Further Results for Starlike Functions Related with Booth Lemniscate

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Abstract

In this paper we investigate an interesting subclass $\mathcal{BS}(\alpha)$ ($0 \leq \alpha < 1$) of starlike functions in the unit disc Δ . The class $\mathcal{BS}(\alpha)$ was introduced by Kargar et al. (Anal Math Phys, 2017. <https://doi.org/10.1007/s13324-017-0187-3>) which is strongly related to the Booth lemniscate. Some geometric properties of this class of analytic functions, including radius of starlikeness of order γ ($0 \leq \gamma < 1$), the image of $f(\{z : |z| < r\})$ when $f \in \mathcal{BS}(\alpha)$, an special example and estimate of bounds for $\operatorname{Re}\{f(z)/z\}$, are studied.

Keywords Booth lemniscate · Radius of starlikeness · Starlike function · Convex function · Subordination

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Also let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ including of functions normalized by $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$ that consists of all univalent functions $f(z)$ in Δ is denoted by $\mathcal{S}$. We denote by $\mathfrak{B}$ the class of functions $w(z)$ analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, ($z \in \Delta$). For two analytic and normalized functions f and g , we say that f is subordinate to g , written $f \prec g$ in Δ , if there exists a function $w \in \mathfrak{B}$ such that $f(z) = g(w(z))$ for all $z \in \Delta$. In special case, if the function g is univalent in Δ , then

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta)).$$

It is easy to see that for any complex numbers $\lambda \neq 0$ and μ , we have:

$$f(z) \prec g(z) \Rightarrow \lambda f(z) + \mu \prec \lambda g(z) + \mu. \quad (1.1)$$

The set of all functions $f \in \mathcal{A}$ that are starlike univalent in Δ will be denoted by $\mathcal{S}^*$ and the set of all functions $f \in \mathcal{A}$ that are convex univalent in Δ will be denoted by $\mathcal{K}$. Robertson (see Robertson 1985) introduced and studied the class $\mathcal{S}^*(\gamma)$ of starlike functions of order $\gamma \leq 1$ as follows

$$\mathcal{S}^*(\gamma) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, z \in \Delta \right\}.$$

We note that if $\gamma \in [0, 1)$, then a function in $\mathcal{S}^*(\gamma)$ is univalent. Also we say that $f \in \mathcal{K}(\gamma)$ (the class of convex functions of order γ) if and only if $zf'(z) \in \mathcal{S}^*(\gamma)$. In particular we put $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$.

Recently, Kargar et al. (2017) introduced and studied a class functions related to the Booth lemniscate as follows.

Definition 1.1 (see Kargar et al. 2017) The function $f \in \mathcal{A}$ belongs to the class $\mathcal{BS}(\alpha)$, $0 \leq \alpha < 1$, if it satisfies the condition

$$\left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{z}{1 - \alpha z^2} \quad (z \in \Delta). \quad (1.2)$$

Recall that (Piejko and Sokół 2013), a one-parameter family of functions given by

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$$F_\alpha(z) := \frac{z}{1 - \alpha z^2} = \sum_{n=1}^{\infty} \alpha^{n-1} z^{2n-1} \quad (z \in \Delta, 0 \leq \alpha \leq 1). \quad (1.3)$$

are starlike univalent when $0 \leq \alpha \leq 1$ and are convex for $0 \leq \alpha \leq 3 - 2\sqrt{2} \approx 0.1715$. We have also $F_\alpha(\Delta) = D(\alpha)$, where

$$D(\alpha) = \left\{ x + iy \in \mathbb{C} : (x^2 + y^2)^2 - \frac{x^2}{(1 - \alpha)^2} - \frac{y^2}{(1 + \alpha)^2} < 0, (0 \leq \alpha < 1) \right\} \quad (1.4)$$

and

$$D(1) = \{x + iy \in \mathbb{C} : (\forall t \in (-\infty, -i/2] \cup [i/2, \infty)) [x + iy \neq it]\}. \quad (1.5)$$

It is clear that the curve

$$(x^2 + y^2)^2 - \frac{x^2}{(1 - \alpha)^2} - \frac{y^2}{(1 + \alpha)^2} = 0 \quad (x, y) \neq (0, 0),$$

is the Booth lemniscate of elliptic type (see Fig. 1, for $\alpha = 1/3$). For more details, see Kargar et al. (2017).

