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Relating Siegel cusp forms to Siegel–Maaß forms

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Abstract

In this paper we generalize a well-known isomorphism between the space of cusp forms of weight k for a Fuchsian subgroup of the first kind $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ and the space of certain Maaß forms of weight k for Γ to an isomorphism between the space of Siegel cusp forms of weight k for a subgroup $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$, which is commensurable with the Siegel modular group $\mathrm{Sp}_n(\mathbb{Z})$, and a suitable space of Siegel–Maaß forms of weight k for Γ.

Keywords: Siegel upper half-space, Siegel modular forms, Maaß forms, Maaß operators

1 Introduction

Let $\mathbb{H} := \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$ denote the upper half-plane and $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind, which acts by fractional linear transformations on $\mathbb{H}$. Let $\mathcal{S}_k(\Gamma)$ denote the space of cusp forms of weight k for Γ and let $\mathcal{H}_k(\Gamma)$ denote the space of real-analytic automorphic forms of weight k for Γ, on which the Maaß Laplacian

$$\Delta_k := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}$$

of weight k acts. Then, it is well-known that there is an isomorphism

$$\mathcal{S}_k(\Gamma) \cong \ker\left(\Delta_k + \frac{k}{2}\left(1 - \frac{k}{2}\right)\mathrm{id}\right) \tag{1.1}$$

of $\mathbb{C}$-vector spaces, induced by the assignment $f \mapsto y^{k/2}f$, where the right-hand side consists of the Maaß forms in $\mathcal{H}_k(\Gamma)$ with eigenvalue $k/2(1-k/2)$ of Δ_k. This identification of two types of automorphic forms for Γ has various useful applications. For example, in the article [2], the isomorphism (1.1) was crucial in relating the sup-norm bound problem for cusp forms of weight k for Γ to bounds for the heat kernel for Δ_k on the quotient space $\Gamma\backslash\mathbb{H}$.

In this paper, we attempt a generalization of the isomorphism (1.1) to the Siegel modular setting, which, to our surprise, we could not find in the literature. Letting $\mathrm{Sym}_n(\mathbb{C})$ be the set of complex symmetric $(n \times n)$-matrices, we let $\mathbb{H}_n := \{Z = X + iY \in \mathrm{Sym}_n(\mathbb{C}) \,|\, Y > 0\}$

denote the Siegel upper half-space of degree $n \geq 1$ and we let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ denote a subgroup acting by generalized fractional linear transformations on $\mathbb{H}_n$, which is commensurable with the Siegel modular group $\mathrm{Sp}_n(\mathbb{Z})$. Then, let $\mathcal{S}_k^n(\Gamma)$ denote the space of Siegel cusp forms of weight k and degree n for Γ and let $\mathcal{H}_k^n(\Gamma)$ be the space of real-analytic automorphic forms of weight k and degree n for Γ, on which the Siegel–Maaß Laplacian

$$\Delta_k := \mathrm{tr}\left(Y\left(\left(Y\frac{\partial}{\partial X}\right)^t\frac{\partial}{\partial X} + \left(Y\frac{\partial}{\partial Y}\right)^t\frac{\partial}{\partial Y}\right) - ikY\frac{\partial}{\partial X}\right) \tag{1.2}$$

of weight k acts. As the main result of this paper, we show in Corollary 5.4 the isomorphism

$$\mathcal{S}_k^n(\Gamma) \cong \ker\left(\Delta_k + \frac{nk}{4}(n-k+1)\mathrm{id}\right), \tag{1.3}$$

of $\mathbb{C}$-vector spaces, induced by the assignment $f \mapsto \det(Y)^{k/2}f$, thereby generalizing the isomorphism (1.1) to the Siegel modular setting. The right-hand side of (1.3) now consists of the Siegel–Maaß forms in $\mathcal{H}_k^n(\Gamma)$ with eigenvalue $nk(n-k+1)/4)$ of Δ_k.

In case $n = 1$, the isomorphism (1.1) is obtained as a by-product of the proof of the symmetry of the Maaß Laplacian Δ_k (see [6]). The most straightforward proof of the symmetry of Δ_k is obtained by constructing a suitable $\mathrm{SL}_2(\mathbb{R})$-invariant differential form using the raising or the lowering operators on $\mathbb{H}$, and then integrating it over the quotient space $\Gamma\backslash\mathbb{H}$ (see [1], p. 135). Generalizations of all these operators as well as their transformation behaviour under the action of the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ to the Siegel modular setting have been provided by Maaß in [3]. However, in spite of all these crucial ingredients being around for a long time, we could not find in the literature a precise proof of the symmetry of the Siegel–Maaß Laplacian Δ_k of weight k. We provide a complete proof of the symmetry of Δ_k in Theorem 5.1, where we construct the appropriate $\mathrm{Sp}_n(\mathbb{R})$-invariant differential form on $\mathbb{H}_n$, which, while calculationally a bit cumbersome, is conceptually a rather straightforward piecing-together of Maaß's calculations. Our main result in Corollary 5.4 is then a consequence of Theorem 5.1.

As indicated above, the generalization of the isomorphism (1.1) will perspectively allow us, among others, to relate the sup-norm bound problem for Siegel cusp forms of weight k and degree n for Γ to bounds for the heat kernel for the Siegel–Maaß Laplacian Δ_k on the quotient space $\Gamma\backslash\mathbb{H}_n$.

This paper is organized as follows: In Sect. 2, we provide a quick summary of the basics of the Siegel upper half-space and Siegel modular forms. In the subsequent two Sects. 3 and 4, we introduce and discuss the transformation behaviour of the Maaß operators K_α, Λ_β, and $\Omega_{\alpha,\beta}$ in the Siegel modular setting that is central to our analysis. This material is already present in Chapter 15 of [3], but due to sub-optimal typesetting, at places, it is hard to decipher. The transformation behaviour of $\Omega_{\alpha,\beta}$ can also be obtained via a somewhat different analysis on $\mathbb{H}_n \times \mathbb{R}/2\pi\mathbb{Z}$ given in Chapter 8 of Maaß's later book [4]. As the two analyses turn out to be essentially equivalent, the transformation behaviours of K_α and Λ_β can also be derived from the alternative method in [4]. However, for the transformation behaviours of K_α and Λ_β, we stick to Maaß's original analysis in [3] as it is more direct and we redo these calculations for the reader's convenience, but to keep the exposition short, we refer to [4] for the transformation behaviour of $\Omega_{\alpha,\beta}$. Finally in Sect. 5, piecing together Maaß's results, we construct the appropriate $\mathrm{Sp}_n(\mathbb{R})$-invariant differential form on $\mathbb{H}_n$ to show the symmetry of the Siegel–Maaß Laplacian Δ_k, and then use it to show the generalization (1.3) of the isomorphism (1.1).

