF entropy measure, Thao and Smarandache (2019) introduced the CORPAS MADM method in the PF environment. Mishra and Rani (2021) introduced five FF entropy measures.

Knowledge of an FS is the amount of precision present in it. Knowledge measure (KM) plays a great role in determining the weight of attributes in an MADM problem involving fuzzy data. Singh et al. (2019) introduced the axiomatic definition of a fuzzy knowledge measure (FKM) and used it in decision-making. They also proposed a fuzzy accuracy measure and utilized it in image processing. Later on, Singh et al. (2020b) also introduced a one-parametric generalization of the FKM and discussed its various applications. For IFSs, there are various studies (Szmidt et al. 2014; Nguyen 2015; Guo 2016; Lalotra and Singh 2018; Das et al. 2018; Guo and Xu 2019; Farhadinia 2020) regarding the knowledge measures (KMs) with their practical applications. Lin et al. (2020) proposed a knowledge measure (KM) for picture fuzzy sets with its utility in decision-making. Some PF KMs with their various applications were introduced by Singh et al (2020a). For hesitant fuzzy sets, Singh and Ganie (2021a) introduced a generalized KM.

The following are the primary factors that prompted us to conduct this research.

- The distance measures for fuzzy and non-standard fuzzy sets have great applicability in many areas such as pattern analysis, decision-making, clustering, etc. But there is no study concerning the distance measures of FFSs available in the literature.
- Most of the fuzzy and non-standard fuzzy distance measures have been proposed at the formula level and fail to satisfy the axiomatic requirements. So, there is not a general method of obtaining the distance measures.
- The fuzzy and non-standard fuzzy knowledge measures are used for computing the attribute weights in MCDM problems. However, the knowledge measures for FFSs have not been proposed.

The main contributions of this paper are as:

- We suggest a novel method of constructing the FF distance measures from t-conorms and propose four FF distance measures with their various properties.
- We propose four weighted FF distance measures.
- We suggest a general method of constructing the FF knowledge measures from the proposed FF distance measures and introduce four FF knowledge measures.
- We compare the suggested FF measures of distance and knowledge with the available PF/FF measures of compatibility.
- We demonstrate the applicability of the suggested measures in pattern analysis and MADM.

The paper is organized as: Sect. 2 is preliminary. Some novel FF distance measures along with desirable properties are given in Sect. 3. Section 4 is devoted to the introduction of some distance-based FF knowledge measures. The comparison of the suggested FF distance measures and knowledge measures with the available PF/FF measures of compatibility is shown in Sect. 5. Section 6 demonstrates the applicability of the suggested distance measures and knowledge measures in pattern analysis and MADM. At last, the conclusion and future study are given in Sect. 7.

2 Preliminaries

Let $W = \{m_1, m_2, \ldots, m_l\}$ be the universe of discourse and $FFS(W)$ be the set of all FFSs of W.

Definition 1 (Yager 2013) A PFS M_1. in W is given by $M_1 = \{(m_j, \mu_{M_1}(m_j), \vartheta_{M_1}(m_j)) | m_j \in W\}$ with $\mu_{M_1}(m_j)$ and $\vartheta_{M_1}(m_j)$ representing, respectively, the membership and non-membership grades of the element m_j in M_1 such that $0 \leq \mu_{M_1}(m_j), \vartheta_{M_1}(m_j) \leq 1$ and $0 \leq \mu^2_{M_1}(m_j) + \vartheta^2_{M_1}(m_j) \leq 1$. Also, $\pi_{M_1}(m_j) = \sqrt{1 - \mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)}$ is the hesitancy grade of the element m_j in M_1.

Definition 2 (Senapati and Yager 2020) A FFS M_1. in W is given by $M_1 = \{(m_j, \mu_{M_1}(m_j), \vartheta_{M_1}(m_j)) | m_j \in W\}$ with $\mu_{M_1}(m_j)$ and $\vartheta_{M_1}(m_j)$ representing, respectively, the membership and non-membership grades of the element m_j in M_1 such that $0 \leq \mu_{M_1}(m_j), \vartheta_{M_1}(m_j) \leq 1$ and $0 \leq \mu^3_{M_1}(m_j) + \vartheta^3_{M_1}(m_j) \leq 1$. Also, $\pi_{M_1}(m_j) = \sqrt[3]{1 - \mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j)}$ is the hesitancy grade of the element m_j in M_1.

Definition 3 (Senapati and Yager 2020) For two FFSs M_1 and M_2 in W, some operations are given as:

1. $M_1 \cup M_2 = \{(m_j, max(\mu_{M_1}(m_j), \mu_{M_2}(m_j)), min(\vartheta_{M_1}(m_j), \vartheta_{M_2}(m_j))) | m_j \in W\}$.
2. $M_1 \cap M_2 = \{(m_j, min(\mu_{M_1}(m_j), \mu_{M_2}(m_j)), max(\vartheta_{M_1}(m_j), \vartheta_{M_2}(m_j))) | m_j \in W\}$.
3. $M_1 \subseteq M_2$ iff $\mu_{M_1}(m_j) \leq \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \geq \vartheta_{M_2}(m_j)$ $\forall m_j \in W$.
4. $(M_1)^c = \{(m_j, \vartheta_{M_1}(m_j), \mu_{M_1}(m_j)) | m_j \in W\}$.

Definition 4 (Peng et al. 2017) A function $S : PFS(W) \times PFS(W) \to [0, 1]$ is called a PF similarity measure if $\forall M_1, M_2$, and $M_3 \in PFS(W)$, we have:

(S1) $0 \le S(M_1, M_2) \le 1$;
(S2) $S(M_1, M_2) = S(M_2, M_1)$;
(S3) $S(M_1, M_2) = 1$ iff $M_1 = M_2$;
(S4) $S(M_1, (M_1)^c) = 0$ iff M_1 is a crisp set;
(S5) If $M_1 \subseteq M_2 \subseteq M_3$, then $S(M_1, M_2) \ge S(M_1, M_3)$ and $S(M_2, M_3) \ge S(M_1, M_3)$.

Definition 5 (Peng et al. 2017) A function $D : PFS(W) \times PFS(W) \to [0, 1]$ is called a PF distance measure if $\forall M_1, M_2$, and $M_3 \in PFS(W)$, we have:
(D1) $0 \le D(M_1, M_2) \le 1$;
(D2) $D(M_1, M_2) = D(M_2, M_1)$;
(D3) $D(M_1, M_2) = 0$ iff $M_1 = M_2$;
(D4) $D(M_1, (M_1)^c) = 1$ iff M_1 is a crisp set;
(D5) If $M_1 \subseteq M_2 \subseteq M_3$, then $D(M_1, M_2) \le D(M_1, M_3)$ and $D(M_2, M_3) \le D(M_1, M_3)$.

Definition 6 (Mishra and Rani 2021) A function $E : FFS(W) \to [0, 1]$ is called a FF entropy measure if $\forall M_1$ and $M_2 \in FFS(W)$, we have:
(E1) $0 \le E(M_1) \le 1$;
(E2) $E(M_1) = 0$ iff M_1 is a crisp set;
(E3) $E(M_1) = 1$ iff $\mu_{M_1}(m_j) = \vartheta_{M_1}(m_j) \forall m_j \in W$;
(E4) $E(M_1) = E((M_1)^c)$;
(E5) $E(M_1) \le E(M_2)$ if $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j) \le \vartheta_{M_2}(m_j) \le \vartheta_{M_1}(m_j)$ or $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j) \ge \vartheta_{M_2}(m_j) \ge \vartheta_{M_1}(m_j) \forall m_j \in W$.

Definition 7 (Singh et al. 2020a) A function $K : PFS(W) \to [0, 1]$ is called a PF knowledge measure if $\forall M_1$ and $M_2 \in PFS(W)$, we have:
(K1) $0 \le K(M_1) \le 1$;
(K2) $K(M_1) = 1$ iff M_1 is a crisp set;
(K3) $K(M_1) = 0$ iff $\mu_{M_1}(m_j) = \vartheta_{M_1}(m_j) \forall m_j \in W$;
(K4) $K(M_1) = K((M_1)^c)$;
(K5) $K(M_1) \ge K(M_2)$ if $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j) \le \vartheta_{M_2}(m_j) \le \vartheta_{M_1}(m_j)$ or $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j) \ge \vartheta_{M_2}(m_j) \ge \vartheta_{M_1}(m_j) \forall m_j \in W$.

Definition 8 (Weber 1983) A function $g : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-conorm if $\forall x, y, z, t \in [0, 1]$

$g(x, y) = g(y, x)$;
$g(x, y) \le g(z, t)$, whenever $x \le z$ and $y \le t$;
$g(x, 0) = x$;
$g(x, g(y, z)) = g(g(x, y), z)$.

In the next section, we introduce some novel distance measures for FFSs along with their properties.

3 New measures of distance for FFSs

Here, we propose some FF measures of distance. First, we define a distance measure in the FF environment.

