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RESEARCH PAPER

On Generalized Pell and Pell–Lucas Numbers

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Abstract
In this paper, we introduce and study a new one-parameter generalization of Pell numbers. We describe their distinct properties also related to matrix representation.

Keywords Fibonacci numbers · Pell numbers · Lucas numbers · Pell–Lucas numbers · k-Fibonacci numbers · k-Pell numbers

Mathematics Subject Classification 11B37 · 11C20 · 15B36 · 05C69

1 Introduction

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with $F_0 = F_1 = 1$. There are the well-known numbers of the Fibonacci type defined by the linear recurrence relation. We list some of them.

Lucas numbers:

$$L_n = L_{n-1} + L_{n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with } L_0 = 2,\ L_1 = 1, \tag{1.1}$$

Pell numbers:

$$P_n = 2P_{n-1} + P_{n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with } P_0 = 0,\ P_1 = 1, \tag{1.2}$$

Pell–Lucas numbers:

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with } Q_0 = Q_1 = 1, \tag{1.3}$$

Let $k \geq 1$ be an integer. Then, we have k-Fibonacci numbers (Falcon and Plaza 2007):

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with } F_{k,0} = 0,\ F_{k,1} = 1, \tag{1.4}$$

and k-Pell numbers (Catarino 2013):

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with } P_{k,0} = 0,\ P_{k,1} = 1. \tag{1.5}$$

In this paper, we introduce and study a new one-parameter generalization of the Pell numbers. Let $k \geq 2$, $n \geq 0$ be integers. We define generalized Pell numbers $P_{k,n}$ recurrently as follows:

$$P_{k,n} = kP_{k,n-1} + (k-1)P_{k,n-2} \quad \text{for } n \geq 2, \tag{1.6}$$

with initial conditions $P_{k,0} = 0$ and $P_{k,1} = 1$. Next, we define generalized Pell–Lucas numbers $Q_{k,n}$ satisfying the recursive recurrence of the form:

$$Q_{k,n} = kQ_{k,n-1} + (k-1)Q_{k,n-2} \quad \text{for } n \geq 2, \tag{1.7}$$

with initial conditions $Q_{k,0} = Q_{k,1} = 2$.

The tables presented below contain initial terms of the sequences $\{P_{k,n}\}$ and $\{Q_{k,n}\}$ for selected values of k (Tables 1 and 2).

As we can see, for $k = 2$ the classical Pell numbers and classical Pell–Lucas numbers are obtained. Sequences $\{P_{4,n}\}$, $\{P_{5,n}\}$, $\{P_{6,n}\}$, $\{P_{7,n}\}$ are listed in The Online Encyclopaedia of Integer Sequences (OEIS Foundation Inc. 2018) under the symbols A015530, A015537, A0155551 and A015564, respectively, while sequences

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Table 1 Initial terms of the generalized Pell numbers $P_{k,n}$

n	0	1	2	3	4	5	6	7	8	9	10
$P_{2,n}$	0	1	2	5	12	29	70	169	408	985	2378
$P_{3,n}$	0	1	3	11	39	139	495	1763	6279	22,363	79,647
$P_{4,n}$	0	1	4	19	88	409	1900	8827	41,008	190,513	885,076
$P_{5,n}$	0	1	5	29	165	941	5365	30,589	174,405	994,381	5,669,525
$P_{6,n}$	0	1	6	41	276	1861	12,546	84,581	570,216	3,844,201	25,916,286
$P_{7,n}$	0	1	7	55	427	3319	25,795	200,479	1,558,123	12,109,735	94,116,883

