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A bijection for tuples of commuting permutations and a log-concavity conjecture

Abdelmalek Abdesselam[1*], Pedro Brunialti[1], Tristan Doan[1] and Philip Velie[1]

*Correspondence: malek@virginia.edu
[1]Department of Mathematics, University of Virginia, P. O. Box 400137, Charlottesville, VA 22904-4137, USA

Abstract

Let $A(\ell, n, k)$ denote the number of ℓ-tuples of commuting permutations of n elements whose permutation action results in exactly k orbits or connected components. We provide a new proof of an explicit formula for $A(\ell, n, k)$ which is essentially due to Bryan and Fulman, in their work on orbifold higher equivariant Euler characteristics. Our proof is self-contained, elementary, and relies on the construction of an explicit bijection, in order to perform the $\ell + 1 \to \ell$ reduction. We also investigate a conjecture by the first author, regarding the log-concavity of $A(\ell, n, k)$ with respect to k. The conjecture generalizes a previous one by Heim and Neuhauser related to the Nekrasov-Okounkov formula.

1 Introduction

For $n \geq 0$, let us denote by $[n]$ the finite set $\{1, \ldots, n\}$, and by $\mathfrak{S}_n$ the symmetric group of permutations of $[n]$. For $\ell \geq 0$, we consider the set of ordered ℓ-tuples of commuting permutations

$$\mathscr{C}_{\ell,n} := \left\{ (\sigma_1, \ldots, \sigma_\ell) \in (\mathfrak{S}_n)^\ell \mid \forall i, j,\ \sigma_i\sigma_j = \sigma_j\sigma_i \right\}.$$

For a tuple $(\sigma_1, \ldots, \sigma_\ell)$ of (non-necessarily commuting) permutations, let $\langle\sigma_1, \ldots, \sigma_\ell\rangle$ be the subgroup they generate inside $\mathfrak{S}_n$. The obvious action of $\mathfrak{S}_n$ on $[n]$ restricts to an action of $\langle\sigma_1, \ldots, \sigma_\ell\rangle$ with a number of orbits which we will denote by $\kappa(\sigma_1, \ldots, \sigma_\ell)$. For $0 \leq k \leq n$, we let $\mathscr{C}_{\ell,n,k}$ be the subset of $\mathscr{C}_{\ell,n}$ made of tuples for which $\kappa(\sigma_1, \ldots, \sigma_\ell) = k$. We finally define our main object of study

$$A(\ell, n, k) := |\mathscr{C}_{\ell,n,k}|,$$

where, as usual, $|\cdot|$ denotes the cardinality of finite sets. Our main result is a new proof of the following theorem giving an explicit, albeit complicated, formula for the $A(\ell, n, k)$.

Theorem 1.1 *We have*

$$A(\ell, n, k) = \frac{n!}{k!} \times \sum_{n_1,\ldots,n_k \geq 1} \mathbb{1}\{n_1 + \cdots + n_k = n\} \times \prod_{i=1}^{k} \left[\frac{B(\ell, n_i)}{n_i} \right],$$

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where $\mathbb{1}\{\cdots\}$ *denotes the indicator function of the condition between braces, and* $B(\ell, \cdot)$ *is the multiplicative function (in the number theory sense, i.e.,* $B(\ell, ab) = B(\ell, a)B(\ell, b)$ *when* a, b *are coprime) which satisfies*

$$B(\ell, q^m) = \frac{(q^\ell - 1)(q^{\ell+1} - 1)\cdots(q^{\ell+m-1} - 1)}{(q-1)(q^2-1)\cdots(q^m-1)},$$

when $m \geq 0$ *and* q *is a prime number.*

Our motivation for considering the previous theorem is the following log-concavity conjecture by the first author.

Conjecture 1.1 (Abdesselam [2]) *For all* $\ell \geq 1$*, all* $n \geq 3$*, and for all* k *such that* $2 \leq k \leq n-1$*,*

$$A(\ell, n, k)^2 \geq A(\ell, n, k-1)\, A(\ell, n, k+1).$$

The case $\ell = 1$, included for esthetic coherence, is not conjectural. Since $A(1, n, k) = c(n, k)$, the unsigned Stirling number of the first kind, the stated log-concavity property is a well known fact (see, e.g., [1] and references therein). The case $\ell = 2$ is a conjecture by Heim and Neuhauser [11] related to the Nekrasov-Okounkov formula [16,19], as will be explained in Sect. 3. The case "$\ell = \infty$" is proved in [2]. The form in which Theorem 1.1 is stated is the one needed for the proof given in [2], and we did not see this precise formulation in the literature. However, we do not claim Theorem 1.1 is new. Indeed, it follows easily from the following identity by Bryan and Fulman [5]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!}\, A(\ell, n, k)\, x^k u^n = \prod_{d_1,\ldots,d_{\ell-1}=1}^{\infty} (1 - u^{d_1\cdots d_{\ell-1}})^{-x d_1^{\ell-2} d_2^{\ell-3}\cdots d_{\ell-2}}, \tag{1}$$