Definition 9 Let $M_1, M_2 \in FFS(W)$, then the function $D_G : FFS(W) \times FFS(W) \to \mathbb{R}$ is called a FF distance measure if:

1. $0 \le D(M_1, M_2) \le 1$;
2. $D(M_1, M_2) = D(M_2, M_1)$;
3. $D(M_1, M_2) = 0$ iff $M_1 = M_2$;
4. $D(M_1, (M_1)^c) = 1$ iff M_1 is a crisp set;
5. If $M_1 \subseteq M_2 \subseteq M_3$, then $D(M_1, M_2) \le D(M_1, M_3)$ and $D(M_2, M_3) \le D(M_1, M_3)$.

Now, we introduce a novel method of generating the FF distance measures from t-conorms.

Definition 10 Let $M_1, M_2 \in FFS(W)$, then we define a function.

$D_G : FFS(W) \times FFS(W) \to \mathbb{R}$

given by.

$$D_G(M_1, M_2) = \frac{1}{l} \sum_{j=1}^{l} g\left(\left| \mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j) \right|, \left| \vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j) \right| \right) \tag{1}$$

where g is a t-conorm.

Theorem 1 *The function* D_G *given in Eq.* (1) is a valid FF distance measure.

Proof To prove that D_G is a FF distance measure, we show that it satisfies the properties given in Definition 9.

(D1) Clearly $0 \le D_G(M_1, M_2) \le 1$.

(D2) $D_G(M_1, M_2) = D_G(M_2, M_1)$ is obvious.

(D3) $D_G(M_1, M_2) = 0 \Leftrightarrow g\left(\left| \mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j) \right|, \left| \vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j) \right| \right) = 0 \forall j, \Leftrightarrow \left| \mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j) \right| = 0$ and $\left| \vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j) \right| = 0 \forall j, \Leftrightarrow \mu^3_{M_1}(m_j) = \mu^3_{M_2}(m_j)$ and $\vartheta^3_{M_1}(m_j) = \vartheta^3_{M_2}(m_j) \forall j, \Leftrightarrow M_1 = M_2$,

(D4) $D_G(M_1, M_1^c) = 1 \Leftrightarrow g\left(\left| \mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j) \right|, \left| \vartheta^3_{M_1}(m_j) - \mu^3_{M_1}(m_j) \right| \right) = 1 \forall j, \Leftrightarrow \left| \mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j) \right| = 1$ and $\left| \vartheta^3_{M_1}(m_j) - \mu^3_{M_1}(m_j) \right| = 1 \forall j, \Leftrightarrow \left| \mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j) \right| = 1 \forall j \Leftrightarrow \mu^3_{M_1}(m_j) = 1$ and $\vartheta^3_{M_1}(m_j) = 0$ or $\mu^3_{M_1}(m_j) = 0$ and $\vartheta^3_{M_1}(m_j) = 1 \forall j, \Leftrightarrow \mu_{M_1}(m_j) = 1$ and $\vartheta_{M_1}(m_j) = 0$ or $\mu_{M_1}(m_j) = 0$ and $\vartheta_{M_1}(m_j) = 1 \forall j, \Leftrightarrow M_1$ is a crisp set.

(D5) Let $M_1 \subseteq M_2 \subseteq M_3$, then $\mu^3_{M_1}(m_j) \le \mu^3_{M_2}(m_j) \le \mu^3_{M_3}(m_j)$ and $\vartheta^3_{M_1}(m_j) \ge \vartheta^3_{M_2}(m_j) \ge \vartheta^3_{M_3}(m_j) \forall j$. Therefore, we get

$$|\mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)| \le |\mu^3_{M_1}(m_j) - \mu^3_{M_3}(m_j)|,$$
$$|\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)| \le |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_3}(m_j)|$$

and

$$|\mu^3_{M_2}(m_j) - \mu^3_{M_3}(m_j)| \le |\mu^3_{M_1}(m_j) - \mu^3_{M_3}(m_j)|,$$
$$|\vartheta^3_{M_2}(m_j) - \vartheta^3_{M_3}(m_j)| \le |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_3}(m_j)|.$$

So,

$$g\left(|\mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)|, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|\right) \le g\left(|\mu^3_{M_1}(m_j) - \mu^3_{M_3}(m_j)|, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_3}(m_j)|\right)$$

And

$$g\left(|\mu^3_{M_2}(m_j) - \mu^3_{M_3}(m_j)|, |\vartheta^3_{M_2}(m_j) - \vartheta^3_{M_3}(m_j)|\right) \le g\left(|\mu^3_{M_1}(m_j) - \mu^3_{M_3}(m_j)|, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_3}(m_j)|\right).$$

Thus, $D_G(M_1, M_2) \le D_G(M_1, M_3)$ and $D_G(M_2, M_3) \le D_G(M_1, M_3)$.

Hence, D_G is a valid FF distance measure.

Theorem 2 *The FF distance measure* D_G *given in Eq.* (1) *has the following properties*:

1. $D_G(M_1^c, M_2^c) = D_G(M_1, M_2) \forall M_1, M_2 \in FFS(W)$,
2. $D_G(M_1, M_2^c) = D_G(M_1^c, M_2) \forall M_1, M_2 \in FFS(W)$,
3. $D_G(M_1, M_1^c) = 0$ *if and only if* $\mu_{M_1}(m_j) = \vartheta_{M_1}(m_j), \forall j$,
4. $D_G(M_1 \cap M_2, M_2) \le D_G(M_1, M_2)$ *for every* $M_1, M_2 \in FFS(W)$,
5. $D_G(M_1 \cup M_2, M_2) \le D_G(M_1, M_2)$ *for every* $M_1, M_2 \in FFS(W)$.

Proof 1.

$$D_G(M_1^c, M_2^c) = \frac{1}{l}\sum_{j=1}^{l} g\left(|\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|, |\mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)|\right)$$
$$= \frac{1}{l}\sum_{j=1}^{l} g\left(|\mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)|, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|\right)$$
$$= D_G(M_1, M_2)$$

2. $D_G(M_1, M_2^c) = \frac{1}{l}\sum_{j=1}^{l} g\left(|\mu^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|, |\vartheta^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)|\right) = \frac{1}{l}\sum_{j=1}^{l} g\left(|\vartheta^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)|, |\mu^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|\right) = D_G(M_1^c, M_2)$.

3. $D_G(M_1, M_1^c) = 0 \Leftrightarrow \frac{1}{l}\sum_{j=1}^{l} g\left(|\mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j)|, |\vartheta^3_{M_1}(m_j) - \mu^3_{M_1}(m_j)|\right) = 0, \Leftrightarrow g\left(|\mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j)|, |\vartheta^3_{M_1}(m_j) - \mu^3_{M_1}(m_j)|\right) = 0, \forall j, \Leftrightarrow |\mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j)| = 0$ and $|\vartheta^3_{M_1}(m_j) - \mu^3_{M_1}(m_j)| = 0 \forall j, \Leftrightarrow |\mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j)| = 0, \forall j, \Leftrightarrow \mu^3_{M_1}(m_j) = \vartheta^3_{M_1}(m_j), \forall j, \Leftrightarrow \mu_{M_1}(m_j) = \vartheta_{M_1}(m_j), \forall j.$,

4. $D_G(M_1 \cap M_2, M_2) = \frac{1}{l}\sum_{j=1}^{l} g(|\min\left(\mu^3_{M_1}(m_j), \mu^3_{M_2}(m_j)\right) - \mu^3_{M_2}(m_j)|, \left|\max\left(\vartheta^3_{M_1}(m_j), \vartheta^3_{M_2}(m_j)\right) - \vartheta^3_{M_2}(m_j)\right|)$.

We have the following cases:

(a) When $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \ge \vartheta_{M_2}(m_j) \forall j$, then

$$\begin{aligned} D_G(M_1 \cap M_2, M_2) &= \frac{1}{l}\sum_{j=1}^{l} g\left(|\mu^3_{M_2}(m_j) - \mu^3_{M_2}(m_j)|, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|\right), \\ &= \frac{1}{l}\sum_{j=1}^{l} g\left(0, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|\right), \\ &\le \frac{1}{l}\sum_{j=1}^{l} g\left(|\mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)|, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|\right), \\ &= D_G(M_1, M_2). \end{aligned}$$

(b) When $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \le \vartheta_{M_2}(m_j) \forall j$, then

$$\begin{aligned} &D_G(M_1 \cap M_2, M_2) \\ &= \frac{1}{l}\sum_{j=1}^{l} g\left(|\mu^3_{M_2}(m_j) - \mu^3_{M_2}(m_j)|, |\vartheta^3_{M_2}(m_j) - \vartheta^3_{M_2}(m_j)|\right), \\ &= \frac{1}{l}\sum_{j=1}^{l} g(0,0) = 0 \le D_G(M_1, M_2). \end{aligned}$$

,

(c) When $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \ge \vartheta_{M_2}(m_j) \forall j$, then.

$$\begin{aligned} &D_G(M_1 \cap M_2, M_2) \\ &= \frac{1}{l}\sum_{j=1}^{l} g\left(|\mu^3_{M_1}(m_j) - \mu^3_{M_2}(m_j)|, |\vartheta^3_{M_1}(m_j) - \vartheta^3_{M_2}(m_j)|\right), \\ &= D_G(M_1, M_2). \end{aligned}$$

(d) When $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \le \vartheta_{M_2}(m_j) \forall j$, then.

$$D_G(M_1 \cap M_2, M_2) = \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_2}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right),$$
$$= \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, 0\right),$$
$$\le \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right),$$
$$= D_G(M_1, M_2).$$

,

5. $D_G(M_1 \cup M_2, M_2) = \frac{1}{l} \sum_{j=1}^{l} g\left(\left|\max\left(\mu_{M_1}^3(m_j), \mu_{M_2}^3(m_j)\right) - \mu_{M_2}^3(m_j)\right|, \left|\min\left(\vartheta_{M_1}^3(m_j), \vartheta_{M_2}^3(m_j)\right) - \vartheta_{M_2}^3(m_j)\right|\right)$.