Table 2 Initial terms of the generalized Pell–Lucas numbers $Q_{k,n}$

n	0	1	2	3	4	5	6	7	8	9	10
$Q_{2,n}$	2	2	6	14	34	82	198	478	1154	2786	6726
$Q_{3,n}$	2	2	10	34	122	434	1546	5506	19,610	69,842	248,746
$Q_{4,n}$	2	2	14	62	290	1346	6254	29,054	134,978	627,074	2,913,230
$Q_{5,n}$	2	2	18	98	562	3202	18,258	104,098	593,522	3,384,002	17,313,532
$Q_{6,n}$	2	2	22	142	962	6482	43,702	294,622	1,986,242	12,212,079	75,258,716
$Q_{7,n}$	2	2	26	194	1514	11,762	91,418	710,498	5,521,994	42,916,946	333,550,586

$\{\frac{1}{2}Q_{2,n}\}$, $\{\frac{1}{2}Q_{4,n}\}$, $\{\frac{1}{2}Q_{5,n}\}$, $\{\frac{1}{2}Q_{6,n}\}$, under the symbols A001333, A086901, A123270 and A123362, respectively.

The recurrence relations (1.6) and (1.7) generate characteristic equation of the form

$$r^2 - kr + 1 - k = 0. \tag{1.8}$$

Since $k \geq 2$, this equation has two roots

$$r_1 = \frac{1}{2}\left(k - \sqrt{k^2 + 4k - 4}\right),\quad r_2 = \frac{1}{2}\left(k + \sqrt{k^2 + 4k - 4}\right). \tag{1.9}$$

Let us observe that for $k \geq 2$ we have $r_1 < 0 < r_2 < 1$ and $|r_1| < r_2$. Moreover, the following interrelationships hold true:

$$r_1 + r_2 = k \tag{1.10}$$

$$r_1 - r_2 = -\sqrt{k^2 + 4k - 4} \tag{1.11}$$

$$r_1 r_2 = 1 - k. \tag{1.12}$$

Next we find generating functions for the generalized Pell numbers $P_{k,n}$. Let us suppose that $P_{k,n}$ are coefficients of a power series with center at the origin and that $f_k(x)$ is the sum of this series, i.e.,

$$f_k(x) = \sum_{n=0}^{\infty} P_{k,n} \cdot x^n. \tag{1.13}$$

Such an analytic function f_k is generating function for this sequence. Taking into account (1.6) and using the initial conditions $P_{k,0} = 0$ and $P_{k,1} = 1$, we get after some calculations

$$\begin{aligned} f_k(x) &= \sum_{n=0}^{\infty} P_{k,n} \cdot x^n \\ &= P_{k,0} + P_{k,1} \cdot x + \sum_{n=2}^{\infty}\left[kP_{k,n-1} + (k-1)P_{k,n-2}\right]x^n \\ &= x + kx\sum_{n=2}^{\infty} P_{k,n-1}x^{n-1} + (k-1)x^2\sum_{n=2}^{\infty} P_{k,n-2}x^{n-2} \\ &= x + kx\sum_{n=1}^{\infty} P_{k,n}x^n + (k-1)x^2\sum_{n=0}^{\infty} P_{k,n}x^n. \end{aligned}$$

Now by using (1.13), we obtain $f_k(x) = x + [kx + (k-1)x^2] \cdot f_k(x)$, and then the generating function for the sequence $(P_{k,n})$ is of the form

$$f_k(x) = \frac{x}{1 - kx - (k-1)x^2}. \tag{1.14}$$

Now we find generating functions for the sequence $\{Q_{k,n}\}$. Let a generating function for this sequence be denoted by $g_k(x)$. Then, we have

$$g_k(x) = \sum_{n=0}^{\infty} G_{k,n} \cdot x^n. \tag{1.15}$$

Using recurrence (1.7) and initial conditions $Q_{k,0} = Q_{k,1} = 2$ we have

$$\begin{aligned} g_k(x) &= \sum_{n=0}^{\infty} Q_{k,n} \cdot x^n \\ &= Q_{k,0} + Q_{k,1} \cdot x + \sum_{n=2}^{\infty}\left[kQ_{k,n-1} + (k-1)Q_{k,n-2}\right]x^n \\ &= 2 + 2x + kx\sum_{n=2}^{\infty} Q_{k,n-1}x^{n-1} + (k-1)x^2\sum_{n=2}^{\infty} Q_{k,n-2}x^{n-2} \\ &= 2 + 2x + kx\sum_{n=1}^{\infty} Q_{k,n}x^n + (k-1)x^2\sum_{n=0}^{\infty} Q_{k,n}x^n. \end{aligned}$$