Lemma 1.1 (see Kargar et al. 2017) *Let $F_\alpha(z)$ be given by (1.3). Then for $0 \leq \alpha < 1$, we have*

$$\frac{1}{\alpha - 1} < \operatorname{Re}\{F_\alpha(z)\} < \frac{1}{1 - \alpha} \quad (z \in \Delta). \quad (1.6)$$

Therefore by definition of subordination and by Lemma 1.1, $f \in \mathcal{A}$ belongs to the class $\mathcal{BS}(\alpha)$, if it satisfies the condition

$$\frac{\alpha}{\alpha - 1} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{2 - \alpha}{1 - \alpha} \quad (z \in \Delta). \quad (1.7)$$

The following lemma will be useful.

Fig. 1 $(x^2 + y^2)^2 - 9x^2/4 - 9y^2/16 = 0$

Lemma 1.2 (see Ruscheweyh and Stankiewicz 1985) *Let $F, G \in \mathcal{H}$ be any convex univalent functions in Δ . If $f \prec F$ and $g \prec G$, then*

$$f * g \prec F * G \quad \text{in } \Delta.$$

In this work, some geometric properties of the class $\mathcal{BS}(\alpha)$ are investigated.

2 Main Results

We start with the following lemma that gives the structural formula for the function of the considered class.

Lemma 2.1 *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{BS}(\alpha)$, $0 \leq \alpha < 1$, if and only if there exists an analytic function q , $q(0) = 0$ and $q \prec F_\alpha$ such that*

$$f(z) = z \exp\left(\int_0^z \frac{q(t)}{t} dt\right). \quad (2.1)$$

The proof is easy. Putting $q = F_\alpha$ in Lemma 2.1 we obtain the function

$$\tilde{f}(z) = z \left(\frac{1 + z\sqrt{\alpha}}{1 - z\sqrt{\alpha}} \right)^{\frac{1}{2\sqrt{\alpha}}}, \quad (2.2)$$

which is extremal function for several problems in the class $\mathcal{BS}(\alpha)$. Moreover, we consider

$$F(z) := \frac{\tilde{f}(z)}{z} = \left(\frac{1 + z\sqrt{\alpha}}{1 - z\sqrt{\alpha}} \right)^{\frac{1}{2\sqrt{\alpha}}} = 1 + z + \frac{1}{2}z^2 + \frac{1}{3}\left(\alpha + \frac{1}{2}\right)z^3 \cdots \quad (2.3)$$

From (1.7) we conclude that $f \in \mathcal{BS}(\alpha)$ is starlike of order $\frac{\alpha}{\alpha - 1} < 0$, hence f may not be univalent in Δ . It may therefore be interesting to consider a problem to find the radius of starlikeness of order γ , $\gamma \in [0, 1)$ (hence univalence) of the class $\mathcal{BS}(\alpha)$, i.e. the largest radius $r_s(\alpha, \gamma)$ such that each function $f \in \mathcal{BS}(\alpha)$ is starlike of order γ in the disc $|z| < r_s(\alpha, \gamma)$. For this purpose we recall the following property of the class $\mathcal{B}$.

Lemma 2.2 (Schwarz lemma) (see Duren 1983) *Let w be analytic in the unit disc Δ , with $w(0) = 0$ and $|w(z)| < 1$ in Δ . Then $|w'(0)| \leq 1$ and $|w(z)| \leq |z|$ in Δ . Strict inequality holds in both estimates unless w is a rotation of the disc: $w(z) = e^{i\theta}z$.*

Theorem 2.1 *Let $\alpha \in (0, 1)$ and $\gamma \in [0, 1)$ be given numbers. If $f \in \mathcal{BS}(\alpha)$, then f is starlike of order γ in the disc $|z| < r_s(\alpha, \gamma) = \frac{\sqrt{1 + 4\alpha(1 - \gamma)} - 1}{2\alpha(1 - \gamma)}$. The result is sharp.*

