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Torsion phenomena for zero-cycles on a product of curves over a number field

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Abstract

For a smooth projective variety X over an algebraic number field k a conjecture of Bloch and Beilinson predicts that the kernel of the Albanese map of X is a torsion group. In this article we consider a product $X = C_1 \times \cdots \times C_d$ of smooth projective curves and show that if the conjecture is true for any subproduct of two curves, then it is true for X. For a product $X = C_1 \times C_2$ of two curves over $\mathbb{Q}$ with positive genus we construct many nontrivial examples that satisfy the weaker property that the image of the natural map $J_1(\mathbb{Q}) \otimes J_2(\mathbb{Q}) \xrightarrow{\varepsilon} \mathrm{CH}_0(C_1 \times C_2)$ is finite, where J_i is the Jacobian variety of C_i. Our constructions include many new examples of non-isogenous pairs of elliptic curves E_1, E_2 with positive rank, including the first known examples of rank greater than 1. Combining these constructions with our previous result, we obtain infinitely many nontrivial products $X = C_1 \times \cdots \times C_d$ for which the analogous map ε has finite image.

Keywords: Zero-cycles, Elliptic curves, Jacobians of curves, Somekawa K-groups

1 Introduction

Let X be a smooth projective variety over an algebraic number field k. Consider the Chow group $\mathrm{CH}_0(X)$ of zero-cycles modulo rational equivalence on X and denote by $F^1(X)$ the subgroup of cycles of degree zero. Let Alb_X denote the Albanese variety of X, and $F^2(X)$ the kernel of the Albanese map

$$\mathrm{alb}_X : F^1(X) \to \mathrm{Alb}_X(k).$$

Set $X_{\overline{\mathbb{Q}}} := X \otimes_k \overline{\mathbb{Q}}$. The following is a famous conjecture of Beilinson [1].

Conjecture 1.1 *The Albanese map $F^1(X_{\overline{\mathbb{Q}}}) \to \mathrm{Alb}_{X_{\overline{\mathbb{Q}}}}(\overline{\mathbb{Q}})$ is injective. That is, $F^2(X_{\overline{\mathbb{Q}}}) = 0$.*

When C is a curve, the Albanese map coincides with the isomorphism between $\mathrm{Pic}^0(C)$ and the k-rational points of the Jacobian J_C of C, and hence $F^2(C) = 0$ in this case. In higher dimensions however there is hardly any evidence for this conjecture.

Next suppose that k is an arbitrary algebraic number field. Using standard push-forward and pull-back arguments for Chow groups, the above conjecture implies the following.

Conjecture 1.2 *The group $F^2(X)$ is torsion.*

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The goal of this article is to obtain some weak evidence for Conjecture 1.2 for a product $X = C_1 \times \cdots \times C_d$ of smooth projective curves over k.

A product formula for Somekawa K-groups and applications

Let J_i be the Jacobian variety of C_i. Under the assumption $X(k) \neq \emptyset$, Raskind and Spiess ([14, (1.4)]) established an isomorphism

$$\mathrm{CH}_0(X) \simeq \mathbb{Z} \oplus \bigoplus_{\nu=1}^{d} \bigoplus_{1 \leq i_1 < \cdots < i_\nu \leq d} K(k; J_{i_1}, \ldots, J_{i_\nu}),$$

which in turn yields an isomorphism (see [19, p. 4])

$$F^2(X) \simeq \bigoplus_{\nu=2}^{d} \bigoplus_{1 \leq i_1 < \cdots < i_\nu \leq d} K(k; J_{i_1}, \ldots, J_{i_\nu}),$$

where $K(k; J_{i_1}, \ldots, J_{i_\nu})$ is the Somekawa K-group attached to $J_{i_1}, \ldots, J_{i_\nu}$ (cf. [17]). For semi-abelian varieties $G_1, \ldots, G_r$ over k, the K-group $K(G_1, \ldots, G_r)$ is a generalization of the Milnor K-group $K_r^M(k)$ of the field k. In fact when $G_1 = \cdots = G_r = \mathbb{G}_m$ there is an isomorphism $K(\mathbb{G}_m, \cdots, \mathbb{G}_m) \simeq K_r^M(k)$. For the latter, there is a well-defined map

$$K_r^M(k) \otimes K_s^M(k) \to K_{r+s}^M(k)$$

for $r, s \geq 0$, given by concatenation of symbols. In Sect. 3 we prove the following analog for Somekawa K-groups attached to abelian varieties.

Proposition 1.3 (cf. Proposition 3.1) *Let $A_1, \ldots, A_r$ be abelian varieties over a perfect field k with $r \geq 3$. Let L/k be a finite extension. There is a well-defined homomorphism*

$$\Phi_L : K(k; A_1, A_2) \otimes A_3(L) \otimes \cdots \otimes A_r(L) \to K(k; A_1, \ldots, A_r).$$

In the simplest case of an element $\{a_1, a_2\}_{k/k} \otimes a_3 \otimes \cdots \otimes a_r$ with $a_i \in A_i(k)$ for all i, the map Φ_L is simply given by concatenation,

$$\Phi_L(\{a_1, a_2\}_{k/k} \otimes a_3 \otimes \cdots \otimes a_r) = \{a_1, \ldots, a_r\}_{k/k}.$$

The general definition is more involved; we omit it here for the purposes of the introduction. We will refer to the homomorphism Φ as a *product formula for Somekawa K-groups attached to abelian varieties*. While the proof of Proposition 1.3 is relatively straightforward, it yields the following important corollary, allowing us to deduce Conjecture 1.2 for a product of arbitrarily many curves, assuming it is known for each subproduct of two curves.

Corollary 1.4 (cf. Corollary 3.3) *Let $d \geq 2$ and $X = C_1 \times \cdots \times C_d$ be a product of smooth projective curves over a number field k with $C_i(k) \neq \emptyset$ for $i \in \{1, \ldots, d\}$. Suppose that for each $1 \leq i < j \leq d$ the group $F^2(C_i \times C_j)$ is torsion. Then $F^2(X)$ is torsion.*

The componentwise subgroup

Proving Conjecture 1.2 for even a single product $X = C_1 \times C_2$ of two non-rational curves C_1, C_2 seems to be far out of reach. To date, the best results towards understanding the structure of $F^2(X)$ are due to Langer and Saito [12] and Langer and Raskind [11], who proved that for the self-product $X = E \times E$ of an elliptic curve E over $\mathbb{Q}$ without and with potential CM respectively, the p-primary torsion subgroup of $\mathrm{CH}_0(X)$ is finite for every prime p coprime to the conductor N of E.

In this article we give evidence for a weaker question. We consider the homomorphism

$$\varepsilon_X : \mathrm{CH}_0(C_1) \otimes \cdots \otimes \mathrm{CH}_0(C_d) \to \mathrm{CH}_0(X)$$

induced by the intersection product. We define *the componentwise subgroup* $F^2(X)_{\mathrm{comp}}$ of $F^2(X)$ to be $F^2(X)_{\mathrm{comp}} := F^2(X) \cap \mathrm{Im}(\varepsilon_X)$ (see Definition 2.3). We propose the following conjecture, which is automatically implied by Conjecture 1.2.

Conjecture 1.5 *The subgroup $F^2(X)_{\mathrm{comp}}$ is finite.*

To the authors' knowledge this article is the first instance where Conjecture 1.5 is stated at this level of generality. In the special case when $X = C_1 \times C_2$ is a product of two curves with $X(k) \neq \emptyset$, the group $F^2(X)_{\mathrm{comp}}$ is precisely the image of the natural map

$$\varepsilon_X : J_1(k) \otimes J_2(k) \to F^2(X).$$

When both C_i are elliptic curves, this conjecture has been considered in [8] and [13], both of which provide new examples of such surfaces X for which $F^2(X)_{\mathrm{comp}}$ is finite. Note that if the base change X_L satisfies Conjecture 1.5 for all finite extensions L/k, then X satisfies Conjecture 1.2 (see Remark 2.17), so Conjecture 1.5 can be seen as a first step towards Conjecture 1.2 for products of curves.

Detecting torsion classes for a product of two curves

In Sect. 4 we construct many nontrivial examples for which Conjecture 1.5 can be verified. For the sake of introduction, we choose to state our results in their simplest form.

Assume the curves C_1 and C_2 each have a distinguished rational point $O_i \in C_i(k)$. Then every k-rational point (P, Q) on $X = C_1 \times C_2$ gives rise to a zero-cycle in $F^2(X)_{\mathrm{comp}}$, namely

$$z_{P,Q} := [P, Q] - [P, O_2] - [O_1, Q] - [O_1, O_2] = \pi_1^\star([P] - [O_1]) \cdot \pi_2^\star([Q] - [O_2]),$$

where $\pi_i : X \to C_i$ is the natural projection. When X is a product of two elliptic curves, such zero-cycles in fact generate $F^2(X)_{\mathrm{comp}}$, but this is generally not true for higher genus curves. The following key proposition can be used to show that certain cycles of this form are torsion.

Proposition 1.6 (cf. Proposition 4.5) *Let H be an elliptic (resp. hyperelliptic) curve, let $P \in H(k)$, and let $\phi_i : H \to C_i$ be a regular map for each $i = 1, 2$. Suppose there exists $W \in H(\overline{k})$, fixed by negation (resp. the hyperelliptic involution), such that $[\phi_i(W)] - [O_i] \in J_i(\overline{k})$ is torsion for $i = 1, 2$. Then the zero-cycle $z_{\phi_1(P),\phi_2(P)}$ is torsion.*

This proposition is the main new tool we use to construct torsion classes in $F^2(X)_{\mathrm{comp}}$. While the result is a relatively direct consequence of Weil reciprocity of the Somekawa K-group $K(k; J_1, J_2)$, its power lies in the fact that its conditions are quite easy to satisfy, allowing us to apply it to large classes of examples.