After taking into account (1.15), we get $g_k(x) = 2 + 2x - 2kx + [kx + (k-1)x^2] \cdot g_k(x)$ and therefore

$$g_k(x) = \frac{2[1-(k-1)x]}{1-kx-(k-1)x^2}. \tag{1.16}$$

2 Explicit Formulas and Its Applications

In this section, we give explicit formulas for the general terms of the sequences $\{P_{k,n}\}$ $\{Q_{k,n}\}$ and next apply them to derive some identities. We start with the following theorem.

Theorem 2.1 *The n-th terms of the sequences $\{P_{k,n}\}$ and $\{Q_{k,n}\}$ are of the form*

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{2.1}$$

$$Q_{k,n} = \frac{2(1-r_2)}{r_1 - r_2} r_1^n + \frac{2(r_1 - 1)}{r_1 - r_2} r_2^n, \tag{2.2}$$

$$P_{k,n} = \frac{1}{2^n\sqrt{k^2+4k-4}} \times \left[\left(k+\sqrt{k^2+4k-4}\right)^n - \left(k-\sqrt{k^2+4k-4}\right)^n\right]$$

$$Q_{k,n} = \frac{\left(2-k+\sqrt{k^2+4k-4}\right)\left(k+\sqrt{k^2+4k-4}\right)^n - \left(2-k-\sqrt{k^2+4k-4}\right)\left(k-\sqrt{k^2+4k-4}\right)^n}{2^n\sqrt{k^2+4k-4}}.$$

respectively, where r_1, r_2 are given in (1.9).

Proof Since the characteristic Eq. (1.8) has two distinct roots r_1, r_2, the numbers r_1^n, r_2^n are linearly invariant and form the basis for the space of all solutions of the Eqs. (1.6) and (1.7). Therefore

$$P_{k,n} = a(r_1)^n + b(r_2)^n \tag{2.3}$$

is the solution of the recurrence (1.6). Putting $P_{k,0} = 0$ and $P_{k,1} = 1$ we obtain $a + b = 0$ and $ar_1 + br_2 = 1$. Solving this system of linear equations, we get $a = -b = \frac{1}{r_1 - r_2}$. After putting these values of a and b in (2.3), we obtain (2.1) as required. We will use the analogical reasoning for the sequence $\{Q_{k,n}\}$. Then, the solution of the recurrence (1.7) is the following

$$Q_{k,n} = c(r_1)^n + d(r_2)^n. \tag{2.4}$$

Giving to n the values $n = 0$ and $n = 1$ and solving the system of equations created in this way, we get the unique solution (c, d) where $c = \frac{2(1-r_2)}{r_1 - r_2}$ and $d = \frac{-2(1-r_1)}{r_1 - r_2}$. Substituting these values in (2.4), we obtain (2.2). □

Now we give the identity showing the relationship between numbers $P_{k,n}$ and $Q_{k,n}$.

Corollary 2.2 *For all integers $k \geq 2$ and $n \geq 0$ we have*

$$Q_{k,n} = 2[P_{k,n} + (k-1)P_{k,n-1}]. \tag{2.5}$$

Proof We will make some transformations of the $Q_{k,n}$ given by the formula (2.2)

$$\begin{aligned} Q_{k,n} &= \frac{2}{r_1 - r_2}\left[r_1^n - r_1^n r_2 + r_1 r_2^n - r_2^n\right] \\ &= \frac{2}{r_1 - r_2}\left[(r_1^n - r_2^n) - r_1 r_2 (r_1^{n-1} - r_2^{n-1})\right] \\ &= 2\left[\frac{r_1^n - r_2^n}{r_1 - r_2} - r_1 r_2 \frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\right]. \end{aligned}$$

