RESEARCH

On the average number of cyclic subgroups of the groups

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \text{ with } n_1, n_2, n_3 \leq x$$

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Abstract

Let $\mathbb{Z}_n$ be the additive group of residue classes modulo n . Let $c(n_1, n_2, n_3)$ denote the number of cyclic subgroups of the group $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}$, where n_1, n_2 and n_3 are arbitrary positive integers. In this paper we obtain an asymptotic formula for the sum $\sum_{n_1, n_2, n_3 \leq x} c(n_1, n_2, n_3)$.

Keywords: Cyclic group, Finite abelian group of rank three, Number of subgroups, Number of cyclic subgroups, Dirichlet series, Perron formula, Asymptotic formula, Error term

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1 Introduction

Let $\mathbb{Z}_n$ denote the additive group of residue classes modulo n . We recall that a finite abelian group G ($|G| > 1$) is of rank r if it is isomorphic to $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$, where $n_1, \dots, n_r \in \mathbb{N} \setminus \{1\}$ and $n_j \mid n_{j+1}$ ($1 \leq j \leq r-1$). Let $\text{rank}(G)$ denote the rank of G and $s(G)$ be the number of subgroups of G . The one variable level function $t_r(n) = \sum_{|G|=n, \text{rank}(G) \leq r} s(G)$ and its summatory function $\sum_{n \leq x} t_r(n)$ were investigated by Bhowmik and Wu [1]. There are many other asymptotic results on the number of subgroups of abelian groups. See, e.g., the papers [3, 9] and their references.

We are interested in the following different problem, not studied in the above papers. Let $r \geq 1$ be a fixed integer. For arbitrary positive integers $n_1, \dots, n_r$, consider the group $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$. Let $s(n_1, \dots, n_r)$ and $c(n_1, \dots, n_r)$ denote the total number of subgroups and the number of cyclic subgroups of the given group $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$, respectively. Here $s(n_1, \dots, n_r)$ and $c(n_1, \dots, n_r)$ are multiplicative functions of r variables.

Suppose $x > 0$ is a real number. For each $r \geq 1$ define the multivariable sums

$$S_r(x) := \sum_{n_1 \leq x, \dots, n_r \leq x} s(n_1, \dots, n_r),$$

$$C_r(x) := \sum_{n_1 \leq x, \dots, n_r \leq x} c(n_1, \dots, n_r).$$

When $r = 1$, the group $\mathbb{Z}_n$ is cyclic and hence $s(n) = c(n) = \tau(n)$, where $\tau(n)$ denotes the number of divisors of n . So, $S_1(x) = C_1(x) = \sum_{n \leq x} \tau(n)$. Therefore, the problem of finding estimates for the sums $S_r(x)$ and $C_r(x)$ can be viewed as a generalization of the well-known Dirichlet divisor problem.

Now consider the case $r \geq 2$. If $(n_i, n_j) = 1$ for all $1 \leq i < j \leq r$, then $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ is cyclic and isomorphic to $\mathbb{Z}_{n_1 \cdots n_r}$, which is a consequence of the Chinese remainder theorem. If otherwise, then $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ has rank at least two.

There are several ways to compute $s(n_1, n_2)$ and $c(n_1, n_2)$. For example, one way is using Goursat's lemma for groups in [2, 8, 13] and another one is using the concept of the fundamental group lattice in [10, 11]. See also the paper [4]. For any $n_1, n_2 \in \mathbb{N}$ we have

$$\begin{aligned} s(n_1, n_2) &= \sum_{d|n_1, e|n_2} (d, e) \\ &= \sum_{d|(n_1, n_2)} \varphi(d) \tau(n_1/d) \tau(n_2/d) \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} c(n_1, n_2) &= \sum_{d|n_1, e|n_2} \varphi((d, e)) \\ &= \sum_{d|(n_1, n_2)} (\mu * \varphi)(d) \tau(n_1/d) \tau(n_2/d). \end{aligned} \quad (1.2)$$

When $r \geq 3$, we have by Theorem 1 of [12],

$$c(n_1, \dots, n_r) = \sum_{d_1|n_1, \dots, d_r|n_r} \frac{\varphi(d_1) \cdots \varphi(d_r)}{\varphi([d_1, \dots, d_r])}. \quad (1.3)$$

However, in the case $r \geq 3$ there is no such a closed formula for $s(n_1, \dots, n_r)$.

W. G. Nowak and L. Tóth [6] studied the average orders of the functions $s(n_1, n_2)$ and $c(n_1, n_2)$. They proved that the asymptotic formulas

$$S_2(x) = x^2 \left(\sum_{j=0}^3 A_j \log^j x \right) + O(x^{\frac{1117}{701} + \varepsilon}) \quad (1.4)$$

and

$$C_2(x) = x^2 \left(\sum_{j=0}^3 B_j \log^j x \right) + O(x^{\frac{1117}{701} + \varepsilon}) \quad (1.5)$$

hold, where A_j and B_j ($0 \leq j \leq 3$) are explicit constants, of which definitions are omitted here. Note that the exponent of the error term is $1117/701 = 1.593437 \dots$. The authors of the present paper proved in [14] that both error terms in the asymptotic formulas (1.4) and (1.5) can be improved into $O(x^{1.5}(\log x)^{6.5})$.

In this paper we shall prove the following theorem, by using a multidimensional Perron formula and the complex integration method.

Theorem 1.1 *The asymptotic formula*

$$C_3(x) = x^3 \left(\sum_{j=0}^7 c_j \log^j x \right) + O(x^{\frac{8}{3}+\varepsilon}) \quad (1.6)$$

holds, where c_j ($0 \leq j \leq 7$) are explicit constants.

Notations Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(a_1, \dots, a_k)$ and $[a_1, \dots, a_k]$ are the gcd and lcm of $a_1, \dots, a_k \in \mathbb{N}$, φ is Euler's totient function, μ is the Möbius function, ζ denotes the Riemann zeta-function, $\tau_k(n)$ denotes the number of ways n can be written as a product of k positive integers ($\tau(n) = \tau_2(n)$).

2 Preliminary lemmas

We shall use the following lemmas.

Lemma 2.1 *Suppose $\Re z > 1$, $\Re w > 1$. Then*

$$C(z, w) := \sum_{m \geq 1} \sum_{n \geq 1} \frac{c(m, n)}{m^z n^w} = \zeta^2(z) \zeta^2(w) \zeta(z+w-1) \zeta^{-2}(z+w).$$

Proof See the paper [6]. □

Lemma 2.2 *Suppose that $r \geq 2$ is a fixed integer and $f(n_1, \dots, n_r)$ is an arithmetical function of r variables such that it is symmetric in the variables $n_1, \dots, n_r$ and its Dirichlet series*

$$F(z_1, \dots, z_r) := \sum_{n_1 \geq 1} \cdots \sum_{n_r \geq 1} \frac{f(n_1, \dots, n_r)}{n_1^{z_1} \cdots n_r^{z_r}}$$

is absolutely convergent for $\Re z_j > \sigma_a$ ($1 \leq j \leq r$) with some $\sigma_a > 0$. Suppose that $x \notin \mathbb{N}$ and T are two large parameters such that $10 \leq T \leq x/2$ and define

$$b_j = \sigma_a + \frac{10^{j-1}}{\log x}, \quad T_j = 10^{j-1} T \quad (1 \leq j \leq r).$$

Then we have

$$\sum_{n_1 \leq x} \cdots \sum_{n_r \leq x} f(n_1, \dots, n_r) = I_f(x, T) + O(x^{r\sigma_a} E_f(x, T)),$$

where

$$I_f(x, T) := \frac{1}{(2\pi i)^r} \int_{b_1-iT_1}^{b_1+iT_1} \cdots \int_{b_r-iT_r}^{b_r+iT_r} F(z_1, \dots, z_r) x^{z_1+\cdots+z_r} \frac{dz_r \cdots dz_1}{z_r \cdots z_1},$$

$$E_f(x, T) := \sum_{n_1 \geq 1} \cdots \sum_{n_r \geq 1} \frac{|f(n_1, \dots, n_r)| (n_1 \cdots n_r)^{-b_1}}{\min_{1 \leq j \leq r} T \left| \log \frac{x}{n_j} \right| + 1}.$$

Proof This is a slight modification of Lemma 2.2 in [14]. □

Lemma 2.3 *Suppose $\ell \geq 0$ is a fixed integer. For $\sigma > 1$ we have*

$$\zeta^{(\ell)}(\sigma + it) \ll \min \left(\frac{1}{(\sigma - 1)^{1+\ell}}, \log^{1+\ell}(|t| + 2) \right).$$

