Free Vibration and Transmission Response Analysis for Torsional Vibration of Circular Annular Plate

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Abstract

In this paper, free vibration and transmission response for the torsional vibration of circular annular plate are presented. To the author's knowledge, few studies can be found for the torsional vibration from wave standpoint. For this purpose, in this study, natural frequencies for the torsional vibration of annular plate with clamped–clamped and free–free boundaries are calculated. The natural frequencies obtained by wave approach are compared with those derived by the classical method. Furthermore, transmissibility curves of the periodic annular model and Fibonacci annular model are analyzed. The finite element simulations are carried out to verify the theoretical results. Finally, the influence of inner radius and length ratio on the transmission response is also discussed. The obtained results are useful for the torsional vibration reduction of machinery structures.

Keywords Circular plate · Classical method · Wave approach · Nature frequency · Vibration response

1 Introduction

Noise and vibration problem of circular annular plate has been a hot spot that makes it attract lots of attention because of the wide applications, such as gear transmission system and aircraft structures. Since most of components within these structures can be regarded as a simple model of annular structure, dynamic properties of these structures in recent years have been the subject of present studies. Therefore, many studies on vibration reduction of annular structure are reported frequently. Bahrami et al. (2015) and Bahrami and Teimourian (2015) deduced the propagation matrix and reflection matrix for the radial vibration of annular circular and sectorial membranes. And they calculated the natural frequency of the radial vibration for these annular plates. Through adopting wave propagation technique, Liu et al. (2016) investigated the free vibration of piezoelectric circular plate for the radial vibration, and the effect of inner radius and piezoelectric effect on the transmission response is also discussed. Bahrami and Teimourian (2017) applied the wave approach to analyze the vibration, wave transmission and

reflection in composite rectangular membranes. Liu et al. (2018) deduced the propagation matrix and reflection matrix for predicting the free vibration for the transverse vibration of composite thin annular plate using wave approach. Furthermore, the transmission response of composite annular structures is also analyzed. Wang et al. (2018) investigated the free vibration of rings via wave approach. The natural frequencies calculated by wave approach are compared with those obtained by classical method and finite element method. They also analyzed the transverse vibration transmissibility of rings propagating from outer to inner and from inner to outer. Huang et al. (2013) presented a systematic approach for the free vibration analysis of rotating thin rings by wave propagation. Li (2008) investigated the free vibration of circular cylindrical shells using wave propagation, and the results show that wave approach has high accurate for long shell with different boundaries. Mei (2012) studied the effects of lumped end mass on vibrations of a Timoshenko beam using a wave-based approach. Sarayi et al. (2018) discussed the free vibration and wave power reflection in Mindlin rectangular plates via exact wave propagation approach. Mei (2019) presented a semi-analytical finite element approach to model guided wave excitation and propagation in damped composite plates.

Recently, many researchers have paid attention to the periodic annular structures. Xu et al. (2012) considered a

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two-dimensional arc-shaped circular structure, finding that start frequency of the band was zero. Torrent et al. (2010) and Carbonell et al. (2013) analyzed the acoustic resonances in two-dimensional radial sonic crystal shell, and the energy band is solved numerically. Shu et al. (2016) studied the flexural wave propagation in a thin circular plate of piezoelectric radial periodic circular plate, finding that the frequency band can be varied by outer controller. Additionally, Shu et al. (2014) also made a detail research on the torsional wave of periodic circular plate. His researches indicated that the low-frequency band was determined by circular plate radius. However, the high-frequency band was caused by the periodicity of circular plate. Ma et al. (2014) studied the band structures of bilayer radial phononic crystal plate with crystal gliding. Shu et al. (2015) analyzed the torsional vibration of piezoelectric radial phononic crystal plate. The effects of structural and piezoelectric parameters on the band gaps were also discussed in detail. The above studies mainly dealt with the vibration characteristic of periodic annular plate. However, there are few reports about torsional vibration for FB structures. Actually, compared with periodic annular plate, the non-periodic annular plate has many unique vibration characteristics.

In our previous study, wave approach has been applied to analyze the radial vibration (Liu et al. 2016) and transverse vibration (Liu et al. 2018) of annular plate. However, to the author's knowledge, few studies on torsional vibration can be found. It will be the first time that wave technique is applied to the torsional vibration of annular structure to obtain the natural characteristic and transmission response. The obtained transmission response will be meaningful for future work correlated with vibration reduction in practical engineering structure such as gear.