We first observe two direct consequences of Proposition 1.6 which are likely known to the experts, but we could not find them in the literature at this level of generality. First, any product $X = E_1 \times E_2$ of isogenous rank 1 elliptic curves over a number field satisfies Conjecture 1.5. Second, let E be an elliptic curve over $\mathbb{Q}$ with potential CM by the ring of integers of a quadratic imaginary field K. Suppose that $E(\mathbb{Q})$ has rank 1 and $E(K)$ has rank 2. Then Conjecture 1.5 holds for the base change $E_K \times E_K$ to K (cf. Proposition 4.8).

The following Corollary gives a less trivial application of Proposition 1.6.

Corollary 1.7 (cf. Proposition 4.7) *Let H be a hyperelliptic curve over k with Jacobian J. Suppose* rank $J(k) = 1$*, and that there exists $P, W \in H(k)$, with W fixed by the hyperelliptic involution, and $[P] - [W] \in J(k)$ is of infinite order. Then $H \times H$ satisfies Conjecture 1.5.*

There are 860 genus 2 curves over $\mathbb{Q}$ in the LMFDB [18] with conductor at most 10000 satisfying the criteria in Corollary 1.7, assuming that analytic rank equals algebraic rank for each of the curves considered (as predicted by the Birch and Swinnerton-Dyer conjecture). To our knowledge these are the first examples of Conjecture 1.5 (and the first pieces of evidence towards Conjecture 1.2) involving products of higher genus curves of positive rank.

The most substantial contribution of Proposition 1.6 is for products $X = E_1 \times E_2$ of elliptic curves with positive rank over $\mathbb{Q}$ and fully rational 2-torsion, a situation which is discussed in Sect. 4.2. For any such pair of elliptic curves, a theorem of Scholten ([15, Theorem 1]) provides an explicit hyperelliptic curve H of genus 2 over k with explicit regular maps $H \xrightarrow{\phi_i} E_i$ for $i = 1, 2$.[1] We thus obtain the following important corollary.

Corollary 1.8 *In the above set-up suppose that the set $\{z_{\phi_1(P),\phi_2(P)} : P \in H(k)\}$ generates a finite index subgroup of $F^2(E_1 \times E_2)_{\text{comp}}$. Then $E_1 \times E_2$ satisfies Conjecture 1.5.*

We present an algorithm (cf. Algorithm 1) that can be used to check the above criteria computationally. Applying this algorithm to a collection of curves from the LMFDB [18] allowed us to find many explicit examples. This includes the first known examples where $F^2(X)_{\text{comp}}$ is finite with E_1, E_2 non-isogenous curves with rank greater than 1. Specifically, for products $X = E_1 \times E_2$ with E_1, E_2 non-isomorphic elliptic curves over $\mathbb{Q}$ with torsion subgroup $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we verify that $F^2(X)_{\text{comp}}$ is finite for 2602 pairs of rank 1 curves (out of 4950 pairs tested), 995 pairs of rank 2 curves (out of 4950), 17 pairs of rank 3 curves (out of 190), 3311 pairs with ranks 1 and 2 respectively (out of 10000), 955 pairs with ranks 1 and 3 respectively (out of 10000), and 615 pairs with ranks 2 and 3 respectively (out of 10000). We also found several self-products $E \times E$ satisfying Conjecture 1.5 with E of rank 2 or 3. We refer to Table 1 for more details.

We note that to our knowledge there are only two previously known constructions of non-isogenous rank 1 pairs with product satisfying Conjecture 1.5. The first one is due to Prasanna and Srinivas, who in an upcoming preprint[2] [13] constructed two such pairs: one pair of two non-isogenous curves with conductor 37, and the other pair with two non-isogenous curves of conductor 91. The second construction is due to the second author, who constructed a 2-parameter family $E_{s,t}$ of elliptic curves over a number field k with the property that for each $s \in k$, and all $t_1, t_2 \in k$ outside a finite subset depending on s, if $E_{s,t_1}(k)$ and $E_{s,t_2}(k)$ both have rank 1, then $E_{s,t_1} \times E_{s,t_2}$ satisfies Conjecture 1.5. In Sect. 4.3 we recall some details of these constructions.

Products of more curves

The proof of Corollary 1.4 yields the following important corollary, which allows us to produce infinitely many nontrivial products for which Conjecture 1.5 is true.

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[1] Constructions of such a curve H were known prior to Scholten, for instance in [3], but we use Scholten's construction because it provides explicit equations.

[2] The authors first learned of this result by private communication in 2018.

Corollary 1.9 *Let E_1, E_2 be any of the rank 1 pairs mentioned above such that $F^2(E_1 \times E_2)_{\text{comp}}$ is finite. Let $d \geq 2$ and $X = C_1 \times \cdots \times C_d$ with each C_i either an elliptic curve isogenous to E_1 or E_2, or a curve with $C_i(\mathbb{Q}) \neq \emptyset$ and $\operatorname{rank}(J_{C_i}(\mathbb{Q})) = 0$. Then $F^2(X)_{\text{comp}}$ is finite.*

The above corollary is one of multiple corollaries we obtain by combining Corollary 1.4 and the various constructions that give finiteness of $F^2(C_1 \times C_2)_{\text{comp}}$.

2 Background

Throughout this article, k will be a perfect field.

Notation 2.1 Let X be a smooth projective variety over k. We will always assume that X has a k-rational point. By a point $x \in X$ we will always mean a closed point and we will denote by $k(x)$ the residue field of x. Given a field extension L/k, we let $X(L)$ denote the set of L-rational points of X with respect to the extension $k \hookrightarrow L$, and X_L denote the base change to L. We use parentheses, as in $(X_1)_L$, if the variety is labelled with an index.

Given $P \in X(k)$ and a finite extension L/k, we also use P to denote the L-rational point of X_L induced by base change. We write $[P]_L$ to denote the divisor on X_L supported at P with multiplicity one; if it is clear from context that we are considering divisors on X_L, we may drop the subscript and just write $[P]$.

2.1 Zero-cycles

Let X be a smooth projective variety over k. The group $\mathrm{CH}_0(X)$ is the quotient of the free abelian group $Z_0(X) := \bigoplus_{x \in X} \mathbb{Z}$ on all closed points of X modulo rational equivalence. The generators of $\mathrm{CH}_0(X)$ will be written as classes $[x]$ of closed points. There is a well-defined degree map

$$\deg : \mathrm{CH}_0(X) \to \mathbb{Z}, [x] \mapsto [k(x) : k].$$

We will denote by $F^1(X)$ the kernel of deg. Moreover, there is a higher dimensional analog of the Abel-Jacobi map,

$$\mathrm{alb}_X : F^1(X) \to \mathrm{Alb}_X(k),$$

called the Albanese map of X, where Alb_X is the Albanese variety of X (dual to the Picard variety $\mathrm{Pic}^0(X)$). When X is a curve, alb_X is precisely the Abel-Jacobi map, and $\mathrm{Alb}_X = J_X$ is the usual Jacobian of X. We will denote by $F^2(X)$ the kernel of alb_X.

From now on we suppose $X = C_1 \times \cdots \times C_d$ is a product of smooth projective curves over k. In this case $\mathrm{Alb}_X = J_1 \times \cdots \times J_d$, where J_i is the Jacobian of C_i, for $i \in \{1, \ldots, d\}$. Let $\pi_i : X \to C_i$ be the projection to the i-th component for $i \in \{1, \ldots, d\}$, which induces a pullback homomorphism, $\pi_i^\star : \mathrm{CH}_0(C_i) \to \mathrm{CH}_0(X)$. We can define a zero-cycle on X given the input of a zero-cycle on each component. Namely, we consider the homomorphism

$$\varepsilon_X : \mathrm{CH}_0(C_1) \otimes \cdots \otimes \mathrm{CH}_0(C_d) \to \mathrm{CH}_0(X) \tag{2.2}$$

defined as follows. Let $x_i \in C_i$ for each $i \in \{1, \ldots, d\}$. Define

$$\varepsilon_X([x_1] \otimes \cdots \otimes [x_d]) = \pi_1^\star([x_1]) \cdot \pi_2^\star([x_2]) \cdot \cdots \cdot \pi_d^\star([x_d]),$$

and extend linearly. Here $\cdot$ is the intersection product.

Definition 2.3 We define the componentwise subgroup $\mathrm{CH}_0(X)_{\mathrm{comp}}$ of $\mathrm{CH}_0(X)$ to be the image of ε_X, and $F^2(X)_{\mathrm{comp}} := \mathrm{CH}_0(X)_{\mathrm{comp}} \cap F^2(X)$.