Proof The case $\ell = 0$ can be found in Chapter 7 of Pan and Pan [7]. The case $\ell \geq 1$ follows from the result of the case $\ell = 0$ and Cauchy's theorem. $\square$

Lemma 2.4 Suppose $\ell \geq 0$ is a fixed integer. Then we have

$$\zeta^{(\ell)}(\sigma + it) \ll \begin{cases} (|t| + 2)^{\frac{1-\sigma}{3}} \log^{1+\ell}(|t| + 2), & \text{if } 1/2 \leq \sigma \leq 1, \\ (|t| + 2)^{\frac{3-4\sigma}{6}} \log^{1+\ell}(|t| + 2), & \text{if } 0 \leq \sigma \leq 1/2. \end{cases}$$

Proof The estimate for the case $\ell = 0$ follows from the bounds

$$\zeta(it) \ll (|t| + 2)^{1/2}, \quad \zeta(1/2 + it) \ll (|t| + 2)^{1/6}, \quad \zeta(1 + it) \ll \log(|t| + 2)$$

and the Phragmen-Lindelöf principle. The estimate for the case $\ell \geq 1$ follows from the result of the case $\ell = 0$ and Cauchy's theorem. $\square$

Lemma 2.5 Suppose $V > 10$ is a large parameter and $41/60 \leq u \leq 1$. Then we have

$$\int_{-V}^V |\zeta(u + iv)|^{10} dv \ll V^{1+\varepsilon}.$$

Proof It is proved in Chapter 8 of Ivić [5] that Lemma 2.5 holds for $u = 41/60$. By Lemma 2.3 we see that Lemma 2.5 trivially holds for $u = 1$. The case $41/60 < u < 1$ follows from the cases $u = 41/60$ and $u = 1$. $\square$

Lemma 2.6 We have the asymptotic formula

$$\sum_{n \leq x} \tau_4(n) = xQ_3(\log x) + O(x^{1/2} \log^5 x),$$

where $Q_3(u)$ is a polynomial in u of degree 3.

Proof Well-known. It follows from the Perron formula and the fourth power moment of $\zeta(s)$ over the critical line. $\square$

Lemma 2.7 Suppose $\ell \geq 0$ is a fixed integer, $11/20 - \varepsilon \leq u \leq 11/20 + \varepsilon$. Then we have

$$\int_{-V}^V |\zeta^{(\ell)}(u + it)|^5 dt \ll V^{1+\varepsilon}.$$

Proof The proof for the case $\ell = 0$ and $u = 11/20$ can be found in Chapter 8 of [5]. The same method works for an arbitrary u with $11/20 - \varepsilon \leq u \leq 11/20 + \varepsilon$. The case $\ell \geq 1$ follows from the result of the case $\ell = 0$ and Cauchy's theorem. $\square$

Lemma 2.8 Suppose $\alpha_j > 2/3$, $\beta_j > 1$ ($j = 1, 2, 3$) are fixed real numbers such that $\alpha_1 + \alpha_2 + \alpha_3 > 2$, $f_j(t)$ ($t \in \mathbb{R}, j = 1, 2, 3$) and $f(t_1, t_2)$ ($t_1, t_2 \in \mathbb{R}$) are complex-valued functions such that

$$f_j(t) \ll \frac{\log^{\beta_j}(|t| + 2)}{(|t| + 2)^{\alpha_j}} \quad (j = 1, 2, 3), \quad f(t_1, t_2) \ll 1. \quad (2.1)$$

Suppose $U > 4$ and $U' > 4$ are positive parameters such that $U \asymp U'$, and define

$$W = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t_1) f_2(t_2) f_3(t_1 + t_2) f(t_1, t_2) dt_1 dt_2,$$

$$W(U, U') = \int_{-U}^U \int_{-U'}^{U'} f_1(t_1) f_2(t_2) f_3(t_1 + t_2) f(t_1, t_2) dt_1 dt_2.$$

Then

(1) we have the estimate

$$W - W(U, U') \ll \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\min(\alpha_1,1)+\alpha_2+\alpha_3-2}} + \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\min(\alpha_2,1)+\alpha_3-2}} + \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\alpha_2+\min(\alpha_3,1)-2}};$$

(2) the integral W is absolutely convergent.

Proof Without loss of generality, we suppose $U \leq U'$. Obviously, from (2.1) we have

$$W - W(U, U') \ll W_1 + W_2 + W_3 + W_4, \quad (2.2)$$

where

$$\begin{aligned} W_1 &:= \int_{\substack{t_1 \geq 0, t_2 \geq 0 \\ \max(|t_1|, |t_2|) \geq U}} \frac{\log^{\beta_1}(|t_1| + 2)}{(|t_1| + 2)^{\alpha_1}} \frac{\log^{\beta_2}(|t_2| + 2)}{(|t_2| + 2)^{\alpha_2}} \frac{\log^{\beta_3}(|t_1 + t_2| + 2)}{(|t_1 + t_2| + 2)^{\alpha_3}} dt_1 dt_2, \\ W_2 &:= \int_{\substack{t_1 \leq 0, t_2 \geq 0 \\ \max(|t_1|, |t_2|) \geq U}} \frac{\log^{\beta_1}(|t_1| + 2)}{(|t_1| + 2)^{\alpha_1}} \frac{\log^{\beta_2}(|t_2| + 2)}{(|t_2| + 2)^{\alpha_2}} \frac{\log^{\beta_3}(|t_1 + t_2| + 2)}{(|t_1 + t_2| + 2)^{\alpha_3}} dt_1 dt_2, \\ W_3 &:= \int_{\substack{t_1 \leq 0, t_2 \leq 0 \\ \max(|t_1|, |t_2|) \geq U}} \frac{\log^{\beta_1}(|t_1| + 2)}{(|t_1| + 2)^{\alpha_1}} \frac{\log^{\beta_2}(|t_2| + 2)}{(|t_2| + 2)^{\alpha_2}} \frac{\log^{\beta_3}(|t_1 + t_2| + 2)}{(|t_1 + t_2| + 2)^{\alpha_3}} dt_1 dt_2, \\ W_4 &:= \int_{\substack{t_1 \geq 0, t_2 \leq 0 \\ \max(|t_1|, |t_2|) \geq U}} \frac{\log^{\beta_1}(|t_1| + 2)}{(|t_1| + 2)^{\alpha_1}} \frac{\log^{\beta_2}(|t_2| + 2)}{(|t_2| + 2)^{\alpha_2}} \frac{\log^{\beta_3}(|t_1 + t_2| + 2)}{(|t_1 + t_2| + 2)^{\alpha_3}} dt_1 dt_2. \end{aligned}$$

So, it suffices for us to prove that for $j = 1, 2, 3, 4$,

$$W_j \ll \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\min(\alpha_1,1)+\alpha_2+\alpha_3-2}} + \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\min(\alpha_2,1)+\alpha_3-2}} + \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\alpha_2+\min(\alpha_3,1)-2}}. \quad (2.3)$$

It is easy to see that $W_1 = W_3$, $W_2 = W_4$.

I. Estimate of W_1 :

We write

$$\begin{aligned} W_1 &= W_{11} + W_{12}, \\ W_{11} &:= \int_{\substack{U \leq t_1 < \infty \\ 0 \leq t_2 \leq t_1}} \frac{\log^{\beta_1}(t_1 + 2)}{(t_1 + 2)^{\alpha_1}} \frac{\log^{\beta_2}(t_2 + 2)}{(t_2 + 2)^{\alpha_2}} \frac{\log^{\beta_3}(t_1 + t_2 + 2)}{(t_1 + t_2 + 2)^{\alpha_3}} dt_1 dt_2, \\ W_{12} &:= \int_{\substack{U \leq t_2 < \infty \\ 0 \leq t_1 \leq t_2}} \frac{\log^{\beta_1}(t_1 + 2)}{(t_1 + 2)^{\alpha_1}} \frac{\log^{\beta_2}(t_2 + 2)}{(t_2 + 2)^{\alpha_2}} \frac{\log^{\beta_3}(t_1 + t_2 + 2)}{(t_1 + t_2 + 2)^{\alpha_3}} dt_1 dt_2. \end{aligned} \quad (2.4)$$

It is easily seen that

$$\begin{aligned} W_{11} &\ll \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1+\alpha_3}} dt_1 \int_0^{t_1} \frac{1}{(t_2 + 2)^{\alpha_2}} dt_2 \\ &\ll \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1+\alpha_3}} \times t_1^{\max(1-\alpha_2, 0)} \log t_1 dt_1 \\ &\ll \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\min(\alpha_2,1)+\alpha_3-2}}. \end{aligned} \quad (2.5)$$

Similarly we have

$$W_{12} \ll \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\min(\alpha_1,1)+\alpha_2+\alpha_3-2}}. \quad (2.6)$$

From (2.4) to (2.6) we see that (2.3) holds for $j = 1$.