We have the following cases:

(a) When $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \ge \vartheta_{M_2}(m_j) \forall j$, then

$D_G(M_1 \cup M_2, M_2) = \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_2}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right) = \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, 0\right) \le \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right) = D_G(M_1, M_2)$,

(b) When $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \le \vartheta_{M_2}(m_j) \forall j$, then

$D_G(M_1 \cup M_2, M_2)$
$= \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right), = D_G(M_1, M_2).$,

(c) When $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \ge \vartheta_{M_2}(m_j) \forall j$, then

$D_G(M_1 \cup M_2, M_2) = \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_2}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_2}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right), = \frac{1}{l}\sum_{j=1}^{l} g(0,0) = 0 \le D_G(M_1, M_2).$,

(d) When $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j)$ and $\vartheta_{M_1}(m_j) \le \vartheta_{M_2}(m_j) \forall j$, then

$D_G(M_1 \cup M_2, M_2) = \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_2}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right), = \frac{1}{l}\sum_{j=1}^{l} g\left(0, \left|\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right), \le \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right), = D_G(M_1, M_2).$

Example 1 Some examples of FF distance measures are given in Table 1.

In most of the decision-making problems, the weights w_j of the elements $m_j, j = 1, 2, \ldots, l$ are taken into consideration, so we introduce the weighted FF distance measures.

Definition 11 Let M_1, $M_2 \in FFS(W)$, then we define a function.

$D_G^W : FFS(W) \times FFS(W) \to \mathbb{R}$

given by.

$D_G^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j g\left(\left|\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\right|, \left|\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|\right)$, (2).

where g is a t-conorm.

Theorem 3 *The function* D_G^W *given in Eq.* (2) is a valid FF distance measure.

Proof Similar to Theorem 1.

Example 2 Some examples of weighted FF distance measures are given in Table 2.

Next, we propose some novel FF measures of knowledge based on the proposed FF distance measures.

4 FF distance-based knowledge measures

The entropy measures are used to compute the amount of ambiguity present in an FFS, whereas the knowledge measures acting as the soft duals of entropy measures are used to calculate the amount of precision in an FFS. Here, we introduce a method of constructing FF knowledge measures from the FF distance measures. First, we define the knowledge measure for FFSs.

Definition 12 A function $K : FFS(W) \to [0, 1]$ is called a FF knowledge measure if $\forall M_1$ and $M_2 \in FFS(W)$, we have:

(K1) $0 \le K(M_1) \le 1$;
(K2) $K(M_1) = 1$ iff M_1 is a crisp set;
(K3) $K(M_1) = 0$ iff $\mu_{M_1}(m_j) = \vartheta_{M_1}(m_j) \forall m_j \in W$;
(K4) $K(M_1) = K((M_1)^c)$;
(K5) $K(M_1) \ge K(M_2)$ if $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j) \le \vartheta_{M_2}(m_j) \le \vartheta_{M_1}(m_j)$ or $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j) \ge \vartheta_{M_2}(m_j) \ge \vartheta_{M_1}(m_j)$ $\forall m_j \in W$.

Now, we give a method of generating the FF knowledge measures from the FF distance measures.

Definition 13 Let $M_1 \in FFS(W)$, then we define a function.

$K_G : FFS(W) \to [0, 1]$

given by

$$K_G(M_1) = 1 - D_G\left(M_1, M_1^c\right) \tag{3}$$

where D_G is a FF distance measure.

Theorem 4 *The function* K_G *defined in Eq. (3) is a valid FF knowledge measure.*

Table 1 Examples of some t-conorm-based FF distance measures

t-conorms	Corresponding FF distance measures
$g(m_1, m_2) = \frac{m_1+m_2-2m_1m_2}{1-m_1m_2}$(Mizumoto 1989)	$D_{G1}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l}\left[\frac{\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert - 2\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}{1 - \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}\right]$
$g(m_1, m_2) = m_1 + m_2 - m_1m_2$(Robert 1995)	$D_{G2}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l}\left[\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert - \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert\right]$
$g(m_1, m_2) = min(1, m_1 + m_2)$(Robert 1995)	$D_{G3}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} min\left(1, \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert\right)$
$g(m_1, m_2) = \frac{m_1+m_2}{1+m_1m_2}$(Mizumoto 1989)	$D_{G4}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l}\left[\frac{\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}{1 + \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}\right]$

Table 2 Weighted distance measures for FFSs

t-conorms	Corresponding weighted FF distance measures
$g(m_1, m_2) = \frac{m_1+m_2-2m_1m_2}{1-m_1m_2}$(Mizumoto 1989)	$D_{G1}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j\left[\frac{\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert - 2\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}{1 - \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}\right]$
$g(m_1, m_2) = m_1 + m_2 - m_1m_2$(Robert 1995)	$D_{G2}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j\left[\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert - \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert\right]$
$g(m_1, m_2) = min(1, m_1 + m_2)$(Robert 1995)	$D_{G3}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j min\left(1, \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert\right)$
$g(m_1, m_2) = \frac{m_1+m_2}{1+m_1m_2}$(Mizumoto 1989)	$D_{G4}^W(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} w_j\left[\frac{\lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert + \lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}{1 + \lvert\mu_{M_1}^3(m_j) - \mu_{M_2}^3(m_j)\rvert\lvert\vartheta_{M_1}^3(m_j) - \vartheta_{M_2}^3(m_j)\rvert}\right]$

Proof To show that the function K_G is a FF measure of knowledge, we show it has the properties of a FF measure of knowledge given in Definition 12.

(K1) Clearly $0 \le K_G(M_1) \le 1$ as $0 \le D_G(M_1, M_1^c) \le 1$.

(K2) $K_G(M_1) = 1 \Longleftrightarrow D_G(M_1, M_1^c) = 0 \Longleftrightarrow M_1$ is a crisp set.

E3) $K_G(M_1) = 0 \Longleftrightarrow D_G(M_1, M_1^c) = 1 \Longleftrightarrow \mu_{M_1}(m_j) = \vartheta_{M_1}(m_j), \forall j$.

(E4) $K_G(M_1^c) = K_G(M_1)$ is obvious.

(E5) Let M_1 be less fuzzy than M_2 i.e., $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j) \le \vartheta_{M_2}(m_j) \le \vartheta_{M_1}(m_j)$ or $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j) \ge \vartheta_{M_2}(m_j) \ge \vartheta_{M_1}(m_j)$.

When $\mu_{M_1}(m_j) \le \mu_{M_2}(m_j) \le \vartheta_{M_2}(m_j) \le \vartheta_{M_1}(m_j)$, then we get

$\left|\mu_{M_1}^3(m_j) - \vartheta_{M_1}^3(m_j)\right| \ge \left|\mu_{M_2}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|$.

$$D_{PYY1}(M_1, M_2) = \frac{1}{2l}\sum_{j=1}^{l}\left(\left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right| + \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| + \left|\pi_{M_1}^2(m_j) - \pi_{M_2}^2(m_j)\right|\right)$$

$$D_{PYY2}(M_1, M_2) = \frac{1}{2l}\sum_{j=1}^{l}\left(\left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j) - \left(\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right)\right|\right)$$

$$D_{PYY3}(M_1, M_2) = \frac{1}{4l}\left\{\begin{array}{c}\sum_{j=1}^{l}\left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right| + \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right| + \left|\pi_{M_1}^2(m_j) - \pi_{M_2}^2(m_j)\right| \\ + \sum_{j=1}^{l}\left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j) - \left(\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right)\right|\end{array}\right\}$$

$$D_{PYY4}(M_1, M_2) = \frac{1}{l}\sum_{j=1}^{l} max\left(\left|\mu_{M_1}^2(m_j) - \mu_{M_2}^2(m_j)\right|, \left|\vartheta_{M_1}^2(m_j) - \vartheta_{M_2}^2(m_j)\right|\right)$$

So, $D_G(M_1, M_1^c) = \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_1}^3(m_j) - \vartheta_{M_1}^3(m_j)\right|, \left|\vartheta_{M_1}^3(m_j) - \mu_{M_1}^3(m_j)\right|\right), \ge \frac{1}{l}\sum_{j=1}^{l} g\left(\left|\mu_{M_2}^3(m_j) - \vartheta_{M_2}^3(m_j)\right|, \left|\vartheta_{M_2}^3(m_j) - \mu_{M_2}^3(m_j)\right|\right), = D_G(M_2, M_2^c)$.