Proof Let $f \in \mathcal{BS}(\alpha)$, $\alpha \in (0, 1)$. Then through (1.2) we have $\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{z}{1-\alpha z^2}$ so there exists $w \in \mathfrak{B}$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{1+w(z)-\alpha w^2(z)}{1-\alpha w^2(z)}\right\}$$

for all $z \in \Delta$. Applying the Schwarz lemma we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &= \operatorname{Re}\left\{1 + \frac{w(z)}{1-\alpha w^2(z)}\right\} = 1 + \operatorname{Re}\left\{\frac{w(z)}{1-\alpha w^2(z)}\right\} \\ &\geq 1 - \frac{|w(z)|}{1-\alpha|w(z)|^2} \geq 1 - \frac{|z|}{1-\alpha|z|^2} = 1 - \frac{r}{1-\alpha r^2},\end{aligned}$$

where $r = |z| < 1$. Let us consider a function $h(r) = 1 - \frac{r}{1-\alpha r^2}$, $r \in (0, 1)$. Note that $h'(r) = -\frac{1+\alpha r^2}{(1-\alpha r^2)^2} < 0$ for all $r \in [0, 1)$ hence h is a strictly decreasing function and it decreases from 1 to $\frac{\alpha}{\alpha-1} < 0$. Therefore the equation $h(r) = \gamma$ has for given α and γ the smallest positive root $r_s(\alpha, \gamma)$ in $(0, 1)$. Therefore f is starlike of order γ in $|z| < r \leq r_s(\alpha, \gamma)$. Note that for the function $\tilde{f}$ given in (2.2) we obtain

$$\operatorname{Re}\frac{\tilde{f}'(z)}{\tilde{f}(z)} = \operatorname{Re}\left\{1 + \frac{z}{1-\alpha z^2}\right\} =: A(z)$$

and $A(-r_s(\alpha, \gamma)) = \gamma$. $\square$

Putting $\gamma = 0$ in Theorem 2.1, we obtain.

Corollary 2.1 Let $\alpha \in (0, 1)$. If $f \in \mathcal{BS}(\alpha)$ then f is starlike univalent in the disc $|z| < r_s(\alpha) = \frac{\sqrt{1+4\alpha}-1}{2\alpha}$. The result is sharp.

Remark 2.1 Note that $\lim_{\alpha \rightarrow 0^+} r_s(\alpha) = \lim_{\alpha \rightarrow 0^+} \frac{2}{\sqrt{1+4\alpha}+1} = 1$. Moreover, it is worth mentioning that $\lim_{\alpha \rightarrow 1^-} r_s(\alpha) = \frac{\sqrt{5}-1}{2} = 0.618\dots$, i.e. this limit is a reciprocal of the golden ratio $\frac{\sqrt{5}+1}{2}$.

Now we consider the following question:

For a given number $r \in (0, 1]$ find $\alpha(r)$ such that for each function $f \in \mathcal{BS}(\alpha(r))$ the image $f(\{z : |z| < r\})$ is a starlike domain.

Theorem 2.2 Let $r \in (0, 1]$ be the given number. If $0 \leq \alpha < \frac{1-r}{r^2}$, then each function $f \in \mathcal{BS}(\alpha)$ maps a disc $|z| < r$ onto a starlike domain.

Proof After using the same argument as in the proof of Theorem 2.1 we conclude that $f \in \mathcal{BS}(\alpha)$ satisfies the equality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{1+w(z)-\alpha w^2(z)}{1-\alpha w^2(z)}\right\}$$

for all $z \in \Delta$ with some $w \in \mathfrak{B}$. Then we have by Schwarz's lemma that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 1 - \frac{|w(z)|}{1-\alpha|w(z)|^2} \geq 1 - \frac{|z|}{1-\alpha|z|^2}.$$

Consequently, for $|z| < r$, we obtain $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 1 - \frac{r}{1-\alpha r^2}$. Let us note that a function $g(\alpha) = 1 - \frac{r}{1-\alpha r^2}$, $\alpha \in [0, 1)$, has positive values for $0 \leq \alpha < \frac{1-r}{r^2}$. Therefore the image of the disc $|z| < r$ is a starlike domain. $\square$