which holds in the ring of formal power series $\mathbb{C}[[x, u]]$. To see how Theorem 1.1 can be derived from (1), first (re)define, for $\ell \geq 1$ and $n \geq 1$,

$$B(\ell, n) := \sum_{s_1|s_2|\cdots|s_{\ell-1}|n} s_1 \cdots s_{\ell-1}, \tag{2}$$

where the sum is over tuples of integers $s_1, \ldots, s_{\ell-1} \geq 1$ which form an "arithmetic flag", namely, such that s_1 divides s_2, s_2 divides s_3,..., $s_{\ell-1}$ divides n. In particular, $B(1, n) = 1$, and $B(2, n) = \sigma(n)$ the divisor sum from number theory. Since the divisor lattice factorizes over the primes, it is clear from the alternative definition (2), that the $B(\ell, \cdot)$ is a mutiplicative function, in the number theory sense. Its computation reduces to the prime power case. If q is a prime and $m \geq 0$, then we have

$$\begin{aligned} B(\ell, q^m) &= \sum_{0 \leq m_1 \leq \cdots \leq m_{\ell-1} \leq m} q^{m_1 + \cdots + m_{\ell-1}} \\ &= \sum_{\lambda \subset (m)^{\ell-1}} q^{|\lambda|} \\ &= \begin{bmatrix} m + \ell - 1 \\ m \end{bmatrix}_q \\ &= \frac{(q^\ell - 1)(q^{\ell+1} - 1)\cdots(q^{\ell+m-1} - 1)}{(q-1)(q^2-1)\cdots(q^m-1)}. \end{aligned}$$

Here, we changed variables to the integer partition $\lambda = (m_{\ell-1}, m_{\ell-2}, \ldots, m_1)$ with weight $|\lambda|$ and whose shape is contained in the rectangular partition $(m)^{\ell-1}$ with $\ell - 1$ parts equal to m. Finally, we used the well known formula for the sum over λ as a Gaussian polynomial or q-binomial coefficient (see, e.g., [18, Prop. 1.7.3]). This shows the equivalence between (2) and the definition given in Theorem 1.1. By changing variables from $s_1, \ldots, s_{\ell-1}$ to $d_1, \ldots, d_\ell$ given by

$$d_1 = s_1,\ d_2 = \frac{s_2}{s_1}, \ldots,\ d_{\ell-1} = \frac{s_{\ell-1}}{s_{\ell-2}},\ d_\ell = \frac{n}{s_{\ell-1}},$$

we can also write

$$B(\ell, n) = \sum_{d_1 \cdots d_\ell = n} d_1^{\ell-1} d_2^{\ell-2} \cdots d_{\ell-1},$$

as a multiple Dirichlet convolution of power functions (see, e.g., [15] where the connection to q-binomial coefficients was also noted). The last formula is also consistent with the extreme $\ell = 0$ case, where $B(0, n) = \mathbb{1}\{n = 1\}$.

We then have the following easy formal power series computations

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{B(\ell, n)}{n} u^n &= \sum_{d_1, \ldots, d_\ell \geq 1} \frac{d_1^{\ell-1} d_2^{\ell-2} \cdots d_{\ell-1}}{d_1 \cdots d_\ell} \times u^{d_1 \cdots d_\ell} \\ &= \sum_{m \geq 1} B(\ell - 1, m) \times \sum_{d_\ell \geq 1} \frac{(u^m)^{d_\ell}}{d_\ell} \\ &= \sum_{m \geq 1} B(\ell - 1, m) \times (-\log(1 - u^m)),\end{aligned}$$

where we introduced the new summation index $m := d_1 \cdots d_{\ell-1}$. Multiplying by x, and taking exponentials gives

$$\exp\left(x \sum_{n=1}^{\infty} \frac{B(\ell, n)}{n} u^n\right) = \prod_{m=1}^{\infty} (1 - u^m)^{-xB(\ell-1,m)}, \tag{3}$$

which is the right-hand side of (1) when collecting factors according to $m := d_1 \cdots d_{\ell-1}$. We have thus shown that (1) can be rewritten as