It is not known whether ε_X is surjective onto $\mathrm{CH}_0(X)$, but we can obtain a surjective map by considering the base change of X to the residue fields of all closed points x of X. Given $x \in X$, denote by ε_x the composition

$$\varepsilon_x : \mathrm{CH}_0((C_1)_{k(x)}) \otimes \cdots \otimes \mathrm{CH}_0((C_d)_{k(x)}) \xrightarrow{\varepsilon_{X_{k(x)}}} \mathrm{CH}_0(X_{k(x)}) \xrightarrow{(\pi_x)_\star} \mathrm{CH}_0(X),$$

where $(\pi_x)_\star$ is the proper push-forward induced by the base change $X_{k(x)} \to X$. Then

$$\bigoplus_{x\in X} \mathrm{CH}_0((C_1)_{k(x)}) \otimes \cdots \otimes \mathrm{CH}_0((C_d)_{k(x)}) \xrightarrow{\oplus\varepsilon_x} \mathrm{CH}_0(X) \tag{2.4}$$

is a surjection. For, every closed point $x \in X$ has a $k(x)$-rational point $(x_1, \ldots, x_d) \in C_1(k(x)) \times \cdots \times C_d(k(x))$ lying above it, so that $\varepsilon_x([x_1] \otimes \cdots \otimes [x_d]) = [x]$.

From now on assume that for each $i = 1, \ldots, d$ there exists a fixed k-rational point $O_i \in C_i(k)$, so that we have an isomorphism

$$\mathrm{alb}_{C_i} : F^1(C_1) = \mathrm{Pic}^0(C_i) \xrightarrow{\simeq} J_i(k),$$

and a decomposition $\mathrm{CH}_0(C_i) \xrightarrow{\simeq} \mathbb{Z} \oplus J_i(k)$, sending the class $[x_i]$ of a closed point to the pair $([k(x_i) : k], [x_i] - [k(x_i) : k][O_i])$. We get a similar decomposition, $\mathrm{CH}_0((C_i)_L) \simeq \mathbb{Z} \oplus J_i(L)$, over any finite extension L/k by using the pullback $[O_i]_L$ of the zero-cycle $[O_i]$. By expanding the tensor products in (2.4), we obtain a surjection

$$\mathbb{Z} \oplus \bigoplus_{\nu=1}^{d} \bigoplus_{1\le i_1<\cdots<i_\nu\le d} \left(\bigoplus_{L/k \text{ finite}} J_{i_1}(k(x)) \otimes \cdots \otimes J_{i_\nu}(k(x)) \right) \twoheadrightarrow \mathrm{CH}_0(X). \tag{2.5}$$

Raskind and Spiess determined the kernel of this map. The relations are given by *Somekawa K-groups* $K(k; J_{i_1}, \ldots, J_{i_\nu})$. Before we recall their precise result, we review some necessary facts about Somekawa K-groups.

2.2 The Somekawa K-group

Let $G_1, \ldots, G_r$ be semi-abelian varieties over k. The Somekawa K-group $K(k; G_1, \ldots, G_r)$ attached to $G_1, \ldots, G_r$ was defined by K. Kato and Somekawa in [17], and it is a generalization of the Milnor K-group $K_r^M(k)$.

We note that the signs need to be dropped from the original definition (see [14, p. 10] for the correction). In what follows we will only need the explicit definition of this K-group for abelian varieties, so we omit the more general cases.

Notation 2.6 Let A be an abelian variety over k. Given a tower $F/L/k$ of field extensions, we denote by $\mathrm{res}_{F/L} : A(L) \to A(F)$ and $N_{F/L} : A(F) \to A(L)$ the usual restriction and norm map for abelian varieties. Here for $N_{F/L}$ to be well-defined we need to assume F/L is a finite extension.

If we want to draw attention to the choice of extension $\phi : L \to F$ (for instance, if we are considering multiple embeddings of L into F), we may use the notation res_ϕ or N_ϕ. The restriction map is in fact defined for an arbitrary field extension F/L; it is the obvious embedding induced by precomposing with the morphism $\mathrm{Spec}\, F \to \mathrm{Spec}\, L$. The norm map is only defined for finite extensions. We note that when F/L is a finite Galois extension with Galois group G, then $N_{F/L}(a) = \sum_{g\in G} ga$, for $a \in A(F)$.

With these norm and restriction maps, an abelian variety A becomes a *Mackey functor* over k (cf. [14, p. 13]). The following definition is due to Kahn.

Definition 2.7 (see [6]) Let $A_1, \ldots, A_r$ be abelian varieties over k. Their Mackey product $(A_1 \otimes^M \cdots \otimes^M A_r)(k)$ is defined to be $\left(\bigoplus_{F/k \text{ finite}} A_1(F) \otimes \cdots \otimes A_r(F)\right)/(\mathbf{PF})$ where $(\mathbf{PF})$ is the subgroup generated by the following type of elements.

Let $F/L/k$ be a tower of finite extensions. Let $a_i \in A_i(F)$ for some $i \in \{1, \ldots, r\}$ and $a_j \in A_j(L)$ for every $j \neq i$. We require

$$\big(a_1 \otimes \cdots \otimes N_{F/L}(a_i) \otimes \cdots \otimes a_r\big) - \big(\mathrm{res}_{F/L}(a_1) \otimes \cdots \otimes a_i \otimes \cdots \otimes \mathrm{res}_{F/L}(a_r)\big) \in (\mathbf{PF}). \tag{2.8}$$

The relation (2.8) is known as *projection formula*. The elements of $(A_1 \otimes^M \cdots \otimes^M A_r)(k)$ are traditionally denoted as symbols $\{a_1, \ldots, a_r\}_{L/k}$, for a given finite extension L/k and $a_i \in A_i(L)$, $i \in \{1, \ldots, r\}$. Using the fact that $N_{F/L} \circ \mathrm{res}_{F/L} = [F : L]$, an immediate consequence of the projection formula is that $[F : L]\{a_1, \ldots, a_r\}_{L/k} = \{\mathrm{res}_{F/L}(a_1), \ldots, \mathrm{res}_{F/L}(a_r)\}_{F/k}$ for any tower $F/L/k$ of finite extensions. In particular, symbols are preserved under isomorphisms of extensions of k.

Below we give the definition of the Somekawa K-group $K(k; A_1, \ldots, A_r)$ attached to abelian varieties $A_1, \ldots, A_r$. For the more general definition we refer to [17, p. 107].

Definition 2.9 ([17, p. 107]) The Somekawa K-group $K(k; A_1, \ldots, A_r)$ attached to $A_1, \ldots, A_r$ is defined to be

$$K(k; A_1, \ldots, A_r) := (A_1 \otimes^M \cdots \otimes^M A_r)(k)/(\mathbf{WR}),$$

where $(\mathbf{WR})$ is the subgroup generated by the following types of elements. Let C be a smooth projective curve over k.

For each closed point $x \in C$, let $\iota_x \in C(k(x))$ be the canonical inclusion $\mathrm{Spec}\, k(x) \to C$, and let the extension $k \to k(x)$ be determined by composing ι_x with the structure morphism $C \to \mathrm{Spec}\, k$.

For every $f \in k(C)^\times$ and every collection of regular maps $g_i : C \to A_i$ for $i \in \{1, \ldots, r\}$ we insist that

$$\sum_{x \in C} \mathrm{ord}_x(f)\{g_1 \circ \iota_x, \ldots, g_r \circ \iota_x\}_{k(x)/k} \in (\mathbf{WR}). \tag{2.10}$$

The generators of $K(k; A_1, \ldots, A_r)$ will again be denoted as symbols $\{a_1, \ldots, a_r\}_{L/k}$. The relation (2.10) is known as *Weil reciprocity*. When $r = 1$ we have an isomorphism $K(k; A) \simeq A(k)$. When k is algebraically closed, this follows by [16, Chapter III, Theorem 1]). For a proof in the general case see [4, Corollary 3.7].

Notation 2.11 We will denote by $\mathrm{Symb}_k(A_1, \ldots, A_r)$ the subgroup of $K(k; A_1, \ldots, A_r)$ generated by symbols $\{a_1, \ldots, a_r\}_{k/k}$ defined over k.

2.3 The norm and restriction maps on K-groups

Let L/k be a finite extension. There is a restriction map on K-groups

$$\mathrm{res}_{L/k} : K(k; A_1, \ldots, A_r) \to K(L; (A_1)_L, \ldots, (A_r)_L)$$

defined as follows. Let F/k be a finite extension. Since we assumed that the base field k is perfect, the primitive element theorem implies that the extension F/k is simple. That is, $F \simeq k[X]/(f(X))$ for some irreducible polynomial $f(X) \in k[X]$ (where restricting the isomorphism to the constant field gives the embedding $k \hookrightarrow F$). This gives an isomorphism

$$F \otimes_k L \simeq L[X]/(f(X)) \simeq \prod_{j=1}^{n} L[X]/(f_j(X)) = \prod_{j=1}^{n} L_j,$$

where $f(X) = \prod_j f_j(X)$ is the factorization of $f(X)$ into irreducible polynomials in $L[X]$. For each j, the given extensions L/k and F/k induce canonical extensions L_j/L (by inclusion of scalars $L \to L[X]/(f_j(X))$) and L_j/F (by $k[X]/(f(X)) \to L[X]/(f(X)) \to L[X]/(f_j(X))$, sending $x \mapsto x$ and constants along the inclusion $k \hookrightarrow L$).

Given $a_i \in A_i(F)$ for $i \in \{1, \dots, r\}$, we define

$$\mathrm{res}_{L/k}(\{a_1, \dots, a_r\}_{F/k}) = \sum_{j=1}^{n} \{\mathrm{res}_{L_j/F}(a_1), \dots, \mathrm{res}_{L_j/F}(a_r)\}_{L_j/L}. \tag{2.12}$$

We refer to [17, (1.3)] for a definition of the restriction map in the imperfect case.