Theorem 2.3 Let $n \geq 2$ be integer. If one of the following conditions holds

- (i) $\frac{1}{\alpha+n(1-\alpha)} < |c| < 1$,
- (ii) $n > \frac{3-\alpha}{1-\alpha}$ and $\frac{1}{\alpha-2+n(1-\alpha)} < |c| < 1$,
- (iii) $n \geq \frac{2-\alpha}{1-\alpha}$ and $|c| > 1$,
- (iv) $n < \frac{2-\alpha}{1-\alpha}$ and $1 < |c| < \frac{1}{2-\alpha+n(\alpha-1)}$,

then the function $g_n(z) = z + cz^n$ does not belong to the class $\mathcal{BS}(\alpha)$.

Proof Let us put $G(z) = \frac{zg'_n(z)}{g_n(z)} - 1 = \frac{1+cnz^{n-1}}{1+cz^{n-1}} - 1$. To prove our assertion it suffices to show that the function G is not subordinate to F_α or equivalently, because of the univalence of the dominant function F_α , that the set $G(\Delta)$ is not included in $F_\alpha(\Delta) = D(\alpha)$. Upon performing simple calculation we find that the set $G(\Delta)$ is the disc with the diameter from the point $x_1 = \frac{|c|(n-1)}{|c|-1}$ to the point $x_2 = \frac{|c|(n-1)}{|c|+1}$. The set $D(\alpha)$ is bounded by the curve

$$(x^2 + y^2)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha)^2} = 0, \quad (x, y) \neq (0, 0).$$

We have $\min_{|z|=1} \operatorname{Re}\{F_\alpha(z)\} = F_\alpha(-1) = \frac{1}{\alpha-1}$ and $\max_{|z|=1} \operatorname{Re}\{F_\alpha(z)\} = F_\alpha(1) = \frac{1}{1-\alpha}$. If one of the conditions (i) – (iv) is satisfied then $\min\{x_1, x_2\} < \frac{1}{\alpha-1}$ or $\max\{x_1, x_2\} > \frac{1}{1-\alpha}$, and then $G(\Delta)$ is not included in $D(\alpha)$. The proof of theorem is completed. $\square$

Recently, one of the interesting problems for mathematician is to find bounds for $\operatorname{Re}\{f(z)/z\}$ (see Kargar et al. 2016; Sim and Kwon 2013). In the sequel, we obtain lower and upper bounds for $\operatorname{Re}\{f(z)/z\}$. We first get the following result for the function $F(z)$ given by (2.3).

Theorem 2.4 The function $F(z)$ of the form (2.3) is convex univalent in Δ .

Proof A simple calculation gives us

$$1 + \frac{zF''(z)}{F'(z)} = 1 + \left(\frac{1}{2\sqrt{\alpha}} - 1\right) \left(\frac{2\sqrt{\alpha}z}{1-\alpha z^2}\right) + \frac{2\sqrt{\alpha}z}{1-\sqrt{\alpha}z}. \quad (2.4)$$

It is sufficient to show that (2.4) has positive real part in the unit disc. From Lemma 1.1 we obtain

$$\begin{aligned} \operatorname{Re}\left\{1 + \frac{zF''(z)}{F'(z)}\right\} &= \operatorname{Re}\left\{1 + \left(\frac{1}{2\sqrt{\alpha}} - 1\right)\left(\frac{2\sqrt{\alpha}z}{1 - \alpha z^2}\right) + \frac{2\sqrt{\alpha}z}{1 - \sqrt{\alpha}z}\right\} \\ &= 1 + 2\sqrt{\alpha}\left(\frac{1}{2\sqrt{\alpha}} - 1\right)\operatorname{Re}\left\{\frac{z}{1 - \alpha z^2}\right\} + 2\sqrt{\alpha}\operatorname{Re}\left\{\frac{z}{1 - \sqrt{\alpha}z}\right\} \\ &> 1 + (1 - 2\sqrt{\alpha})\left(\frac{1}{\alpha - 1}\right) - \frac{2\sqrt{\alpha}}{1 + \sqrt{\alpha}} =: K(\alpha) \quad (0 \leq \alpha < 1). \end{aligned}$$

It is easily seen that $K'(\alpha) = \frac{1}{(\alpha-1)^2} > 0$. Thus $K(\alpha) \geq K(0) = 0$, and hence $F(z)$ is convex univalent function. $\square$

In the proof of the next theorem we will use the following result concerning the convexity of the boundary of $D(\alpha)$.