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!} A(\ell, n, k)\, x^k u^n = \exp\left(x \sum_{n=1}^{\infty} \frac{B(\ell, n)}{n} u^n\right). \tag{4}$$

Extracting coefficients of monomials in x and u, on both sides, immediately yields Theorem 1.1. In the article [5], x is assumed to be the Euler characteristic of a manifold. However, their proof of (1) holds if x merely is a formal variable. Their work was aiming at generalizing the "stringy" orbifold Euler characteristic [3,8], from sums over pairs of commuting permutations, to commuting tuples of arbitrary length ℓ. Another motivation for their work was the study by Hopkins, Kuhn, and Ravenel [13] of a hierarchy of cohomology theories where the ℓ-th level seemed to crucially involve ℓ-tuples of commuting elements of a finite group such as $\mathfrak{S}_n$. The group-theoretic proof by Bryan and Fulman involved a delicate analysis of conjugacy classes in wreath products. Another proof one can find in the literature is the one by White [20]. It uses the remark that $\mathscr{C}_{\ell,n}$ is in bijection with $\mathrm{Hom}(\mathbb{Z}^\ell, \mathfrak{S}_n)$, namely, the set of group homomorphisms from the additive group $\mathbb{Z}^\ell$ to the symmetric group $\mathfrak{S}_n$, i.e., $\mathbb{Z}^\ell$ actions on a set of n elements. The proof by White also uses the fact that $B(\ell, n)$ is the number of subgroups of $\mathbb{Z}^\ell$ of index n (a remark by Stanley already mentioned in [5]) and the main part of the argument is the computation of this number using Hermite normal forms, i.e., Gaussian elimination over the integers. Note that $B(\ell, n)$ is a well studied quantity, see, e.g., [14, Ch. 15] as well as the article by Solomon [17] where work on $B(\ell, n)$ is traced back to the time of Hermite and Eisenstein. Also note that a proof of the $x = 1$ evaluation of the $\ell = 3$ case of (1) was also given in [4]. Our proof, given in the next section, is elementary and in the spirit of bijective enumerative combinatorics. In Lemma 2.1, we reduce the $A(\ell, n, k)$ to the $k = 1$ case of transitive actions, via a polymer gas representation [6], in the language of statistical mechanics, or the exponential formula in enumerative combinatorics, often mentioned as the general slogan "sums over all objects are exponentials of sums over connected objects". The main argument is a reduction of $A(\ell + 1, n, 1)$ to the computation of $A(\ell, n, 1)$. We condition the sum over tuples $(\sigma_1, \dots, \sigma_{\ell+1})$, first on the number r of orbits for the sub-tuple $(\sigma_1, \dots, \sigma_\ell)$ and then on the set partition $X = \{X_1, \dots, X_r\}$ of $[n]$ given by that orbit decomposition. With r and X fixed, we then construct a bijection

$$(\sigma_1, \dots, \sigma_{\ell+1}) \longmapsto (\widetilde{\sigma}, \gamma, \tau, z) \tag{5}$$

where $\widetilde{\sigma}$ is a transitive ℓ-tuple of commuting permutations on the subset X_1 containing the element $1 \in [n]$. By γ we denote a permutation of $[r]$ which is such that $\gamma(1) = 1$. The τ is a certain collection of bijective maps between blocks X_i. Finally, the crucial ingredient is z which is an element of X_1. One can intuitively understand our proof as counting possibly flat or degenerate discrete $(\ell + 1)$-dimensional tori with n points. As is familiar in topology, one can build such a torus by gluing both ends of a cylinder. However, we are allowed to perform a twist when doing this gluing and this is determined by z. Namely, $(\sigma_{\ell+1})^r$, the "Poincaré return map" to X_1, does not necessarily fix 1 but may send it to some $z \neq 1$. We remark that it is possible to explicitly iterate the bijection involved in the $\ell + 1$ to ℓ reduction, but given the complexity of the resulting recursive combinatorial data, we will refrain from doing this here.

2 Proofs

We first take care of the reduction to the transitive action case.