2 Basic notations and definitions

For $n \in \mathbb{N}_{>0}$ and a commutative ring R, let $\mathrm{M}_n(R)$ denote the set of $(n \times n)$-matrices with entries in R and $\mathrm{Sym}_n(R)$ the set of symmetric matrices in $\mathrm{M}_n(R)$. The Siegel upper half-space $\mathbb{H}_n$ of degree n is then defined by

$$\mathbb{H}_n := \{Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \,|\, X, Y \in \mathrm{Sym}_n(\mathbb{R}) : \; Y > 0\}.$$

The symplectic group $\mathrm{Sp}_n(\mathbb{R})$ of degree n is defined by

$$\mathrm{Sp}_n(\mathbb{R}) := \{\gamma \in \mathrm{M}_{2n}(\mathbb{R}) \,|\, \gamma^t J_n \gamma = J_n\},$$

where $J_n \in \mathrm{M}_{2n}(\mathbb{R})$ is the skew-symmetric matrix

$$J_n := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

with $\mathbb{1}_n$ denoting the identity matrix of $\mathrm{M}_n(\mathbb{R})$. The group $\mathrm{Sp}_n(\mathbb{R})$ acts by the symplectic action

$$\mathbb{H}_n \ni Z \mapsto \gamma Z = (AZ + B)(CZ + D)^{-1} \qquad \left(\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})\right) \tag{2.1}$$

on $\mathbb{H}_n$. Under this action $\mathrm{Im}(Z)$ transforms as

$$\mathrm{Im}(\gamma Z) = (CZ + D)^{-t}\, \mathrm{Im}(Z)(C\overline{Z} + D)^{-1}, \tag{2.2}$$

which, on taking determinants on both sides, gives

$$\det(\mathrm{Im}(\gamma Z)) = \frac{\det(\mathrm{Im}(Z))}{|\det(CZ + D)|^2}.$$

Similarly, taking matrix-differentials on both sides of the symplectic action (2.1), it is easy to see that under this action, the matrix-differential form $\mathrm{d}Z$ transforms as

$$\mathrm{d}(\gamma Z) = (CZ + D)^{-t}\, \mathrm{d}Z\, (CZ + D)^{-1}. \tag{2.3}$$

The arclength $\mathrm{d}s_n^2$ and the volume form $\mathrm{d}\mu_n$ on $\mathbb{H}_n$ in terms of $Z = (z_{j,k})_{1 \le j \le k \le n} \in \mathbb{H}_n$ are given by

$$\mathrm{d}s_n^2(Z) = \mathrm{tr}(Y^{-1}\, \mathrm{d}Z\, Y^{-1}\, \mathrm{d}\overline{Z}) \qquad (Z = X + iY),$$

$$\mathrm{d}\mu_n(Z) = \frac{\bigwedge_{1 \le j \le k \le n} \mathrm{d}x_{j,k} \wedge \mathrm{d}y_{j,k}}{\det(Y)^{n+1}} \qquad (z_{j,k} = x_{j,k} + iy_{j,k}).$$

From Eqs. (2.2) and (2.3) it is obvious that the arclength $\mathrm{d}s_n^2$ and the volume form $\mathrm{d}\mu_n$ on $\mathbb{H}_n$ given by the above equations are invariant under the symplectic action. Corresponding to this metric, we have the Laplace–Beltrami operator Δ on $\mathbb{H}_n$, called the Siegel Laplacian, which is also invariant under the symplectic action.

Definition 2.1 Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$, i.e., the intersection $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ is a finite index subgroup of Γ as well as of $\mathrm{Sp}_n(\mathbb{Z})$. We let $\gamma_j \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$ denote a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$. Then, a *Siegel modular form of weight k and degree n for* Γ is a function $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfying the following conditions:

(i) f is holomorphic;
(ii) $f(\gamma Z) = \det(CZ + D)^k f(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$;
(iii) given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, the quantities $\det(C_j Z + D_j)^{-k} f(\gamma_j Z)$ are bounded in the region $\{Z = X + iY \in \mathbb{H}_n \mid Y \geq Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$.

Moreover, a Siegel modular form f as above is called a *Siegel cusp form of weight k and degree n for* Γ if condition (iii) above is strengthened to the condition

(iii') given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 \gg 0$, the quantities $\det(C_j Z + D_j)^{-k} f(\gamma_j Z)$ become arbitrarily small in the region $\{Z = X + iY \in \mathbb{H}_n \mid Y \geq Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$.

Remark 2.2 The sets of Siegel modular forms and Siegel cusp forms of weight k and degree n for Γ have the structure of $\mathbb{C}$-vector spaces, which we denote by $\mathcal{M}_k^n(\Gamma)$ and $\mathcal{S}_k^n(\Gamma)$, respectively, and which turn out to be finite dimensional. Moreover, the space $\mathcal{S}_k^n(\Gamma)$ is equipped with the so-called Petersson inner product given by

$$\langle f, g \rangle := \int_{\Gamma \backslash \mathbb{H}_n} \det(Y)^k f(Z) \overline{g}(Z) \, \mathrm{d}\mu_n(Z) \qquad (f, g \in \mathcal{S}_k^n(\Gamma)),$$

making $\mathcal{S}_k^n(\Gamma)$ into a hermitian inner product space.

3 Siegel–Maaß Laplacian of weight (α, β)

In this section, we will recall from [3] various differential operators acting on smooth complex valued functions defined on $\mathbb{H}_n$. In particular, we will define the Siegel–Maaß Laplacian of weight (α, β), where $\alpha, \beta \in \mathbb{R}$. Letting $\alpha = k/2$ and $\beta = -k/2$ will then lead us to the Siegel–Maaß Laplacian Δ_k mentioned in formula (1.2) in the introduction. We point out that the Siegel Laplacian Δ mentioned in the previous section and the Siegel–Maaß Laplacian Δ_k are related by the formula

$$\Delta_k = \Delta - ik \,\mathrm{tr}\left(Y \frac{\partial}{\partial X}\right)$$

with the symmetric $(n \times n)$-matrix $\partial/\partial X$ of partial derivatives being defined below.

Given $Z = X + iY \in \mathbb{H}_n$, we start by introducing the following symmetric $(n \times n)$-matrices of partial derivatives:

$$\text{(i)} \quad \left(\frac{\partial}{\partial X}\right)_{j,k} := \frac{1+\delta_{j,k}}{2}\frac{\partial}{\partial x_{j,k}},$$

$$\text{(ii)} \quad \left(\frac{\partial}{\partial Y}\right)_{j,k} := \frac{1+\delta_{j,k}}{2}\frac{\partial}{\partial y_{j,k}},$$

$$\text{(iii)} \quad \frac{\partial}{\partial Z} := \frac{1}{2}\left(\frac{\partial}{\partial X} - i\frac{\partial}{\partial Y}\right),$$

$$\text{(iv)} \quad \frac{\partial}{\partial \overline{Z}} := \frac{1}{2}\left(\frac{\partial}{\partial X} + i\frac{\partial}{\partial Y}\right),$$

where $\delta_{j,k}$ is the Kornecker delta symbol.

Definition 3.1 Following Maaß [3], we define, using the above notations, for arbitrary real numbers $\alpha, \beta \in \mathbb{R}$, the following $(n \times n)$-matrices of differential operators acting on smooth complex valued functions on $\mathbb{H}_n$:

$$\text{(i)} \quad K_\alpha := (Z - \overline{Z})\frac{\partial}{\partial Z} + \alpha \mathbb{1}_n,$$

$$\text{(ii)} \quad \Lambda_\beta := (Z - \overline{Z})\frac{\partial}{\partial \overline{Z}} - \beta \mathbb{1}_n,$$

$$\text{(iii)} \quad \Omega_{\alpha,\beta} := \Lambda_{\beta-(n+1)/2}K_\alpha + \alpha(\beta - (n+1)/2)\mathbb{1}_n,$$

$$\text{(iv)} \quad \widetilde{\Omega}_{\alpha,\beta} := K_{\alpha-(n+1)/2}\Lambda_\beta + \beta(\alpha - (n+1)/2)\mathbb{1}_n.$$

Next, we want to expand $\Omega_{\alpha,\beta}$ and $\widetilde{\Omega}_{\alpha,\beta}$ in terms of $Z, \overline{Z}, \partial/\partial Z$, and $\partial/\partial\overline{Z}$. For that we need the following lemma.