Lemma 2.3 (see Piejko and Sokół 2013) *Suppose that F_α is given by (1.3). If $0 \leq \alpha \leq 3 - 2\sqrt{2} \approx 0.1715$, then the curve $F_\alpha(e^{i\varphi})$, $\varphi \in [0, 2\pi)$, is convex. If $\alpha \in (3 - 2\sqrt{2}, 1)$, then the curve $F_\alpha(e^{i\varphi})$, $\varphi \in [0, 2\pi)$, is concave. Moreover, in both cases this curve is symmetric with respect to both axes.*

Theorem 2.5 *If a function f belongs to the class $\mathcal{BS}(\alpha)$, $0 \leq \alpha \leq 3 - 2\sqrt{2}$, then*

$$\frac{f(z)}{z} \prec F(z) \quad (z \in \Delta), \quad (2.5)$$

where $F(z)$ is given by (2.3).

Proof Let $0 \leq \alpha \leq 3 - 2\sqrt{2}$ and let f be in the class $\mathcal{BS}(\alpha)$. Then we have

$$\phi(z) := \frac{zf'(z)}{f(z)} - 1 \prec F_\alpha(z) \quad (z \in \Delta), \quad (2.6)$$

where F_α is given by (1.3). It is well known that the normalized function

$$l(z) = \log \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (z \in \Delta),$$

belongs to the class $\mathcal{K}$ and for $f \in \mathcal{A}$ we get

$$\phi(z) * l(z) = \int_0^z \frac{\phi(t)}{t} dt \quad \text{and} \quad F_\alpha(z) * l(z) = \int_0^z \frac{F_\alpha(t)}{t} dt. \quad (2.7)$$

By Lemma 2.3 we deduce that the function F_α is convex. Thus applying Lemma 1.2 in (2.6) we obtain

$$\phi(z) * l(z) \prec F_\alpha(z) * l(z) \quad (z \in \Delta). \quad (2.8)$$

Now from (2.7) and (2.8), we can obtain

$$\int_0^z \frac{\phi(t)}{t} dt \prec \int_0^z \frac{F_\alpha(t)}{t} dt \quad (z \in \Delta).$$

Thus

$$\frac{f(z)}{z} = \exp \int_0^z \frac{\phi(t)}{t} dt \prec \int_0^z \frac{F_\alpha(t)}{t} dt = \frac{\tilde{f}(z)}{z}.$$

This completes the proof of theorem. $\square$

Here by combining Theorem 2.4, Theorem 2.5 and (1.1), we get:

Theorem 2.6 *Let $f \in \mathcal{BS}(\alpha)$, $0 \leq \alpha \leq 3 - 2\sqrt{2}$ and $|z| = r < 1$. Then*

$$\left(\frac{1 - r\sqrt{\alpha}}{1 + \sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} \leq \operatorname{Re}\left(\frac{f(z)}{z}\right) \leq \left(\frac{1 + r\sqrt{\alpha}}{1 - r\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} \quad (z \in \Delta). \quad (2.9)$$

The result is sharp.

Proof By the subordination principle, we have:

$$f(z) \prec g(z) \Rightarrow f(|z| < r) \subset g(|z| < r) \quad (0 \leq r < 1).$$

From Theorem 2.4, since $F(z)$ is convex univalent in Δ , and it is real for real z , thus it maps the disc $|z| = r < 1$ onto a convex set symmetric which respect to the real axis laying between $F(-r) - 1$ and $F(r) - 1$. Now the assertion is obtained from Theorem 2.5. $\square$

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