Moreover, there is a norm map

$$N_{L/k} : K(L; (A_1)_L, \dots, (A_r)_L) \to K(k; A_1, \dots, A_r)$$

sending a symbol $\{b_1, \dots, b_r\}_{F/L}$ to $\{b_1, \dots, b_r\}_{F/k}$, with the extension F/k determined by composing the embeddings $k \hookrightarrow L \hookrightarrow F$. As usual, we have an equality $N_{L/k} \circ \mathrm{res}_{L/k} = [L : k]$.

2.4 *K*-group of a product of curves

From now on we assume that $X = C_1 \times \cdots \times C_d$ is a product of smooth projective curves over k, with fixed k-rational points $O_i \in C_i(k)$ for each $i \in \{1, \dots, d\}$. To each nonempty subset $\{i_1, \dots, i_\nu\} \subseteq \{1, \dots, d\}$, we have a Somekawa K-group $K(k; J_{i_1}, \dots, J_{i_\nu})$ associated to the Jacobians $J_{i_1}, \dots, J_{i_\nu}$.

In this setting, Raskind and Spiess ([14, Corollary 2.4.1]) established an isomorphism

$$\widehat{\varepsilon} : \mathbb{Z} \oplus \bigoplus_{\nu=1}^{d} \bigoplus_{1 \le i_1 < \cdots < i_\nu \le d} K(k; J_{i_1}, \dots, J_{i_\nu}) \xrightarrow{\simeq} \mathrm{CH}_0(X). \tag{2.13}$$

The map $\widehat{\varepsilon}$ is quite explicit. Given a finite extension $k \hookrightarrow L$ and a symbol $\{a_{i_1}, \dots, a_{i_\nu}\}_{L/k} \in K(k; J_{i_1}, \dots, J_{i_\nu})$, define $z_i \in \mathrm{CH}_0((C_i)_L)$ by $z_i := a_i$ if $i \in \{i_1, \dots, i_\nu\}$, and $z_i := [O_i]_L$ otherwise. Then

$$\begin{aligned} \widehat{\varepsilon}(\{a_{i_1}, \dots, a_{i_\nu}\}_{L/k}) &:= ((\pi_{L/k})_\star \circ \varepsilon_{X_L})(z_1 \otimes \cdots \otimes z_d) \\ &= (\pi_{L/k})_\star(\pi_1^\star(z_1) \cdot \pi_2^\star(z_2) \cdot \cdots \cdot \pi_d^\star(z_d)), \end{aligned}$$

where $\mathrm{CH}_0(X_L) \xrightarrow{(\pi_{L/k})_\star} \mathrm{CH}_0(X)$ is the pushforward map induced by base change along L/k.

One can then define a finite filtration on $\mathrm{CH}_0(X)$ by setting $\mathrm{Fil}^0(X) = \mathrm{CH}_0(X)$, and

$$\mathrm{Fil}^n(X) := \widehat{\varepsilon}\left(\bigoplus_{\nu=n}^{d} \bigoplus_{1 \le i_1 < \cdots < i_\nu \le d} K(k; J_{i_1}, \dots, J_{i_\nu}) \right)$$

for $1 \leq n \leq d$ (see [19, Example 2.2]). This gives identifications $\mathrm{Fil}^1(X) = F^1(X) = \ker(\deg)$, $\mathrm{Fil}^2(X) = F^2(X) = \ker(\mathrm{alb}_X)$, and we obtain an isomorphism

$$\bigoplus_{\nu=2}^{d} \bigoplus_{1\leq i_1<\cdots<i_\nu\leq d} K(k;J_{i_1},\ldots,J_{i_\nu}) \simeq F^2(X). \tag{2.14}$$

Recall that $\mathrm{Symb}_k(J_{i_1},\ldots,J_{i_\nu})$ denotes the subgroup of $K(k;J_{i_1},\ldots,J_{i_\nu})$ generated by symbols of the form $\{a_{i_1},\ldots,a_{i_\nu}\}_{k/k}$ (cf. Notation 2.11). The following lemma follows easily by the explicit description of the map $\widehat{\varepsilon}$.

Lemma 2.15 *The isomorphism* $\widehat{\varepsilon}$ *induces identifications*

$$\mathrm{CH}_0(X)_{\mathrm{comp}} \simeq \mathbb{Z} \oplus \bigoplus_{\nu=1}^{d} \bigoplus_{1\leq i_1<\cdots<i_\nu\leq d} \mathrm{Symb}_k(J_{i_1},\ldots,J_{i_\nu})$$

and

$$F^2(X)_{\mathrm{comp}} \simeq \bigoplus_{\nu=2}^{d} \bigoplus_{1\leq i_1<\cdots<i_\nu\leq d} \mathrm{Symb}_k(J_{i_1},\ldots,J_{i_\nu}). \tag{2.16}$$

Proof We only prove the first claim. The second follows in a similar manner. It is clear by definition that an element in $\mathbb{Z}$ or in $\mathrm{Symb}_k(J_{i_1},\ldots,J_{i_\nu})$ maps into $\mathrm{CH}_0(X)_{\mathrm{comp}}$. Conversely, write an arbitrary element $z_1\otimes\cdots\otimes z_d \in \mathrm{CH}_0(C_1)\otimes\cdots\otimes\mathrm{CH}_0(C_d)$ as

$$((z_1-\deg(z_1)[O_1])+\deg(z_1)[O_1])\otimes\cdots\otimes((z_d-\deg(z_d)[O_d])+\deg(z_d)[O_d])$$

and expand. Each term in the expansion will have in the i-th component either a multiple of $[O_i]$ or an element of $J_i(k)$, and so its image under ε_X will be either in $\widehat{\varepsilon}(\mathbb{Z})$ or in $\widehat{\varepsilon}(\mathrm{Symb}_k(J_{i_1},\ldots,J_{i_\nu}))$ for some choice of $i_1,\ldots,i_\nu$.

Remark 2.17 All the above discussion shows that if the base change X_L satisfies the weaker Conjecture 1.5 for all finite extensions L/k, then $F^2(X)$ is generated by torsion elements, so X satisfies Conjecture 1.2

3 Product formula and applications

Proposition 3.1 *Let* $A_1,\ldots,A_r$ *be abelian varieties over a perfect field* k *with* $r\geq 3$*. Let* L/k *be a finite extension. There is a well-defined homomorphism*

$$\begin{aligned}\Psi_k:\ & K(k;A_1,A_2)\otimes A_3(k)\otimes\cdots\otimes A_r(k)\to K(k;A_1,\ldots,A_r)\\ & \{a_1,a_2\}_{F/k}\otimes a_3\otimes\cdots\otimes a_r\mapsto\{a_1,a_2,\mathrm{res}_{F/k}(a_3)\ldots,\mathrm{res}_{F/k}(a_r)\}_{F/k},\end{aligned}$$

which extends for every finite extension L/k *to a homomorphism*

$$\Phi_L: K(k;A_1,A_2)\otimes A_3(L)\otimes\cdots\otimes A_r(L)\to K(k;A_1,\ldots,A_r)$$

given by $\Phi_L = N_{L/k}\circ\Psi_L\circ(\mathrm{res}_{L/k}\otimes 1_{A_3}\otimes\cdots\otimes 1_{A_r})$*, where* $N_{L/k}$ *and* $\mathrm{res}_{L/k}$ *are the norm and restriction maps on Somekawa* K*-groups (see Sect. 2.3).*

Proof It is enough to show that the homomorphism Ψ_k is well-defined. We first verify the projection formula (2.8). Let $F/K/k$ be a tower of finite extensions, and suppose $a_1\in A_1(F)$, $a_2\in A_2(K)$. Moreover, let $a_i\in A_i(k)$ for $3\leq i\leq r$. We compute

$$\begin{aligned}
&\Psi_k(\{a_1, \mathrm{res}_{F/K}(a_2)\}_{F/k} \otimes a_3 \otimes \cdots \otimes a_r) \\
&\quad = \{a_1, \mathrm{res}_{F/K}(a_2), \mathrm{res}_{F/k}(a_3), \ldots, \mathrm{res}_{F/k}(a_r)\}_{F/k} \\
&\quad = \{a_1, \mathrm{res}_{F/K}(a_2), \mathrm{res}_{F/K}(\mathrm{res}_{K/k}(a_3)), \ldots, \mathrm{res}_{F/K}(\mathrm{res}_{K/k}(a_r))\}_{F/k} \\
&\quad \overset{\textbf{(PF)}}{=} \{N_{F/K}(a_1), a_2, \mathrm{res}_{K/k}(a_3), \ldots, \mathrm{res}_{K/k}(a_r)\}_{K/k} \\
&\quad = \Psi_k(\{N_{F/K}(a_1), a_2\}_{K/k} \otimes a_3 \otimes \cdots \otimes a_r).
\end{aligned}$$

The symmetric relation for $a_1 \in A_1(K)$, $a_2 \in A_2(F)$ is analogous.