Lemma 3.2 *Let $C, D\colon \mathbb{H}_n \longrightarrow \mathrm{M}_n(\mathbb{C})$ be smooth matrix valued functions depending on Z and $\overline{Z}$. Then, the following matrix operator identities hold:*
(i) Assuming that $\partial C/\partial Z = 0$ and $\partial D/\partial Z = 0$, we have

$$\frac{\partial}{\partial Z}(CZ + D)^t = \left((CZ + D)\frac{\partial}{\partial Z}\right)^t + \frac{1}{2}(n+1)C^t.$$

(ii) Assuming that $\partial C/\partial\overline{Z} = 0$ and $\partial D/\partial\overline{Z} = 0$, we have

$$\frac{\partial}{\partial \overline{Z}}(C\overline{Z} + D)^t = \left((C\overline{Z} + D)\frac{\partial}{\partial \overline{Z}}\right)^t + \frac{1}{2}(n+1)C^t.$$

Proof Since part (ii) follows from part (i) by conjugation, we prove only (i). Let Φ be a matrix depending on Z and $\overline{Z}$ such that the product $(CZ + D)^t\Phi$ makes sense. Then, writing the (j, k)-th entry of the matrix $\partial/\partial Z(CZ + D)^t\Phi$ as the sum

$$\left(\frac{\partial}{\partial Z}(CZ + D)^t\Phi\right)_{j,k} = \sum_{l,m=1}^{n}\left(\frac{\partial}{\partial Z}\right)_{j,l}\left((CZ + D)^t_{l,m}\Phi_{m,k}\right)$$

and noting that $\partial Z/\partial z_{j,l} = (1-\delta_{j,l})E_{j,l} + E_{l,j}$, where $E_{j,k} \in \mathrm{M}_n(\mathbb{C})$ is the matrix with its (j,k)-th entry being 1 and the remaining entries being 0, elementary calculations lead us to the operator identity

$$\frac{\partial}{\partial Z}(CZ+D)^t = \left((CZ+D)\frac{\partial}{\partial Z}\right)^t + \frac{1}{2}(n+1)C^t,$$

which is what we needed to prove. □

Corollary 3.3 *For $Z \in \mathbb{H}_n$, the following operator identities hold:*

(i) $$\frac{\partial}{\partial Z}(Z-\overline{Z}) = \left((Z-\overline{Z})\frac{\partial}{\partial Z}\right)^t + \frac{1}{2}(n+1)\mathbb{1}_n,$$

(ii) $$\frac{\partial}{\partial \overline{Z}}(Z-\overline{Z}) = \left((Z-\overline{Z})\frac{\partial}{\partial \overline{Z}}\right)^t - \frac{1}{2}(n+1)\mathbb{1}_n.$$

Proof As $\partial\overline{Z}/\partial Z = 0$, we can choose $C = \mathbb{1}_n$ and $D = -\overline{Z}$ in Lemma 3.2 (i), from which the first claimed formula follows. The second formula follows analogously. □

Using the above corollary, one can expand $\Omega_{\alpha,\beta}$ and $\widetilde{\Omega}_{\alpha,\beta}$ as

$$\Omega_{\alpha,\beta} = (Z-\overline{Z})\left((Z-\overline{Z})\frac{\partial}{\partial \overline{Z}}\right)^t\frac{\partial}{\partial Z} + \alpha(Z-\overline{Z})\frac{\partial}{\partial \overline{Z}} - \beta(Z-\overline{Z})\frac{\partial}{\partial Z},$$

$$\widetilde{\Omega}_{\alpha,\beta} = (Z-\overline{Z})\left((Z-\overline{Z})\frac{\partial}{\partial Z}\right)^t\frac{\partial}{\partial \overline{Z}} + \alpha(Z-\overline{Z})\frac{\partial}{\partial \overline{Z}} - \beta(Z-\overline{Z})\frac{\partial}{\partial Z}.$$

Then, $\Omega_{\alpha,\beta}$ and $\widetilde{\Omega}_{\alpha,\beta}$ are related by the identity

$$\widetilde{\Omega}_{\alpha,\beta} = (Z-\overline{Z})\big((Z-\overline{Z})^{-1}\Omega_{\alpha,\beta}\big)^t.$$

Definition 3.4 The operator $\Delta_{\alpha,\beta} := -\operatorname{tr}(\Omega_{\alpha,\beta}) = -\operatorname{tr}(\widetilde{\Omega}_{\alpha,\beta})$ is called the *Siegel–Maaß Laplacian of weight* (α,β).

4 Transformation behaviour of Maaß operators

Recall that the symplectic action of $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ on the point $Z \in \mathbb{H}_n$ is given by

$$\gamma Z = (AZ+B)(CZ+D)^{-1} = (CZ+D)^{-t}(AZ+B)^t;$$

to avoid cumbersome notation, we will use in this paper sometimes the shorthand $Z^\gamma := \gamma Z$. In this section, we will then study the transformation behaviour of the Maaß operators introduced in Definition 3.1 by expressing the operators K_α^γ, Λ_β^γ, $\Omega_{\alpha,\beta}^\gamma$ obtained by replacing $Z, \overline{Z}$ by $Z^\gamma, \overline{Z}^\gamma$ in K_α, Λ_β, $\Omega_{\alpha,\beta}$, respectively, as they operate on smooth complex valued functions defined on $\mathbb{H}_n$.

We begin by investigating how the matrix differential operators $\partial/\partial Z$ and $\partial/\partial\overline{Z}$ transform under the symplectic action of γ on Z. From equation (2.3), we know that the differential $\mathrm{d}Z$ transforms like

$$\mathrm{d}Z^\gamma = (CZ+D)^{-t}\,\mathrm{d}Z\,(CZ+D)^{-1}.$$

Therefore, as the differential of a smooth function $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ depending only on Z is given by $\mathrm{d}\varphi = \mathrm{tr}(\partial\varphi/\partial Z\,\mathrm{d}Z)$, we have

$$\frac{\partial\varphi}{\partial Z} = (CZ+D)^{-1}\frac{\partial\varphi}{\partial Z^{\gamma}}(CZ+D)^{-t},$$

i.e., the operator $\partial/\partial Z$ transforms as

$$\frac{\partial}{\partial Z^{\gamma}} = (CZ+D)\left((CZ+D)\frac{\partial}{\partial Z}\right)^{t}. \tag{4.1}$$

By conjugation, the operator $\partial/\partial\overline{Z}$ transforms as

$$\frac{\partial}{\partial \overline{Z}^{\gamma}} = (C\overline{Z}+D)\left((C\overline{Z}+D)\frac{\partial}{\partial \overline{Z}}\right)^{t}. \tag{4.2}$$

Next we need to know how to differentiate $\det(Z-\overline{Z})$ and $\det(CZ+D)$ with respect to Z, which we carry out in the next two lemmas.

Lemma 4.1 *The matrix identity*

$$\frac{\partial \det(Z-\overline{Z})}{\partial Z} = \det(Z-\overline{Z})(Z-\overline{Z})^{-1}$$

holds.

Proof By writing $\mathrm{Im}(Z) = Y = (y_{j,k})_{1\leq j\leq k\leq n} \in \mathrm{Sym}_n(\mathbb{R})$ as before, we obtain from equation (140) in subsection 2.8.2 of [5] the equality

$$\frac{\partial \det(Y)}{\partial y_{j,k}} = \det(Y)(2-\delta_{j,k})(Y^{-1})_{j,k}.$$

Now since $((1+\delta_{j,k})/2)(2-\delta_{j,k}) = 1$, we have $\partial\det(Y)/\partial Y = \det(Y)Y^{-1}$, which is equivalent to the identity stated in the lemma. □

Lemma 4.2 *Let* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$. *Then, the matrix identity*

$$\frac{\partial \det(CZ+D)}{\partial Z} = \det(CZ+D)(CZ+D)^{-1}C = \det(CZ+D)C^{t}(CZ+D)^{-t}$$

holds.