Thus, $K_G(M_1) \ge K_G(M_2)$.

Similarly, when $\mu_{M_1}(m_j) \ge \mu_{M_2}(m_j) \ge \vartheta_{M_2}(m_j) \ge \vartheta_{M_1}(m_j)$, we get $K_G(M_1) \ge K_G(M_2)$. Hence the function K_G given in Eq. (3) is a valid FF knowledge measure.

With the help of Eq. (3) and based on the suggested FF measures of distance, some FF measures of knowledge are given in Table 3 below.

Now, we compare the suggested FF measures of distance and knowledge with some available PF/FF measures of information.

5 Comparative analysis

In this section, we show that our suggested FF measures of distance and knowledge give better results than most of the available PF/FF measures of information.

5.1 Comparison of the proposed FF distance measures with various available PF measures of similarity/distance

To contrast the performance of the suggested FF measures of distance, we first list the PF measures of similarity/distance available in the literature.

Distance measures (Peng et al. 2017):

$$D_{PYY5}(M_1,M_2)=\frac{2}{l}\sum_{j=1}^{l}\frac{max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}{1+max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}$$

$$D_{PYY6}(M_1,M_2)=\frac{2\sum_{j=1}^{l}max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}{\sum_{j=1}^{l}1+max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}$$

$$D_{PYY7}(M_1,M_2)=1-x\frac{\sum_{j=1}^{l}\min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}-y\frac{\sum_{j=1}^{l}\min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)},$$
$$x+y=1,x,y\in[0,1];$$

$$D_{PYY8}(M_1,M_2)=1-\frac{x}{l}\frac{\sum_{j=1}^{l}\min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}-\frac{y}{l}\frac{\sum_{j=1}^{l}\min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)},$$
$$x+y=1,x,y\in[0,1];$$

$$D_{PYY9}(M_1,M_2)=1-\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$$

$$D_{PYY10}(M_1,M_2)=1-\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$$

$$D_{PYY11}(M_1,M_2)=1-\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$$

$$D_{PYY12}(M_1,M_2)=1-\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$$

Similarity measures (Peng et al. 2017):

$$S_{PYY1}(M_1,M_2)=1-\frac{1}{2l}\sum_{j=1}^{l}\left(\begin{array}{c}\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|+\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\\+\left|\pi_{M_1}^2(m_j)-\pi_{M_2}^2(m_j)\right|\end{array}\right)$$

$$S_{PYY2}(M_1,M_2)=1-\frac{1}{2l}\sum_{j=1}^{l}\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)-\left(\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right)\right|\right)$$

$$S_{PYY3}(M_1,M_2)=1-\frac{1}{4l}\left\{\begin{array}{c}\sum_{j=1}^{l}\left(\begin{array}{c}\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|+\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\\+\left|\pi_{M_1}^2(m_j)-\pi_{M_2}^2(m_j)\right|\end{array}\right)\\+\sum_{j=1}^{l}\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)-\left(\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right)\right|\end{array}\right\}$$

$$S_{PYY4}(M_1,M_2)=1-\frac{1}{l}\sum_{j=1}^{l}max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)$$

$$S_{PYY5}(M_1,M_2)=\frac{1}{l}\sum_{j=1}^{l}\frac{1-max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}{1+max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}$$

$$S_{PYY6}(M_1,M_2)=\frac{\sum_{j=1}^{l}1-max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}{\sum_{j=1}^{l}1+max\left(\left|\mu_{M_1}^2(m_j)-\mu_{M_2}^2(m_j)\right|,\left|\vartheta_{M_1}^2(m_j)-\vartheta_{M_2}^2(m_j)\right|\right)}$$

$$S_{PYY7}(M_1,M_2)=x\frac{\sum_{j=1}^{l}\min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}+y\frac{\sum_{j=1}^{l}\min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)},$$
$$x+y=1,x,y\in[0,1]$$

$$S_{PYY8}(M_1,M_2)=\frac{x}{l}\frac{\sum_{j=1}^{l}\min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)}-\frac{y}{l}\frac{\sum_{j=1}^{l}\min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}\max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)},$$
$$x+y=1,x,y\in[0,1]$$

$$S_{PYY9}(M_1,M_2)=\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$$

$$S_{PYY10}(M_1,M_2)=\frac{\sum_{j=1}^{l}min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}{\sum_{j=1}^{l}max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)}$$

$$S_{PYY11}(M_1,M_2)=\frac{1}{l}\sum_{j=1}^{l}\frac{min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1-max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$$

$$S_{PYY12}(M_1,M_2)=\frac{\sum_{j=1}^{l}min\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+min\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}{\sum_{j=1}^{l}max\left(\mu_{M_1}^2(m_j),\mu_{M_2}^2(m_j)\right)+\left(1+max\left(\vartheta_{M_1}^2(m_j),\vartheta_{M_2}^2(m_j)\right)\right)}$$

We now consider three different cases of FFSs with each case consisting of two different FFSs. The compatibility values of these three different cases computed by the existing PF distance/similarity measures including the suggested FF distance measures are shown in Table 4.

From Table 4, we have

Table 3 Some suggested FF knowledge measures

Proposed FF distance measures	Corresponding FF knowledge measures
D_{G1}(Proposed)	$K_{G1}(M_1)=\frac{1}{l}\sum_{j=1}^{l}\frac{2\left(\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert-\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert^2\right)}{1-\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert^2}$
D_{G2}(Proposed)	$K_{G2}(M_1)=\frac{1}{l}\sum_{j=1}^{l}2\left(\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert-\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert^2\right)$
D_{G3}(Proposed)	$K_{G3}(M_1)=\frac{1}{l}\sum_{j=1}^{l}min\left(1,2\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert\right)$
D_{G4}(Proposed)	$K_{G4}(M_1)=\frac{1}{l}\sum_{j=1}^{l}\frac{2\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert}{1+\lvert\mu_{M_1}^3(m_j)-\vartheta_{M_1}^3(m_j)\rvert^2}$

Table 4 Computed values of various PF/FF distance/ similarity measures

FF distance/similarity measures	Case I $M_1=\{(0.5,0.5)\}$ $M_2=\{(0.0,0.0)\}$	Case II $M_1=\{(0.4,0.3)\}$ $M_2=\{(0.5,0.3)\}$	Case III $M_1=\{(0.4,0.3)\}$ $M_2=\{(0.5,0.2)\}$
D_{PYY1} (Peng et al. 2017)	0.50000	**0.0900**	**0.0900**
D_{PYY2} (Peng et al. 2017)	**0**	0.0450	0.0700
D_{PYY3} (Peng et al. 2017)	0.2500	0.1350	0.1850
D_{PYY4} (Peng et al. 2017)	0.2500	**0.0900**	**0.0900**
D_{PYY5} (Peng et al. 2017)	0.4000	**0.1651**	**0.1651**
D_{PYY6} (Peng et al. 2017)	0.4000	**0.1651**	**0.1651**
D_{PYY7} (Peng et al. 2017)	**1.0000**	0.1080	0.4969
D_{PYY8} (Peng et al. 2017)	**1.0000**	0.1080	0.4969
D_{PYY9} (Peng et al. 2017)	**1.0000**	**0.3600**	**0.3600**
D_{PYY0} (Peng et al. 2017)	**1.0000**	**0.3600**	**0.3600**
D_{PYY11} (Peng et al. 2017)	0.4000	0.0776	0.1157
D_{PYY12} (Peng et al. 2017)	0.4000	0.0776	0.1157
S_{PYY1}(Peng et al. 2017)	0.5000	**0.9100**	**0.9100**
S_{PYY2}(Peng et al. 2017)	**1.0000**	0.9550	0.9300
S_{PYY3}(Peng et al. 2017)	0.7500	0.8650	0.8150
S_{PYY4}(Peng et al. 2017)	0.7500	**0.9100**	**0.9100**
S_{PYY5}(Peng et al. 2017)	0.6000	**0.8349**	**0.8349**
S_{PYY6}(Peng et al. 2017)	0.6000	**0.8349**	**0.8349**
S_{PYY7}(Peng et al. 2017)	**0**	**-0.5080**	**-0.1191**
S_{PYY8}(Peng et al. 2017)	**0**	**-0.5080**	**-0.1191**
S_{PYY9}(Peng et al. 2017)	**0**	**0.6400**	**0.6400**
S_{PYY10}(Peng et al. 2017)	**0**	**0.6400**	**0.6400**
S_{PYY11}(Peng et al. 2017)	0.60000	0.9224	0.8843
S_{PYY12} (Peng et al. 2017)	0.60000	0.9224	0.8843
D_{G1}(Proposed)	0.2222	0.0610	0.0778
D_{G2}(Proposed)	0.2344	0.0610	0.0778
D_{G3}(Proposed)	0.2500	0.0610	0.0800
D_{G4}(Proposed)	0.2462	0.0610	0.0799