Now, after applying (1.12) and (2.1), we immediately obtain the required result (2.5). □

As a consequence of Binet's formulas for the sequences $\{P_{k,n}\}$ and $\{Q_{k,n}\}$, the following results can be obtained:

Corollary 2.3 *For all integers $k \geq 2$ and $n \geq 0$*

In particular case, if $k = 2$, we have the well-known formulas for the Pell sequence and the Pell–Lucas sequence, respectively.

$$P_n = \frac{\sqrt{2}}{4}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right],$$

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

Lemma 2.4 *For all integers $k \geq 2$ and $n \geq 0$ we have*

$$\lim_{n\to\infty} \frac{P_{k,n+1}}{P_{k,n}} = r_2.$$

Proof By using (2.1) we have

$$\lim_{n\to\infty} \frac{P_{k,n+1}}{P_{k,n}} = \lim_{n\to\infty} \frac{r_2^{n+1} - r_1^{n+1}}{r_2^n - r_1^n}$$

The relation $|r_1| < |r_2|$ implies that $\lim_{n\to\infty}\left(\frac{r_1}{r_2}\right)^n = 0$ and therefore $\lim_{n\to\infty} \frac{P_{k,n+1}}{P_{k,n}} = r_2$. □

For $k = 2$, it can be obtained that $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = 1 + \sqrt{2}$ for the classical Pell sequence. The number $1 + \sqrt{2}$ is known as silver ratio. For the sequence $P_{3,n}$ is $\lim_{n\to\infty} \frac{P_{3,n+1}}{P_{3,n}} = \frac{1}{2}(3 + \sqrt{17})$.

By Lemma 2.4, we obtain that the radius of convergence of the series given in (1.13) is equal to $\frac{1}{r_2} > 1$. Therefore, we get the following equality for all integers $k \geq 2$

$$f_k(x) = \sum_{n=0}^{\infty} P_{k,n} \cdot x^n \quad \text{for} \quad x \in \left(-\frac{1}{r_2}, \frac{1}{r_2}\right).$$

The generating function can now be used to give another form of the general term of the sequence $(P_{k,n})$.

Theorem 2.5 *For all integers* $k \geq 2$ and $n \geq 1$ *we have*

$$P_{k,n} = \frac{f_k^{(n)}(0)}{n!},$$

where $f^{(n)}$ *denotes as usual the n-th-order derivative of the function f(x).*

Proof The desired result is a simple consequence of the well-known relationship between coefficients of the Maclaurin series for the function and a value at the point 0 of its *n*th derivative. □

The Binet's formulas (2.1) and (2.2) can be very useful to derive some identities for $\{P_{k,n}\}$ and $\{Q_{k,n}\}$. We give them in the following theorem.