II. Estimate of W_4 :

We write

$$\begin{aligned} W_4 &= W_{41} + W_{42}, \\ W_{41} &:= \int_{\substack{U \leq t_1 < \infty \\ 0 \leq t_2 \leq t_1}} \frac{\log^{\beta_1}(t_1+2)}{(t_1+2)^{\alpha_1}} \frac{\log^{\beta_2}(t_2+2)}{(t_2+2)^{\alpha_2}} \frac{\log^{\beta_3}(t_1-t_2+2)}{(t_1-t_2+2)^{\alpha_3}} dt_1 dt_2, \\ W_{42} &:= \int_{\substack{U \leq t_2 < \infty \\ 0 \leq t_1 \leq t_2}} \frac{\log^{\beta_1}(t_1+2)}{(t_1+2)^{\alpha_1}} \frac{\log^{\beta_2}(t_2+2)}{(t_2+2)^{\alpha_2}} \frac{\log^{\beta_3}(t_2-t_1+2)}{(t_2-t_1+2)^{\alpha_3}} dt_1 dt_2. \end{aligned} \quad (2.7)$$

We estimate W_{41} first. We have

$$\begin{aligned} W_{41} &\ll \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1}} dt_1 \int_0^{t_1} \frac{1}{(t_2+2)^{\alpha_2}} \frac{1}{(t_1-t_2+2)^{\alpha_3}} dt_2 \\ &= \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1}} dt_1 \int_0^{t_1/2} \frac{1}{(t_2+2)^{\alpha_2}} \frac{1}{(t_1-t_2+2)^{\alpha_3}} dt_2 \\ &\quad + \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1}} dt_1 \int_{t_1/2}^{t_1} \frac{1}{(t_2+2)^{\alpha_2}} \frac{1}{(t_1-t_2+2)^{\alpha_3}} dt_2 \\ &\ll \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1+\alpha_3}} dt_1 \int_0^{t_1/2} \frac{1}{(t_2+2)^{\alpha_2}} dt_2 \\ &\quad + \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1+\alpha_2}} dt_1 \int_{t_1/2}^{t_1} \frac{1}{(t_1-t_2+2)^{\alpha_3}} dt_2 \\ &\ll \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1+\alpha_3}} \times t_1^{\max(1-\alpha_2,0)} \log t_1 dt_1 \\ &\quad + \int_U^\infty \frac{\log^{\beta_1+\beta_2+\beta_3} t_1}{t_1^{\alpha_1+\alpha_2}} dt_1 \times t_1^{\max(1-\alpha_3,0)} \log t_1 \\ &\ll \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\min(\alpha_2,1)+\alpha_3-2}} + \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\alpha_2+\min(\alpha_3,1)-2}}. \end{aligned} \quad (2.8)$$

Similarly we have

$$W_{42} \ll \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\min(\alpha_2,1)+\alpha_2+\alpha_3-2}} + \frac{\log^{\beta_1+\beta_2+\beta_3+1} U}{U^{\alpha_1+\alpha_2+\min(\alpha_3,1)-2}}. \quad (2.9)$$

From (2.7) to (2.9) we see that (2.3) holds for $j = 4$.

Now the first assertion of Lemma 2.8 follows from (2.2) and (2.3). The second assertion of Lemma 2.8 is obviously a consequence of the first assertion. $\square$

Lemma 2.9 Suppose $M(s) = AM_1(s)M_2(s)$ such that $M_1(s)$ is meromorphic in the region $|s - s_0| \leq \eta$, $s = s_0$ is a pole of order $r \geq 2$, and $M_2(s)$ is analytic in $|s - s_0| \leq \eta$, where $\eta > 0$ is a positive real number, $A \neq 0$ is a constant. Write

$$M_1(s) = \sum_{n=-r}^{\infty} a_n(s - s_0)^n.$$

Then we have

$$\text{Res}_{s=s_0} M(s) = A \sum_{m=0}^{r-1} a_{-m-1} \frac{M_2^{(m)}(s_0)}{m!}.$$

Proof It follows from the theory of complex variable functions. $\square$

Lemma 2.10 *Let $k, \ell \geq 2$ be two fixed integers and $f_j(s)$ ($j = 1, \dots, k$) be functions which are at least ℓ -times differentiable. Then for each $0 \leq j \leq \ell$ we have*

$$(f_1(s) \cdots f_k(s))^{(j)} = \sum_{\substack{n_1 + \dots + n_k = j \\ n_1, \dots, n_k \geq 0}} \frac{j!}{n_1! \cdots n_k!} f_1^{(n_1)}(s) \cdots f_k^{(n_k)}(s).$$

Proof Well-known. □

3 On the Dirichlet series of $c(n_1, n_2, n_3)$

In this section we shall study the Dirichlet series

$$C(s_1, s_2, s_3) := \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \frac{c(n_1, n_2, n_3)}{n_1^{s_1} n_2^{s_2} n_3^{s_3}} \quad (\Re s_j > 1, j = 1, 2, 3). \quad (3.1)$$

We have the following proposition.

Proposition 3.1 *Suppose that $s_j = \sigma_j + it_j$, $\sigma_j > 1$ ($j = 1, 2, 3$). Then we have*

$$\begin{aligned} C(s_1, s_2, s_3) &= \zeta^2(s_1) \zeta^2(s_2) \zeta^2(s_3) \zeta(s_1 + s_2 - 1) \zeta(s_1 + s_3 - 1) \\ &\quad \times \zeta(s_2 + s_3 - 1) \zeta(s_1 + s_2 + s_3 - 2) H(s_1, s_2, s_3), \end{aligned} \quad (3.2)$$

where $H(s_1, s_2, s_3)$ can be written as a triple Dirichlet series, which is absolutely convergent when $\sigma_1, \sigma_2, \sigma_3$ satisfy the following conditions:

$$\begin{aligned} \sigma_j &> 1/3 \quad (j = 1, 2, 3), \quad 2\sigma_j + \sigma_\ell > 2 \quad (1 \leq j \neq \ell \leq 3), \\ \sigma_1 + \sigma_2 + \sigma_3 + \sigma_j &> 3 \quad (j = 1, 2, 3), \\ 2(\sigma_1 + \sigma_2 + \sigma_3) - \sigma_j &> 4 \quad (j = 1, 2, 3). \end{aligned} \quad (3.3)$$

Proof Noting that $c(n_1, n_2, n_3)$ is multiplicative and (1.3) holds, we have that

$$C(s_1, s_2, s_3) = \zeta(s_1) \zeta(s_2) \zeta(s_3) \prod_p \left(\sum_{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3} \frac{\varphi(p^{\alpha_1}) \varphi(p^{\alpha_2}) \varphi(p^{\alpha_3})}{\varphi(p^{\max(\alpha_1, \alpha_2, \alpha_3)}) p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}} \right). \quad (3.4)$$

Write

$$\mathbb{N}_0^3 = D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5, \quad (3.5)$$

where

$$\begin{aligned} D_0 &:= \{(0, 0, 0)\}, \quad D_1 := \bigcup_{j=1}^7 D_{1j}, \\ D_{11} &:= \mathbb{N} \times \{0\} \times \{0\}, \quad D_{12} := \{0\} \times \mathbb{N} \times \{0\}, \\ D_{13} &:= \{0\} \times \{0\} \times \mathbb{N}, \quad D_{14} := \{(k, k, 0) : k \in \mathbb{N}\}, \\ D_{15} &:= \{(k, 0, k) : k \in \mathbb{N}\}, \quad D_{16} := \{(0, k, k) : k \in \mathbb{N}\}, \\ D_{17} &:= \{(k, k, k) : k \in \mathbb{N}\}, \quad D_2 := \mathbb{N}^3 \setminus D_{17}, \\ D_3 &:= \mathbb{N} \times \mathbb{N} \times \{0\} \setminus D_{14} = \{(\alpha, \beta, 0) : \alpha, \beta \in \mathbb{N}, \alpha \neq \beta\}, \\ D_4 &:= \mathbb{N} \times \{0\} \times \mathbb{N} \setminus D_{15}, \quad D_5 := \{0\} \times \mathbb{N} \times \mathbb{N} \setminus D_{16}. \end{aligned}$$

For each $1 \leq j \leq 7$, we define

$$D_{1j}(p; s_1, s_2, s_3) := \sum_{(\alpha_1, \alpha_2, \alpha_3) \in D_{1j}} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})\varphi(p^{\alpha_3})}{\varphi(p^{\max(\alpha_1, \alpha_2, \alpha_3)})p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}}.$$

