Some Inequalities for Analytic Functions in the Unit Disc

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Abstract

In Ozaki (Sci Rep Tokyo Bunrika Daigaku Sect A 4:45–87, 1941) proved that if $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is analytic in $\mathbb{D}$, $f'(z) \neq 0$ on $|z| = r < 1$, then the total variation of $\arg\{f(z)\}$ on $|z| = r$ is not greater than the total variation of $\arg\{df(z)\}$ on $|z| = r$, namely

$$\forall 0 < r < 1 : \int_{|z|=r} |\mathrm{d} \arg\{f(z)\}| \leq \int_{|z|=r} |\mathrm{d} \arg\{df(z)\}|.$$

We prove that without the integrals the above inequality implies $f(z) = z$. Furthermore, we prove that if $f'(z) \prec ((1+z)/(1-z))^2$, then $f(z)$ is starlike in $|z| < 0.24 \dots$

Keywords Univalent functions · Starlike · Convex · Starlike of order alpha

Mathematics Subject Classification Primary 30C45 · Secondary 30C80

1 Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the class of functions $f(z) \in \mathcal{H}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in \mathbb{D}). \quad (1.1)$$

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E . Let $f(z) \in \mathcal{H}$ and let $f(z)$ be univalent in $\mathbb{D}$. Then $f(z)$ maps $\mathbb{D}$ onto a convex domain if and only if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{D}). \quad (1.2)$$

Such function f is said to be convex in $\mathbb{D}$ (or briefly convex). Let $\mathcal{K}$ denote the subclass of $\mathcal{H}$ consisting of functions satisfying 1.2 and normalized by $f(0) = 0, f'(0) = 1$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{S}$ is said to be starlike of order α if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{D}), \quad (1.3)$$

for some $0 \leq \alpha < 1$. Condition (1.3) may be written as

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad (z \in \mathbb{D}), \quad (1.4)$$

where $\prec$ denotes the subordination. If $f(z)$ and $g(z)$ are analytic in $\mathbb{D}$, then we say that $f(z)$ is subordinate to $g(z)$ if there exists a function $w(z)$, which is analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{D}).$$

Therefore, $f(z) \prec g(z)$ implies $f(\mathbb{D}) \subset g(\mathbb{D})$. Moreover, if the function $g(z)$ is univalent in $\mathbb{D}$, then

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$$f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}) \iff f(z) \prec g(z). \quad (1.5)$$

We denote by $\mathcal{S}^*(\alpha)$ the class of functions starlike of order α . An univalent function $f \in \mathcal{S}$ belongs to the class of close-to-convex functions $\mathcal{C}$ if and only if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays). Equivalently, if $f \in \mathcal{A}$ satisfies

$$\Re \left\{ \frac{zf'(z)}{e^{i\alpha}g(z)} \right\} > 0, \quad z \in \mathbb{D} \quad (1.6)$$

for some $g(z) \in \mathcal{S}^*$ and some $\alpha \in (-\pi/2, \pi/2)$, then $f(z)$ is close-to-convex (with respect to $g(z)$) in $\mathbb{D}$ and denoted by $f(z) \in \mathcal{C}$. It is known that $\mathcal{C} \subset \mathcal{S}$ and it is known that if $f(z) \in \mathcal{A}$ and $f(z)$ satisfies (1.3), then $f(z)$ is univalent in $\mathbb{D}$.

A function $f(z)$ which is analytic in a domain $D \subset \mathbb{C}$ is called p -valent in D if for every complex number w , the equation $f(z) = w$ has at most p roots in D and there will be a complex number w_0 such that the equation $f(z) = w_0$, has exactly p roots in D . In Ozaki (1935) proved that if $f(z)$ of the form $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and is analytic in a convex domain $D \subset \mathbb{C}$ and for some real α we have

$$\Re \{ \exp(i\alpha) f^{(p)}(z) \} > 0 \quad z \in D, \quad (1.7)$$

then $f(z)$ is at most p -valent in D . Ozaki's condition is a generalization of the well-known Noshiro–Warschawski univalence condition with $p = 1$ (Noshiro 1934; Warschawski 1935). Let $\mathcal{A}(p)$ be the class of functions of the form

$$f(z) = z + \sum_{n=p}^{\infty} a_n z^n, \quad (z \in \mathbb{D}), \quad (1.8)$$

where $a_p \neq 0$ and $p \in \mathbb{N} = \{1, 2, \dots\}$.