This paper is organized as follows. A brief introduction is summarized in Sect. 1. Section 2 gives the process of solving natural frequencies by classical method. The propagation matrix and reflection matrix are deduced for obtaining the natural frequencies by wave approach. These natural frequencies calculated by wave approach are compared with those by classical method. In Sect. 3, the transfer matrix of periodic annular model and FB annular model is deduced. Transmission response of the two models with different inner radius and length ratio is discussed. Section 4 gives the conclusion.

2 Theoretical Basis

2.1 Governing Equation of Torsional Vibration

Consider a single annular plate with inner radius r_0 and outer radius r_a as shown in Fig. 1. The positive-going and negative-going wave vectors are $a_1^+, a_1^-, b_1^+, b_1^-$. The thickness

Fig. 1 Annular plate model

is h_1 . The Young modulus, density and Poisson's ratio are E_1 , ρ_1 and σ_1 , respectively. Governing equation for the torsional vibration of annular plate can be defined as (Shu et al. 2014)

$$\frac{\partial^2 \varphi_{(r)}}{\partial r^2} + \frac{3}{r} \frac{\partial \varphi_{(r)}}{\partial r} + k^2 \varphi_{(r)} = t_{(r,t)} \quad (1)$$

where $\varphi_{(r)}$ is the torsional angle, $k = w/c$ is the wavenumber, $c = \sqrt{G/\rho}$ is the wave speed, $G = E/(2 + 2\nu)$ is the shear modulus and $t_{(r,t)}$ is the torque.

Setting the torque of the torsional vibration in Eq. (1) to zero gives the torsional solution

$$\varphi_{(r)} = A \frac{1}{r} J_1(kr) + B \frac{1}{r} Y_1(kr). \quad (2)$$

The shear stress is given by

$$\tau_{(r)} = G \{ A_1^+ [-2J_1(kr)/r + kY_0(kr)] + A_1^- [-2Y_1(kr)/r + kJ_0(kr)] \} \quad (3)$$

where $J_1(kr)$ is the first kind Bessel function, $Y_1(kr)$ is the second kind Bessel function and A_1^+ , A_1^- are constants.

2.2 Classical Method

Applying clamped–clamped boundary condition to the annular plate, one has

$$\begin{cases} \varphi(r_0) = 0 \\ \varphi(r_a) = 0 \end{cases} \quad (4)$$

Substituting Eq. (2) into Eq. (4) obtains

$$\begin{vmatrix} J_1(kr_0)/r_0 & Y_1(kr_0)/r_0 \\ J_1(kr_a)/r_a & Y_1(kr_a)/r_a \end{vmatrix} = 0. \quad (5)$$

Applying free–free boundary condition gives

$$\begin{cases} \tau_{(r)}(r_0) = 0 \\ \tau_{(r)}(r_a) = 0 \end{cases} \quad (6)$$

Substituting Eq. (10) into Eq. (13) yields

$$G \begin{vmatrix} -2J_1(kr_0)/r_0 + kJ_0(kr_0) & -2Y_1(kr_0)/r_0 + kY_0(kr_0) \\ -2J_1(kr_a)/r_a + kJ_0(kr_a) & -2Y_1(kr_a)/r_a + kY_0(kr_a) \end{vmatrix} = 0. \quad (7)$$

Through calculating the eigenvalues of Eqs. (5) and (7), one can obtain the natural frequencies of annular plate with the case of clamped–clamped boundary and free–free boundary.

2.3 Wave Approach

The torsional solution of annular plate can also be expressed in terms of cylindrical waves as

$$\varphi_{(r)} = A_1^+ \frac{H_0^{(2)}(k_1 r)}{r} + A_1^- \frac{H_0^{(1)}(k_1 r)}{r} = g_1^+ + g_1^-. \quad (8)$$

The clamped–clamped boundary condition at both ends is

$$\begin{cases} \varphi(r_0) = g_1^+(r_0) + g_1^-(r_0) = 0 \\ \varphi(r_a) = g_1^+(r_a) + g_1^-(r_a) = 0 \end{cases}. \quad (9)$$

Thus, the reflection matrix can be written as

$$\begin{cases} R_A = [-1] \\ R_C = [-1] \end{cases}. \quad (10)$$

Figure 1 clearly indicates the positive-going and negative-going wave vectors. And these wave vectors are related by

$$\begin{cases} b_1^+ = f_1^+(r_a - r_0)a_1^+ \\ a_1^- = f_1^-(r_a - r_0)b_1^- \end{cases}. \quad (11)$$