Also we define for $j = 2, 3, 4, 5$,

$$D_j(p; s_1, s_2, s_3) := \sum_{(\alpha_1, \alpha_2, \alpha_3) \in D_j} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})\varphi(p^{\alpha_3})}{\varphi(p^{\max(\alpha_1, \alpha_2, \alpha_3)})p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}}.$$

For $j = 1, 2, 3$, it is easy to see that

$$\begin{aligned} D_{11}(p; s_1, s_2, s_3) &= \sum_{\alpha_1 \in \mathbb{N}} \frac{1}{p^{\alpha_1 s_1}} = \frac{p^{-s_1}}{1 - p^{-s_1}}, \\ D_{12}(p; s_1, s_2, s_3) &= \sum_{\alpha_2 \in \mathbb{N}} \frac{1}{p^{\alpha_2 s_2}} = \frac{p^{-s_2}}{1 - p^{-s_2}}, \\ D_{13}(p; s_1, s_2, s_3) &= \sum_{\alpha_3 \in \mathbb{N}} \frac{1}{p^{\alpha_3 s_3}} = \frac{p^{-s_3}}{1 - p^{-s_3}}. \end{aligned} \quad (3.6)$$

For $j = 4$ we have

$$\begin{aligned} D_{14}(p; s_1, s_2, s_3) &= \sum_{\alpha_1 \in \mathbb{N}} \frac{\varphi(p^{\alpha_1})}{p^{\alpha_1 s_1 + \alpha_1 s_2}} = \sum_{\alpha_1 \in \mathbb{N}} \frac{p^{\alpha_1} - p^{\alpha_1 - 1}}{p^{\alpha_1 s_1 + \alpha_1 s_2}} \\ &= \frac{p^{-(s_1 + s_2 - 1)}}{1 - p^{-(s_1 + s_2 - 1)}} - \frac{p^{-(s_1 + s_2)}}{1 - p^{-(s_1 + s_2 - 1)}}. \end{aligned} \quad (3.7)$$

Similarly, we have

$$D_{15}(p; s_1, s_2, s_3) = \frac{p^{-(s_1 + s_3 - 1)}}{1 - p^{-(s_1 + s_3 - 1)}} - \frac{p^{-(s_1 + s_3)}}{1 - p^{-(s_1 + s_3 - 1)}} \quad (3.8)$$

and

$$D_{16}(p; s_1, s_2, s_3) = \frac{p^{-(s_2 + s_3 - 1)}}{1 - p^{-(s_2 + s_3 - 1)}} - \frac{p^{-(s_2 + s_3)}}{1 - p^{-(s_2 + s_3 - 1)}}. \quad (3.9)$$

For $j = 7$ we have

$$\begin{aligned} D_{17}(p; s_1, s_2, s_3) &= \sum_{\alpha \in \mathbb{N}} \frac{\varphi^2(p^\alpha)}{p^{\alpha(s_1 + s_2 + s_3)}} = \sum_{\alpha \in \mathbb{N}} \frac{p^{2\alpha} - 2p^{2\alpha - 1} + p^{2\alpha - 2}}{p^{\alpha(s_1 + s_2 + s_3)}} \\ &= \frac{p^{-(s_1 + s_2 + s_3 - 2)}}{1 - p^{-(s_1 + s_2 + s_3 - 2)}} - 2 \frac{p^{-(s_1 + s_2 + s_3 - 1)}}{1 - p^{-(s_1 + s_2 + s_3 - 2)}} + \frac{p^{-(s_1 + s_2 + s_3)}}{1 - p^{-(s_1 + s_2 + s_3 - 2)}}. \end{aligned} \quad (3.10)$$

For simplicity, we write

$$\begin{aligned} u_1 &:= s_1, \quad u_2 := s_2, \quad u_3 := s_3, \quad u_4 := s_1 + s_2 - 1, \\ u_5 &:= s_1 + s_3 - 1, \quad u_6 := s_2 + s_3 - 1, \quad u_7 := s_1 + s_2 + s_3 - 2. \end{aligned}$$

So, from (3.6) to (3.10), we can write

$$1 + \sum_{j=1}^7 D_{1j}(p; s_1, s_2, s_3) = 1 + \sum_{j=1}^7 \frac{p^{-u_j}}{1 - p^{-u_j}} + B_1(p; s_1, s_2, s_3), \quad (3.11)$$

where

$$B_1(p; s_1, s_2, s_3) := -\frac{p^{-(s_1+s_2)}}{1-p^{-(s_1+s_2-1)}} - \frac{p^{-(s_1+s_3)}}{1-p^{-(s_1+s_3-1)}} - \frac{p^{-(s_2+s_3)}}{1-p^{-(s_2+s_3-1)}} \\ - 2\frac{p^{-(s_1+s_2+s_3-1)}}{1-p^{-(s_1+s_2+s_3-2)}} + \frac{p^{-(s_1+s_2+s_3)}}{1-p^{-(s_1+s_2+s_3-2)}}.$$

Obviously, we have

$$B_1(p; s_1, s_2, s_3) \ll p^{-\sigma_1-\sigma_2} + p^{-\sigma_1-\sigma_3} + p^{-\sigma_2-\sigma_3} + p^{-\sigma_1-\sigma_2-\sigma_3+1}. \quad (3.12)$$

It is easy to see that

$$\prod_{\ell=1}^7 (1-p^{-u_\ell}) \\ = 1 - \sum_{\ell=1}^7 p^{-u_\ell} + \sum_{m_1=2}^7 (-1)^{m_1} \sum_{1 \leq \ell_1 < \ell_2 < \dots < \ell_{m_1} \leq 7} p^{-(u_{\ell_1} + \dots + u_{\ell_{m_1}})}. \quad (3.13)$$

By the expression $1/(1-\xi) = \sum_{m=0}^{\infty} \xi^m$ ($|\xi| < 1$), we have

$$1 + \sum_{j=1}^7 \frac{p^{-u_j}}{1-p^{-u_j}} = 1 + \sum_{j=1}^7 p^{-u_j} + \sum_{j=1}^7 \sum_{m=2}^{\infty} p^{-mu_j}. \quad (3.14)$$

So, from (3.13) and (3.14) we get

$$\left(\prod_{\ell=1}^7 (1-p^{-u_\ell}) \right) \left(1 + \sum_{j=1}^7 \frac{p^{-u_j}}{1-p^{-u_j}} \right) \\ = 1 + \sum_{m_1=2}^7 (-1)^{m_1} \sum_{1 \leq \ell_1 < \ell_2 < \dots < \ell_{m_1} \leq 7} p^{-(u_{\ell_1} + \dots + u_{\ell_{m_1}})} - \sum_{j=1}^7 \sum_{\ell=1}^7 p^{-u_\ell - u_j} \\ + \sum_{j=1}^7 \sum_{m_1=2}^7 (-1)^{m_1} \sum_{1 \leq \ell_1 < \ell_2 < \dots < \ell_{m_1} \leq 7} p^{-(u_j + u_{\ell_1} + \dots + u_{\ell_{m_1}})} \\ + \sum_{j=1}^7 \sum_{m=2}^{\infty} p^{-mu_j} \times \prod_{\ell=1}^7 (1-p^{-u_\ell}) \\ = 1 - \sum_{j=1}^7 \sum_{\ell=1}^7 p^{-u_\ell - u_j} + \sum_{j=1}^7 \sum_{m=2}^{\infty} p^{-mu_j} \times \prod_{\ell=1}^7 (1-p^{-u_\ell}) \\ + O \left(\sum_{1 \leq j \neq \ell \leq 7} p^{-\Re(u_\ell + u_j)} \right) \\ = 1 + O \left(\sum_{1 \leq j \neq \ell \leq 7} p^{-\Re(u_\ell + u_j)} + \sum_{j=1}^7 p^{-3\Re u_j} \right). \quad (3.15)$$

Now we estimate $D_2(p; s_1, s_2, s_3)$. We write

$$D_2(p; s_1, s_2, s_3) = \sum_{j=1}^3 \sum_{\substack{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \\ \alpha_j \text{ largest}}} \frac{\varphi(p^{\alpha_1}) \varphi(p^{\alpha_2}) \varphi(p^{\alpha_3})}{\varphi(p^{\alpha_j}) p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}} \\ = \sum_{j=1}^3 D_{2j}(p; s_1, s_2, s_3), \quad (3.16)$$

say. We only estimate $D_{23}(p; s_1, s_2, s_3)$. The estimates for $D_{21}(p; s_1, s_2, s_3)$ and $D_{22}(p; s_1, s_2, s_3)$ are similar.