Bold values indicate unreasonable results

1. The PF distance measures $D_{PYY1}, D_{PYY4}, D_{PYY5}, D_{PYY6}, D_{PYY9}$, and$D_{PYY10}$ gives the same distance for the two distinct cases (Case II and Case III).
2. The PF distance D_{PYY2} gives "0" as the distance between the two different PFSs (Case I) and thus fails to satisfy the axiom (D3) of the PF distance measure given in Definition 5.
3. The PF distance $D_{PYY7}, D_{PYY8}, D_{PYY9}$, and D_{PYY10} gives "1" as the distance between the two different PFSs (Case I) although they are not a complement to each other.
4. The PF similarity measures $S_{PYY1}, S_{PYY4}, S_{PYY5}, S_{PYY6}, S_{PYY9}$, and$S_{PYY10}$ give the same degree of similarity for the two distinct cases (Case II and Case III).
5. The PF similarity measure S_{PYY2} gives "1" as a similarity degree for the two different PFSs (Case I) and thus fails to satisfy the axiom (S3) of the PF measure of similarity given in Definition 4.
6. The PF similarity measures $S_{PYY7}, S_{PYY8}, S_{PYY9}$, and S_{PYY10} gives "0" as similarity degree for the two different PFSs (Case I) although they are not a complement to each other.
7. The similarity degree of the different PFSs (Case II and III) by the similarity measures S_{PYY7} and S_{PYY8} comes out to be negative, which is unreasonable.
8. The proposed FF distance measures $D_{Gj}, 1 \leq j \leq 4$ outperform the majority of the available PF measures of distance/similarity.

Thus, it follows, that the suggested FF distance measures are more robust and effective than the available PF distance/similarity measures.

Next, we compare the suggested FF knowledge measures with the available PF/FF measures of entropy/knowledge.

5.2 Comparison of the suggested FF measures of knowledge with the available PF/FF measures of entropy/knowledge

To contrast the performance of the newly introduced FF measures of knowledge, we first list the PF/FF entropy/knowledge measures available in the literature.

PF entropy measures (Peng et al. 2017):

$$E_{PYY1}(M_1) = \frac{1}{l}\sum_{j=1}^{l} \frac{\pi^2_{M_1}(m_j) + 1 - \left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|}{\pi^2_{M_1}(m_j) + 1 + \left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|}$$

$$E_{PYY2}(M_1) = \frac{\sum_{j=1}^{l}\left(1 - \left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|\right)}{\sum_{j=1}^{l}\left(1 + \left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|\right)}$$

$$E_{PYY3}(M_1) = 1 - \frac{1}{l}\sum_{j=1}^{l}\left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|$$

$$E_{PYY4}(M_1) = \frac{1}{l}\sum_{j=1}^{l} \frac{min\left(\mu^2_{M_1}(m_j), \vartheta^2_{M_1}(m_j)\right)}{max\left(\mu^2_{M_1}(m_j), \vartheta^2_{M_1}(m_j)\right)}$$

$$E_{PYY5}(M_1) = \frac{1}{(\sqrt{2}-1)l} \sum_{j=1}^{l}\left(sin\frac{1 + \mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)}{4}\pi + sin\frac{1 - \mu^2_{M_1}(m_j) + \vartheta^2_{M_1}(m_j)}{4}\pi - 1\right)$$

$$E_{PYY6}(M_1) = \frac{1}{(\sqrt{2}-1)l} \sum_{j=1}^{l}\left(cos\frac{1 + \mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)}{4}\pi + cos\frac{1 - \mu^2_{M_1}(m_j) + \vartheta^2_{M_1}(m_j)}{4}\pi - 1\right)$$

$$E_{PYY7}(M_1) = \frac{1}{l}\sum_{j=1}^{l} cot\left(\frac{\pi}{4} + \frac{\left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|}{4\left(1 + \pi^2_{M_1}(m_j)\right)}\pi\right)$$

$$E_{PL8}(M_1) = \frac{1}{l}\sum_{j=1}^{l} tan\left(\frac{\pi}{4} - \frac{\left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|}{4\left(1 + \pi^2_{M_1}(m_j)\right)}\pi\right)$$

PF entropy measure (Xue et al. 2018):

$$E_{XXZT}(M_1) = \frac{1}{l}\sum_{j=1}^{l}\left[1 - \left(\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right)\left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|\right]$$

PF entropy measure (Thao and Smarandache 2019):

$$E_{TS}(M_1) = \frac{1}{l}\sum_{j=1}^{l}\left[1 - \left|\mu^2_{M_1}(m_j) - \frac{1}{3}\right| - \left|\vartheta^2_{M_1}(m_j) - \frac{1}{3}\right|\right]$$

PF entropy measure (Yang and Hussain 2018).

$$E_{YH}(M_1) = 1 - \sqrt{\frac{1}{l}\sum_{j=1}^{l}\left(\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right)^2}$$

PF knowledge measures (Singh et al. 2020a)

$$K_{SSG1}(M_1) = \sqrt{\frac{1}{l}\sum_{j=1}^{l}\left(\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right)^2}$$

$$K_{SSG2}(M_1) = \frac{1}{l}\sum_{j=1}^{l}\left|\mu^2_{M_1}(m_j) - \vartheta^2_{M_1}(m_j)\right|$$

FF entropy measures (Mishra and Rani 2021)

$$E_{MR1}(M_1) = \frac{1}{(\sqrt{2}-1)l} \sum_{j=1}^{l}\left(sin\left(\frac{\pi\left(1 + \mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j)\right)}{4}\right) + sin\left(\frac{\pi\left(1 - \mu^3_{M_1}(m_j) + \vartheta^3_{M_1}(m_j)\right)}{4}\right) - 1\right)$$

$$E_{MR2}(M_1) = \frac{1}{(\sqrt{2}-1)l} \sum_{j=1}^{l}\left(cos\left(\frac{\pi\left(1 + \mu^3_{M_1}(m_j) - \vartheta^3_{M_1}(m_j)\right)}{4}\right) + cos\left(\frac{\pi\left(1 - \mu^3_{M_1}(m_j) + \vartheta^3_{M_1}(m_j)\right)}{4}\right) - 1\right)$$

$$E_{MR3}(M_1) = \frac{1}{2l}\sum_{j=1}^{l}\left(sin\left(\frac{\left(\mu^3_{M_1}(m_j) + 1 - \vartheta^3_{M_1}(m_j)\right)}{2}\right)\pi + sin\left(\frac{\left(\vartheta^3_{M_1}(m_j) + 1 - \mu^3_{M_1}(m_j)\right)}{2}\right)\pi\right)$$

$$E_{MR4}(M_1) = \frac{-1}{l \times ln2} \sum_{j=1}^{l}\left(\begin{array}{c}\mu^3_{M_1}(m_j)ln\left(\mu^3_{M_1}(m_j)\right) + \vartheta^3_{M_1}(m_j)ln\left(\vartheta^3_{M_1}(m_j)\right) \\ -\left(1 - \pi^3_{M_1}(m_j)\right)ln\left(1 - \pi^3_{M_1}(m_j)\right) - \pi^3_{M_1}(m_j)ln2\end{array}\right)$$

Now, using linguistic hedges, we show the effectiveness of the suggested FF measures of knowledge.

Definition 14 (Senapati and Yager 2020) For any $M_1 \in FFS(W)$, its modifier $(M_1)^{\lambda}, \lambda > 0$ is defined as.

$$(M_1)^{\lambda} = \left\{ \left(m_j, (\mu_{M_1}(m_j))^{\lambda}, \left(1 - \left(1 - \vartheta^3_{M_1}(m_j)\right)^{\lambda}\right)^{\frac{1}{3}} \right) | m_j \in W \right\}$$

Then, we have the following FFSs:

M_1: LARGE; $(M_1)^2$: very LARGE; $(M_1)^3$: quite very LARGE; $(M_1)^4$: very very LARGE; $(M_1)^{\frac{1}{2}}$: more or less LARGE.

Since an FF entropy measure, E computes the ambiguity content in an FFS, so it has to satisfy the following requirement:

$$E\left((M_1)^{\frac{1}{2}}\right) > E(M_1) > E\left((M_1)^2\right) > E\left((M_1)^3\right) > E\left((M_1)^4\right) \tag{4}$$

Also, an FF knowledge measure K acts as a soft dual of an FF entropy measure and calculates the amount of precision in an FFS, so it has to satisfy the following requirement:

$$K\left((M_1)^{\frac{1}{2}}\right) < K(M_1) < K\left((M_1)^2\right) < K\left((M_1)^3\right) < K\left((M_1)^4\right) \tag{5}$$

We now consider an example related to the ambiguity computation of the above-mentioned FFSs.

Example 4 Let $M_1 \in FFS(W)$ be given as:

$M_1 = \{(m_1, 0.33, 0.47), (m_2, 0.45, 0.72), (m_3, 0.21, 0.60), (m_4, 0.80, 35), (m_5, 0.48, 0.56)\}$.

With the help of Definition 14, we construct the FFSs $(M_1)^{\frac{1}{2}}, (M_1)^2, (M_1)^3$, and $(M_1)^4$. The ambiguity content of these FFSs using the suggested FF knowledge measures and the existing ones is shown in Table 5.