We next verify Weil reciprocity (2.10). Let C be a smooth projective curve over k and suppose we have regular maps $g_i : C \to A_i$ for $i = 1, 2$. Let $a_i \in A_i(k)$ for $i \in \{3, \ldots, r\}$. Let $C \xrightarrow{\pi} \operatorname{Spec} k$ be the structure morphism. For each $i \in \{3, \ldots, r\}$ define $g_i : C \to A_i$ to be the constant map $g_i := a_i \circ \pi$. Let $f \in k(C)^\times$. We have

$$\begin{aligned}
&\Phi_k\left(\sum_{x \in C} \mathrm{ord}_x(f)\{g_1 \circ \iota_x, g_2 \circ \iota_x\}_{k(x)/k} \otimes a_3 \otimes \cdots \otimes a_r\right) \\
&\quad = \sum_{x \in C} \mathrm{ord}_x(f)\{g_1 \circ \iota_x, g_2 \circ \iota_x, \mathrm{res}_{k(x)/k}(a_3), \ldots, \mathrm{res}_{k(x)/k}(a_r)\}_{k(x)/k} \\
&\quad = \sum_{x \in C} \mathrm{ord}_x(f)\{g_1 \circ \iota_x, g_2 \circ \iota_x, g_3 \circ \iota_x, \ldots, g_r \circ \iota_x\}_{k(x)/k} \overset{\textbf{(WR)}}{=} 0.
\end{aligned}$$

Remark 3.2 Proposition 3.1 can be thought of as the analog of the product formula for Milnor K-groups. Kahn and Yamazaki (cf. [7]) in fact defined a tensor product structure in the category of homotopy invariant Nisnevich sheaves with transfers, which when evaluated at $\mathrm{Spec}(k)$ coincides with a Somekawa K-group. Such a property therefore is not surprising.

Corollary 1.4 follows now as an easy corollary to Proposition 3.1. We restate this in greater generality here.

Corollary 3.3 *Let $X = C_1 \times \cdots \times C_d$ be a product of smooth projective curves over a perfect field k with $C_i(k) \neq \emptyset$ for $i \in \{1, \ldots, d\}$. Suppose that for each $1 \leq i < j \leq d$ the group $F^2(C_i \times C_j)$ is torsion. Then $F^2(X)$ is torsion. In the special case when k is an algebraic number field, we conclude that if Conjecture 1.2 is true for any subproduct of two curves, then it is true for X.*

Proof The assumption that $F^2(C_i \times C_j)$ is torsion for each $1 \leq i < j \leq d$ means precisely that the group $\bigoplus_{1 \leq i < j \leq d} K(k; J_i, J_j)$ is torsion. If $d = 2$ then we are done, so let us assume $d \geq 3$. Using the identification (2.14), it is enough to show that for every $3 \leq \nu \leq d$ the group $K(k; J_{i_1}, \ldots, J_{i_\nu})$ is torsion. Thus, the claim comes down to showing that if $A_1, \ldots, A_r$ are abelian varieties over k such that the Somekawa K-group $K(k; A_1, A_2)$ is torsion, then so is the group $K(k; A_1, \ldots, A_r)$.

Let $\{a_1, \ldots, a_r\}_{L/k} \in K(k; A_1, \ldots, A_r)$, where L/k is a finite extension and $a_i \in A_i(L)$ for $i \in \{1, \ldots, r\}$. If $\widehat{L}/k$ is the Galois closure of L/k, then

$$[\widehat{L} : L]\{a_1, \ldots, a_r\}_{L/k} \overset{\textbf{(PF)}}{=} \{\mathrm{res}_{\widehat{L}/L}(a_{i_1}), \mathrm{res}_{\widehat{L}/L}(a_1), \ldots, \mathrm{res}_{\widehat{L}/L}(a_r)\}_{\widehat{L}/k}.$$

Thus, to prove $\{a_1, a_2, \ldots, a_r\}_{L/k}$ is torsion, it is enough to consider the case that L/k is a Galois extension with Galois group G.

We consider the homomorphism

$$\Phi_L : K(k; A_1, A_2) \otimes A_3(L) \otimes \cdots \otimes A_r(L) \to K(k; A_1, \ldots, A_r)$$

of Proposition 3.1. To show that the symbol $\{a_1, a_2, \ldots, a_r\}_{L/k}$ is torsion, it suffices to show that an integer multiple of it lies in the image of Φ_L. We have,

$$\begin{aligned}&\Phi_L(\{a_1, a_2\}_{L/k} \otimes a_3 \otimes \cdots \otimes a_r) = \\ &N_{L/k} \circ \Psi_L \circ (\mathrm{res}_{L/k} \otimes 1_{A_3} \otimes \cdots \otimes 1_{A_r})(\{a_1, a_2\}_{L/k} \otimes a_3 \otimes \cdots \otimes a_r).\end{aligned}$$

Since L/k is finite Galois, we have an isomorphism of L-algebras,

$$L \otimes_k L \simeq \prod_{\sigma \in G} L.$$

In particular, in the definition of the restriction map on Somekawa K-groups (see Sect. 2.3) we have $L_j \simeq L$ for all $j = 1, \ldots, [L : k]$, and hence

$$\mathrm{res}_{L/k}(\{a_1, a_2\}_{L/k}) = \sum_{j=1}^{[L:k]} \{a_1, a_2\}_{L/L} = [L : k]\{a_1, a_2\}_{L/L}.$$

We conclude that

$$\begin{aligned}\Phi_L(\{a_1, a_2\}_{L/k} \otimes a_3 \otimes \cdots \otimes a_r) &= N_{L/k}([L : k]\{a_1, \ldots, a_r\}_{L/L}) \\ &= [L : k]\{a_1, \ldots, a_r\}_{L/k}.\end{aligned}$$

Remark 3.4 The assumption $C_i(k) \neq \emptyset$ can be removed, if we instead assume that for some finite extension L/k, we have $C_i(L) \neq \emptyset$ for all i and Conjecture 1.2 holds for every product $(C_i)_L \times (C_j)_L$ with $1 \leq i < j \leq d$. This follows easily by the formula $N_{L/k} \circ \mathrm{res}_{L/k} = [L : k]$.

The previous corollary shows that Conjecture 1.2 holds for arbitrary products of curves if it holds for pairwise products; the following corollary gives the analogous statement for the weaker Conjecture 1.5.

Corollary 3.5 *Let $X = C_1 \times \cdots \times C_d$ be a product of smooth projective curves over a perfect field k with $C_i(k) \neq \emptyset$ for $i \in \{1, \ldots, d\}$. Suppose that for each $1 \leq i < j \leq d$ the group $F^2(C_i \times C_j)_{\mathrm{comp}}$ is finite. Then $F^2(X)_{\mathrm{comp}}$ is finite.*

Proof Since $C_i(k) \neq \emptyset$ for all i the Raskind-Spiess isomorphism applies, and so by (2.16), the assumption that $F^2(C_i \times C_j)_{\mathrm{comp}}$ is finite for all $1 \leq i < j \leq d$ amounts to saying that $\mathrm{Symb}_k(J_i, J_j)$ is finite. The corollary then follows by Proposition 3.1 after noticing the surjectivity of the homomorphism

$$\Psi_k : \mathrm{Symb}_k(J_{i_1}, J_{i_2}) \otimes J_{i_3}(k) \otimes \cdots \otimes J_{i_\nu}(k) \to \mathrm{Symb}_k(J_{i_1}, \ldots, J_{i_\nu}).$$

In the following section, we will give many applications of Corollary 3.5.

4 Finiteness of $F^2(C_1 \times C_2)_{\mathrm{comp}}$

The goal of this section is to determine pairs of curves C_1, C_2 such that $F^2(C_1 \times C_2)_{\mathrm{comp}}$ is finite. Note that if C is a curve then $F^2(C \times C)_{\mathrm{comp}}$ is not a priori finite, so the case $C_1 = C_2$ is often nontrivial.

Convention 4.1 Let C_1, C_2 be curves over k with Jacobians J_1 and J_2, and let $X = C_1 \times C_2$. We will suppose there are fixed rational points $O_1 \in C_1(k)$ and $O_2 \in C_2(k)$; if C_i is an elliptic curve, we will always take O_i to be the identity of $C_i(k)$.

There are two main principles we use in order to examine whether $F^2(X)_{\mathrm{comp}}$ is finite. Lemma 4.2 allows us to reduce the problem to checking a finite list of relations, and Proposition 4.5 gives us a way to produce these relations.

Lemma 4.2 *Let B be a list of* $(\operatorname{rank} J_1(k))(\operatorname{rank} J_2(k))$ *linearly independent elements of* $J_1(k) \otimes J_2(k)$*. If $\varepsilon_X(t)$ is torsion for all $t \in B$, then $F^2(X)_{\mathrm{comp}}$ is finite.*

Proof By (2.16), $F^2(X)_{\mathrm{comp}}$ is the image of the finitely generated abelian group $J_1(k) \otimes J_2(k)$ under ε_X. Since B generates a full-rank subgroup of $J_1(k) \otimes J_2(k)$, every element of $J_1(k) \otimes J_2(k)$ maps to torsion in $F^2(X)_{\mathrm{comp}}$.

We list two consequences of Lemma 4.2. The first is immediate.

Corollary 4.3 *If $J_1(k)$ or $J_2(k)$ has rank 0, then $F^2(X)_{\mathrm{comp}}$ is finite.*

Corollary 4.4 *Let C_1, C_2, C_2' be curves over k. If $F^2(C_1 \times C_2)_{\mathrm{comp}}$ is finite and the Jacobians of C_2 and C_2' are isogenous, then $F^2(C_1 \times C_2')_{\mathrm{comp}}$ is finite.*

Proof This follows by covariant functoriality of the Somekawa K-group (cf. [17, p. 4]). Namely, let J_2 and J_2' be the Jacobians of C_2 and C_2' respectively; if $\phi : J_2 \to J_2'$ is an isogeny, then it induces a group homomorphism $K(k; J_1, J_2) \to K(k; J_1, J_2')$ sending a symbol $\{a, b\}_{L/k}$ to $\{a, \phi(b)\}_{L/k}$.