We write

$$D_{23}(p; s_1, s_2, s_3) = \sum_{j=1}^5 D_{23j}(p; s_1, s_2, s_3), \quad (3.17)$$

where

$$\begin{aligned} D_{231}(p; s_1, s_2, s_3) &:= \sum_{1 \leq \alpha_1 < \alpha_2 < \alpha_3} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})}{p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}}, \\ D_{232}(p; s_1, s_2, s_3) &:= \sum_{1 \leq \alpha_2 < \alpha_1 < \alpha_3} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})}{p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}}, \\ D_{233}(p; s_1, s_2, s_3) &:= \sum_{1 \leq \alpha_1 < \alpha_2 = \alpha_3} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})}{p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}}, \\ D_{234}(p; s_1, s_2, s_3) &:= \sum_{1 \leq \alpha_2 < \alpha_1 = \alpha_3} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})}{p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}}, \\ D_{235}(p; s_1, s_2, s_3) &:= \sum_{1 \leq \alpha_1 = \alpha_2 < \alpha_3} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})}{p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}}. \end{aligned}$$

We have the estimate

$$\sum_{m=M}^{\infty} \xi^m \ll |\xi|^M, \quad (M \geq 0, |\xi| < C < 1). \quad (3.18)$$

So, from (3.18) we get

$$\begin{aligned} D_{231}(p; s_1, s_2, s_3) &= \sum_{\alpha_1=1}^{\infty} \sum_{\alpha_2=\alpha_1+1}^{\infty} \sum_{\alpha_3=\alpha_2+1}^{\infty} \frac{(p^{\alpha_1} - p^{\alpha_1-1})(p^{\alpha_2} - p^{\alpha_2-1})}{p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}} \\ &\ll \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1 s_1 - \alpha_1}} \sum_{\alpha_2=\alpha_1+1}^{\infty} \frac{1}{p^{\alpha_2 s_2 - \alpha_2}} \sum_{\alpha_3=\alpha_2+1}^{\infty} \frac{1}{p^{\alpha_3 s_3}} \\ &\ll \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1 s_1 - \alpha_1}} \sum_{\alpha_2=\alpha_1+1}^{\infty} \frac{1}{p^{\alpha_2 s_2 - \alpha_2}} \times \frac{1}{p^{(\alpha_2+1)s_3}} \\ &\ll \frac{1}{p^{s_3}} \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1 s_1 - \alpha_1}} \sum_{\alpha_2=\alpha_1+1}^{\infty} \frac{1}{p^{\alpha_2(s_2 + s_3 - 1)}} \\ &\ll \frac{1}{p^{s_3}} \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1 s_1 - \alpha_1}} \times \frac{1}{p^{(\alpha_1+1)(s_2 + s_3 - 1)}} \\ &\ll \frac{1}{p^{\sigma_2 + 2\sigma_3 - 1}} \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1(\sigma_1 + \sigma_2 + \sigma_3 - 2)}} \\ &\ll \frac{1}{p^{\sigma_2 + 2\sigma_3 - 1}} \times \frac{1}{p^{\sigma_1 + \sigma_2 + \sigma_3 - 2}} = \frac{1}{p^{\sigma_1 + 2\sigma_2 + 3\sigma_3 - 3}}. \end{aligned} \quad (3.19)$$

Similarly we have

$$D_{232}(p; s_1, s_2, s_3) \ll \frac{1}{p^{2\sigma_1 + \sigma_2 + 3\sigma_3 - 3}}. \quad (3.20)$$

We have

$$\begin{aligned}
 D_{233}(p; s_1, s_2, s_3) &= \sum_{\alpha_1=1}^{\infty} \frac{\varphi(p^{\alpha_1})}{p^{\alpha_1 s_1}} \sum_{\alpha_2=\alpha_1+1}^{\infty} \frac{\varphi(p^{\alpha_2})}{p^{\alpha_2(s_2+s_3)}} \\
 &\ll \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1(\sigma_1-1)}} \sum_{\alpha_2=\alpha_1+1}^{\infty} \frac{1}{p^{\alpha_2(\sigma_2+\sigma_3-1)}} \\
 &\ll \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1(\sigma_1-1)}} \times \frac{1}{p^{(\alpha_1+1)(\sigma_2+\sigma_3-1)}} \\
 &\ll \frac{1}{p^{\sigma_2+\sigma_3-1}} \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1(\sigma_1+\sigma_2+\sigma_3-2)}} \\
 &\ll \frac{1}{p^{\sigma_1+2\sigma_2+2\sigma_3-3}}.
 \end{aligned} \tag{3.21}$$

Similarly we have

$$D_{234}(p; s_1, s_2, s_3) \ll \frac{1}{p^{2\sigma_1+\sigma_2+2\sigma_3-3}}. \tag{3.22}$$

Finally, we estimate $D_{235}(p; s_1, s_2, s_3)$. We have

$$\begin{aligned}
 D_{235}(p; s_1, s_2, s_3) &= \sum_{\alpha_1=1}^{\infty} \frac{\varphi^2(p^{\alpha_1})}{p^{\alpha_1(s_1+s_2)}} \sum_{\alpha_3=\alpha_1+1}^{\infty} \frac{1}{p^{\alpha_3 s_3}} \\
 &\ll \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1(\sigma_1+\sigma_2-2)}} \sum_{\alpha_3=\alpha_1+1}^{\infty} \frac{1}{p^{\alpha_3 \sigma_3}} \\
 &\ll \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1(\sigma_1+\sigma_2-2)}} \times \frac{1}{p^{(\alpha_1+1)\sigma_3}} \\
 &\ll \frac{1}{p^{\sigma_3}} \sum_{\alpha_1=1}^{\infty} \frac{1}{p^{\alpha_1(\sigma_1+\sigma_2+\sigma_3-2)}} \\
 &\ll \frac{1}{p^{\sigma_3}} \times \frac{1}{p^{\sigma_1+\sigma_2+\sigma_3-2}} = \frac{1}{p^{\sigma_1+\sigma_2+2\sigma_3-2}}.
 \end{aligned} \tag{3.23}$$

From (3.17) to (3.23) we get

$$D_{23}(p; s_1, s_2, s_3) \ll \frac{1}{p^{\sigma_1+\sigma_2+2\sigma_3-2}} + \frac{1}{p^{\sigma_1+2\sigma_2+2\sigma_3-3}} + \frac{1}{p^{2\sigma_1+\sigma_2+2\sigma_3-3}}. \tag{3.24}$$

Similarly we have

$$D_{22}(p; s_1, s_2, s_3) \ll \frac{1}{p^{\sigma_1+2\sigma_2+\sigma_3-2}} + \frac{1}{p^{\sigma_1+2\sigma_2+2\sigma_3-3}} + \frac{1}{p^{2\sigma_1+2\sigma_2+\sigma_3-3}}. \tag{3.25}$$

and

$$D_{21}(p; s_1, s_2, s_3) \ll \frac{1}{p^{2\sigma_1+\sigma_2+\sigma_3-2}} + \frac{1}{p^{2\sigma_1+2\sigma_2+\sigma_3-3}} + \frac{1}{p^{2\sigma_1+\sigma_2+2\sigma_3-3}}. \tag{3.26}$$

From (3.16), (3.24–3.26) we have

$$D_2(p; s_1, s_2, s_3) \ll \frac{p^{\sigma_1} + p^{\sigma_2} + p^{\sigma_3}}{p^{2\sigma_1+2\sigma_2+2\sigma_3-3}} + \frac{p^{-\sigma_1} + p^{-\sigma_2} + p^{-\sigma_3}}{p^{\sigma_1+\sigma_2+\sigma_3-2}}. \tag{3.27}$$

We estimate $D_3(p; s_1, s_2, s_3)$ now. We have

$$\begin{aligned} D_3(p; s_1, s_2, s_3) &= \sum_{1 \leq \alpha < \beta} \frac{\varphi(p^\alpha)}{p^{\alpha s_1 + \beta s_2}} + \sum_{1 \leq \beta < \alpha} \frac{\varphi(p^\beta)}{p^{\alpha s_1 + \beta s_2}} \\ &\ll \sum_{\alpha=1}^{\infty} \frac{p^\alpha}{p^{\alpha s_1}} \sum_{\beta=\alpha+1}^{\infty} \frac{1}{p^{\beta s_2}} + \sum_{\beta=1}^{\infty} \frac{p^\beta}{p^{\beta s_2}} \sum_{\alpha=\beta+1}^{\infty} \frac{1}{p^{\alpha s_1}} \\ &\ll \sum_{\alpha=1}^{\infty} \frac{p^\alpha}{p^{\alpha \sigma_1}} \times \frac{1}{p^{(\alpha+1)\sigma_2}} + \sum_{\beta=1}^{\infty} \frac{p^\beta}{p^{\beta \sigma_2}} \times \frac{1}{p^{(\beta+1)\sigma_1}} \\ &\ll \frac{1}{p^{\sigma_1+2\sigma_2-1}} + \frac{1}{p^{2\sigma_1+\sigma_2-1}}. \end{aligned} \quad (3.28)$$