From Table 5, we have the following:

$$E_{PYY1}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY1}(M_1) > E_{PYY1}\left((M_1)^2\right) > E_{PYY1}\left((M_1)^3\right) > E_{PYY1}\left((M_1)^4\right);$$

$$E_{PYY2}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY2}(M_1) > E_{PYY2}\left((M_1)^2\right) > E_{PYY2}\left((M_1)^3\right) > E_{PYY2}\left((M_1)^4\right);$$

$$E_{PYY3}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY3}(M_1) > E_{PYY3}\left((M_1)^2\right) > E_{PYY3}\left((M_1)^3\right) > E_{PYY3}\left((M_1)^4\right);$$

$$E_{PYY4}\left((M_1)^{\frac{1}{2}}\right) > E_{PYY4}(M_1) > E_{PYY4}\left((M_1)^2\right) < E_{PYY4}\left((M_1)^3\right) > E_{PYY4}\left((M_1)^4\right);$$

$$E_{PYY5}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY5}(M_1) > E_{PYY5}\left((M_1)^2\right) > E_{PYY5}\left((M_1)^3\right) > E_{PYY5}\left((M_1)^4\right);$$

$$E_{PYY6}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY6}(M_1) > E_{PYY6}\left((M_1)^2\right) > E_{PYY6}\left((M_1)^3\right) > E_{PYY6}\left((M_1)^4\right);$$

$$E_{PYY7}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY7}(M_1) > E_{PYY7}\left((M_1)^2\right) > E_{PYY7}\left((M_1)^3\right) > E_{PYY7}\left((M_1)^4\right);$$

$$E_{PYY8}\left((M_1)^{\frac{1}{2}}\right) < E_{PYY8}(M_1) > E_{PYY8}\left((M_1)^2\right) > E_{PYY8}\left((M_1)^3\right) > E_{PYY8}\left((M_1)^4\right);$$

$$E_{XXZT}\left((M_1)^{\frac{1}{2}}\right) < E_{XXZT}(M_1) < E_{XXZT}\left((M_1)^2\right) < E_{XXZT}\left((M_1)^3\right) < E_{XXZT}\left((M_1)^4\right);$$

$$E_{TS}\left((M_1)^{\frac{1}{2}}\right) < E_{TS}(M_1) > E_{TS}\left((M_1)^2\right) > E_{TS}\left((M_1)^3\right) > E_{TS}\left((M_1)^4\right);$$

$$E_{YH}\left((M_1)^{\frac{1}{2}}\right) < E_{YH}(M_1) > E_{YH}\left((M_1)^2\right) > E_{YH}\left((M_1)^3\right) > E_{YH}\left((M_1)^4\right);$$

$$K_{SSG1}\left((M_1)^{\frac{1}{2}}\right) > K_{SSG1}(M_1) < K_{SSG1}\left((M_1)^2\right) < K_{SSG1}\left((M_1)^3\right) < K_{SSG1}\left((M_1)^4\right);$$

$$K_{SSG2}\left((M_1)^{\frac{1}{2}}\right) > K_{SSG2}(M_1) < K_{SSG2}\left((M_1)^2\right) < K_{SSG2}\left((M_1)^3\right) < K_{SSG2}\left((M_1)^4\right);$$

$$E_{MR1}\left((M_1)^{\frac{1}{2}}\right) < E_{MR1}(M_1) > E_{MR1}\left((M_1)^2\right) > E_{MR1}\left((M_1)^3\right) > E_{MR1}\left((M_1)^4\right);$$

$$E_{MR2}\left((M_1)^{\frac{1}{2}}\right) < E_{MR2}(M_1) > E_{MR2}\left((M_1)^2\right) > E_{MR2}\left((M_1)^3\right) > E_{MR2}\left((M_1)^4\right);$$

$$E_{MR3}\left((M_1)^{\frac{1}{2}}\right) < E_{MR3}(M_1) > E_{MR3}\left((M_1)^2\right) > E_{MR3}\left((M_1)^3\right) > E_{MR3}\left((M_1)^4\right);$$

$$E_{MR4}\left((M_1)^{\frac{1}{2}}\right) < E_{MR4}(M_1) > E_{MR4}\left((M_1)^2\right) > E_{MR4}\left((M_1)^3\right) > E_{MR4}\left((M_1)^4\right);$$

$$K_{G1}\left((M_1)^{\frac{1}{2}}\right) < K_{G1}(M_1) < K_{G1}\left((M_1)^2\right) < K_{G1}\left((M_1)^3\right) < K_{G1}\left((M_1)^4\right);$$

$$K_{G2}\left((M_1)^{\frac{1}{2}}\right) < K_{G2}(M_1) < K_{G2}\left((M_1)^2\right) < K_{G2}\left((M_1)^3\right) < K_{G2}\left((M_1)^4\right);$$

$$K_{G3}\left((M_1)^{\frac{1}{2}}\right) < K_{G3}(M_1) < K_{G3}\left((M_1)^2\right) < K_{G3}\left((M_1)^3\right) < K_{G3}\left((M_1)^4\right);$$

$$K_{G4}\left((M_1)^{\frac{1}{2}}\right) < K_{G4}(M_1) < K_{G4}\left((M_1)^2\right) < K_{G4}\left((M_1)^3\right) < K_{G4}\left((M_1)^4\right).$$

Thus, it follows that all the available PF/FF measures of entropy $E_{PYYj}, 1 \le j \le 8$, $E_{XXZT}, E_{TS}, E_{YH}, E_{MRj}, 1 \le j \le 4$, and the PF knowledge measures $K_{SSGj}, j = 1, 2$, does not satisfy the requirements given in Eq. (4) and Eq. (5), respectively. However, all our suggested FF knowledge measures $K_{Gj}, j = 1, 2, 3, 4$ follow the desired requirement given in Eq. (5). This shows that from a linguistic hedge perspective, the suggested measures of knowledge are robust than the available ones.

Next, we show the utility of the suggested FF measures of knowledge and distance in pattern recognition and decision-making.

6 Application of the proposed measures

In this section, we demonstrate the application of the suggested measures in pattern analysis and MCDM.

6.1 Pattern analysis

Here, we show that the suggested FF distance measures can be used for solving the problems related to pattern classification. In pattern analysis, an unfamiliar pattern is categorized into one of the known patterns using some measures of compatibility viz., similarity measures,

Table 5 Values of various PF/FF entropy/knowledge measures regarding Example 4

Entropy/Knowledge measures	$(M_1)^{\frac{1}{2}}$	M_1	$(M_1)^2$	$(M_1)^3$	$(M_1)^4$
E_{PYY1} (Peng et al. 2017)	0.6761	0.6900	0.5062	0.4183	0.3050
E_{PYY2} (Peng et al. 2017)	0.5596	0.5762	0.3602	0.2706	0.1873
E_{PYY3} (Peng et al. 2017)	0.7176	0.7311	0.5296	0.4260	0.3155
E_{PYY4} (Peng et al. 2017)	0.4849	0.3864	0.1497	0.1677	0.0836
E_{PYY5} (Peng et al. 2017)	1.3222	1.3272	1.3087	1.2909	1.2769
E_{PYY6} (Peng et al. 2017)	12.2186	12.2385	12.1659	12.0963	12.0414
E_{PYY7} (Peng et al. 2017)	0.7209	0.7412	0.5750	0.4785	0.3640
E_{PYY8} (Peng et al. 2017)	0.7209	0.7412	0.5750	0.4785	0.3640
E_{XXZT} (Xue et al. 2018)	0.8592	0.9902	1.2429	1.4081	1.5283
E_{TS} (Thao and Smarandache 2019)	0.6670	0.6782	0.5296	0.4224	0.3155
E_{YH} (Yang and Hussain 2018)	0.6248	0.6880	0.4943	0.3612	0.2731
K_{SSG1}(Singh et al. 2020a)	0.3752	0.3120	0.5057	0.6388	0.7269
K_{SSG2} (Singh et al. 2020a)	0.2824	0.2689	0.4704	0.5740	0.6845
E_{MR1} (Mishra et al. 2021)	0.8831	0.9266	0.8602	0.7717	0.6867
E_{MR2} (Mishra et al. 2021)	0.8831	0.9266	0.8602	0.7717	0.6867
E_{MR3} (Mishra et al. 2021)	0.1829	0.1919	0.1782	0.1598	0.1422
E_{MR4} (Mishra et al. 2021)	0.8455	0.8650	0.7101	0.6056	0.5229
K_{G1} (Proposed)	0.3308	0.3341	0.4812	0.5222	0.6151
K_{G2} (Proposed)	0.3596	0.3661	0.5320	0.5790	0.6759
K_{G3} (Proposed)	0.3954	0.4364	0.6269	0.6918	0.7788
K_{G4} (Proposed)	0.3762	0.3906	0.5675	0.6110	0.7061

Table 6 Calculated values of various FF/PF compatibility measures regarding Example 5