Proof By writing $Z = (z_{j,k})_{1\leq j\leq k\leq n} \in \mathrm{Sym}_n(\mathbb{C})$, we compute, using the chain rule of differentiation

$$\frac{\partial \det(CZ+D)}{\partial z_{j,k}} = \mathrm{tr}\left(\left(\frac{\partial \det(CZ+D)}{\partial(CZ+D)}\right)^{t}\frac{\partial(CZ+D)}{\partial z_{j,k}}\right). \tag{4.3}$$

By equation (49) in subsection 2.1.2 of [5], we note

$$\frac{\partial \det(CZ+D)}{\partial(CZ+D)} = \det(CZ+D)(CZ+D)^{-t}. \tag{4.4}$$

Then, plugging (4.4) into (4.3), we arrive at

$$\frac{\partial \det(CZ+D)}{\partial z_{j,k}} = \det(CZ+D)\,\mathrm{tr}\left((CZ+D)^{-1}\frac{\partial(CZ+D)}{\partial z_{j,k}}\right)$$
$$= \det(CZ+D)\sum_{l,m=1}^{n}(CZ+D)^{-1}_{l,m}\frac{\partial(CZ+D)_{m,l}}{\partial z_{j,k}}.$$

Now, entrywise partial differentiation with respect to the entries $z_{j,k}$ of Z followed by an elementary calculation with taking care of the ensuing Kronecker delta symbols leads us to the matrix identity

$$\frac{\partial \det(CZ+D)}{\partial Z} = \det(CZ+D)(CZ+D)^{-1}C = \det(CZ+D)C^t(CZ+D)^{-t},$$

which is what we needed to prove. □

Lemmas 4.1 and 4.2 prepare the groundwork for calculating the transformation behaviour of the Maaß operators, which we undertake one by one in the subsequent three propositions.

Proposition 4.3 *Let* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ *and* $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ *be a smooth function. Then, the operator* K_α^γ *obtained by replacing* $Z \in \mathbb{H}_n$ *in* K_α *by* $Z^\gamma = \gamma Z$ *is related to the operator* K_α *by the identity*

$$K_\alpha^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\big)$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t.$$

Proof From the definition of K_α^γ, we have

$$K_\alpha^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\big)$$
$$= \left((Z^\gamma - \overline{Z}^\gamma)\frac{\partial}{\partial Z^\gamma} + \alpha\mathbb{1}_n\right)\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z).$$

Then, expanding $\partial/\partial Z^\gamma$ by means of equation (4.1) gives

$$K_\alpha^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\big) = \alpha\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\mathbb{1}_n$$
$$+ (C\overline{Z}+D)^{-t}(Z-\overline{Z})\frac{\partial}{\partial Z}\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\big)(CZ+D)^t. \quad (4.5)$$

Now, focusing on the second line of the above equality and using Lemma 4.2, we get

$$\frac{\partial}{\partial Z}\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\big)$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\left(\alpha\varphi(Z)C^t(CZ+D)^{-t} + \frac{\partial\varphi}{\partial Z}\right).$$

Multiplying the above equation from the left by $(Z-\overline{Z})$ gives

$$(Z-\overline{Z})\frac{\partial}{\partial Z}\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\varphi(Z)\big)$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta\left(\alpha\varphi(Z)(ZC^t - \overline{Z}C^t)(CZ+D)^{-t} + (Z-\overline{Z})\frac{\partial\varphi}{\partial Z}\right).$$

Now writing $(ZC^t - \overline{Z}C^t) = (CZ+D)^t - (C\overline{Z}+D)^t$ and using the definition of K_α on the right-hand side of the above equation, we have

$$(Z-\overline{Z})\frac{\partial}{\partial Z}\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big)$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \big(K_\alpha\varphi(Z) - \alpha\varphi(Z)(C\overline{Z}+D)^t(CZ+D)^{-t}\big).$$

Therefore, multiplying on the left by $(C\overline{Z}+D)^{-t}$ and on the right by $(CZ+D)^t$, we obtain

$$(C\overline{Z}+D)^{-t}(Z-\overline{Z})\frac{\partial}{\partial Z}\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big)(CZ+D)^t$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \big((C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t - \alpha\varphi(Z)\mathbb{1}_n\big).$$

Combining the last equality with Eq. (4.5), leads to the identity

$$K_\alpha^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big)$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (C\overline{Z}+D)^{-t}K_\alpha\varphi(Z)(CZ+D)^t,$$

which is what we had set out to prove. □

Proposition 4.4 *Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ and $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ be a smooth function. Then, the operator Λ_β^γ obtained by replacing $Z \in \mathbb{H}_n$ in Λ_β by $Z^\gamma = \gamma Z$ is related to the operator Λ_β by the identity*

$$\Lambda_\beta^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big)$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (CZ+D)^{-t}\Lambda_\beta\varphi(Z)(C\overline{Z}+D)^t.$$

Proof Since $\overline{K}_\beta = -\Lambda_\beta$, the required identity follows from Proposition 4.3 by complex conjugation. □

Proposition 4.5 *Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ and $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ be a smooth function. Then, the operator $\Omega_{\alpha,\beta}^\gamma$ obtained by replacing $Z \in \mathbb{H}_n$ in $\Omega_{\alpha,\beta}$ by $Z^\gamma = \gamma Z$ is related to the operator $\Omega_{\alpha,\beta}$ by the identity*

$$\Omega_{\alpha,\beta}^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big)$$
$$= \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (CZ+D)^{-t}\Omega_{\alpha,\beta}\varphi(Z)(CZ+D)^t. \quad (4.6)$$

Proof The proof of this proposition can be obtained by suitably combining Propositions 4.3 and 4.4 according to the definition of the operator $\Omega_{\alpha,\beta}$. Alternatively, the claimed transformation formula can also be found on p. 120 of [4]. □

Remark 4.6 Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ and $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ be a smooth function. Taking traces on both sides of Eq. (4.6) leads to the following transformation behaviour of the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$

$$\Delta_{\alpha,\beta}^\gamma\big(\det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \varphi(Z)\big) = \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta \Delta_{\alpha,\beta}\varphi(Z).$$

Now, if the smooth function φ satisfies the functional equation

$$\varphi(Z^{\gamma}) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z),$$

the transformation behaviour of $\Delta_{\alpha,\beta}$ leads to the identity

$$\Delta^{\gamma}_{\alpha,\beta}\varphi(Z^{\gamma}) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\Delta_{\alpha,\beta}\varphi(Z).$$

Definition 4.7 Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$, i.e., the intersection $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ is a finite index subgroup of Γ as well as of $\mathrm{Sp}_n(\mathbb{Z})$. We let $\gamma_j \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \dots, h)$ denote a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$. We then let $\mathcal{V}^n_{\alpha,\beta}(\Gamma)$ denote the space of all functions $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfying the following conditions:

(i) φ is real-analytic;
(ii) $\varphi(\gamma Z) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$;
(iii) given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, there exist $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that the inequalities

$$|\det(C_jZ+D_j)^{-\alpha}\det(C_j\overline{Z}+D_j)^{-\beta}\varphi(\gamma_jZ)| \leq M\,\mathrm{tr}(Y)^N$$

hold in the region $\{Z = X+iY \in \mathbb{H}_n \,|\, Y \geq Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \dots, h)$.

Remark 4.8 For $\varphi \in \mathcal{V}^n_{\alpha,\beta}(\Gamma)$, we set

$$\|\varphi\|^2 := \int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}|\varphi(Z)|^2\,\mathrm{d}\mu_n(Z),$$

whenever it is defined. In this way we obtain the Hilbert space

$$\mathcal{H}^n_{\alpha,\beta}(\Gamma) := \left\{\varphi \in \mathcal{V}^n_{\alpha,\beta}(\Gamma) \,\middle|\, \|\varphi\| < \infty\right\}$$

equipped with the inner product

$$\langle\varphi,\psi\rangle = \int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}\varphi(Z)\overline{\psi}(Z)\,\mathrm{d}\mu_n(Z) \qquad (\varphi,\psi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)).$$

We note that in order to enable $\|\varphi\| < \infty$, the exponent $N \in \mathbb{N}$ in part (iii) of Definition 4.7 has to be 0. Moreover, we note that Remark 4.6 shows that the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$ acts on the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$.