Similarly we have

$$D_4(p; s_1, s_2, s_3) \ll \frac{1}{p^{\sigma_1+2\sigma_3-1}} + \frac{1}{p^{2\sigma_1+\sigma_3-1}} \quad (3.29)$$

and

$$D_5(p; s_1, s_2, s_3) \ll \frac{1}{p^{\sigma_2+2\sigma_3-1}} + \frac{1}{p^{2\sigma_2+\sigma_3-1}}. \quad (3.30)$$

We write

$$\sum_{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})\varphi(p^{\alpha_3})}{\varphi(p^{\max(\alpha_1, \alpha_2, \alpha_3)})p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}} = \prod_{j=1}^7 (1 - p^{-u_j})^{-1} \times h(p; s_1, s_2, s_3), \quad (3.31)$$

where

$$\begin{aligned} h(p; s_1, s_2, s_3) &= \prod_{j=1}^7 (1 - p^{-u_j}) \times \sum_{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3} \frac{\varphi(p^{\alpha_1})\varphi(p^{\alpha_2})\varphi(p^{\alpha_3})}{\varphi(p^{\max(\alpha_1, \alpha_2, \alpha_3)})p^{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}} \\ &= \prod_{j=1}^7 (1 - p^{-u_j}) \times (1 + \sum_{j=1}^7 D_{1j}(p; s_1, s_2, s_3) + \sum_{j=2}^5 D_j(p; s_1, s_2, s_3)) \\ &= 1 + h^*(p; s_1, s_2, s_3), \end{aligned} \quad (3.32)$$

say. From (3.11), (3.12), (3.15), (3.27)–(3.30) we get

$$\begin{aligned} h^*(p; s_1, s_2, s_3) &\ll \sum_{1 \leq j \neq \ell \leq 7} \frac{1}{p^{\Re(u_j + u_\ell)}} + \sum_{j=1}^7 p^{-3\Re u_j} + \frac{1}{p^{\sigma_1 + \sigma_2 + \sigma_3 - 1}} \\ &\quad + \frac{p^{\sigma_1} + p^{\sigma_2} + p^{\sigma_3}}{p^{2\sigma_1 + 2\sigma_2 + 2\sigma_3 - 3}} + \frac{p^{-\sigma_1} + p^{-\sigma_2} + p^{-\sigma_3}}{p^{\sigma_1 + \sigma_2 + \sigma_3 - 2}} \\ &\quad + \sum_{1 \leq j \neq \ell \leq 3} \frac{1}{p^{2\sigma_j + \sigma_\ell - 1}}. \end{aligned} \quad (3.33)$$

Now we see from (3.4), (3.31)–(3.33) that Proposition 3.1 holds and $H(s_1, s_2, s_3)$ can be written as a triple Dirichlet series which is absolutely convergent under the condition (3.3). $\square$

4 On an infinite sum

Suppose $x \notin \mathbb{N}$ and T are two large parameters such that $5 \leq T \leq x/2$. For each $1 \leq j \leq 3$, define the infinite sum

$$E_j(x, T) := \sum_{n_1, n_2, n_3 \geq 1} \frac{c(n_1, n_2, n_3)(n_1 n_2 n_3)^{-1-1/\log x}}{T |\log \frac{x}{n_j}| + 1}.$$

Since $c(n_1, n_2, n_3)$ is symmetric for n_1, n_2, n_3 , we have

$$E_1(x, T) = E_2(x, T) = E_3(x, T) =: E(x, T),$$

say.

Write

$$E(x, T) = \sum_{n_3=1}^{\infty} \frac{n_3^{-1-1/\log x}}{T |\log \frac{x}{n_3}| + 1} \times G(n_3, 1/\log x), \quad (4.1)$$

where

$$G(n_3, \sigma) := \sum_{n_1, n_2 \geq 1} c(n_1, n_2, n_3)(n_1 n_2)^{-1-\sigma}, \quad 0 < \sigma < 1.$$

4.1 Upper bound of $G(n_3, \sigma)$

In this subsection we shall give an upper bound of $G(n_3, \sigma)$.

If $n_3 = 1$, then $c(n_1, n_2, 1) = c(n_1, n_2)$, so from Lemmas 2.1 and 2.3 we have

$$G(1, \sigma) \ll \sigma^{-5}. \quad (4.2)$$

Now suppose $n_3 \geq 2$. We begin with the formula (1.3). By noting that $\varphi(n) \gg n/\log n$ and $[m, n] = mn/(m, n)$ we have

$$\begin{aligned} c(n_1, n_2, n_3) &= \sum_{d_j | n_j (1 \leq j \leq 3)} \frac{\varphi(d_1)\varphi(d_2)\varphi(d_3)}{\varphi([d_1, d_2, d_3])} \\ &\ll \sum_{d_j | n_j (1 \leq j \leq 3)} \frac{\varphi(d_1)\varphi(d_2)\varphi(d_3)}{[d_1, d_2, d_3]} \times \log[d_1, d_2, d_3] \\ &\ll \sum_{d_j | n_j (1 \leq j \leq 3)} \frac{d_1 d_2 d_3}{[d_1, d_2, d_3]} \times \log(d_1 d_2 d_3) \\ &= \sum_{d_j | n_j (1 \leq j \leq 3)} \frac{d_1 d_2 d_3}{[[d_1, d_2], d_3]} \times \log(d_1 d_2 d_3) \\ &= \sum_{d_j | n_j (1 \leq j \leq 3)} \frac{d_1 d_2 ([d_1, d_2], d_3)}{[d_1, d_2]} \times \log(d_1 d_2 d_3) \\ &= \sum_{d_j | n_j (1 \leq j \leq 3)} \frac{d_1 d_2 ([d_1, d_2], d_3)}{[d_1, d_2]} \times \log(d_1 d_2) \\ &\quad + \sum_{d_j | n_j (1 \leq j \leq 3)} \frac{d_1 d_2 ([d_1, d_2], d_3)}{[d_1, d_2]} \times \log d_3. \end{aligned} \quad (4.3)$$

Inserting the estimate (4.3) into the expression of $G(n_3, \sigma)$, we have

$$G(n_3, \sigma) \ll G_1(n_3, \sigma) + G_2(n_3, \sigma), \quad (4.4)$$

where

$$\begin{aligned} G_1(n_3, \sigma) &:= \sum_{d_3|n_3} \sum_{n_1, n_2=1}^{\infty} \sum_{d_1|n_1, d_2|n_2} \frac{d_1 d_2 ([d_1, d_2], d_3) \log(d_1 d_2)}{[d_1, d_2] (n_1 n_2)^{1+\sigma}}, \\ G_2(n_3, \sigma) &:= \sum_{d_3|n_3} \log d_3 \sum_{n_1, n_2=1}^{\infty} \sum_{d_1|n_1, d_2|n_2} \frac{d_1 d_2 ([d_1, d_2], d_3)}{[d_1, d_2] (n_1 n_2)^{1+\sigma}}. \end{aligned} \quad (4.5)$$

We estimate $G_1(n_3, \sigma)$ first. By writing $n_j = d_j \ell_j$ ($j = 1, 2$) we have

$$\begin{aligned} G_1(n_3, \sigma) &= \sum_{d_3|n_3} \sum_{d_1, d_2, \ell_1, \ell_2 \geq 1} \frac{d_1 d_2 ([d_1, d_2], d_3) \log(d_1 d_2)}{[d_1, d_2] (d_1 \ell_1 d_2 \ell_2)^{1+\sigma}} \\ &= \sum_{d_3|n_3} \sum_{d_1, d_2 \geq 1} \frac{d_1 d_2 ([d_1, d_2], d_3) \log(d_1 d_2)}{[d_1, d_2] (d_1 d_2)^{1+\sigma}} \sum_{\ell_1, \ell_2 \geq 1} \frac{1}{(\ell_1 \ell_2)^{1+\sigma}} \\ &= \zeta^2(1+\sigma) \sum_{d_3|n_3} \sum_{d_1, d_2 \geq 1} \frac{d_1 d_2 ([d_1, d_2], d_3) \log(d_1 d_2)}{[d_1, d_2] (d_1 d_2)^{1+\sigma}} \\ &= \zeta^2(1+\sigma) \sum_{d|n_3} d \sum_{\substack{d_1, d_2 \geq 1 \\ d|[d_1, d_2]}} \frac{d_1 d_2 \log(d_1 d_2)}{[d_1, d_2] (d_1 d_2)^{1+\sigma}}. \end{aligned} \quad (4.6)$$