	(M_1, M)	(M_2, M)	(M_3, M)	Result
S_{PYY1} (Peng et al. 2017)	0.9792	0.9829	0.9824	M_2
S_{PYY2} (Peng et al. 2017)	0.9833	0.9869	0.9851	M_2
S_{PYY3} (Peng et al. 2017)	0.9813	0.9849	0.9837	M_2
S_{PYY4} (Peng et al. 2017)	0.9792	0.9829	0.9824	M_2
S_{PYY5} (Peng et al. 2017)	0.9595	0.9665	0.9655	M_2
S_{PYY6} (Peng et al. 2017)	0.0204	0.0168	0.0173	M_1
S_{PYY7} (Peng et al. 2017)	0.8242	0.8560	0.8431	M_2
S_{PYY8} (Peng et al. 2017)	0.7494	0.7152	0.7237	M_1
S_{PYY9} (Peng et al. 2017)	0.5923	0.5503	0.5639	M_1
S_{PYY10}(Peng et al. 2017)	0.8188	0.8547	0.8348	M_2
S_{PYY11} (Peng et al. 2017)	0.9643	0.9719	0.9694	M_2
S_{PYY12} (Peng et al. 2017)	0.9654	0.9728	0.9690	M_2
D_{PYY1} (Peng et al. 2017)	0.0208	0.0171	0.0176	M_2
D_{PYY2} (Peng et al. 2017)	0.0167	0.0131	0.0149	M_2
D_{PYY3} (Peng et al. 2017)	0.0187	0.0151	0.0163	M_2
D_{PYY4} (Peng et al. 2017)	0.0208	0.0171	0.0176	M_2
D_{PYY5} (Peng et al. 2017)	0.0405	0.0335	0.0345	M_2
D_{PYY6} (Peng et al. 2017)	0.0408	0.0336	0.0347	M_2
D_{PYY7} (Peng et al. 2017)	0.1758	0.1440	0.1569	M_2
D_{PYY8} (Peng et al. 2017)	0.2506	0.2848	0.2763	M_2
D_{PYY9} (Peng et al. 2017)	0.4077	0.4497	0.4361	M_1
D_{PYY0} (Peng et al. 2017)	0.1812	0.1453	0.1652	M_2
D_{PYY11} (Peng et al. 2017)	0.0357	0.0281	0.0306	M_2
D_{PYY12} (Peng et al. 2017)	0.0346	0.0272	0.0310	M_2
D_{G1}(Proposed)	0.0173	0.0128	0.0146	M_2
D_{G2} (Proposed)	0.0174	0.0128	0.0146	M_2
D_{G3} (Proposed)	0.0175	0.0128	0.0147	M_2
D_{G4} (Proposed)	0.0175	0.0128	0.0147	M_2

distance measures, correlation measures, etc. We also contrast our results with the available measures of compatibility.

Now, we solve some problems related to pattern analysis in the examples given below.

Example 5 (Jiang et al. 2019) Consider the patterns M_1, M_2, M_3, and M expressed in the form of FFSs in W as:

$$M_1 = \{(m_1, 0.34, 0.34), (m_2, 0.19, 0.48), (m_3, 0.02, 0.12)\},$$

$$M_2 = \{(m_1, 0.35, 0.33), (m_2, 0.20, 0.47), (m_3, 0.00, 0.14)\},$$

$$M_3 = \{(m_1, 0.33, 0.35), (m_2, 0.21, 0.46), (m_3, 0.01, 0.13)\},$$

$$M = \{(m_1, 0.37, 0.31), (m_2, 0.23, 0.44), (m_3, 0.04, 0.10)\}.$$

Table 7 Calculated values of various FF/PF compatibility measures regarding Example 6

	(M_1, M)	(M_2, M)	(M_3, M)	Result
S_{PYY1} (Peng et al. 2017)	0.8860	0.8000	0.7840	M_1
S_{PYY2} (Peng et al. 2017)	0.9280	0.9030	0.8590	M_1
S_{PYY3} (Peng et al. 2017)	0.9070	0.8515	0.8215	M_1
S_{PYY4} (Peng et al. 2017)	0.8860	0.8260	0.7940	M_1
S_{PYY5} (Peng et al. 2017)	0.8075	0.7291	0.6868	M_1
S_{PYY6} (Peng et al. 2017)	0.1023	0.1482	0.1708	M_3
S_{PYY7} (Peng et al. 2017)	0.6027	0.3147	0.3429	M_1
S_{PYY8} (Peng et al. 2017)	0.4911	0.3094	0.4036	M_1
S_{PYY9} (Peng et al. 2017)	0.6498	0.4024	0.3749	M_1
S_{PYY10} (Peng et al. 2017)	0.6418	0.3627	0.3041	M_1
S_{PYY11} (Peng et al. 2017)	0.8870	0.8130	0.7707	M_1
S_{PYY12} (Peng et al. 2017)	0.8873	0.8026	0.7545	M_1
D_{PYY1} (Peng et al. 2017)	0.1140	0.2000	0.2160	M_1
D_{PYY2} (Peng et al. 2017)	0.0720	0.0970	0.1410	M_1
D_{PYY3} (Peng et al. 2017)	0.0930	0.1485	0.1785	M_1
D_{PYY4} (Peng et al. 2017)	0.1140	0.1740	0.2060	M_1
D_{PYY5} (Peng et al. 2017)	0.1925	0.2709	0.3132	M_1
D_{PYY6} (Peng et al. 2017)	0.2047	0.2964	0.3416	M_1
D_{PYY7} (Peng et al. 2017)	0.3973	0.6853	0.6571	M_1
D_{PYY8} (Peng et al. 2017)	0.5089	0.6906	0.5964	M_1
D_{PYY9} (Peng et al. 2017)	0.3502	0.5976	0.6251	M_1
D_{PYY0} (Peng et al. 2017)	0.3582	0.6373	0.6959	M_1
D_{PYY11} (Peng et al. 2017)	0.1130	0.1870	0.2293	M_1
D_{PYY12} (Peng et al. 2017)	0.1127	0.1974	0.2455	M_1
D_{G1} (Proposed)	0.0904	0.1381	0.1679	M_1
D_{G2} (Proposed)	0.0917	0.1415	0.1743	M_1
D_{G3} (Proposed)	0.0932	0.1466	0.1842	M_1
D_{G4} (Proposed)	0.0930	0.1448	0.1803	M_1

The problem is to see with which pattern $M_j, j = 1, 2, 3,$ the pattern M has maximum resemblance. For this purpose, we use the suggested FF distance measures along with the

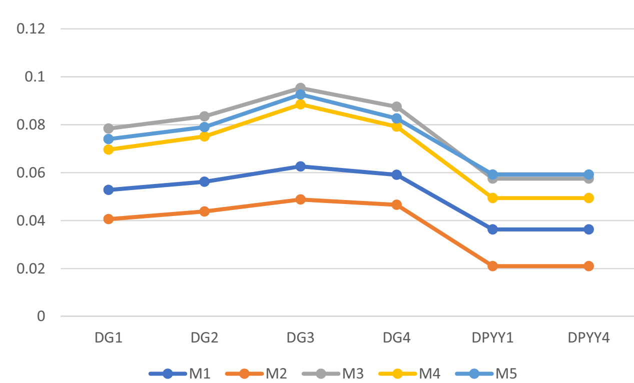

Fig. 1 Ranking results

Table 8 Computed values of the distance of each alternative from the FF ideal solution

	(M_1, M^*)	(M_2, M^*)	(M_3, M^*)	(M_4, M^*)	(M_5, M^*)
D_{G1}	0.0528	0.0406	0.0784	0.0696	0.0740
D_{G2}	0.0562	0.0438	0.0835	0.0751	0.0790
D_{G3}	0.0626	0.0488	0.0953	0.0885	0.0926
D_{G4}	0.0591	0.0466	0.0875	0.0792	0.0826

Table 9 Ranking of alternatives

	Ranking
D_{G1}(Proposed)	$M_2 > M_1 > M_4 > M_5 > M_3$
D_{G2} (Proposed)	$M_2 > M_1 > M_4 > M_5 > M_3$
D_{G3} (Proposed)	$M_2 > M_1 > M_4 > M_5 > M_3$
D_{G4} (Proposed)	$M_2 > M_1 > M_4 > M_5 > M_3$
D_{PYY1}(Peng et al. 2017)	$M_2 > M_1 > M_4 > M_5 > M_3$
D_{PYY4}(Peng et al. 2017)	$M_2 > M_1 > M_4 > M_5 > M_3$

existing measures of compatibility. The computed values are shown in Table 6.

From Table 6, it is clear that M should be assigned to M_2 as shown by most of the PF/FF distance/similarity measures including the suggested FF measures of distance.