Lemma 2.1 *We have*

$$A(\ell, n, k) = \frac{n!}{k!} \times \sum_{n_1, \dots, n_k \geq 1} \mathbb{1}\{n_1 + \cdots + n_k = n\} \times \prod_{i=1}^{k} \left(\frac{A(\ell, n_i, 1)}{n_i!} \right).$$

Proof For a tuple $(\sigma_1, \dots, \sigma_\ell)$ in $\mathscr{C}_{\ell,n,k}$, let $\Pi(\sigma_1, \dots, \sigma_\ell)$ denote the unordered set partition of $[n]$ given by the orbits of the action of the subgroup $\langle \sigma_1, \dots, \sigma_\ell \rangle$. We condition the sum over tuples in $\mathscr{C}_{\ell,n,k}$, according to this set partition. We also sum over orderings of the blocks of that partition (with k blocks), and compensate for this overcounting by dividing by $k!$. This gives

$$A(\ell, n, k) = \frac{1}{k!} \times \sum_{(X_1, \dots, X_k)} \sum_{(\sigma_1, \dots, \sigma_\ell) \in \mathscr{C}_{\ell,n,k}} \mathbb{1} \left\{ \Pi(\sigma_1, \dots, \sigma_\ell) = \{X_1, \dots, X_k\} \right\},$$

where the sum is over ordered tuples of subsets $(X_1, \dots, X_r)$, where the X_i are nonempty, pairwise disjoint, and together have union equal to $[n]$. For $1 \leq i \leq k$ and $1 \leq j \leq \ell$, we let $\sigma_j^{(i)}$ be the restriction and corestriction of σ_j to the subset X_i which must be stable by σ_j. For fixed $X_1, \dots, X_k$, the sum over tuples $(\sigma_1, \dots, \sigma_\ell)$ clearly amounts to summing independently over the tuples $(\sigma_1^{(i)}, \ldots \sigma_\ell^{(i)})$ in each X_i, $1 \le i \le k$. The tuple $(\sigma_1^{(i)}, \ldots \sigma_\ell^{(i)})$ is made of commuting permutations of X_i whose action on the latter must be transitive. The number of such tuples only depends on the size $|X_i|$ of the set X_i, and not its location within $[n]$. As a result, we have

$$\begin{aligned} A(\ell, n, k) &= \frac{1}{k!} \times \sum_{(X_1,\ldots,X_k)} A(\ell, |X_1|, 1) \cdots A(\ell, |X_k|, 1) \\ &= \frac{1}{k!} \times \sum_{n_1,\ldots,n_k \ge 1} \mathbb{1}\{n_1 + \cdots + n_k = n\} \times \frac{n!}{n_1! \cdots n_k!} \times \prod_{i=1}^{k} A(\ell, n_i, 1), \end{aligned}$$

where the multinomial coefficient accounts for the number of tuples of disjoint sets $(X_1, \ldots, X_k)$ with fixed cardinalities $n_1, \ldots, n_k$. □

We now move on to the main part of the proof, i.e., the $\ell + 1$ to ℓ reduction and showing that

$$A(\ell + 1, n, 1) = \sum_{rs=n} A(\ell, s, 1) \times \frac{n!}{r! \times s!^r} \times (r-1)! \times s!^{r-1} \times s, \tag{6}$$

where the sum is over pairs of integers $r, s \ge 1$ whose product is n. Let $(\sigma_1, \ldots, \sigma_{\ell+1}) \in \mathscr{C}_{\ell+1,n,1}$ denote a $(\ell+1)$-tuple of commuting permutations being counted on the left-hand side. We let $X = \{X_1, \ldots, X_r\} := \Pi(\sigma_1, \ldots, \sigma_\ell)$ be the set of orbits determined by the first ℓ permutations. For a fixed set partition X of $[n]$, define $\mathscr{C}^X_{\ell+1,n,1} \subset \mathscr{C}_{\ell+1,n,1}$ as the set of $(\ell + 1)$-tuples which produce the given X by the above definition. We organize the count by conditioning on X, i.e., writing

$$A(\ell + 1, n, 1) = \sum_X \left|\mathscr{C}^X_{\ell+1,n,1}\right|,$$

and then computing the terms in the last sum by constructing an explicit bijection between $\mathscr{C}^X_{\ell+1,n,1}$ and a set of combinatorial data whose cardinality is easy to derive. We will use an automatic numbering of the blocks of X by ordering them according to their minimal element, with respect to the ordered set $[n]$. We let X_1 be the block containing the element $1 \in [n]$, and number the other blocks so that

$$1 = \min X_1 < \min X_2 < \cdots < \min X_r.$$

Lemma 2.2 *Let f be an element of $\langle \sigma_{\ell+1} \rangle$, i.e., a power of $\sigma_{\ell+1}$, and let $\alpha, \beta \in [r]$. If $\exists x \in X_\alpha, f(x) \in X_\beta$, then $\forall y \in X_\alpha, f(y) \in X_\beta$.*