Theorem 2.6 (Catalan's identity) *For any positive integer r we have*

$$P_{k,n-r}P_{k,n+r} - P_{k,n}^2 = -(1-k)^{n-r}P_{k,r}^2, \tag{2.6}$$

$$Q_{k,n-r}Q_{k,n+r} - Q_{k,n}^2 = -8(1-k)^{n+1-r}P_{k,r}^2. \tag{2.7}$$

Proof First we will prove (2.6). By using (2.1) and (1.12), we obtain

$$\begin{aligned}
&P_{k,n-r}P_{k,n+r} - P_{k,n}^2 \\
&= \frac{r_2^{n-r} - r_1^{n-r}}{r_2 - r_1} \cdot \frac{r_2^{n+r} - r_1^{n+r}}{r_2 - r_1} - \left(\frac{r_2^n - r_1^n}{r_2 - r_1}\right)^2 \\
&= \frac{2(r_1r_2)^n - r_2^{n-r}r_1^{n+r} - r_1^{n-r}r_2^{n+r}}{(r_2 - r_1)^2} \\
&= \frac{(r_1r_2)^n}{(r_2 - r_1)^2}\left[2 - \left(\frac{r_1}{r_2}\right)^r - \left(\frac{r_2}{r_1}\right)^r\right] \\
&= \frac{(1-k)^n}{(r_2 - r_1)^2}\left[2 - \frac{r_1^{2r} + r_2^{2r}}{(r_1r_2)^r}\right] \\
&= \frac{-(1-k)^n}{(1-k)^r(r_2 - r_1)^2} \times \left[r_1^{2r} - 2(r_1r_2)^r + r_2^{2r}\right] \\
&= -(1-k)^{n-r}\left(\frac{r_2^r - r_1^r}{r_2 - r_1}\right)^2.
\end{aligned}$$

Repeated application of (2.1) gives the required result. In the proof of (2.7), we use the Binet's formula (2.2) for $Q_{k,n}$. Putting $c = \frac{2(1-r_2)}{r_1 - r_2}$ and $d = \frac{2(r_1 - 1)}{r_1 - r_2}$, we have

$$\begin{aligned}
&Q_{k,n-r}Q_{k,n+r} - Q_{k,n}^2 \\
&= \left(cr_1^{n-r} + dr_2^{n-r}\right)\left(cr_1^{n+r} + dr_2^{n+r}\right) - \left(cr_1^n + dr_2^n\right)^2 \\
&= c^2r_1^{2n} + cd(r_1^{n-r}r_2^{n+r} + r_1^{n+r}r_2^{n-r}) \\
&\quad + d^2r_2^{2n} - c^2r_1^{2n} - 2cdr_1^nr_2^n - d^2r_2^{2n} \\
&= cd(r_1r_2)^n\left[\left(\frac{r_1}{r_2}\right)^r + \left(\frac{r_2}{r_1}\right)^r - 2\right] \\
&= -4(1-r_1)(1-r_2)\frac{(r_1r_2)^n}{(r_1 - r_2)^2} \\
&\quad \cdot \frac{(r_1)^{2r} - 2(r_1r_2)^r + (r_2)^{2r}}{(r_1r_2)^r} \\
&= -4[1 - (r_1 + r_2) + r_1r_2](r_1r_2)^{n-r} \times \left(\frac{r_1^r - r_2^r}{r_1 - r_2}\right)^2.
\end{aligned}$$

Now using (1.10) and (1.12) we obtain $Q_{k,n-r}Q_{k,n+r} - Q_{k,n}^2 = -8(1-k)^{n+1-r}P_{k,r}^2$ as required. □

For $k = 2$, the following identities can be deduced:

$$P_{n-r}P_{n+r} - P_n^2 = (-1)^{n+1-r}P_r^2,$$
$$Q_{n-r}Q_{n+r} - Q_n^2 = 8(-1)^{n+2-r}P_r^2.$$

First of them is the well-known Catalan's identity for Pell numbers. Giving to r the value 1, we can write two identities $P_{n-1}P_{n+1} - P_n^2 = (-1)^n$ and $Q_{n-1}Q_{n+1} - Q_n^2 = 8(-1)^{n+1}$ known as Simpson formulas for the Pell numbers (Horadam 1971) and for the Pell–Lucas numbers (Horadam and Mahon 1985).

Now we give next identities in the following theorem.

Theorem 2.7 *Let m, n be positive integers and* $m \geq n$. *Then, we have*

$$P_{k,m}P_{k,n+1} - P_{k,m+1}P_{k,n} = (1-k)^nP_{k,m-n} \tag{2.8}$$

and

$$Q_{k,m}Q_{k,n+1} - Q_{k,m+1}Q_{k,n} = 8(1-k)^{n+1}P_{k,m-n}. \tag{2.9}$$

Proof Similarly that before we use (2.1). After some calculation, we obtain

$$\begin{aligned}
&P_{k,m}P_{k,n+1} - P_{k,m+1}P_{k,n} \\
&= \frac{-1}{(r_2 - r_1)^2}\left[r_1^{n+1}r_2^m + r_1^mr_2^{n+1} - r_1^nr_2^{m+1} - r_1^{m+1}r_2^n\right] \\
&= \frac{1}{(r_2 - r_1)^2}\left[r_1^nr_2^m(r_2 - r_1) - r_1^mr_2^n(r_2 - r_1)\right] \\
&= (r_1r_2)^n\frac{r_2^{m-n} - r_1^{m-n}}{r_2 - r_1}.
\end{aligned}$$