2 Main result

If

$$\left| \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} \right| \leq n + 1 \quad (z \in \mathbb{D})$$

then $f(z) = z + a_{n+1}z^{n+1} + \dots$ is close to convex, (Avhadiev and Aksent'ev 1973). Umezawa in Umezawa (1955) proved that

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \sqrt{6} \quad (|z| \leq 1), \quad (2.1)$$

implies the univalence of $f(z)$ in $|z| \leq 1$. Notice also here that in Ozaki (1941) proved that if $f(z) = z + a_2 z^2 +$

$a_3 z^3 + \dots$ is analytic in $\mathbb{D}$, with $f(z)f'(z)/z \neq 0$ there, and if either

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) \geq -\frac{1}{2}$$

or

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) \leq \frac{3}{2} \quad (2.2)$$

holds throughout $\mathbb{D}$, then f is univalent and convex in at least one direction in $\mathbb{D}$. It has been generalized in Ogawa (1961), Shah (1973). The number $\sqrt{6}$ in (2.1), was improved to 3.05... in Kudryashov (1973). Notice that the condition

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \alpha z \quad (z \in \mathbb{D}),$$

$0 \leq \alpha < 2.832\dots$ is sufficient for starlikeness, (Miller and Mocanu 2000, p.273). It is known that for $f(z) \in \mathcal{A}$

$$\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > -1, \quad z \in \mathbb{D} \implies \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1}{2}, \quad z \in \mathbb{D}.$$

In Ozaki (1941) proved that if $f(z) \in \mathcal{H}$, $f'(z) \neq 0$ on $|z| = r < 1$, then the total variation of $\arg\{f(z)\}$ on $|z| = r$ is not greater than the total variation of $\arg\{df(z)\}$ on $|z| = r$, namely

$$\forall 0 < r < 1 : \int_{|z|=r} |d \arg\{f(z)\}| \leq \int_{|z|=r} |d \arg\{df(z)\}| \quad (2.3)$$

or by a modification in the above inequality,

$$\forall 0 < r < 1 : \int_0^{2\pi} \left| \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \right| d\theta \leq \int_0^{2\pi} \left| 1 + \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} \right| d\theta, \quad (2.4)$$

where $z = re^{i\theta}$ and $0 \leq \theta \leq 2\pi$. The first our theorem describes a further relation between

$$\frac{zf'(z)}{f(z)} \quad \text{and} \quad \frac{zf''(z)}{f'(z)}.$$

For some new conditions for starlikeness and strongly starlikeness of order alpha, we refer to our recent paper (Nunokawa and Sokół (2017)).

Theorem 2.1 Let $f(z) \in \mathcal{H}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $f'(z) \neq 0$ in $\mathbb{D}$, and suppose that there exists r_0 , $0 < r_0 \leq 1$, such that

$$\left| \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \geq \left| 1 + \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} \right|, \quad |z| < r_0. \quad (2.5)$$

Then $f(z) = z$ in $\mathbb{D}$.

Proof We have

$$\left[1 + \frac{zf''(z)}{f'(z)} \right]_{z=0} = 1, \quad \left[1 + \frac{zf'(z)}{f(z)} \right]_{z=0} = 1$$

so there exists a $r_1, r_2, r_1, r_2 \in (0, 1)$, such that

$$1 + \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0, \quad \text{for } |z| < r_1 \quad (2.6)$$

and

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad \text{for } |z| < r_2.$$

From the hypothesis of Theorem 2.1, we have

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1 + \Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0, \\ \text{for } |z| < R = \min\{r_0, r_1, r_2\}$$

and therefore, for all $r, 0 < r < R$

$$\int_{|z|=r} \left| \Re \frac{zf'(z)}{f(z)} \right| d\theta = \int_{|z|=r} \Re \frac{zf'(z)}{f(z)} d\theta \\ \geq \int_{|z|=r} \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} d\theta = \int_{|z|=r} \Re \left| 1 + \frac{zf''(z)}{f'(z)} \right| d\theta,$$

where $z = \rho e^{i\theta}$, and so

$$\forall 0 < r < R : \int_{|z|=r} \left| \Re \frac{zf'(z)}{f(z)} \right| d\theta \geq \int_{|z|=r} \Re \left| 1 + \frac{zf''(z)}{f'(z)} \right| d\theta.$$