Definition 4.9 Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$. The elements of the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$ are called *automorphic forms of weight* (α,β) *and degree n for* Γ. Moreover, if $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ is an eigenform of $\Delta_{\alpha,\beta}$, it is called a *Siegel–Maaß form of weight* (α,β) *and degree n for* Γ.

Corollary 4.10 *Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$. Then, we have for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$*

(i) $K^\gamma_\alpha \varphi(Z^\gamma) = \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (C\overline{Z}+D)^{-t} K_\alpha\varphi(Z)(CZ+D)^t$,
(ii) $\Lambda^\gamma_\beta \varphi(Z^\gamma) = \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (CZ+D)^{-t} \Lambda_\beta\varphi(Z)(C\overline{Z}+D)^t$,
(iii) $\Omega^\gamma_{\alpha,\beta} \varphi(Z^\gamma) = \det(CZ+D)^\alpha \det(C\overline{Z}+D)^\beta (CZ+D)^{-t} \Omega_{\alpha,\beta}\varphi(Z)(CZ+D)^t$.

Proof The proof is an immediate consequence of Propositions 4.3–4.5 and the definition of the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$. □

5 Symmetry of the Siegel–Maaß Laplacian of weight (α, β)

Let $\mathrm{d}Z := (\mathrm{d}z_{j,k})_{1\leq j,k\leq n}$ denote the $(n \times n)$-matrix consisting of differential forms of degree 1 and let $[\mathrm{d}Z] := \bigwedge_{1\leq j\leq k\leq n} \mathrm{d}z_{j,k}$ denote the differential form of degree $n(n+1)/2$ at $Z \in \mathbb{H}_n$. We introduce an $(n \times n)$-matrix $\{\mathrm{d}Z\}$ consisting of differential forms of degree $(n(n+1)/2 - 1)$, namely

$$\{\mathrm{d}Z\}_{j,k} := \frac{1+\delta_{j,k}}{2}\,\varpi_{j,k},$$

where $\varpi_{j,k}$ is defined by

$$\varpi_{j,k} := \varepsilon_{j,k} \bigwedge_{\substack{1\leq l\leq m\leq n \\ (l,m)\neq(j,k)}} \mathrm{d}z_{l,m} \qquad (1 \leq j \leq k \leq n)$$

in case $j \leq k$ and $\varpi_{j,k} = \varpi_{k,j}$ in case $j > k$ with the sign $\varepsilon_{j,k} = \pm 1$ determined by $\mathrm{d}z_{j,k} \wedge \varpi_{j,k} = [\mathrm{d}Z]$. It is easy to see that

$$\mathrm{d}Z \wedge \{\mathrm{d}Z\} = \frac{1}{2}(n+1)[\mathrm{d}Z]\mathbb{1}_n.$$

Let now $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$. Since we have $\mathrm{d}Z^\gamma = (CZ+D)^{-t}\,\mathrm{d}Z\,(CZ+D)^{-1}$ and $[\mathrm{d}Z^\gamma] = \det(CZ+D)^{-(n+1)}[\mathrm{d}Z]$, we derive from the relation

$$\mathrm{d}Z^\gamma \wedge \{\mathrm{d}Z^\gamma\} = \frac{1}{2}(n+1)[\mathrm{d}Z^\gamma]\mathbb{1}_n$$

that the matrix $\{\mathrm{d}Z\}$ has the transformation behaviour

$$\{\mathrm{d}Z^\gamma\} = \det(CZ+D)^{-(n+1)}(CZ+D)\{\mathrm{d}Z\}(CZ+D)^t.$$

Next we shall use these differential forms to show that the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$ acts as a symmetric operator on the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$.

Theorem 5.1 *Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and let $\varphi, \psi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ be compactly supported. Then, we have the formula*

$$\langle -\Delta_{\alpha,\beta}\varphi, \psi\rangle = \int_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\right)\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\psi\rangle.$$

In particular, this formula establishes the relation

$$\langle \Delta_{\alpha,\beta}\varphi, \psi\rangle = \langle \varphi, \Delta_{\alpha,\beta}\psi\rangle,$$

which shows that the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$ *acts as a symmetric operator on the Hilbert space* $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$.

Proof We start by considering the differential form

$$\omega(Z) := \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)(Z-\overline{Z})\{\mathrm{d}Z\}\right)\wedge[\mathrm{d}\overline{Z}].$$

Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma$. Then, the transformation formulas

$$\begin{aligned}
&\text{(a)}\quad \det(Z^\gamma-\overline{Z}^\gamma)^{\alpha+\beta-(n+1)}\\
&\qquad = \det(CZ+D)^{-(\alpha+\beta-(n+1))}\det(C\overline{Z}+D)^{-(\alpha+\beta-(n+1))}\det(Z-\overline{Z})^{\alpha+\beta-(n+1)},\\
&\text{(b)}\quad \overline{\psi}(Z^\gamma) = \det(CZ+D)^{\beta}\det(C\overline{Z}+D)^{\alpha}\overline{\psi}(Z),\\
&\text{(c)}\quad \mathrm{tr}\left(\Lambda^\gamma_\beta\varphi(Z^\gamma)(Z^\gamma-\overline{Z}^\gamma)\{\mathrm{d}Z^\gamma\}\right)\\
&\qquad = \det(CZ+D)^{\alpha-(n+1)}\det(C\overline{Z}+D)^{\beta}\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)(Z-\overline{Z})\{\mathrm{d}Z\}\right),\\
&\text{(d)}\quad [\mathrm{d}\overline{Z}^\gamma] = \det(C\overline{Z}+D)^{-(n+1)}[\mathrm{d}\overline{Z}]
\end{aligned}$$

show that $\omega(Z^\gamma) = \omega(Z)$ for all $\gamma \in \Gamma$, i.e., $\omega(Z)$ is a Γ-invariant differential form on $\mathbb{H}_n$, and hence can be considered as a differential form on the quotient space $\Gamma\backslash\mathbb{H}_n$. Since the automorphic forms φ, ψ are real-analytic, the differential form ω is a smooth differential form. Therefore, by Stokes' theorem, we have

$$\int\limits_{\Gamma\backslash\mathbb{H}_n}\mathrm{d}\omega(Z) = \int\limits_{\partial\Gamma\backslash\mathbb{H}_n}\omega(Z).$$

As φ, ψ are compactly supported, the integral on the right-hand side of the above equation vanishes, which gives

$$\int\limits_{\Gamma\backslash\mathbb{H}_n}\mathrm{d}\omega(Z) = 0. \tag{5.1}$$

As we shall see, by explicitly computing $\mathrm{d}\omega(Z)$, the vanishing of the above integral will lead to the formula claimed in the theorem.