Then, the conclusion can be directly obtained by (1.12) and (2.1). □

Note that for $k = 2$ d'Ocagne's identity for Pell numbers appears:

$$P_mP_{n+1} - P_{m+1}P_n = (-1)^n P_{m-n}.$$

Here, next identities for generalized Pell numbers and generalized Pell–Lucas numbers will be shown. They will allow to express the sum of the first terms of the sequences $\{P_{k,n}\}$ and $\{Q_{k,n}\}$ in concise form.

Theorem 2.8 *For all integers $k \geq 2$, $n \geq 0$ we have*

$$\sum_{i=0}^{n} P_{k,i} = \frac{1}{2(k-1)} \times \left[(k-1)P_{k,n} + P_{k,n+1} - 1\right] \tag{2.10}$$

and

$$\sum_{i=0}^{n} Q_{k,i} = \frac{1}{2(k-1)} \left[(k-1)Q_{k,n} + Q_{k,n+1} + 2k - 4\right]. \tag{2.11}$$

Proof Note that

$$\sum_{i=0}^{n} P_{k,i} = \frac{1}{r_2 - r_1}\left[\sum_{i=0}^{n} r_2^n - \sum_{i=0}^{n} r_1^n\right].$$

Thus, by summing up the geometric partial sums, we obtain

$$\begin{aligned}\sum_{i=0}^{n} P_{k,i} &= \frac{1}{r_2 - r_1}\left(\frac{1 - r_2^{n+1}}{1 - r_2} - \frac{1 - r_1^{n+1}}{1 - r_1}\right)\\ &= \frac{(1-r_2)^{n+1}(1-r_1) - (1-r_1)^{n+1}(1-r_2)}{(r_2 - r_1)(1-r_1)(1-r_2)}\\ &= \frac{(r_2 - r_1) - \left(r_2^{n+1} - r_1^{n+1}\right) + r_1r_2\left(r_2^n - r_1^n\right)}{(r_2 - r_1)(1-r_1)(1-r_2)}\\ &= \frac{1}{(1-r_1)(1-r_2)} \times \left[1 - \frac{r_2^{n+1} - r_1^{n+1}}{r_2 - r_1} + r_1r_2\frac{r_2^n - r_1^n}{r_2 - r_1}\right].\end{aligned}$$

Now, again by using (1.12) and (2.1) and by noting that $(1-r_1)(1-r_2) = 2(1-k)$, the result is obtained. For the proof of the identity (2.11) we use Binet's formula (2.2) for $Q_{k,n}$ with $c = \frac{2(1-r_2)}{r_1 - r_2}$ and $d = \frac{2(r_1 - 1)}{r_1 - r_2}$. Then, we obtain

$$\begin{aligned}\sum_{i=0}^{n} Q_{k,i} &= \sum_{i=0}^{n}(cr_1^n + dr_2^n) = c\sum_{i=0}^{n} r_1^n + d\sum_{i=0}^{n} r_2^n\\ &= c\frac{1 - r_2^{n+1}}{1 - r_2} + d\frac{1 - r_1^{n+1}}{1 - r_1}\\ &= \frac{c(1 - r_1^{n+1})(1-r_2) + d(1 - r_2^{n+1})(1-r_1)}{(1-r_1)(1-r_2)}\\ &= \frac{(c+d) - (cr_2 + dr_1) - (cr_1^{n+1} + dr_2^{n+1}) + r_1r_2(cr_1^n + dr_2^n)}{(1-r_1)(1-r_2)}.\end{aligned}$$