Proof Since such y is in the same $\langle \sigma_1, \ldots, \sigma_\ell \rangle$-orbit as x, there exists a permutation $g \in \langle \sigma_1, \ldots, \sigma_\ell \rangle$, such that $y = g(x)$. Since $\sigma_1, \ldots, \sigma_{\ell+1}$ commute, then g must commute with f, and therefore $f(y) = f(g(x)) = g(f(x))$. This shows that $f(y)$ is in the same $\langle \sigma_1, \ldots, \sigma_\ell \rangle$-orbit as $f(x)$, namely, X_β. □

The last lemma allows us, from an $f \in \langle \sigma_{\ell+1} \rangle$, to construct a map $\widehat{f} : [r] \to [r]$ defined by $\widehat{f}(\alpha) = \beta$, whenever $\exists x \in X_\alpha, f(x) \in X_\beta$. This construction satisfies $\widehat{\mathrm{Id}} = \mathrm{Id}$, and $\widehat{f \circ g} = \widehat{f} \circ \widehat{g}$, namely, it gives a group homomorphism from $\langle \sigma_{\ell+1} \rangle$ to $\mathfrak{S}_r$. We apply this to $f = \sigma_{\ell+1}$ and consider the cycle decomposition of the permutation $\widehat{\sigma_{\ell+1}}$, and focus on the cycle containing the element $1 \in [r]$, namely $(\alpha_1\ \alpha_2\ \cdots\ \alpha_t)$, with $\alpha_1 = 1$. We clearly have

$$\sigma_{\ell+1}(X_1) \subset X_{\alpha_2},\ \sigma_{\ell+1}(X_{\alpha_2}) \subset X_{\alpha_3},\ \cdots,\ \sigma_{\ell+1}(X_{\alpha_{t-1}}) \subset X_{\alpha_t},\ \sigma_{\ell+1}(X_{\alpha_t}) \subset X_1.$$

Hence $X_1 \cup X_{\alpha_2} \cup \cdots \cup X_{\alpha_t}$ is stable by $\sigma_{\ell+1}$, in addition to being stable by $\langle \sigma_1, \ldots, \sigma_\ell \rangle$ since, each of the X blocks are. Given that the $(\ell+1)$-tuple of permutations $(\sigma_1, \ldots, \sigma_{\ell+1})$ is assumed to act transitively, this can only happen if the previous union of X blocks is all of $[n]$, i.e., if $t = r$. For notational convenience, we define the permutation $\gamma \in \mathfrak{S}_r$, by letting $\gamma(i) = \alpha_i$ for all $i \in [r]$. In particular, $\gamma(1) = 1$, by construction. We now have,

$$\begin{aligned} &\sigma_{\ell+1}(X_1) \subset X_{\gamma(2)},\ \sigma_{\ell+1}(X_{\gamma(2)}) \subset X_{\gamma(3)},\ \cdots, \\ &\sigma_{\ell+1}(X_{\gamma(r-1)}) \subset X_{\gamma(r)},\ \sigma_{\ell+1}(X_{\gamma(r)}) \subset X_1. \end{aligned} \tag{7}$$

Since $\sigma_{\ell+1}$ is injective, it follows that

$$|X_1| \le |X_{\gamma(2)}| \le \cdots \le |X_{\gamma(r)}| \le |X_1|,$$

and, therefore, all the X blocks must have the same cardinality say s, so that $n = rs$, namely, r must divide n. The above argument also produces bijective maps

$$\tau_i : X_{\gamma(i)} \longrightarrow X_{\gamma(i+1)},$$

for $1 \le i \le r-1$, obtained by restriction (and corestriction) of $\sigma_{\ell+1}$. We collect them into a tuple $\tau = (\tau_1, \ldots, \tau_{r-1})$. We now define the ℓ-tuple of permutations of the first block X_1 given by $\widetilde{\sigma} = (\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_\ell)$ where, for all $j \in [\ell]$, $\widetilde{\sigma}_j$ is obtained from σ_j by restricting it to the subset X_1. It is easy to see that $\widetilde{\sigma}$ is an ℓ-tuple of commuting permutations of the set X_1, which altogether act transitively on it. Finally, we define the element $z = (\sigma_{\ell+1})^r(1)$ of the block X_1. This concludes the definition of the map mentioned in (5) which to a tuple $(\sigma_1, \ldots, \sigma_{\ell+1}) \in \mathscr{C}_{\ell+1,n,1}$ associates the data $(\widetilde{\sigma}, \gamma, \tau, z)$. Once we establish that this construction is bijective, the reduction formula (6) will follow easily. Indeed, after identification of X_1 with $[s]$, we see that there are $A(\ell, s, 1)$ possible choices for $\widetilde{\sigma}$. Deciding on the permutation γ, which fixes 1, results in $(r-1)!$ choices. The number of possibilities for the bijective maps in τ accounts for a factor $s!^{r-1}$, and there are s possibilities for z. Summing over the unordered set partition X can be done with the multinomial coefficient $n!/s!^r$ for ordered set partitions and correcting for the overcounting by dividing by $r!$, as in the proof of Lemma 2.1. All that remains in order to complete the proof of (6) is to show our map (5) is indeed bijective.