For the computation of $\mathrm{d}\omega(Z)$, we set $\rho := \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)$, $P := \Lambda_\beta\varphi(Z)$, and $Q := (Z-\overline{Z})$. Then, we obtain

$$\omega(Z) = \rho\,\mathrm{tr}(P\,Q\,\{\mathrm{d}Z\})\wedge[\mathrm{d}\overline{Z}] = \sum_{j,k,l=1}^{n}\rho\,p_{j,k}\,q_{k,l}\,\{\mathrm{d}Z\}_{l,j}\wedge[\mathrm{d}\overline{Z}].$$

Taking exterior derivatives on both sides leads to

$$\begin{aligned} \mathrm{d}\omega(Z) &= \sum_{j,k,l=1}^{n} \frac{\partial}{\partial z_{l,j}}(\rho\, p_{j,k}\, q_{k,l})\, \mathrm{d}z_{l,j} \wedge \frac{1+\delta_{l,j}}{2} \varpi_{l,j} \wedge [\mathrm{d}\overline{Z}] \\ &= \sum_{j,k,l=1}^{n} \frac{1+\delta_{l,j}}{2} \frac{\partial}{\partial z_{l,j}}(\rho\, p_{j,k}\, q_{k,l})\, [\mathrm{d}Z] \wedge [\mathrm{d}\overline{Z}] \\ &= \sum_{j,k,l=1}^{n} \left(\frac{\partial}{\partial Z}\right)_{l,j} (\rho\, p_{j,k}\, q_{k,l})\, [\mathrm{d}Z] \wedge [\mathrm{d}\overline{Z}]. \end{aligned} \tag{5.2}$$

Now a term by term differentiation in the last expression on the right-hand side of the above equation allows us to write it as the sum of the three traces

$$\sum_{j,k,l=1}^{n} \left(\frac{\partial}{\partial Z}\right)_{l,j} (\rho\, p_{j,k}\, q_{k,l}) = \mathrm{tr}\left(\frac{\partial \rho}{\partial Z} P\,Q\right) + \rho\, \mathrm{tr}\left(\frac{\partial}{\partial Z} P\,Q\right) + \rho\, \mathrm{tr}\left(P^{t} \frac{\partial}{\partial Z} Q\right), \tag{5.3}$$

which we calculate one by one next.
(i) We begin by considering

$$\frac{\partial \rho}{\partial Z} = \frac{\partial}{\partial Z}\big(\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\big),$$

which, by Lemma 4.1, calculates to

$$\begin{aligned} \frac{\partial \rho}{\partial Z} &= (\alpha+\beta-(n+1))\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}(Z-\overline{Z})^{-1}\, \overline{\psi}(Z) \\ &\quad + \det(Z-\overline{Z})^{\alpha+\beta-(n+1)} \frac{\partial \overline{\psi}(Z)}{\partial Z}. \end{aligned}$$

Now multiplying both sides of the above equation on the right by $P\,Q = \Lambda_{\beta}\varphi(Z)(Z-\overline{Z})$ and taking the trace gives

$$\begin{aligned} &\mathrm{tr}\left(\frac{\partial \rho}{\partial Z} P\,Q\right) \\ &= \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big((\alpha+\beta-(n+1))\,\mathrm{tr}\left((Z-\overline{Z})^{-1}\overline{\psi}(Z)\Lambda_{\beta}\varphi(Z)(Z-\overline{Z})\right) \\ &\quad + \mathrm{tr}\left(\frac{\partial \overline{\psi}(Z)}{\partial Z}\Lambda_{\beta}\varphi(Z)(Z-\overline{Z})\right)\Big), \end{aligned}$$

which, upon rearranging the terms inside the traces on the right-hand side by cyclically permuting them, becomes

$$\begin{aligned} \mathrm{tr}\left(\frac{\partial \rho}{\partial Z} P\,Q\right) &= \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big((\alpha+\beta-(n+1))\,\mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)\overline{\psi}(Z)\right) \\ &\quad + \mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)(Z-\overline{Z})\frac{\partial \overline{\psi}(Z)}{\partial Z}\right)\Big). \end{aligned} \tag{5.4}$$

(ii) Next, we consider the second trace

$$\rho\, \mathrm{tr}\left(\frac{\partial}{\partial Z} P\,Q\right) = \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\, \mathrm{tr}\left(\frac{\partial}{\partial Z}\Lambda_{\beta}\varphi(Z)(Z-\overline{Z})\right)$$

in Eq. (5.3), which, again through rearrangement of the terms inside the trace by a cyclical permutation, takes the form

$$\rho\,\mathrm{tr}\left(\frac{\partial}{\partial Z}P\,Q\right)=\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\,\mathrm{tr}\left((Z-\overline{Z})\frac{\partial}{\partial Z}\Lambda_{\beta}\varphi(Z)\overline{\psi}(Z)\right). \tag{5.5}$$

(iii) Finally, we consider the third trace

$$\rho\,\mathrm{tr}\left(P^{t}\frac{\partial}{\partial Z}Q\right)=\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\overline{\psi}(Z)\,\mathrm{tr}\left(\left(\Lambda_{\beta}\varphi(Z)\right)^{t}\left(\frac{\partial}{\partial Z}(Z-\overline{Z})\right)\mathbb{1}_{n}\right)$$

in Eq. (5.3). By the first operator identity in Corollary 3.3, we have the matrix identity

$$\left(\frac{\partial}{\partial Z}(Z-\overline{Z})\right)\mathbb{1}_{n}=\left((Z-\overline{Z})\frac{\partial}{\partial Z}\right)^{t}\mathbb{1}_{n}+\frac{1}{2}(n+1)\mathbb{1}_{n}=\frac{1}{2}(n+1)\mathbb{1}_{n},$$

which gives, upon rearrangement of the scalar quantities, the identity

$$\rho\,\mathrm{tr}\left(P^{t}\frac{\partial}{\partial Z}Q\right)=\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\frac{1}{2}(n+1)\,\mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)\overline{\psi}(Z)\right). \tag{5.6}$$

Now, adding up Eqs. (5.4)–(5.6), it follows from Eq. (5.3) that

$$\begin{aligned}&\sum_{j,k,l=1}^{n}\left(\frac{\partial}{\partial Z}\right)_{l,j}(\rho\,p_{j,k}\,q_{k,l})\\&\quad=\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Bigg((\alpha+\beta-(n+1)/2)\,\mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)\overline{\psi}(Z)\right)\\&\qquad+\mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)(Z-\overline{Z})\frac{\partial\overline{\psi}(Z)}{\partial Z}\right)+\mathrm{tr}\left((Z-\overline{Z})\frac{\partial}{\partial Z}\Lambda_{\beta}\varphi(Z)\overline{\psi}(Z)\right)\Bigg).\end{aligned}$$

Rearranging terms on the right-hand side of the last expression, leads to

$$\begin{aligned}&\sum_{j,k,l=1}^{n}\left(\frac{\partial}{\partial Z}\right)_{l,j}(\rho\,p_{j,k}\,q_{k,l})\\&\quad=\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Bigg(\mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)\Big((Z-\overline{Z})\frac{\partial}{\partial Z}+\beta\mathbb{1}_{n}\Big)\overline{\psi}(Z)\right)\\&\qquad+\mathrm{tr}\left((Z-\overline{Z})\frac{\partial}{\partial Z}+(\alpha-(n+1)/2)\mathbb{1}_{n}\right)\Lambda_{\beta}\varphi(Z)\overline{\psi}(Z)\Bigg).\end{aligned}$$

Identifying the operator $(Z-\overline{Z})\partial/\partial Z+\beta\mathbb{1}_{n}$ on the right-hand side of the above equation as $-\overline{\Lambda}_{\beta}$ and the operator $(Z-\overline{Z})\partial/\partial Z+(\alpha-(n+1)/2)\mathbb{1}_{n}$ as $K_{\alpha-(n+1)/2}$, we can rewrite the right-hand side of the above equation as

$$\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big(-\mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)\,\overline{\Lambda}_{\beta}\overline{\psi}(Z)\right)+\mathrm{tr}\left(K_{\alpha-(n+1)/2}\,\Lambda_{\beta}\varphi(Z)\overline{\psi}(Z)\right)\Big),$$

which, by definition of $\widetilde{\Omega}_{\alpha,\beta}$, is equal to

$$\begin{aligned}&\det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\\&\quad\Big(\mathrm{tr}\left(\widetilde{\Omega}_{\alpha,\beta}-\beta(\alpha-(n+1)/2)\mathbb{1}_{n}\right)\varphi(Z)\overline{\psi}(Z)-\mathrm{tr}\left(\Lambda_{\beta}\varphi(Z)\overline{\Lambda}_{\beta}\overline{\psi}(Z)\right)\Big).\end{aligned}$$