From Eq. (9), the positive-going and negative-going wave can be obtained as

$$f_1^+(r_a - r_0) = \frac{r_0 H_0^{(2)}(k_1 r_a)}{r_a H_0^{(2)}(k_1 r_0)} \quad (12)$$

$$f_1^-(r_a - r_0) = \frac{r_a H_0^{(1)}(k_1 r_0)}{r_0 H_0^{(1)}(k_1 r_a)}. \quad (13)$$

2.4 Verification

In this section, the derived propagation and reflection matrices are combined to provide a concise approach for calculating the natural frequency of the annular plate. The relationship between the wave's vectors is related by

$$\begin{cases} b_1^+ = f_1^+(r_a - r_0)a_1^+ & a_1^+ = R_A a_1^- \\ a_1^- = f_1^-(r_a - r_0)b_1^- & b_1^- = R_C b_1^+ \end{cases}. \quad (14)$$

Equation (14) can be reduced to

$$F(f) = [f_1^+(r_a - r_0)R_A f_1^-(r_a - r_0)R_C - 1]b_1^+ = 0. \quad (15)$$

The material is selected as resin. The material and structural parameters are given as follows:

Material parameters: Young modulus is $E = 0.435 \times 10^{10}$ Pa, density is $\rho = 1180$ kg/m³ and Poisson's coefficient is $\sigma = 0.3679$.

Structural parameters: Inner radius is $r_0 = 0.08$ m, external radius is $r_a = 0.16$ m and thickness is $h = 0.003$ m.

In this section, numerical examples are presented to validate the proposed classical method and wave approach. Through substituting material and structural parameters into Bessel/Hankel Eqs. (5), (7) and (15) and setting the real and imaginary parts of Eq. (15) to zero, one can obtain the natural frequencies by solving the three characteristic equations. In order to clearly present the value of the natural frequency, the real and imaginary characteristic curves of the annular plate are plotted in Fig. 2. It shows the obtained natural frequencies for torsional vibration with the case of clamped–clamped and free–free boundaries. It can be found that the real and imaginary characteristic curves meet zero simultaneously in X-axis. The natural frequencies obtained by classical method and wave approach significantly intersect at multiple points, which verifies the correctness of this calculation.

The transfer matrix can be regarded as a 2×2 unit matrix, while torsional wave propagates in an annular plate. According to the exciting force matrix and transfer matrix, the torsional transmission response curves can be depicted by using formula $dB = 20 \log(d_m/d_c)$, shown in Fig. 2d. It can be seen that the natural frequencies are found in the transmission response curves marked in Fig. 2d. That is to say, the natural frequencies of torsional vibration are shown in Fig. 2d, while torsional exciting force is applied to the annular plate. The natural frequency characteristic curves are depicted in Fig. 2c by using classical method. The natural frequencies shown in Fig. 2c are exactly the same as those presented in Fig. 2d, which verifies the correctness of the numerical calculation.

Through model the annular plate and mesh this solid model, the mode shapes are obtained with the case of free-clamped boundary by ANSYS software. The mode shapes are presented in Fig. 3, from which it can be observed that the first five order modes calculated by the numerical results are in good agreement with FEM simulation. Additionally, Fig. 3 is clearly employed to describe the mode shape, from which one can find the minimum and maximum deformations of the first five mode shapes for the torsional vibration.

Fig. 2 Real and imaginary characteristic curves and transmission response curves

3 Transmission Response of Periodic and FB Model

3.1 Structural Model and Transfer Matrix

Consider a composite periodic annular plate with eight layers ABABABAB, shown in Fig. 4. Here, the white and gray regions are selected as resin and steel materials. Similarly, Fig. 5 depicts a composite FB annular plate with eight layers ABAABABA. The inner and outer annular plate is r_1 and r_2 , respectively. The length ratio is a_1 and a_2 , respectively. Thickness is h .

The interface of resin and steel is given as a, b, c, d, e, f, m, p, q as shown in Fig. 4. Then, Fig. 5 shows the interface of resin and steel with b, c, d, e, f, m. The resin's Young modulus, density and Poisson's coefficient are E_1 , ρ_1 and σ_1 , respectively. The torsional wave velocity is $c_1 = [E_1/\rho_1(1 - \sigma_1^2)]^{0.5}$. The wavenumber is $k_1 = w/c_1 = 2\pi f/c_1$. The steel's Young modulus, density and Poisson's coefficient are E_2 , ρ_2 and σ_2 , respectively. The torsional wave velocity is $c_2 = [E_2/\rho_2(1 - \sigma_2^2)]^{0.5}$. The wavenumber is $k_2 = w/c_2 = 2\pi f/c_2$.