Injectivity: We will show how the tuple $(\sigma_1, \ldots, \sigma_{\ell+1})$ is determined by the data $(\widetilde{\sigma}, \gamma, \tau, z)$, and the a priori knowledge of the fixed partition X. By construction, for all j, $1 \le j \le \ell$, the restriction of σ_j to X_1 must be

$$\sigma_j|_{X_1} = \widetilde{\sigma}_j. \tag{8}$$

Strictly speaking, there is also a change of codomain involved (from X_1 to $[n]$), but we ignored this and will continue to do this for the next similar statements. We must also have, for all i, $1 \le i \le r-1$,

$$\sigma_{\ell+1}|_{X_{\gamma(i)}} = \tau_i. \tag{9}$$

From the commutation relation $\sigma_j \circ (\sigma_{\ell+1})^i = (\sigma_{\ell+1})^i \circ \sigma_j$, restricted to X_1, we deduce that for all i, $2 \le i \le r$, we must have

$$\sigma_j \circ \tau_{i-1} \circ \cdots \circ \tau_1 = \tau_{i-1} \circ \cdots \circ \tau_1 \circ \widetilde{\sigma}_j$$

i.e.,

$$\sigma_j|_{X_{\gamma(i)}} = \tau_{i-1} \circ \cdots \circ \tau_1 \circ \widetilde{\sigma}_j \circ \tau_1^{-1} \circ \cdots \circ \tau_{i-1}^{-1}. \tag{10}$$

Hence $\sigma_1, \ldots, \sigma_\ell$ are known, while $\sigma_{\ell+1}$ is almost entirely determined. We are only missing the restriction of $\sigma_{\ell+1}$ on the last block $X_{\gamma(r)}$. Since z is in the orbit X_1 of the element 1 for the action of $\sigma_1, \ldots, \sigma_\ell$, or equivalently $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_\ell$, there exists $g \in \langle \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_\ell \rangle$, such that $g(1) = z$. We claim that we must have

$$\sigma_{\ell+1}|_{X_{\gamma(r)}} = g \circ \tau_1^{-1} \circ \cdots \circ \tau_{r-1}^{-1}. \tag{11}$$

Indeed, let $x \in X_{\gamma(r)}$, then $x = (\sigma_{\ell+1})^{r-1}(y)$ for some $y \in X_1$. Again, by transitivity on X_1, there exists $h \in \langle \sigma_1, \ldots, \sigma_\ell \rangle$ such that $y = h(1)$. As a consequence of the Abelian property of the group $\langle \sigma_1, \ldots, \sigma_{\ell+1} \rangle$, we must have

$$\begin{aligned}
\sigma_{\ell+1}(x) &= (\sigma_{\ell+1})^r \circ h(1) \\
&= h \circ (\sigma_{\ell+1})^r(1) \\
&= h(z) \\
&= h(g(1)) \\
&= g(h(1)) \\
&= g(y) \\
&= g \circ \tau_1^{-1} \circ \cdots \circ \tau_{r-1}^{-1}(x).
\end{aligned}$$

We now have recovered the restrictions of all $\ell + 1$ permutations σ_j on all blocks X_i of the decomposition of $[n]$, from the output of our map, which shows that it is injective.