In total, we get

$$\sum_{j,k,l=1}^{n}\left(\frac{\partial}{\partial Z}\right)_{l,j}(\rho\, p_{j,k}\, q_{k,l})$$
$$= \det(Z-\overline{Z})^{\alpha+\beta-(n+1)}\Big(-\Delta_{\alpha,\beta}\,\varphi(Z)\overline{\psi}(Z) - \mathrm{tr}\left(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\right)$$
$$- n\beta(\alpha-(n+1)/2)\,\varphi(Z)\overline{\psi}(Z)\Big).$$

Thus, substituting $\sum\limits_{j,k,l=1}^{n}(\partial/\partial Z)_{l,j}(\rho\, p_{j,k}\, q_{k,l})$ back into equation (5.2), we arrive at

$$\mathrm{d}\omega(Z) = \det(Z-\overline{Z})^{\alpha+\beta}\Big(-\Delta_{\alpha,\beta}\,\varphi(Z)\overline{\psi}(Z) - \mathrm{tr}\left(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\right)$$
$$- n\beta(\alpha-(n+1)/2)\,\varphi(Z)\overline{\psi}(Z)\Big)\frac{[\mathrm{d}Z]\wedge[\mathrm{d}\overline{Z}]}{\det(Z-\overline{Z})^{n+1}}.$$

Now, noting that the volume form

$$\det(Z-\overline{Z})^{\alpha+\beta}\,\frac{[\mathrm{d}Z]\wedge[\mathrm{d}\overline{Z}]}{\det(Z-\overline{Z})^{n+1}}$$

is just a constant multiple of $\det(Y)^{\alpha+\beta}\mathrm{d}\mu_n(Z)$, it follows readily from the vanishing result (5.1) that

$$\langle -\Delta_{\alpha,\beta}\varphi,\psi\rangle$$
$$= \int\limits_{\Gamma\backslash\mathbb{H}_n}\det(Y)^{\alpha+\beta}\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\right)\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\psi\rangle,$$

which is the claimed formula.

Using the latter formula, we compute

$$\langle\varphi,-\Delta_{\alpha,\beta}\psi\rangle$$
$$= \overline{\langle -\Delta_{\alpha,\beta}\psi,\varphi\rangle}$$
$$= \int\limits_{\Gamma\backslash\mathbb{H}_n}\det(Y)^{\alpha+\beta}\,\overline{\mathrm{tr}\left(\Lambda_\beta\psi(Z)\overline{\Lambda}_\beta\overline{\varphi}(Z)\right)}\,\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\overline{\langle\psi,\varphi\rangle}$$
$$= \int\limits_{\Gamma\backslash\mathbb{H}_n}\det(Y)^{\alpha+\beta}\,\mathrm{tr}\left(\Lambda_\beta\varphi(Z)\overline{\Lambda}_\beta\overline{\psi}(Z)\right)\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\psi\rangle$$
$$= \langle -\Delta_{\alpha,\beta}\varphi,\psi\rangle,$$

which proves the claimed symmetry of the Siegel–Maaß Laplacian $\Delta_{\alpha,\beta}$. □

Corollary 5.2 *Let* $\Gamma\subset\mathrm{Sp}_n(\mathbb{R})$ *be a subgroup commensurable with* $\mathrm{Sp}_n(\mathbb{Z})$ *and let* $\varphi\in\mathcal{H}^n_{\alpha,\beta}(\Gamma)$ *be a Siegel–Maaß form of weight* (α,β) *and degree* n *for* Γ*. Then, if* φ *is an eigenform of* $\Delta_{\alpha,\beta}$ *with eigenvalue* λ*, we have* $\lambda\in\mathbb{R}$ *and* $\lambda\geq n\beta(\alpha-(n+1)/2)$*.*

Furthermore, φ *has eigenvalue* $\lambda=\beta(\alpha-(n+1)/2)$ *if and only if* $\varphi(Z)=\det(Y)^{-\beta}f(Z)$*, where* $f:\mathbb{H}_n\longrightarrow\mathbb{C}$ *is a holomorphic function satisfying*

$$f(\gamma Z) = \det(CZ+D)^{\alpha-\beta}f(Z)$$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. *Moreover, if* $\beta < 0$, *then* f *is a Siegel cusp form of weight* $\alpha - \beta$ *and degree* n *for* Γ.

Proof Since $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ is an eigenform of $\Delta_{\alpha,\beta}$ with eigenvalue λ, i.e., we have $(\Delta_{\alpha,\beta} + \lambda\,\mathrm{id})\varphi = 0$, we compute using Theorem 5.1

$$\begin{aligned}\lambda\langle\varphi,\varphi\rangle &= \langle -\Delta_{\alpha,\beta}\varphi,\varphi\rangle \\ &= \int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}\,\mathrm{tr}\left(|\Lambda_\beta\varphi(Z)|^2\right)\mathrm{d}\mu_n(Z) + n\beta(\alpha-(n+1)/2)\langle\varphi,\varphi\rangle.\end{aligned}$$

This immediately implies that $\lambda \in \mathbb{R}$. Furthermore, since $\mathrm{tr}(|\Lambda_\beta\varphi(Z)|^2) \geq 0$, we conclude that

$$\lambda \geq n\beta(\alpha-(n+1)/2).$$

To prove the second part of the corollary, we observe that the above equation shows that the equality $\lambda = n\beta(\alpha-(n+1)/2)$ is equivalent to

$$\int\limits_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta}\,\mathrm{tr}\left(|\Lambda_\beta\varphi(Z)|^2\right)\mathrm{d}\mu_n(Z) = 0.$$

Since $\mathrm{tr}(|\Lambda_\beta\varphi(Z)|^2) \geq 0$, the above integral vanishes if and only if $\mathrm{tr}(|\Lambda_\beta\varphi(Z)|^2) = 0$. Now, as the matrix

$$\Lambda_\beta\varphi(Z) = (Z-\overline{Z})\frac{\partial\varphi}{\partial\overline{Z}} - \beta\varphi(Z)\mathbb{1}_n$$

is similar to the complex symmetric matrix

$$S(Z) := 2i\,Y^{1/2}\frac{\partial\varphi}{\partial\overline{Z}}Y^{1/2} - \beta\varphi(Z)\mathbb{1}_n,$$

as we have the relation $\Lambda_\beta\varphi(Z) = Y^{1/2}S(Z)Y^{-1/2}$, the matrix $|\Lambda_\beta\varphi(Z)|^2$ becomes similar to the positive semidefinite hermitian matrix $S(Z)\overline{S}(Z)$, which is diagonalizable with non-negative real eigenvalues. Therefore, the condition $\mathrm{tr}(S(Z)\overline{S}(Z)) = \mathrm{tr}(|\Lambda_\beta\varphi(Z)|^2) = 0$ is equivalent to the vanishing of all the eigenvalues of $S(Z)\overline{S}(Z)$, which is equivalent to the vanishing of $S(Z)$ and hence of $\Lambda_\beta\varphi(Z)$. All in all, this proves that the equality $\lambda = n\beta(\alpha-(n+1)/2)$ is equivalent to the vanishing condition $\Lambda_\beta\varphi = 0$.