A simple calculation gives $c + d = 2$ and $cr_2 + dr_1 = 2(k-1)$. Therefore, after using (1.12) and once more the equality $(1-r_1)(1-r_2) = 2(1-k)$ and Binet's formula for $Q_{k,n}$ we get

$$\begin{aligned}\sum_{i=0}^{n} Q_{k,i} &= \frac{1}{1-k} + 2 - \frac{1}{2(1-k)}Q_{k,n+1} + \frac{1}{2}Q_{k,n}\\ &= \frac{1}{2(k-1)}\left[(k-1)Q_{k,n} + Q_{k,n+1} + 2k - 4\right].\end{aligned}$$

The proof is completed. □

As a consequence of (2.10) and (2.11), we can give the following identities for Pell numbers and Pell–Lucas numbers:

$$\sum_{i=0}^{n} P_i = \frac{1}{2}(P_n + P_{n+1} - 1)$$

$$\sum_{i=0}^{n} Q_i = \frac{1}{2}(Q_n + Q_{n+1}).$$

3 Application of Matrices

One of the most popular methods for study of the sequences defined recursively is to use matrices. Application of the so-called *matrix generator* or *generating matrix* not only allows us to derive many identities for given sequence but also provides relatively simple proofs of them. It is well known that the numbers of the Fibonacci sequence are generated by so-called *Q-matrix*

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

since by taking successive positive powers of Q one can obtain

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

The authorship of the name *Q-matrix* is assigned to King (1960) by Gould (1981) which in his article included more historical details about the Q-matrix, its other versions and its generalizations. In 1979, Silvester (1979) derived the properties of the Fibonacci numbers from its matrix representation. Bicknell (1975) showed that

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

is the generating matrix for the Pell sequence. A constructive method of finding all possible matrix generators of Pell sequence was given by Ercolano (1979) who showed that all these matrices are related to each other. Also in recent decades many authors use the matrix generators method to study a variety of sequences defined by recursion. For example, in Bednarz et al. (2013) and Włoch et al. (2013) the authors construct matrix representations of Fibonacci numbers generalized in the distance sense, while in Catarino (2013), Catarino and Vasco (2013) two-by-two matrices are applied to derive Cassini identities for k-Pell and k-Lucas sequences. In another recent paper Spivey (2006), the sum property of determinants is used to give a new proof of the well-known identity for Fibonacci sequence $F_mF_n - F_{m-r}F_{n+r} = (-1)^{m-r}F_rF_{n+r-m}$. The next example of using matrix generators can be found in the paper Dasdemir (2011).

The permanent of a matrix $A = [a_{i,j}]_{n\times n}$ is defined as follows:

$$perA = \sum_{\sigma\in S_n} a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$$

where S_n denotes the symmetric group of all permutations of the indices $\{1,2,\ldots,n\}$, see, e.g., Minc (1978). The permanent of a square matrix is very similar to its determinant. The difference is that, in the case of permanent, all the signs occurring in the Laplace expansion of minors, are positive. The matrix $A = [a_{i,j}]_{m\times n}$ with row vectors $r_1, r_2, \ldots, r_m$ and column vectors $c_1, c_2, \ldots, c_n$ is called *contractible on column* k if its column c_k has exactly two nonzero elements. Similarly, A is called *contractible on row* k provided its row r_k has exactly two nonzero elements. Let A be contractible on column k with two different nonzero entries $a_{i,k}$ and $a_{j,k}$ where $i\neq j$. Let us transform matrix A by replacing its ith row r_i with $a_{j,k}r_i + a_{i,k}r_j$ and then deleting row r_j and column c_k. This new $(m-1)\times(n-1)$ matrix is called *contraction of A on column k relative to rows i and j* (Brualdi 2006).