Let $a = (d_1, d_2)$, $d_1 = ab$, $d_2 = ac$. Then $(b, c) = 1$ and $[d_1, d_2] = abc$. So from (4.6) we have

$$\begin{aligned} G_1(n_3, \sigma) &= \zeta^2(1+\sigma) \sum_{d|n_3} d \sum_{\substack{a, b, c \geq 1, (b, c)=1 \\ d|abc}} \frac{a^2 bc \log a^2 bc}{abc (a^2 bc)^{1+\sigma}} \\ &\leq \zeta^2(1+\sigma) \sum_{d|n_3} d \sum_{\substack{a, b, c \geq 1 \\ d|abc}} \frac{a \log a^2 bc}{(a^2 bc)^{1+\sigma}} \\ &\leq 2\zeta^2(1+\sigma) \sum_{d|n_3} d \sum_{\substack{a, b, c \geq 1 \\ d|abc}} \frac{\log abc}{(abc)^{1+\sigma}} \\ &= 2\zeta^2(1+\sigma) \sum_{d|n_3} d \sum_{\substack{n \geq 1 \\ d|n}} \frac{\tau_3(n) \log n}{n^{1+\sigma}} \\ &= 2\zeta^2(1+\sigma) \sum_{d|n_3} d \sum_{n \geq 1} \frac{\tau_3(nd) \log nd}{(dn)^{1+\sigma}} \\ &\ll \zeta^2(1+\sigma) \sum_{d|n_3} \tau_3(d) \sum_{n \geq 1} \frac{\tau_3(n) (\log n + \log d)}{n^{1+\sigma}} \\ &\ll \sigma^{-2} \sum_{d|n_3} \tau_3(d) (A(\sigma) + B(\sigma) \log d), \end{aligned} \quad (4.7)$$

where we used the estimate $\tau_3(nd) \leq \tau_3(n)\tau_3(d)$ and Lemma 2.3, and where

$$A(\sigma) := \sum_{n \geq 1} \frac{\tau_3(n) \log n}{n^{1+\sigma}}, \quad B(\sigma) := \sum_{n \geq 1} \frac{\tau_3(n)}{n^{1+\sigma}} = \zeta^3(1+\sigma).$$

By Lemma 2.3 we get

$$B(\sigma) \ll \sigma^{-3}. \quad (4.8)$$

It is easy to see that

$$A(\sigma) = -B'(\sigma) = -3\zeta^2(1+\sigma)\zeta'(1+\sigma).$$

By Lemma 2.3 again we get

$$A(\sigma) \ll \sigma^{-4}. \quad (4.9)$$

From (4.7) to (4.9) we get

$$\begin{aligned} G_1(n_3, \sigma) &\ll \sigma^{-2} \sum_{d|n_3} \tau_3(d) (\sigma^{-4} + \sigma^{-3} \log d) \\ &\ll \sigma^{-2} \sum_{d|n_3} \tau_3(d) (\sigma^{-4} + \sigma^{-3} \log n_3) \\ &\ll \sigma^{-6} \tau_4(n_3) \log n_3. \end{aligned} \quad (4.10)$$

$G_2(n_3, \sigma)$ can be bounded in the same way. We have

$$G_2(n_3, \sigma) \ll \sigma^{-5} \tau_4(n_3) \log n_3. \quad (4.11)$$

From (4.4), (4.10) and (4.11) we get

$$G(n_3, \sigma) \ll \sigma^{-6} \tau_4(n_3) \log n_3 \quad (n_3 \geq 2). \quad (4.12)$$

4.2 Upper bound of $E(x, T)$

Inserting (4.2) and (4.12) into (4.1), we get

$$E(x, T) \ll \frac{\log^4 x}{T} + \log^6 x \times \sum_{n_3=2}^{\infty} \frac{n_3^{-1-1/\log x} \tau_4(n_3) \log n_3}{T |\log \frac{x}{n_3}| + 1}. \quad (4.13)$$

Let

$$\begin{aligned} U_1 &:= \sum_{n_3 \leq x/2} \frac{n_3^{-1-1/\log x} \tau_4(n_3) \log n_3}{T |\log \frac{x}{n_3}| + 1}, \\ U_2 &:= \sum_{x/2 < n_3 \leq 3x/2} \frac{n_3^{-1-1/\log x} \tau_4(n_3) \log n_3}{T |\log \frac{x}{n_3}| + 1}, \\ U_3 &:= \sum_{n_3 > 3x/2} \frac{n_3^{-1-1/\log x} \tau_4(n_3) \log n_3}{T |\log \frac{x}{n_3}| + 1}. \end{aligned} \quad (4.14)$$

We have by Lemma 2.3 that

$$\begin{aligned} U_1 + U_3 &\ll \frac{1}{T} \sum_{n=1}^{\infty} \frac{\tau_4(n) \log n}{n^{1+1/\log x}} \\ &= \frac{1}{T} \sum_{n=1}^{\infty} \frac{\tau_4(n) \log n}{n^{1+\sigma}} \Big|_{\sigma=1/\log x} \\ &= \frac{1}{T} \left(\sum_{n=1}^{\infty} \frac{-\tau_4(n)}{n^{1+\sigma}} \right)' \Big|_{\sigma=1/\log x} \\ &= \frac{1}{T} (-\zeta^4(1+\sigma))' \Big|_{\sigma=1/\log x} \\ &= \frac{1}{T} (-4\zeta^3(1+\sigma)\zeta'(1+\sigma)) \Big|_{\sigma=1/\log x} \\ &\ll \frac{\log^5 x}{T}. \end{aligned} \quad (4.15)$$

Now we bound U_2 . We write

$$\begin{aligned} U_2 &= U_{21} + U_{22} + U_{23}, \\ U_{21} &:= \sum_{x/2 < n_3 \leq xe^{-1/T}} \frac{n_3^{-1-1/\log x} \tau_4(n_3) \log n_3}{T |\log \frac{x}{n_3}| + 1}, \\ U_{22} &:= \sum_{xe^{-1/T} < n_3 \leq xe^{1/T}} \frac{n_3^{-1-1/\log x} \tau_4(n_3) \log n_3}{T |\log \frac{x}{n_3}| + 1}, \\ U_{23} &:= \sum_{xe^{1/T} < n_3 \leq 3x/2} \frac{n_3^{-1-1/\log x} \tau_4(n_3) \log n_3}{T |\log \frac{x}{n_3}| + 1}. \end{aligned} \quad (4.16)$$

When $x/2 < n_3 \leq xe^{-1/T}$, we have $T |\log \frac{x}{n_3}| = T \log \frac{x}{n_3} \geq 1$. Thus

$$\begin{aligned} U_{21} &\ll \frac{\log x}{xT} \sum_{x/2 < n_3 \leq xe^{-1/T}} \frac{\tau_4(n_3)}{\log \frac{x}{n_3}} \\ &\ll \frac{\log x}{xT} \sum_{x/T \ll \ell \leq x/2} \frac{x \tau_4([x] - \ell)}{\ell} \quad (n_3 = [x] - \ell) \\ &\ll \frac{\log^2 x}{T} \max_{x/T \ll L \leq x/4} \frac{1}{L} \sum_{L < \ell \leq 2L} \tau_4([x] - \ell) \\ &= \frac{\log^2 x}{T} \max_{x/T \ll L \leq x/4} \frac{1}{L} \sum_{[x]-2L \leq n < [x]-L} \tau_4(n) \\ &\ll \frac{\log^2 x}{T} \max_{x/T \ll L \leq x/4} \frac{1}{L} \left(L \log^3 x + x^{1/2} \log^5 x \right) \\ &\ll \max_{x/T \ll L \leq x/4} \left(\frac{\log^5 x}{T} + \frac{\log^2 x}{TL} \times x^{1/2} \log^5 x \right) \\ &\ll \frac{\log^5 x}{T} + x^{-1/2} \log^7 x, \end{aligned} \quad (4.17)$$

where in the second line we used the estimate $\log(x/([x] - \ell)) \gg \ell/x$ and in the fifth line we used Lemma 2.6. Similarly we have

$$U_{23} \ll \frac{\log^5 x}{T} + x^{-1/2} \log^7 x. \quad (4.18)$$

When $xe^{-1/T} < n_3 \leq xe^{1/T}$, we have $T |\log \frac{x}{n_3}| \leq 1$. Thus by Lemma 2.6 again we get

$$U_{22} \ll \frac{\log x}{x} \sum_{xe^{-1/T} < n_3 \leq xe^{1/T}} \tau_4(n_3) \ll \frac{\log^4 x}{T} + x^{-1/2} \log^6 x. \quad (4.19)$$

Combining (4.13)–(4.19) we get

Proposition 4.1 *Suppose $x \notin \mathbb{N}$ and T are two large parameters such that $5 \leq T \leq x/2$. Then*

$$E(x, T) \ll T^{-1} \log^{11} x + x^{-1/2} \log^{13} x. \quad (4.20)$$

5 Proof of the theorem

In this section, we shall prove Theorem 1.1.