Finiteness of $F^2(C_1 \times C_2)_{\mathrm{comp}}$ implies that $\{P_1, P_2\}_{k/k} \in \mathrm{Symb}_k(J_1, J_2)$ is torsion for all $P_1 \in J_1(k)$ and $P_2 \in J_2(k)$, and so

$$\varepsilon_X(P_1 \otimes \phi(P_2)) = \{P_1, \phi(P_2)\}_{k/k} \in \mathrm{Symb}_k(J_1, J_2')$$

is torsion. Since the map $J_2(k) \to J_2'(k)$ induced by ϕ has finite cokernel, elements of the form $P_1 \otimes \phi(P_2)$ generate a full-rank subgroup of $J_1(k) \otimes J_2(k)$.

Given an elliptic or hyperelliptic curve H, let $P \mapsto \overline{P}$ denote negation or the hyperelliptic involution, respectively.

Proposition 4.5 *Let H be an elliptic or hyperelliptic curve, let $P \in H(\overline{k})$, and let $\phi_i : H \to C_i$ be a morphism for each $i = 1, 2$. Suppose that $\phi_i(P) \in C_i(k)$ for each $i = 1, 2$, and also suppose there exists $W \in H(\overline{k})$ with $W = \overline{W}$ and such that $[\phi_i(W)] - [O_i] \in J_i(\overline{k})$ is torsion for $i = 1, 2$. Then the symbol*

$$\{[\phi_1(P)] - [O_1], [\phi_2(P)] - [O_2]\}_{k/k} \in \mathrm{Symb}_k(J_1, J_2)$$

is torsion.

Proof Let L/k be a finite extension that contains subfields isomorphic to the fields of definition of W and of P, so that we can identify $P, \overline{P}$, and W with elements of $H(L)$. Then $[P] + [\overline{P}] - 2[W]$ is a principal divisor on H_L.

Define $g_i : H_L \to (J_i)_L$ to act on L-rational points by $R \mapsto [\phi_i(R)] - [\phi_i(W)]$. For $i = 1, 2$, set $W_i = \phi_i(W)$ and $P_i := \phi_i(P)$. Note that the relation $[\overline{P}] - [W] = -[P] + [W]$ on H

induces the relation

$$[\phi_i(\overline{P})] - [\phi_i(W)] = -[\phi_i(P)] + [\phi_i(W)] = -[P_i] + [W_i]$$

in $J_i(L)$ by proper pushforward of Chow groups. The principal divisor $[P] + [\overline{P}] - 2[W]$ then induces a Weil reciprocity relation (2.10) on the group $K(L; (J_1)_L, (J_2)_L)$:

$$\begin{aligned} 0 &= \{[P_1] - [W_1],\ [P_2] - [W_2]\}_{L/L} + \{-[P_1] + [W_1],\ -[P_2] + [W_2]\}_{L/L} - 2\{0,\ 0\}_{L/L} \\ &= 2\{[P_1] - [W_1],\ [P_2] - [W_2]\}_{L/L}. \end{aligned}$$

Since $[W_i] - [O_i] \in J_{1L}(L)$ is torsion for each $i = 1, 2$, there exists $n > 0$ such that

$$\begin{aligned} 0 &= n\{[P_1] - [W_1], [P_2] - [W_2]\}_{L/L} \\ &= n\{[P_1] - [O_1] + [O_1] - [W_1], [P_2] - [O_2] + [O_2] - [W_2]\}_{L/L} \\ &= n\{[P_1] - [O_1], [P_2] - [O_2]\}_{L/L}. \end{aligned}$$

Applying the norm map $N_{L/k} : K(L; (J_1)_L, (J_2)_L) \to K(k; J_1, J_2)$ (see Sect. 2.3), we can conclude that

$$0 = 2n\{[P_1]_L - [O_1]_L, [P_2]_L - [O_2]_L\}_{L/k}.$$

Since $P_i \in C_i(k)$ by assumption, the divisors $[P_i]_L - [O_i]_L \in J_i(L)$ are in the image of $\mathrm{res}_{L/k} : J_i(k) \to J_i(L)$. Hence, the projection formula (2.8) gives

$$\begin{aligned} 0 &= n\{\mathrm{res}_{L/k}([P_1] - [O_1]), \mathrm{res}_{L/k}([P_2] - [O_2])\}_{L/k} \\ &= n\{[P_1] - [O_1], N_{L/k}(\mathrm{res}_{L/k}([P_2] - [O_2]))\}_{k/k} \\ &= n[L : k]\{[P_1] - [O_1], [P_2] - [O_2]\}_{k/k}. \end{aligned}$$

Remark 4.6 Most of the examples we explore below use Proposition 4.5; that is, the key relation comes from Weil reciprocity for an elliptic or hyperelliptic curve. See Remark 4.18 for examples that use Weil reciprocity for a curve that is neither elliptic nor hyperelliptic.

4.1 Isogenous pairs

We list some consequences of Proposition 4.5 that can be obtained by setting H equal to one of the factors of X.

4.1.1 Rank 1 hyperelliptic curves

Proposition 4.7 *Let C_1, C_2 be hyperelliptic curves over k with isogenous Jacobians J_1 and J_2. Suppose* $\mathrm{rank}\, J_1(k) = 1$, *and there exist $P, W \in C_1(k)$ with W a Weierstrass point and $[P] - [W]$ of infinite order. Then $F^2(C_1 \times C_2)_{\mathrm{comp}}$ is finite.*

Proof For the case $C_2 = C_1$, we can apply Proposition 4.5 and Lemma 4.2 using $\phi_1 = \phi_2$ the identity map on H, and $O_1 = O_2 = W$. The general case follows by Corollary 4.4.

In light of this result, Corollaries 4.3 and 3.5, we can conclude that $F^2(C_1 \times \cdots \times C_d)_{\mathrm{comp}}$ is finite if all J_i with $\mathrm{rank}\, J_i(k) \neq 0$ come from a single isogeny class with rank 1 over k.

While Proposition 4.7 does apply when C_1, C_2 are isogenous elliptic curves (take W to be the identity of $C_1(k)$), it yields a more interesting result when the genus of C_1 is at least 2. In the LMFDB [18] there are 868 genus 2 curves over $\mathbb{Q}$ that have Jacobian of rank

1, conductor at most 10000, and a rational Weierstrass point W. For all but 8 of these, there exists another rational point P such that $[P]-[W]$ has infinite order. These provide the first nontrivial examples of products X with $F^2(X)_{\mathrm{comp}}$ finite involving positive rank curves of higher genus.

4.1.2 Self-product of a rank 2 CM elliptic curve

Let E be an elliptic curve over $\mathbb{Q}$ such that $E_{\overline{\mathbb{Q}}}$ has complex multiplication by the full ring of integers $\mathcal{O}_K$ of a quadratic imaginary field K. It follows that K is of the form $K = \mathbb{Q}(\sqrt{-D})$ with $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$. We write $\mathcal{O}_K = \mathbb{Z}[\omega]$ and we denote by $[\omega] : E_K \to E_K$ the extra endomorphism. Suppose that $E(\mathbb{Q})$ has rank 1 and let P be a point of infinite order. It follows by [5, Lemma 4.5] that the point $[\omega](P) \in E(K)$ is $\mathbb{Z}$-linearly independent from P, and thus $E(K)$ has rank at least 2. In all cases we have computed, $E(K)$ actually has rank exactly 2.

Proposition 4.8 *Suppose that E satisfies the above assumptions and assume additionally that* $\mathrm{rank}(E(\mathbb{Q})) = 1$ *and* $\mathrm{rank}(E(K)) = 2$*. Let S be a set consisting of* E_K *together with arbitrarily many curves over K with Jacobian of rank* 0*. Then* $F^2(C_1 \times \cdots \times C_d)_{\mathrm{comp}}$ *is finite for all* $C_1, \cdots, C_d \in S$*. (In particular,* $F^2(E_K \times E_K)_{\mathrm{comp}}$ *is finite.)*

Proof By Corollary 3.5 it suffices to show that $F^2(C_i \times C_j)_{\mathrm{comp}}$ is finite for all $1 \le i < j \le d$. If either J_i or J_j has rank 0 this follows by Corollary 4.3. The only remaining case is when $C_i = C_j = E_K$. Since $P \in E(\mathbb{Q})$ and $[\omega](P) \in E(K) \setminus E(\mathbb{Q})$ are $\mathbb{Z}$-linearly independent, they generate a finite index subgroup of $E(K)$. By Lemma 4.2, it is enough to show that the symbols

$$\{P, P\}_{K/K}, \ \{[\omega](P), P\}_{K/K}, \ \{P, [\omega](P)\}_{K/K}, \ \{[\omega](P), [\omega](P)\}_{K/K}$$

are all torsion. This follows by applying Proposition 4.5 to the four choices of $\phi_1, \phi_2 \in \{[1], [\omega]\}$.

4.2 Non-isogenous pairs: a new construction

Let E_1, E_2 be elliptic curves over k with fully k-rational 2-torsion, so that there exist Weierstrass forms

$$E_1 \simeq E_{\alpha,\beta} : y^2 = x(x-\alpha)(x-\beta), \qquad E_2 \simeq E_{\gamma,\delta} : y^2 = x(x-\gamma)(x-\delta)$$

for some $\alpha, \beta, \gamma, \delta \in k^\times$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. In [15, Theorem 1], Scholten defines the curve

$$H_{\alpha,\beta,\gamma,\delta} : (\alpha\delta - \beta\gamma)y^2 = ((\alpha-\beta)x^2 - (\gamma-\delta))(\alpha x^2 - \gamma)(\beta x^2 - \delta) \tag{4.9}$$

over k, and provides explicit degree 2 k-rational maps $f_1 : H_{\alpha,\beta,\gamma,\delta} \to E_{\alpha,\beta}$ and $f_2 : H_{\alpha,\beta,\gamma,\delta} \to E_{\gamma,\delta}$. These maps each send the Weierstrass points of $H_{\alpha,\beta,\gamma,\delta}$ to 2-torsion points.