We shall use contractions to prove that numbers $P_{k,n}$ can be expressed as permanents of some matrix. For integer $n\geq 1$, let $D_n(k) = [d_{i,j}]_{n\times n}$ denote tridiagonal matrix whose entries on the main diagonal are equal to k, entries on the subdiagonal are 1, on the super-diagonal equal to $k-1$ and the other entries equal 0. In other words, nonzero entries of this matrix are the following $d_{i,i} = k$, $d_{i,i+1} = k-1$, $d_{i+1,i} = 1$, $i = 1,2,\ldots,n$. This definition gives

$$D_n(k) = \begin{bmatrix} k & k-1 & 0 & \ldots & 0 & 0 \\ 1 & k & k-1 & \ldots & 0 & 0 \\ 0 & 1 & k & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & k \end{bmatrix}.$$

Let us recall the useful result

Lemma 3.1 (Brualdi 2006) *Let A be a nonnegative integral matrix of order $n>1$ and B be a contraction of A. Then $perA = perB$.*

Then, we have the following theorem.

Theorem 3.2 *Let $n\geq 1$, $k\geq 2$ be integers. Then $perD_n(k) = P_{k,n+1}$.*

Proof For $n=1$ we have $perD_1(k) = per[k] = kP_{k,1} + (k-1)P_{k,0} = P_{k,2}$. For $n=2$ we obtain

$$D_2(k) = \begin{bmatrix} k & k-1 \\ 1 & k \end{bmatrix}$$

and $perD_2(k) = k^2 + k - 1 = kP_{k,2} + (k-1)P_{k,1} = P_{k,3}$. All contractions of the matrices that will appear in what follows will be contractions on the column 1 relative to rows 1 and 2. Let $D_n^{(i)}$ denote the matrix being ith such contraction of the matrix $D_n(k)$. Of course, $D_n^{(i)}$ has dimensions $(n-i)\times(n-i)$. Let $n\geq 2$. Then, first contraction of the matrix $D_n(k)$ has a form

$$D_n^{(1)} = \begin{bmatrix} k^2+k-1 & k(k-1) & 0 & \ldots & 0 & 0 \\ 1 & k & k-1 & \ldots & 0 & 0 \\ 0 & 1 & k & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & k \end{bmatrix} = \begin{bmatrix} P_{k,3} & (k-1)P_{k,2} & 0 & \ldots & 0 & 0 \\ 1 & k & k-1 & \ldots & 0 & 0 \\ 0 & 1 & k & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & k \end{bmatrix}.$$

Contraction of $D_n^{(1)}$ becomes

$$D_n^{(2)} = \begin{bmatrix} kP_{k,3}+(k-1)P_{k,2} & (k-1)P_{k,3} & \ldots & 0 & 0 \\ 1 & k & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & k \end{bmatrix} = \begin{bmatrix} P_{k,4} & (k-1)P_{k,3} & \ldots & 0 & 0 \\ 1 & k & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & k \end{bmatrix}.$$

In general, for $1\leq s\leq n-4$, we obtain

$$D_n^{(s)} = \begin{bmatrix} P_{k,s+2} & (k-1)P_{k,s+1} & \ldots & 0 & 0 \\ 1 & k & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & k \end{bmatrix}.$$

Therefore, $D_n^{(n-3)}$ is the following 3×3 matrix

$$D_n^{(n-3)} = \begin{bmatrix} P_{k,n-1} & (k-1)P_{k,n-2} & 0 \\ 1 & k & k-1 \\ 0 & 1 & k \end{bmatrix}$$

and after the last contraction appears

$$D_n^{(n-2)} = \begin{bmatrix} kP_{k,n-1} + (k-1)P_{k,n-2} & (k-1)P_{k,n-2} \\ 1 & k \end{bmatrix} = \begin{bmatrix} P_{k,n} & (k-1)P_{k,n-1} \\ 1 & k \end{bmatrix}.$$

Hence, by Lemma 3.1 and by definition of generalized Pell numbers, we get

$$perD_n(k) = perD_n^{(n-2)} = kP_{k,n} + (k-1)P_{k,n-1} = P_{k,n+1}.$$

□

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