Surjectivity: We start from the data $(\widetilde{\sigma}, \gamma, \tau, z)$ and construct $(\sigma_1, \ldots, \sigma_{\ell+1}) \in \mathscr{C}^X_{\ell+1,n,1}$ which maps to it. This time, we use the equations (8), (9), (10), (11) as definitions of $\sigma_1, \ldots, \sigma_{\ell+1}$ as maps $[n] \to [n]$. The use of (11) requires some care, namely showing the uniqueness of g. Let $\widetilde{H} := \langle \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_p \rangle$. The hypothesis on the tuple $\widetilde{\sigma}$ is that it is made of ℓ commuting permutations of the set X_1, such that the permutation action of $\widetilde{H}$ on X_1 is transitive. Suppose $g_1(1) = g_2(1) = z$ for some $g_1, g_2 \in \widetilde{H}$. If $x \in X_1$, then $\exists h \in \widetilde{H}$, $h(1) = x$. By the Abelian property of $\widetilde{H}$, we have

$$g_i(x) = g_i \circ h(1) = h \circ g_i(1) = h(z),$$

for $i = 1$ as well as $i = 2$, and thus $g_1(x) = g_2(x)$. Since x is arbitrary, we have $g_1 = g_2$. This justifies the use of (11) as a definition of a map. We now have constructed the maps $\sigma_1, \ldots, \sigma_{\ell+1}$. It is immediate, from (8) and (10), that $\sigma_1, \ldots, \sigma_\ell$ are bijective within each $X_{\gamma(i)}$, $1 \le i \le r$, and therefore over all of $[n]$. One easily checks also the commutation relations $\sigma_j \circ \sigma_{j'} = \sigma_{j'} \circ \sigma_j$, $1 \le j, j' \le \ell$, on each X block, and therefore on $[n]$. From (9), we see that $\sigma_{\ell+1}$ is injective on each $X_{\gamma(i)}$, $1 \le i \le r-1$, and the images of these restrictions are disjoint because γ is a permutation. From (11), it holds that $\sigma_{\ell+1}|_{X_{\gamma(r)}} : X_{\gamma(r)} \to X_1$ is bijective. As a result, $\sigma_{\ell+1} : [n] \to [n]$ is bijective. From (9) and (10), we also obtain

$$\sigma_j \circ \sigma_{\ell+1}|_{X_{\gamma(i)}} = \tau_i \circ \cdots \circ \tau_1 \circ \widetilde{\sigma}_j \circ \tau_1^{-1} \circ \cdots \circ \tau_{i-1}^{-1} = \sigma_{\ell+1} \circ \sigma_j|_{X_{\gamma(i)}},$$

for all i, j such that $1 \le j \le \ell$ and $1 \le i \le r-1$. Finally, for all j, $1 \le j \le \ell$, the restrictions of $\sigma_j \circ \sigma_{\ell+1}$ and $\sigma_{\ell+1} \circ \sigma_j$ on $X_{\gamma(r)}$ coincide, because g and $\widetilde{\sigma}_j$ must commute. We have now checked that $(\sigma_1, \ldots, \sigma_{\ell+1})$ is a commuting tuple of permutations of $[n]$. The corresponding action is transitive because (7) holds by construction and $\widetilde{\sigma}$ is assumed to act transitively on X_1. Checking that the produced tuple $(\sigma_1, \ldots, \sigma_{\ell+1}) \in \mathscr{C}^X_{\ell+1,n,1}$ indeed maps to $(\widetilde{\sigma}, \gamma, \tau, z)$ is straightforward. Therefore, our map is surjective.

In order to finish the proof of Theorem 1.1, we define $C(\ell, n) := \frac{A(\ell,n,1)}{(n-1)!}$. Since $A(1, n, 1) = (n-1)!$ counts cyclic permutations of n elements, we have $C(1, n) = 1 =$

$B(1, n)$. The, now established, recursion (6) implies that C satisfies

$$C(\ell+1, n) = \sum_{rs=n} s\, C(\ell, s).$$

By a trivial induction on ℓ, $C(\ell, n)$ must coincide with $B(\ell, n)$ defined, e.g., in (2). We plug $A(\ell, n, 1) = (n-1)! \times B(\ell, n)$ in the result of Lemma 2.1, and Theorem 1.1 follows.

3 On conjecture 1.1

As mentioned in the introduction, the case $\ell = 1$ of Conjecture 1.1 is well established. The opposite extreme "$\ell = \infty$" is settled in the companion article [2]. Let us now focus on the $\ell = 2$ case, and relate it to an already large body of literature, in particular, the work of Heim, Neuhauser, and many others. Since, for $\ell = 2$, $B(\ell-1, m) = B(1, m) = 1$, the Bryan-Fulman identity (1) simply reads

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!} A(2, n, k) x^k u^n = \prod_{m=1}^{\infty} (1-u^m)^{-x}.$$