Composing f_1 with an isomorphism $E_{\alpha,\beta} \to E_1$, and f_2 with an isomorphism $E_{\gamma,\delta} \to E_2$, we obtain maps $\phi_i : H_{\alpha,\beta,\gamma,\delta} \to E_i$, which map Weierstrass points to torsion points. This means that any $P \in H(k)$ will allow us to apply Proposition 4.5.

Proposition 4.10 *Let* $\alpha, \beta, \gamma, \delta \in k^\times$ *with* $\alpha \neq \beta$ *and* $\gamma \neq \delta$*, and set* $E_1 : y^2 = x(x-\alpha)(x-\beta)$ *and* $E_2 : y^2 = x(x-\gamma)(x-\delta)$*. Let B be the list of points in* $E_1(k) \times E_2(k)$ *output*

Algorithm 1: Computing elements of $E_1(k) \otimes E_2(k)$ that map to torsion in $F^2(E_1 \times E_2)$.

Input : $\alpha, \beta, \gamma, \delta \in k^\times$ with $\alpha \neq \beta$ and $\gamma \neq \delta$.
Output: A list of pairs $(P_1, P_2) \in E_1(k) \times E_2(k)$, for $E_1 : y^2 = x(x-\alpha)(x-\beta)$ and $E_2 : y^2 = x(x-\gamma)(x-\delta)$.

1 Find a basis $R_1, \ldots, R_r$ for a full-rank subgroup of $E_1(k)$, and a basis $S_1, \ldots, S_s$ for a full-rank subgroup of $E_2(k)$.
2 Initialize empty lists B (of points in $E_1(k) \times E_2(k)$) and T (of vectors in $\mathbb{Q}^{rs}$).
3 Define $H := H_{\alpha,\beta,\gamma,\delta}$ as in (4.9) and the corresponding maps $\phi_i : H \to E_i$.
4 Search for rational points $P \in H(k)$.
5 **for** *each point* $P \in H(k)$ *found* **do**
6 Find $a_1, \ldots, a_r, d_1 \in \mathbb{Z}$, $d_1 \neq 0$, such that $a_1R_1 + \cdots + a_rR_r = d_1\phi_1(P)$ in $E_1(k)$.
7 Find $b_1, \ldots, b_s, d_2 \in \mathbb{Z}$, $d_2 \neq 0$, such that $b_1S_1 + \cdots + b_sS_s = d_2\phi_2(P)$ in $E_2(k)$.
8 Define the vector $v(P) \in \mathbb{Q}^{rs}$ to be

$$\frac{1}{d_1d_2}(a_1, \ldots, a_r) \otimes (b_1, \ldots, b_s) = \frac{1}{d_1d_2}(a_1b_1, a_1b_2, \ldots, a_1b_s, a_2b_1, \ldots, a_2b_s, \ldots, a_rb_s).$$

9 **if** $T \cup \{v(P)\}$ *is linearly independent* **then**
10 Append $v(P)$ to T.
11 Append $(\phi_1(P), \phi_2(P))$ to B.
12 **end**
13 **end**
14 Return B.

by Algorithm 1. Then $(P_1 \otimes P_2 : (P_1, P_2) \in B)$ *is a linearly independent list of elements of* $E_1(k) \otimes E_2(k)$*, all of which map to torsion in* $F^2(E_1 \times E_2)$ *under* ε_X*.*

Proof An element $(P_1, P_2) \in B$ is of the form $(\phi_1(P), \phi_2(P))$ for some $P \in H(k)$. Recall that the maps ϕ_i send Weierstrass points of H to 2-torsion points of E_1 and E_2. Hence, by Proposition 4.5, $\varepsilon_X(P_1 \otimes P_2)$ is torsion. Now $v(P)$, defined in line 8, is the image of $(\phi_1(P), \phi_2(P))$ under the bilinear map

$$E_1(k) \times E_2(k) \to (E_1(k) \otimes E_2(k)) \otimes \mathbb{Q} \simeq \mathbb{Q}^{rs}$$

induced by sending $(R_i, S_j) \mapsto R_i \otimes S_j$ for each $1 \leq i \leq r$ and $1 \leq j \leq s$. We can see from line 11 and the preceding lines that B is constructed so that its image T in $(E_1(k) \otimes E_2(k)) \otimes \mathbb{Q}$ (and therefore its image in $E_1(k) \otimes E_2(k)$) is a linearly independent list.

Applying Lemma 4.2 gives the following immediate consequence.

Corollary 4.11 *Assume the setup of Proposition 4.10. If the length of* B *is equal to* $(\operatorname{rank} E_1(k))(\operatorname{rank} E_2(k))$*, then* $F^2(E_1 \times E_2)_{\mathrm{comp}}$ *is finite.*

4.2.1 Computations

The authors used Magma to implement a version of Algorithm 1 [10], with a few modifications as described in Sect. 4.2.2.

By running this algorithm and applying Corollary 4.11, the authors were able to find a substantial collection of pairs of curves $E_1 \times E_2$ such that $F^2(E_1 \times E_2)_{\mathrm{comp}}$ is finite, including the first known examples of pairs of non-isomorphic curves with rank greater than 1. Specifically, they used the LMFDB [18] to obtain a list of the first m_1 elliptic curves

Table 1 Counting pairs of elliptic curves over $\mathbb{Q}$ with $F^2(E_1 \times E_2)_{\text{comp}}$ finite.

# of E_1	rank $E_1(\mathbb{Q})$	# of E_2	rank $E_2(\mathbb{Q})$	total # of pairs	# $F^2(X)_{\text{comp}}$ finite
100	1	100	1	4950	2602
100	2	100	2	4950	995
100	2	$= E_1$		100	70
20	3	20	3	190	17
20	3	$= E_1$		20	8
100	1	100	2	10,000	3311
500	1	20	3	10,000	955
500	2	20	3	10,000	615

over $\mathbb{Q}$ of rank r_1 and torsion subgroup $C_2 \times C_2$ (ordered by conductor), and the first m_2 elliptic curves over $\mathbb{Q}$ of rank r_2 and torsion subgroup $C_2 \times C_2$, for various choices of m_1, r_1, m_2, r_2. The torsion subgroup $C_2 \times C_2$ was chosen as it is the simplest torsion structure for which the curves have fully rational 2-torsion, though the algorithm would work equally well on any torsion structure of the form $C_2 \times C_{2n}$. For each E_1 from the first list and E_2 from the second, they ran algorithm 1; for each pair with an output of length $r_1 r_2$, the pair (E_1, E_2) was added to a list of pairs for which $F^2(E_1 \times E_2)_{\text{comp}}$ is known to be finite.

The findings are summarized in Table 1, with m_1, r_1, m_2, r_2 appearing as "# of E_1," "rank $E_1(\mathbb{Q})$," "# of E_2," and "rank $E_2(\mathbb{Q})$," respectively. For rows with rank $E_1(\mathbb{Q})$ = rank $E_2(\mathbb{Q})$, only one pair out of (E_1, E_2) or (E_2, E_1) was tested, and diagonal pairs (E_1, E_1) were not included. For rows containing "$= E_1$" in the "# of E_2" column, only pairs (E_1, E_1) were tested.

4.2.2 Improvements on algorithm 1

The authors' code makes some changes to algorithm 1, including the following.

- The curve $H_{\alpha,\beta,\gamma,\delta}$ depends on the choice of $\alpha, \beta, \gamma, \delta$, and is not invariant under isomorphism. In fact, while the curves

$$E_{\alpha,\beta} \simeq E_{\beta,\alpha} \simeq E_{-\alpha,\beta-\alpha} \simeq E_{\beta-\alpha,-\alpha} \simeq E_{-\beta,\alpha-\beta} \simeq E_{\alpha-\beta,-\beta} \tag{4.12}$$

  are isomorphic, these typically produce six[3] non-isomorphic hyperelliptic curves

$$H_{\alpha,\beta,\gamma,\delta},\ H_{\beta,\alpha,\gamma,\delta},\ H_{-\alpha,\beta-\alpha,\gamma,\delta},\ H_{\beta-\alpha,-\alpha,\gamma,\delta},\ H_{-\beta,\alpha-\beta,\gamma,\delta},\ H_{\alpha-\beta,-\beta,\gamma,\delta}. \tag{4.13}$$

  Each of these six curves H comes with maps $\phi_1 : H \to E_1$ and $\phi_2 : H \to E_2$. Therefore, we can extend algorithm 1 to search for points P on all six of these curves.
- As written, line 1 requires a full set of Mordell-Weil generators for $E_1(k)$ and $E_2(k)$. This is not a problem if generators are given (for instance if the curves are obtained from the LMFDB), but if they are not already known then computing these sets can potentially be a significant bottleneck for the runtime. Fortunately, a full set of

[3]There may be fewer, for example because of accidental isomorphisms, or because $\alpha\delta - \beta\gamma = 0$ so that (4.9) defines a degenerate curve. There are of course many other tuples $(\alpha', \beta', \gamma', \delta')$ with isomorphisms $E_{\alpha',\beta'} \simeq E_{\alpha,\beta}$ and $E_{\gamma',\delta'} \simeq E_{\gamma,\delta}$, but one can check in each case that the resulting curve $H_{\alpha',\beta',\gamma',\delta'}$ is isomorphic to one of the six curves from (4.13), and that the corresponding maps are compatible in an appropriate sense.

generators is not necessary. Instead, one can initialize empty lists B_i of elements of $E_i(k)$ for $i = 1, 2$. For each $P \in H(k)$, one checks whether line 6 is possible using the current points in B_1. If not, first append $\phi_1(P)$ to B_1, and then proceed. One can do the same for B_2 and line 7.