Then, from Ozaki's theorem (2.4), we get

$$\forall 0 < r < R : \int_{|z|=r} \left| \Re \frac{zf'(z)}{f(z)} \right| d\theta = \int_{|z|=r} \Re \left| 1 + \frac{zf''(z)}{f'(z)} \right| d\theta.$$

Therefore, we have

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} = 0, \quad |z| < R$$

or

$$\Re \left\{ z \left[\log \frac{zf'(z)}{f(z)} \right]' \right\} = 0, \quad |z| < R.$$

An interpretation above equality provides

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \equiv c, \quad \text{for } 0 < |z| < R$$

where c is a real constant. Putting $z = 0$ gives us $c = 1$ or

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \equiv 1 \quad \text{for } 0 < |z| < R.$$

Applying Cauchy–Riemann's differential equations, we easily obtain

$$\frac{zf'(z)}{f(z)} \equiv 1 \quad \text{for } 0 < |z| < R$$

so, applying the Theorem of identity of analytic functions, we have $f(z) = z$ for $z \in \mathbb{D}$. It completes the proof. $\square$

Theorem 2.2 Let $f(z) \in \mathcal{A}$, $f'(z) \neq 0$ in $\mathbb{D}$, and suppose that

$$f'(z) \prec \left(\frac{1+z}{1-z} \right)^2 \quad z \in \mathbb{D}. \quad (2.7)$$

Then $f(z)$ is starlike in $|z| < r_0$, where $r_0 = 0.246964 \dots$ is the smallest positive root of the equation

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right) = \frac{\pi}{2} \left\{ \beta + \frac{2}{\pi} \tan^{-1}(\beta) \right\} \quad (2.8)$$

and $\beta = 0.38344486 \dots$ satisfies the condition

$$\tan^{-1}(\beta) = \frac{\pi(1-2\beta)}{2}. \quad (2.9)$$

Proof Subordination (2.7) follows that

$$\forall z \in \mathbb{D}, \quad |\arg\{f'(z)\}| < 2 \sin^{-1} \frac{2|z|}{1+|z|^2}. \quad (2.10)$$

In Nunokawa (1993), Nunokawa and Sokół (2017) it was proved that if $p(z) \in \mathcal{H}$, $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{D}$ and if there exists a point z_0 , $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \pi\beta/2 \quad \text{in } |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \pi\beta/2$$

for some $\beta \in (0, 2]$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta, \quad (2.11)$$

where $k \geq 1$, when $\arg\{p(z_0)\} = \pi\beta/2$ and $k \leq -1$, when $\arg\{p(z_0)\} = -\pi\beta/2$.

To apply this result for the proof of Theorem 2.2 assume that $p(z) = f(z)/z$. For the value $\beta = 0.38344486 \dots$ if there exists a point z_0 , $|z_0| = r_0 < 1$, such that

$$|\arg\{p(z)\}| < \frac{\pi}{2}\beta, \quad \text{for } |z| < r_0$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\beta.$$

Then from (2.11), for the case $\arg\{p(z_0)\} = \pi\beta/2$, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where $k \geq 1$. Because

$$\begin{aligned}\tan^{-1}\beta &\leq \arg\left\{1 + \frac{z_0 p'(z_0)}{p(z_0)}\right\} = \arg\{1 + i\beta k\} < \frac{\pi}{2}, \\ \arg\{p(z_0)\} &= \frac{\pi}{2}\beta \in \left[0, \frac{\pi}{2}\right),\end{aligned}$$

we have

$$\begin{aligned}\arg\{f'(z_0)\} &= \arg\{p(z_0) + z_0 p'(z_0)\} \\ &= \arg\left\{p(z_0)\left[1 + \frac{z_0 p'(z_0)}{p(z_0)}\right]\right\} \\ &= \arg\{p(z_0)\} + \arg\left\{1 + \frac{z_0 p'(z_0)}{p(z_0)}\right\} \\ &= \frac{\pi\beta}{2} + \arg\{1 + i\beta k\} \\ &\geq \frac{\pi\beta}{2} + \tan^{-1}\beta \\ &\geq 2\sin^{-1}\left\{\frac{2r_0}{1+r_0^2}\right\}.\end{aligned}$$