Example 6 Consider the patterns M_1, M_2, M_3, and M expressed in the form of FFSs in W as

$$M_1 = \{(m_1, 0.4, 0.3), (m_2, 0.5, 0.3), (m_3, 0.4, 0.3), (m_4, 0.7, 0), (m_5, 0.6, 0.1)\},$$

$$M_2 = \{(m_1, 0.7, 0.1), (m_2, 0.2, 0.3), (m_3, 0.2, 0.1), (m_4, 0.1, 0.4), (m_5, 0.3, 0.3)\},$$

$$M_3 = \{(m_1, 0.1, 0.3), (m_2, 0.4, 0.3), (m_3, 0.3, 0.4), (m_4, 0.2, 0.5), (m_5, 0.5, 0.3)\},$$

$$M = \{(m_1, 0.6, 0.2), (m_2, 0.3, 0.4), (m_3, 0.4, 0.3), (m_4, 0.7, 0.1), (m_5, 0.4, 0.2)\}.$$

The problem is to see with which pattern $M_j, j = 1, 2, 3,$ the pattern M has maximum resemblance. For this purpose, we use the suggested FF distance measures along with the existing measures of compatibility. The computed values are shown in Table 7.

From Table 7, it follows that M should be assigned to M_1 as shown by most of the PF/FF distance/similarity measures including the suggested FF measures of distance.

Thus, from Examples 5 and 6, we conclude that in terms of pattern recognition, the suggested FF measures of distance are consistent with the existing distance/similarity measures.

6.2 Multicriteria decision-making

Here, we show that the suggested FF measures of knowledge and distance are useful for solving multicriteria decision-making (MCDM) problems involving uncertainty and ambiguity. The main hurdle in an MCDM problem is the computation of criteria weights and we use the suggested knowledge measures for this purpose. For determining the best alternative, we take the help of the suggested distance measures. First, we give the algorithm for solving an MCDM problem having n alternatives $M_j, j = 1, 2, \ldots, n$ and m criteria $N_k, k = 1, 2, \ldots, m$ with $w_k, k = 1, 2, \ldots, m$ as criteria weights where $0 \le w_k \le 1$ and $\sum_{k=1}^{m} w_k = 1$.

Algorithm Step 1: Formulate the decision matrix $D = \left[\left(\mu_{jk}, \vartheta_{jk}\right)\right]_{n\times m}$ expressing the information of the available alternatives with respect to the criteria.

Step 2: Formulate the normalized decision matrix $E = \left[\left(\mu'_{jk}, \vartheta'_{jk}\right)\right]_{n\times m}$ where,

$$\left(\mu'_{jk}, \vartheta'_{jk}\right) = \begin{cases} \left(\mu_{jk}, \vartheta_{jk}\right), \textit{if } N_k \text{ is a benefit criteria} \\ \left(\vartheta_{jk}, \mu_{jk}\right), \textit{if } N_k \text{ is a cost criteria} \end{cases}.$$

Step 3: Compute the criteria weights $w_k, k = 1, 2, \ldots, m$ as:

$$w_k = \frac{1 - K(N_k)}{m - \sum_{k=1}^{m} K(N_k)}, k = 1, 2, \ldots, m.$$

Here, K is a FF knowledge measure.

Step 4: Determine the FF ideal solution $M^* = \left\{\left(\mu_1^*, \vartheta_1^*\right), \left(\mu_2^*, \vartheta_2^*\right), \ldots, \left(\mu_m^*, \vartheta_m^*\right)\right\}$ where $\mu_k^* = \max_j \mu_{jk}$ and $\vartheta_k^* = \min_j \vartheta_{jk}, k = 1, 2, \ldots, m$.

Step 5: Compute the distance of each alternative $M_j, j = 1, 2, \ldots, n$ from the FF ideal solution M^* using the suggested weighted FF distance measures.

Step 6: Rank the alternatives as $M_j > M_t$ if $D(M_j, M^*) < D(M_t, M^*)$, where D is a FF distance measure and $1 \leq j, t \leq n$.

Now, we solve an MCDM problem in the example given below.

Example 5 (Singh and Ganie 2021b) Consider the problem of purchasing a house out of the five houses $M_j, j = 1, 2, 3, 4, 5$ by considering the following criteria:

N_1 : Ceiling height, N_2 : Design, N_3 : Location, N_4 : Purchase price, N_5 : Ventilation.

The information about the five houses with respect to the above-mentioned five criteria is expressed in the form of FFSs as shown by the decision matrix D below:

$$D = \begin{pmatrix} \langle 0.7, 0.5\rangle & \langle 0.6, 0.8\rangle & \langle 0.4, 0.7\rangle & \langle 0.8, 0.3\rangle & \langle 0.6, 0.5\rangle \\ \langle 0.6, 0.6\rangle & \langle 0.7, 0.3\rangle & \langle 0.2, 0.7\rangle & \langle 0.4, 0.6\rangle & \langle 0.1, 0.7\rangle \\ \langle 0.29, 0.8\rangle & \langle 0.21, 0.9\rangle & \langle 0.6, 0.8\rangle & \langle 0.71, 0.3\rangle & \langle 0.1, 0.3\rangle \\ \langle 0.2, 0.9\rangle & \langle 0.2, 0.8\rangle & \langle 0.1, 0.6\rangle & \langle 0.5, 0.6\rangle & \langle 0.4, 0.7\rangle \\ \langle 0.3, 0.9\rangle & \langle 0.32, 0.9\rangle & \langle 0.4, 0.8\rangle & \langle 0.6, 0.6\rangle & \langle 0.3, 0.4\rangle \end{pmatrix}$$

As the criteria N_4 is a cost attribute, so the normalized decision matrix E with the help of Step 2 is given below:

$$E = \begin{pmatrix} \langle 0.7, 0.5\rangle & \langle 0.6, 0.8\rangle & \langle 0.4, 0.7\rangle & \langle 0.3, 0.8\rangle & \langle 0.6, 0.5\rangle \\ \langle 0.6, 0.6\rangle & \langle 0.7, 0.3\rangle & \langle 0.2, 0.7\rangle & \langle 0.6, 0.4\rangle & \langle 0.1, 0.7\rangle \\ \langle 0.29, 0.8\rangle & \langle 0.21, 0.9\rangle & \langle 0.6, 0.8\rangle & \langle 0.3, 0.71\rangle & \langle 0.1, 0.3\rangle \\ \langle 0.2, 0.9\rangle & \langle 0.2, 0.8\rangle & \langle 0.1, 0.6\rangle & \langle 0.6, 0.5\rangle & \langle 0.4, 0.7\rangle \\ \langle 0.3, 0.9\rangle & \langle 0.32, 0.9\rangle & \langle 0.4, 0.8\rangle & \langle 0.6, 0.6\rangle & \langle 0.3, 0.4\rangle \end{pmatrix}$$

With the help of Step 3 and using the suggested knowledge measure K_{G1} given in Table 3, we obtain the criteria weights as:

$w_1 = 0.1675, w_2 = 0.1250, w_3 = 0.1897, w_4 = 0.2464,$ and $w_5 = 0.2714$.

Next, using Step 4, the FF ideal solution M^* is given as:

$$M^* = \{\langle 0.7, 0.5\rangle, \langle 0.7, 0.3\rangle, \langle 0.6, 0.6\rangle, \langle 0.6, 0.4\rangle, \langle 0.6, 0.3\rangle\}.$$

The computed values of the distance of each alternative $M_j, j = 1, 2, 3, 4, 5$ from the FF ideal solution M^* using the suggested weighted distance measures $D^w_{Gj}, j = 1, 2, 3, 4$ given in Table 2 are shown in Table 8.

The final ranking of alternatives with the help of Step 6 is shown in Table 9 and Fig. 1.

From Table 9, we conclude that M_2 is the most feasible alternative as all the suggested FF distance measures and the existing PF distance measures D_{PYY1} and D_{PYY4} indicate the same. This shows that the suggested distance measures are consistent with the existing distance measures (Tables 8 and 9).

7 Conclusion

This paper has presented a novel method of constructing some distance measures and knowledge measures for FFSs with the help of t-conorms. First with the help of t-conorms, four distance measures for FFSs have been proposed and then with the help of proposed FF distance measures, four new FF knowledge measures have been introduced. The suggested measures of distance are more effective than most of the available PF distance/similarity measures as far as the distance/similarity degree between different PFSs/FFSs are concerned. Most of the existing PF distance/similarity measures have given unsatisfactory results while computing the distance/similarity between different PFSs/FFSs and also some measures have failed to satisfy all the axiomatic requirements. However, the proposed FF distance measures have produced satisfactory results without any counterintuitive situation. Further, the suggested measures of knowledge for FFSs are more robust than the available PF/FF entropy/knowledge measures from the linguistic hedge aspect. The applicability of the suggested FF distance measures has been shown in classification problems and the results are contrasted with the existing measures. Also, the suggested measures have been used for solving a multicriteria decision-making problem and the results are consistent with the available measures.

In the future, we will show the applicability of the suggested measures of distance for FFSs in medical diagnosis and clustering. We will also extend the proposed method of obtaining distance and knowledge measures to some recent generalizations of FSs such as interval-valued fuzzy sets (Turksen 1986; Shyi-Ming Chen 1997; Chen et al. 1997; Chen and Hsiao 2000), interval type-2 fuzzy sets (Mendel et al. 2006; Chen and Lee 2011; Chen et al. 2013; Chen and Hong 2014), picture fuzzy sets (Cuong and Kreinovich 2013), spherical fuzzy sets (Mahmood et al. 2019), complex fuzzy sets (Ramot et al. 2002), etc.

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Data availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The author declares that there is no conflict of interest.

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