With regard to the torsional vibration, the displacement and stress satisfy the continuity at different interfaces. The specific transfer matrix of torsional vibration can be deduced as follows.

Applying the geometric continuity at $r_b = r_1 + a_1$ leads to

$$\begin{cases} \varphi_{r1} = \varphi_{r2} \\ \tau_{r1} = \tau_{r2} \end{cases} \quad (16)$$

Hence, Eq. (16) can be arranged as

$$\mathbf{H}_1 \begin{bmatrix} A_{11}^+ \\ A_{11}^- \end{bmatrix} = \mathbf{K}_1 \begin{bmatrix} B_{12}^+ \\ B_{12}^- \end{bmatrix} \quad (17)$$

Applying the continuity at $r_c = r_1 + a_1 + a_2$ yields

$$\begin{cases} \varphi_{r2} = \varphi_{r3} \\ \tau_{r2} = \tau_{r3} \end{cases} \quad (18)$$

Equation (18) can be organized as

$$\mathbf{H}_2 \begin{bmatrix} A_{21}^+ \\ A_{21}^- \end{bmatrix} = \mathbf{K}_2 \begin{bmatrix} B_{12}^+ \\ B_{12}^- \end{bmatrix} \quad (19)$$

Combining Eqs. (17) and (19) yields

Fig. 3 Mode shapes of torsional vibration

Fig. 4 Periodic annular plate model

$$\begin{bmatrix} A_{21}^+ \\ A_{21}^- \end{bmatrix} = \mathbf{T}_{31} \begin{bmatrix} A_{11}^+ \\ A_{11}^- \end{bmatrix} \quad (20)$$

where $\mathbf{T}_{31} = \mathbf{H}_2^{-1} \mathbf{K}_2 \mathbf{K}_1^{-1} \mathbf{H}_1$ is the transfer matrix of torsional vibration between the third and the first annular plate model.

For the periodic annular plate model in Fig. 4 at the interface d, e, f, m, p, one can easily obtain

Fig. 5 FB annular plate model

$$\mathbf{T}_{n1} = \mathbf{K}_{n-1}^{-1} \mathbf{H}_{(n-1)(n-3)} \mathbf{H}_{(n-3)(n-5)} \cdots \mathbf{T}_{53} \mathbf{T}_{31} \quad (21)$$

where $\mathbf{T}_{n1}$ is the total transfer matrix of torsional vibration between the nth and the first annular plate model.

In this case, transfer matrix of periodic model can be described as

$$\mathbf{T}_{81} = \mathbf{K}_{71}^{-1} \mathbf{H}_{75} \mathbf{H}_{53} \mathbf{T}_{31}. \quad (22)$$

For the FB annular plate model in Fig. 5 at the interface d, e, f, m, one can smoothly obtain the total transfer matrix as follows.

For layer 3 and layer 4, one can easily obtain $\varphi_3 = \varphi_4$. According to the continuity of interface d and e, the transfer matrix for layer 4 and layer 6 gives

$$\varphi_6 = T_{64}\varphi_4. \quad (23)$$

Similarly, the transfer matrix between layer 6 and layer 8 at interface f and m gives

$$\varphi_8 = T_{86}\varphi_6. \quad (24)$$

Finally, combining Eqs. (17), (20), (23) and (24) yields

$$\varphi_8 = \mathbf{T}_{86}\mathbf{T}_{64}\mathbf{H}_2^{-1}\mathbf{K}_2\mathbf{K}_1^{-1}\mathbf{H}_1\varphi_1 = \mathbf{T}_{81}\varphi_1 \quad (25)$$

where $\mathbf{T}_{81} = \mathbf{T}_{86}\mathbf{T}_{64}\mathbf{H}_2^{-1}\mathbf{K}_2\mathbf{K}_1^{-1}\mathbf{H}_1$ is the total transfer matrix of FB model.

3.2 Results Analysis and Discussion

Material parameter of the two annular plate models is given in Table 1. Material parameter for the RESIN I is given in Sect. 2.4, and structural parameters are also given in the following.

Here, $a_1 = 0.03$ m, $a_2 = 0.006$ m, $r_1 = 0.01$ m, $r_2 = 0.154$ m, $h = 0.003$ m.

Adopting the torsional transfer matrix of periodic model in Eq. (22) and FB model in Eq. (25), the vibration response

of outer radius can be picked up with the case of loading excitation force at inner radius. Furthermore, the torsional transmission response of periodic model and FB model can be calculated numerically by using MATLAB software.