This contradicts condition (2.10). For the case $\arg\{p(z_0)\} = -\pi\beta/2$, applying the same method as above we can get

$$\arg\{f'(z_0)\} \leq -2\sin^{-1}\left\{\frac{2r_0}{1+r_0^2}\right\}.$$

This is also contradicts condition (2.10). Therefore, we have

$$\left|\arg\left\{\frac{f(z)}{z}\right\}\right| < \frac{\pi}{2}\beta \quad z \in \mathbb{D}.$$

Applying the above results, we have

$$\begin{aligned}\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| &= \left|\arg\{f'(z)\} + \arg\left\{\frac{z}{f(z)}\right\}\right| \\ &\leq |\arg\{f'(z)\}| + \left|\arg\left\{\frac{z}{f(z)}\right\}\right| \\ &< \frac{\pi}{2}\left(\beta + \frac{2}{\pi}\tan^{-1}\beta\right) + \frac{\pi\beta}{2} \\ &= \frac{\pi}{2}\left(2\beta + \frac{2}{\pi}\tan^{-1}\beta\right).\end{aligned}$$

Putting

$$2\beta + \frac{2}{\pi}\tan^{-1}\beta = 1 \quad (2.12)$$

or

$$\tan^{-1}\beta = \frac{\pi(1-2\beta)}{2}$$

and

$$\frac{\pi}{4}\left(\beta + \frac{2}{\pi}\tan^{-1}\beta\right) = \sin^{-1}\left\{\frac{2r_0}{1+r_0^2}\right\}, \quad (2.13)$$

then we have

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2} \quad \text{for } |z| < r_0.$$

From (2.12) and (2.13), we can get $\beta = 0.38344486\dots$ and $r_0 = 0.246964\dots$

It completes the proof of Theorem 2.2. $\square$

Notice that if we take $p(z) = f'(z)$ in the Theorem 2.2, then we obtain the following corollary.

Corollary 2.3 Let $p(z) \in \mathcal{H}$, $p(z) = 1 + p_1z + \dots$, $p(z) \neq 0$ in $\mathbb{D}$, and suppose that

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^2 \quad z \in \mathbb{D}, \quad (2.14)$$

then

$$\left|\arg\left\{\frac{zp(z)}{\int_0^z p(t)dt}\right\}\right| < \frac{\pi}{2} \quad \text{for } |z| < r_0, \quad (2.15)$$

where $r_0 = 0.246964\dots$ is described in the hypothesis of Theorem 2.2.

Theorem 2.4 Let $f(z) \in \mathcal{A}$, $(zf'(z))' \neq 0$ in $\mathbb{D}$, and suppose that

$$(zf'(z))' \prec \left(\frac{1+z}{1-z}\right)^2 \quad z \in \mathbb{D}. \quad (2.16)$$

Then

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2} \quad \text{for } |z| < r_0, \quad (2.17)$$

where $r_0 = 0.246964\dots$ is described in the hypothesis of Theorem 2.2. Condition (2.17) means that $f(z)$ is convex in $|z| < r_0$.

Proof The proof becomes trivial if we put $p(z) = (zf'(z))'$ in Corollary 2.3. $\square$

Theorem 2.5 Let $f(z) \in \mathcal{H}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and suppose that there exists a starlike function $g(z)$ of order α for which

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}, \quad z \in \mathbb{D}. \quad (2.18)$$

Then $f(z)$ is starlike of order at least α in $\mathbb{D}$.

Proof The above theorem is trivial, because (2.17) follows that

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in \mathbb{D}.$$

and so

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in \mathbb{D}.$$

This is exactly (1.4), so $f(z) \in \mathcal{S}^*(\alpha)$. $\square$

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