With this modification, the values of $r = \#B_1$ and $s = \#B_2$ are regularly updated, and T must be recomputed from B whenever this occurs. It is therefore helpful to define new lists to store the coefficients of $\phi_1(P)$ (resp. $\phi_2(P)$) generated in line 6 (resp. line 7) for each P considered, in order to more quickly recompute $\nu(P) \in \mathbb{Q}^{rs}$ for the new values of r and s. Note that the ranks of $E_1(k)$ and $E_2(k)$ must be known in advance in order to know when we reach $r = \operatorname{rank} E_1(k)$ and $s = \operatorname{rank} E_2(k)$.

- Rather than returning a list of points $(\phi_1(P), \phi_2(P)) \in E_1(k) \times E_2(k)$, the code returns a list of tuples (P, ϕ_1, ϕ_2), so that Weil reciprocity (2.10) can be verified.

4.3 Non-isogenous pairs: previous constructions

We are aware of two previous constructions of pairs E_1, E_2 of non-isogenous rank 1 curves over k with $F^2(E_1 \times E_2)_{\text{comp}}$ finite.

Both constructions use the fact that it suffices to find non-torsion points $P_1 \in E_1(k)$ and $P_2 \in E_2(k)$ with $\{P_1, P_2\}_{k/k}$ torsion (Lemma 4.2).

4.3.1 Modular parametrizations

Prasanna and Srinivas considered two pairs of non-isogenous rank 1 curves over $\mathbb{Q}$: one pair of curves of conductor 91,

$$E_1^{(91)} : y^2 + y = x^3 + x, \qquad E_2^{(91)} : y^2 + y = x^3 + x^2 - 7x + 5,$$

and one pair of curves of conductor 37,

$$E_1^{(37)} : y^2 + y = x^3 - x, \qquad E_2^{(37)} : y^2 + y = x^3 + x^2 - 23x - 50.$$

They prove the following proposition.

Proposition 4.14 ([13], in preparation) *Let $N = 37$ or 91. Let $X = E_1^{(N)} \times E_2^{(N)}$ as defined above. Then $F^2(X)_{\text{comp}}$ is finite.*

Note that for $N = 37$ or 91, every elliptic curve of conductor N is isogenous to either $E_1^{(N)}$ or $E_2^{(N)}$ (see for example [18]). So together with Corollary 4.4 we can conclude that $F^2(E_1 \times E_2)_{\text{comp}}$ is finite for any two curves E_1, E_2 of conductor N.

We make a few remarks on the approach. For each of $N = 37$ and $N = 91$, they show that the two modular parametrizations $X_0(N) \to E_i^{(N)}$ factor through a common hyperelliptic curve H. For $N = 37$, $X_0(37)$ is itself a genus 2 hyperelliptic curve. On the other hand, $X_0(91)$ is not hyperelliptic, but it has an Atkin-Lehner involution w_{91}, and the modular parametrizations of $E_1^{(91)}$ and $E_2^{(91)}$ both factor through the genus 2 hyperelliptic curve $X_0(91)/\langle w_{91} \rangle$. Since the two elliptic curves have rank 1, the theory of Heegner points can be used to compute a point $P \in H(\overline{\mathbb{Q}})$ such that the image of P under each $f_i : H \to E_i^{(N)}$ is a rational point of infinite order.

As in Proposition 4.5, one can obtain the desired relation in $F^2(E_1^{(N)} \times E_2^{(N)})$ by applying Weil reciprocity to a principal divisor on H. Rather than using a Weierstrass point as in

Proposition 4.5, however, Prasanna and Srinivas find another point $Q \in H(\overline{\mathbb{Q}})$ such that $f_i(Q) \in E_i^{(N)}(\mathbb{Q})$, and apply Weil reciprocity to the divisor $[P] + [\overline{P}] - [Q] - [\overline{Q}]$.

4.3.2 Rational curves in a Kummer surface

Given a number field k, define a 2-parameter family of curves over k by

$$E_{s,t} : y^2 = x^3 - 3t^2x + 2t^3 + (1-s-3t)^2 s. \tag{4.15}$$

Note that the curve $E_{s,t}$ has a k-point $P_{s,t} := (1-s-2t, 1-s-3t)$. For each $s \in k$, let D_s denote the locus of points $t \in k$ at which the discriminant of $E_{s,t}$ vanishes, or the point $P_{s,t}$ is torsion in $E_{s,t}(k)$. It can be shown (as in the proof of [8, Proposition 5.3]) that D_s is finite for all $s \in k \setminus \{0\}$. In a previous paper, the second author proved the following.

Proposition 4.16 ([8, Corollary 1.4]) *Let $s \in k$. For all $t_1, t_2 \in k \setminus D_s$ such that $E_{s,t_1}(k)$ and $E_{s,t_2}(k)$ have rank 1, $F^2(E_{s,t_1} \times E_{s,t_2})_{\text{comp}}$ is finite.*

Together with Corollaries 3.5, 4.3, and 4.4, this result can be used to create large sets of elliptic curves for which any product X of curves from the set has finite $F^2(X)_{\text{comp}}$.

Corollary 4.17 *Fix $s \in k$, and set*

$$S_s = \{E_{s,t} : t \in k \setminus D_s, \ \operatorname{rank} E_{s,t}(k) = 1\}.$$

Suppose $C_1, \dots, C_d$ are elliptic curves over k such that for each i, either $\operatorname{rank} C_i(k) = 0$ *or C_i is isogenous to a curve in S_s. Then $F^2(C_1 \times \cdots C_d)_{\text{comp}}$ is finite.*

As an example, setting $s = 1$ and searching through elliptic curves of height at most 60^6, the second author computed 27062 pairwise non-isomorphic rank 1 elliptic curves in S_1. More generally, for any fixed value of $s \in (k \setminus k^2) \cup \{1\}$, the one-parameter family $E_{s,t}$ (considered as an elliptic curve over $k(t)$) has generic rank 1, with $P_{s,t}$ a point of infinite order [8, Proposition 5.3]. It is believed that any such family should have infinitely many specializations of rank 1. If this is true, then for any $s \in (k \setminus k^2) \cup \{1\}$, the set S_s will contain an infinite set of rank 1 curves, pairwise non-isogenous over $\mathbb{Q}$, all of whose pairwise products have finite $F^2(X)_{\text{comp}}$.

We mention a few words about the construction. Let E_1 and E_2 be rank 1 elliptic curves over k, and consider the degree 2 map $\pi : E_1 \times E_2 \to Y$ obtained by taking the quotient by negation. The family $E_{s,t}$ is defined in order to guarantee existence of a rational curve in Y passing through both $\pi(P_{s,t_1}, P_{s,t_2})$ and $\pi(O_1, O_2)$.

The paper [8] then proceeds to take a principal divisor on this rational curve and pull it back along π to produce the desired relation in $F^2(E_1 \times E_2)$. We will provide another explanation for this relation. Note that any rational curve $\mathbb{P}^1 \to Y$ defines a hyperelliptic curve[4] H mapping to $E_1 \times E_2$ by taking the fiber product:

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d] & E_1 \times E_2 \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r] & Y, \end{tikzcd}$$

[4] The rational curves defined in [8] correspond to hyperelliptic curves of genus 5.

with $f(\overline{P}) = -f(P)$ for all $P \in H(\overline{k})$. By composing with projection maps, we find maps $\phi_1 : H \to E_1$ and $\phi_2 : H \to E_2$ satisfying $\phi_i(\overline{P}) = -\phi_i(P)$; in particular, Weierstrass points of H must map to 2-torsion points on each E_i. If $\pi(P_1, P_2)$ is in the image of the rational curve $\mathbb{P}^1 \to Y$, then (P_1, P_2) is in the image of f, so Proposition 4.5 can be used to verify that $\{P_1, P_2\}_{k/k}$ is torsion.

Remark 4.18 All examples discussed so far can be accounted for using Weil reciprocity on an elliptic or hyperelliptic curve, but there are examples using other curves. In his thesis [9], the second author provides some rank 1 pairs E_1, E_2 with $F^2(E_1 \times E_2)_{\text{comp}}$ finite, not accounted for using any of the above techniques, for which the desired relation in $F^2(E_1 \times E_2)$ comes from Weil reciprocity on a bielliptic curve. Using the Cremona labelling [2], these examples include (37a1, 43a1), (37a1, 57a1), (37a1, 77a1), (53a1, 58a1), (61a1, 65a1), (61a1, 65a2), (65a2, 79a1) ([9, Sect. 3.3.1]), and (37a1, 53a1) ([9, Proposition 3.4.1]).

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Data availability The code used to produce the data in Table 1 is available at https://github.com/jonathanrlove/zero-cycles/ [10], and can also be made available by the second author upon request. All other relevant data are available within the paper.

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