Figure 6a clearly indicates that the vibration response approximately attenuates -38 dB, while torsional wave propagates in this single annular plate. Generally speaking, there is no vibration attenuation when wave propagates in a single model. This phenomenon reveals that the vibration attenuation -38 dB is mainly caused by the expansion of the radius as shown in Shu et al. 2014.

While torsional wave propagates in periodic model, the transmission response shows a big vibration attenuation band, namely bands A and B, depicted in Fig. 6. It should be noted that compared with the single model, the periodic model is more like to generate vibration attenuation band, which is very significant for torsional vibration reduction.

Compared with the periodic model, there are some newly emerging resonance frequencies 17.6 kHz and 33.1 kHz, while torsional wave propagates in FB model. This phenomenon indicates that the previous order of periodic model has been changed, which causes the resonance frequency of 17.6 kHz and 33.1 kHz. It should be noted that the newly emerging resonance frequency is disadvantage for vibration reduction. It can also be found that there is a newly emerging vibration attenuation band C for the FB model, while there is no vibration attenuation band for the previous periodic model. Additionally, compared with the periodic model, the frequency band obtained by FB model moves toward the low-frequency region, which is very meaningful for the torsional vibration reduction. Therefore, it can be clearly concluded that the FB model is easier to obtain a better low-frequency region but it has two resonance frequencies. In this section, periodic model and FB model are presented to illustrate the advantage and disadvantage of the vibration characteristic of the two models. According to the practice

Table 1 Material parameter

Material	Density ρ (kg/m ³)	Young modulus E (Pa)	Poisson's ratio ν
II (steel)	7850	2.11×10^{11}	0.33

(a) Theoretical result

(b) Finite element result

Fig. 6 Torsional transmission response of periodic model and FB model

engineering requirement, one can determine which model should be selected for the torsional vibration reduction.

Through adopting ANSYS 14.5 finite element software to simulate the torsional vibration transmission response, the harmonic response analysis has been carried out with the case of full algorithm and sparse solution. Loading torsional force on inner boundary and keeping the outer boundary free, the finite element simulation results can be obtained as shown in Fig. 6b. Comparing theoretical results with finite element simulations, the frequency bands of torsional vibration are in good agreement with each other which verifies the correctness of this calculation.

3.3 Influence Factors

3.3.1 Inner Radius

In this section, the influence of inner radius for the periodic model and FB model on the vibration transmission response is calculated, shown in Fig. 7a and b. It is depicted from Fig. 7 that with the inner radius increase, vibration reduction bands almost have no change. However, the vibration transmission response reduces heavily. Therefore, it can be concluded that inner radius is a main factor for the torsional vibration reduction.

Fig. 7 Influence of inner radius on transmission response. **a** Periodic model, **b** FB model

Fig. 8 Influence of length ratio on transmission response. **a** Periodic model, **b** FB model

3.3.2 Length Ratio

Figure 8 presents the influence of length ratio of these models on the vibration transmission response. For the periodic model, with the length ratio increasing, vibration reduction frequency bands almost have no change, while the vibration transmission response reduces heavily. The phenomena are similar to the case of influence of inner radius. For the FB model, with the length ratio decrease, transmission response reduction decibel changes rarely. And the transmission response reduction bands move toward the low frequency.

In summary, the length ratio has great influence on the vibration transmission response. The length ratio has different influence rules on the periodic model and FB model. Due to the advantage and disadvantage of the two models, one can determine which model is adopted for the reduction structures in the practice engineering requirements.

4 Conclusion

This work analytically presents the torsional vibration characteristic of the annular plate by using classical method and wave approach. From wave standpoint, free vibration and transmission response are analyzed. Additionally, the parameter's influence is also discussed in detail. The above researches show that:

1. Wave approach can be applied for the analysis of free vibration of annular plate. The natural frequencies calculated by wave approach agree well with the classical method results.
2. Compared with the periodic model, it can be found that the torsional vibration response of FB model appears two resonance frequencies in the attenuation band. This phenomenon is mainly caused by the disrupted order of the annular plate structures.
3. With regard to the periodic model and FB model, inner radius and length ratio have great influence on the transmission response. That is to say, one can achieve the required reduction band through selecting different parameters and models of the annular plate structures.

For the purposes of reducing torsional vibration, free vibration and transmission response of the annular plate are investigated numerically. This work is significant for torsional vibration reduction of machinery structures because the annular plate can be simplified a model of rotary body.

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