On the other hand, the so-called D'Arcais polynomials $P_n(x)$ are defined [7] by the generating function identity

$$\prod_{m=1}^{\infty} (1-u^m)^{-x} = \sum_{n=0}^{\infty} P_n(x) u^n.$$

The D'Arcais polynomials can therefore be expressed in terms of commuting pairs of permutations

$$P_n(x) = \frac{1}{n!} \sum_{k=0}^{n} A(2, n, k)\, x^k. \tag{12}$$

We are not aware of the commuting permutation interpretation (12) of D'Arcais polynomials having been used in the number theory literature reviewed, e.g., in [11], and we hope it could be of help in this area. If one shifts the variable x by one, one gets the standard formulation of the Nekrasov-Okounkov formula [16,19]

$$\prod_{m=1}^{\infty} (1-u^m)^{-x-1} = \sum_{n=0}^{\infty} Q_n(x) u^n$$

where

$$Q_n(x) = \sum_{\lambda \vdash n} \prod_{\square \in \lambda} \left(1 + \frac{x}{h(\square)^2}\right).$$

Namely, the sum is over integer partitions λ of n. The product is over cells in the usual Ferrers-Young diagram of the partition λ, and $h(\square)$ denotes the hook length number of that cell. Clearly $Q_n(x) = P_n(x+1)$ and therefore, the log-concavity (of the coefficients of) the polynomial P_n would imply that of Q_n as well as the unimodality of the latter which was conjectured by Heim and Neuhauser as well as Amdeberhan (see [11] and references therein). As a strengthening of this unimodality conjecture, the log-concavity of the $P_n(x)$'s, i.e., the $\ell = 2$ case of Conjecture 1.1 was stated as Challenge 3 in [11]. The authors also reported on checking this numerically for all $n \le 1500$. While the log-concavity in the $\ell = 1$ case can be derived using the real-rootedness of the relevant polynomial, namely, the Pochhammer symbol, this approach cannot work for $\ell = 2$. Indeed, D'Arcais polynomials can have roots off the real axis, as was shown in [9]. For recent progress towards such log-concavity properties in the $\ell = 2$ case, see [12,21].

Using Mathematica, we checked that Conjecture 1.1 is true for $\ell = 3, 4, 5$ for all $n \le 100$. One can also test the conjecture by considering the dilute polymer gas regime, in the terminology of statistical mechanics (see, e.g., [6]), i.e., when k is close to n and most orbits are singletons, as in the next proposition. Note that the latter can also be deduced from [10, Proposition 4]. The $\ell = 2$ case was, in fact, already proved in [10, Corollary 4].

Proposition 3.1 *The inequality in Conjecture 1.1 holds for all $\ell \ge 1$, and $n \ge 3$, when $k = n - 1$.*

Proof Let

$$\Delta(\ell, n) := A(\ell, n, n-1)^2 - A(\ell, n, n)\, A(\ell, n, n-2).$$

From Theorem 1.1, we easily deduce

$$\begin{aligned} A(\ell, n, n) &= 1 \\ A(\ell, n, n-1) &= \binom{n}{2} (2^\ell - 1) \\ A(\ell, n, n-2) &= \binom{n}{3} (3^\ell - 1) + \binom{n}{4} 3(2^\ell - 1)^2. \end{aligned}$$

Therefore

$$\Delta(\ell, n) = \left[\binom{n}{2}^2 - 3\binom{n}{4}\right](2^\ell - 1)^2 - \binom{n}{3}(3^\ell - 1).$$

As mentioned before, the conjecture is known for $\ell = 1$, so now we focus on $\ell \ge 2$. If $\ell \ge 3$, then $2\left(\frac{1}{2}\right)^\ell + \left(\frac{3}{4}\right)^\ell \le \frac{43}{64}$, the $\ell = 3$ value. Therefore, for $\ell \ge 3$, we have $4^\ell \ge 2 \times 2^\ell + 3^\ell$ which implies

$$4^\ell - 2 \times 2^\ell + 1 \ge 3^\ell - 1.$$

The last inequality being also true for $\ell = 2$, we have that for all $\ell \ge 2$, the inequality $(2^\ell - 1)^2 \ge 3^\ell - 1$ holds. Hence

$$\begin{aligned} \Delta(\ell, n) &\ge \left[\binom{n}{2}^2 - 3\binom{n}{4} - \binom{n}{3}\right](2^\ell - 1)^2 \\ &= \frac{1}{24} n(n-1)(3n^2 + 5n - 10)(2^\ell - 1)^2. \end{aligned}$$

Since $n \ge 3$ implies $3n^2 + 5n - 10 \ge 32 > 0$, we have $\Delta(\ell, n) > 0$. □

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