Continuing, we now set $f(Z) := \det(Y)^\beta\varphi(Z)$, and compute

$$\frac{\partial f}{\partial\overline{Z}} = \beta\det(Y)^{\beta-1}\frac{\partial\det(Y)}{\partial\overline{Z}}\varphi(Z) + \det(Y)^\beta\frac{\partial\varphi}{\partial\overline{Z}}.$$

Since we have

$$\frac{\partial\det(Y)}{\partial\overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial X} + i\frac{\partial}{\partial Y}\right)\det(Y) = \frac{i}{2}\frac{\partial\det(Y)}{\partial Y} = \frac{i}{2}\det(Y)Y^{-1},$$

the above equality becomes

$$\begin{aligned}\frac{\partial f}{\partial \overline{Z}} &= \frac{i\beta}{2}\det(Y)^{\beta}Y^{-1}\varphi(Z) + \det(Y)^{\beta}\frac{\partial \varphi}{\partial \overline{Z}} \\ &= -\frac{i}{2}\det(Y)^{\beta}Y^{-1}\Big(-\beta\varphi(Z)\mathbb{1}_n + 2iY\frac{\partial \varphi}{\partial \overline{Z}}\Big) \\ &= -\frac{i}{2}\det(Y)^{\beta}Y^{-1}\Big((Z-\overline{Z})\frac{\partial \varphi}{\partial \overline{Z}} - \beta\varphi(Z)\mathbb{1}_n\Big) \\ &= -\frac{i}{2}\det(Y)^{\beta}Y^{-1}\Lambda_{\beta}\varphi(Z).\end{aligned}$$

In total, this shows that $\partial f/\partial \overline{Z} = 0$, i.e., the function f is holomorphic, if and only if $\Lambda_{\beta}\varphi(Z) = 0$, which, by the previous argument, is equivalent to $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ being a Siegel–Maaß form with eigenvalue $\lambda = \beta(\alpha - (n+1)/2)$.

Furthermore, as the function $\varphi \in \mathcal{H}^n_{\alpha,\beta}(\Gamma)$ has the transformation behaviour

$$\varphi(\gamma Z) = \det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z)$$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma$, the function $f(Z) = \det(Y)^{\beta}\varphi(Z) = \det(\mathrm{Im}(Z))^{\beta}\varphi(Z)$ has the transformation behaviour

$$\begin{aligned}f(\gamma Z) &= \det(\mathrm{Im}(\gamma Z))^{\beta}\varphi(\gamma Z) \\ &= \left(\frac{\det(\mathrm{Im}(Z))}{|\det(CZ+D)|^2}\right)^{\beta}\det(CZ+D)^{\alpha}\det(C\overline{Z}+D)^{\beta}\varphi(Z) \\ &= \det(CZ+D)^{\alpha-\beta}\det(\mathrm{Im}(Z))^{\beta}\varphi(Z) \\ &= \det(CZ+D)^{\alpha-\beta}f(Z),\end{aligned}$$

as claimed.

Finally, letting $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j\end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1,\ldots,h)$ be a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$, Remark 4.8 shows that given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, the quantities

$$|\det(C_jZ+D_j)^{-\alpha}\det(C_j\overline{Z}+D_j)^{-\beta}\varphi(\gamma_jZ)|$$

have to be bounded in the region $\{Z = X+iY \in \mathbb{H}_n \mid Y \geq Y_0\}$. Therefore, if $\beta < 0$, this implies that given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 \gg 0$, the quantities

$$\begin{aligned}&|\det(C_jZ+D_j)^{-(\alpha-\beta)}f(\gamma_jZ)| \\ &\quad = |\det(C_jZ+D_j)^{-\alpha}\det(C_j\overline{Z}+D_j)^{-\beta}\det(\mathrm{Im}(\gamma_jZ))^{\beta}\varphi(\gamma_jZ)|\end{aligned}$$

will become arbitrarily small in the region $\{Z = X+iY \in \mathbb{H}_n \mid Y \geq Y_0\}$. In other words, f is indeed a Siegel cusp form of weight $\alpha-\beta$ and degree n for Γ.

With all this, the proof of the corollary is complete. □

Remark 5.3 For $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and $\alpha = k/2$, $\beta = -k/2$ with $k \in \mathbb{N}_{>0}$, we denote the Hilbert space $\mathcal{H}^n_{\alpha,\beta}(\Gamma)$ simply by $\mathcal{H}^n_k(\Gamma)$. Similarly, we write for the operator $\Omega_{\alpha,\beta}$ simply Ω_k, which becomes

$$\Omega_k = (Z-\overline{Z})\left((Z-\overline{Z})\frac{\partial}{\partial\overline{Z}}\right)^t\frac{\partial}{\partial Z} + \frac{k}{2}(Z-\overline{Z})\frac{\partial}{\partial\overline{Z}} + \frac{k}{2}(Z-\overline{Z})\frac{\partial}{\partial Z}$$
$$= -Y\left(\left(Y\frac{\partial}{\partial X}\right)^t\frac{\partial}{\partial X} + \left(Y\frac{\partial}{\partial Y}\right)^t\frac{\partial}{\partial Y}\right) + ikY\frac{\partial}{\partial X}.$$

Finally, we write for the operator $\Delta_{\alpha,\beta}$ simply Δ_k and call it the Siegel–Maaß Laplacian of weight k; it is given as

$$\Delta_k = \mathrm{tr}\left(Y\left(\left(Y\frac{\partial}{\partial X}\right)^t\frac{\partial}{\partial X} + \left(Y\frac{\partial}{\partial Y}\right)^t\frac{\partial}{\partial Y}\right) - ikY\frac{\partial}{\partial X}\right).$$

We note that the transformation behaviour of a Siegel–Maaß form φ of weight k and degree n for Γ takes the form

$$\varphi(\gamma Z) = \left(\frac{\det(CZ+D)}{\det(C\overline{Z}+D)}\right)^{k/2}\varphi(Z),$$

where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma$.

In the last corollary, we summarize the main results about Siegel–Maaß forms of weight k and degree n for Γ.

Corollary 5.4 *Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$ and let $\varphi \in \mathcal{H}^n_k(\Gamma)$ be a Siegel–Maaß form of weight k and degree n for Γ. Then, if φ is an eigenform of Δ_k with eigenvalue λ, we have $\lambda \in \mathbb{R}$ and*

$$\lambda \geq \frac{nk}{4}(n-k+1),$$

with equality attained if and only if the function φ is of the form $\varphi(Z) = \det(Y)^{k/2}f(Z)$ for some Siegel cusp form $f \in \mathcal{S}^n_k(\Gamma)$ of weight k and degree n for Γ. In other words, there is an isomorphism

$$\mathcal{S}^n_k(\Gamma) \cong \ker\left(\Delta_k + \frac{nk}{4}(n-k+1)\mathrm{id}\right)$$

of $\mathbb{C}$-vector spaces, induced by the assignment $f \mapsto \det(Y)^{k/2}f$.

Proof The proof is an immediate consequence of Corollray 5.2 by setting $\alpha = k/2$ and $\beta = -k/2$. □

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References

1. Bump, D.: Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (1998)
2. Friedman, J.S., Jorgenson, J., Kramer, J.: Uniform sup-norm bounds on average for cusp forms of higher weights. Arbeitstagung Bonn: Progr. Math., Vol. 319. Birkhäuser/Springer, Cham **2016**, 127–154 (2013)
3. Maaß, H.: Lectures on Siegel's Modular Functions, TIFR Lectures on Mathematics, Vol. 3. Tata Institute of Fundamental Research, Mumbai (1954–1955)
4. Maaß, H.: Siegel's Modular Forms and Dirichlet Series. Course given at the University of Maryland, 1969-1970, Lecture Notes in Mathematics, Vol. 216, Springer (1971)
5. Petersen, K.B., Pedersen, M.S.: The Matrix Cookbook. Technical University of Denmark. Version 20121115 (2012)
6. Roelcke, W.: Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. II. Math. Ann. **168**, 261–324 (1967)

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