Suppose $x \geq 20$, $x \notin \mathbb{N}$. Let $10 \leq T \leq x/2$ be a parameter to be determined later. Define

$$b_j = 1 + \frac{10^{j-1}}{\log x}, \quad T_j = 10^{j-1} T \quad (1 \leq j \leq 3).$$

By Lemma 2.2 with $r = 3$ and $\sigma_a = 1$ we have

$$C_3(x) = I(x, T) + O(x^3 E(x, T)), \quad (5.1)$$

where

$$I(x, T) = \frac{1}{(2\pi i)^3} \int_{b_1 - iT_1}^{b_1 + iT_1} \int_{b_2 - iT_2}^{b_2 + iT_2} \int_{b_3 - iT_3}^{b_3 + iT_3} \frac{C(s_1, s_2, s_3) x^{s_1 + s_2 + s_3}}{s_1 s_2 s_3} ds_3 ds_2 ds_1,$$

$C(s_1, s_2, s_3)$ and $E(x, T)$ being defined in the last section. Here $E(x, T)$ can be bounded by Proposition 4.1, so we only need to evaluate $I(x, T)$. We shall evaluate $I(x, T)$ in the order s_3, s_2, s_1 .

5.1 Evaluation of $I(x, t)$ for s_3

In this subsection we shall evaluate the integral $I(x, T)$ for the variable s_3 . Consider the rectangle domain formed by the four points $s_3 = b_3 \pm iT_3, s_3 = 11/20 \pm iT_3$. From Proposition 3.1 we see easily that in this domain the integral function

$$\begin{aligned} g(s_1, s_2, s_3) &:= \frac{C(s_1, s_2, s_3) x^{s_1 + s_2 + s_3}}{s_1 s_2 s_3} \\ &= \zeta^2(s_1) \zeta^2(s_2) \zeta^2(s_3) \zeta(s_1 + s_2 - 1) \zeta(s_1 + s_3 - 1) \\ &\quad \times \zeta(s_2 + s_3 - 1) \zeta(s_1 + s_2 + s_3 - 2) H(s_1, s_2, s_3) \frac{x^{s_1 + s_2 + s_3}}{s_1 s_2 s_3} \end{aligned} \quad (5.2)$$

has four poles, which are $s_3^{(0)} = 1, s_3^{(1)} = 2 - s_1, s_3^{(2)} = 2 - s_2, s_3^{(3)} = 3 - s_1 - s_2$, respectively. Here $s_3 = 1$ is a pole of order 2 and the other three poles are simple. By the residue theorem we get

$$I(x, T) = \sum_{j=0}^3 I_j(x, T) + H_1(x, T) + H_2(x, T) - H_3(x, T), \quad (5.3)$$

where

$$\begin{aligned} I_j(x, T) &= \frac{1}{(2\pi i)^2} \int_{b_1 - iT_1}^{b_1 + iT_1} \int_{b_2 - iT_2}^{b_2 + iT_2} \operatorname{Res}_{s_3 = s_3^{(j)}} g(s_1, s_2, s_3) ds_2 ds_1 \quad (j = 0, 1, 2, 3), \\ H_1(x, T) &= \frac{1}{(2\pi i)^3} \int_{b_1 - iT_1}^{b_1 + iT_1} \int_{b_2 - iT_2}^{b_2 + iT_2} \int_{11/20 + iT_3}^{b_3 + iT_3} g(s_1, s_2, s_3) ds_3 ds_2 ds_1, \\ H_2(x, T) &= \frac{1}{(2\pi i)^3} \int_{b_1 - iT_1}^{b_1 + iT_1} \int_{b_2 - iT_2}^{b_2 + iT_2} \int_{11/20 - iT_3}^{11/20 + iT_3} g(s_1, s_2, s_3) ds_3 ds_2 ds_1, \\ H_3(x, T) &= \frac{1}{(2\pi i)^3} \int_{b_1 - iT_1}^{b_1 + iT_1} \int_{b_2 - iT_2}^{b_2 + iT_2} \int_{11/20 - iT_3}^{b_3 - iT_3} g(s_1, s_2, s_3) ds_3 ds_2 ds_1. \end{aligned}$$

5.1.1 Estimates of $H_j(x, T)$ ($j = 1, 2, 3$)

We estimate $H_1(x, T)$ first. In this case we have $s_1 = b_1 + it_1$ with $|t_1| \leq T, s_2 = b_2 + it_2$ with $|t_2| \leq 10T$ and $s_3 = \sigma_3 + i100T$ with $11/20 \leq \sigma_3 \leq b_3$. By the condition (3.3) in Proposition 3.1 we see that

$$H(s_1, s_2, s_3) \ll 1.$$

By Lemmas 2.3 and 2.4 we have

$$\begin{aligned}\zeta^2(s_1)\zeta^2(s_2)\zeta(s_1+s_2-1) &\ll \log^5 x, \\ \zeta^2(s_3)\zeta(s_1+s_3-1)\zeta(s_2+s_3-1)\zeta(s_1+s_2+s_3-2) &\ll T^{\frac{5}{3}\max(1-\sigma_3,0)} \log^5 T.\end{aligned}$$

From the above estimates and (5.2) we have

$$g(s_1, s_2, s_3) \ll \frac{x^2 \log^{10} x}{T(|t_1|+1)(|t_2|+1)} \times x^{\sigma_3} T^{\frac{5}{3}\max(1-\sigma_3,0)}. \quad (5.4)$$

So we get

$$\begin{aligned}H_1(x, T) &\ll \frac{x^2 \log^{10} x}{T} \int_{-T}^T \frac{dt_1}{|t_1|+1} \int_{-T_2}^{T_2} \frac{dt_2}{|t_2|+1} \left(\int_{11/20}^1 x^{\sigma_3} T^{\frac{5}{3}(1-\sigma_3)} d\sigma_3 + \int_1^{b_3} x^{\sigma_3} d\sigma_3 \right) \\ &\ll \frac{x^3 \log^{12} x}{T} + x^{\frac{51}{20}} T^{-\frac{1}{4}} \log^{12} x.\end{aligned} \quad (5.5)$$

Similarly, we have

$$H_3(x, T) \ll \frac{x^3 \log^{12} x}{T} + x^{\frac{51}{20}} T^{-\frac{1}{4}} \log^{12} x. \quad (5.6)$$

Now we estimate $H_2(x, T)$. In this case we have $s_1 = b_1 + it_1$ with $|t_1| \leq T$, $s_2 = b_2 + it_2$ with $|t_2| \leq 10T$ and $s_3 = 11/20 + it_3$ with $|t_3| \leq T_3 = 100T$. Similar to (5.4) we have

$$\begin{aligned}g(s_1, s_2, s_3) &\ll x^{\frac{51}{20}} \log^5 x \\ &\times \frac{|\zeta^2(s_3)\zeta(s_1+s_3-1)\zeta(s_2+s_3-1)\zeta(s_1+s_2+s_3-2)|}{(|t_1|+1)(|t_2|+2)(|t_3|+1)}\end{aligned} \quad (5.7)$$

So we have

$$H_2(x, T) \ll x^{\frac{51}{20}} \log^5 x \times H_2(T), \quad (5.8)$$

where

$$H_2(T) := \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} dt_2 \int_{-T_3}^{T_3} \frac{|\zeta^2(s_3)\zeta(s_1+s_3-1)\zeta(s_2+s_3-1)\zeta(s_1+s_2+s_3-2)|}{(|t_1|+1)(|t_2|+2)(|t_3|+1)} dt_3.$$

By Hölder's inequality we get

$$H_2(T) \leq (H_{21}(T))^{\frac{2}{5}} (H_{22}(T))^{\frac{1}{5}} (H_{23}(T))^{\frac{1}{5}} (H_{24}(T))^{\frac{1}{5}}, \quad (5.9)$$

where

$$\begin{aligned}H_{21}(T) &:= \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} dt_2 \int_{-T_3}^{T_3} \frac{|\zeta(11/20 + it_3)|^5}{(|t_1|+1)(|t_2|+2)(|t_3|+1)} dt_3, \\ H_{22}(T) &:= \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} dt_2 \int_{-T_3}^{T_3} \frac{|\zeta(11/20 + 1/\log x + i(t_1 + t_3))|^5}{(|t_1|+1)(|t_2|+2)(|t_3|+1)} dt_3, \\ H_{23}(T) &:= \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} dt_2 \int_{-T_3}^{T_3} \frac{|\zeta(11/20 + 10/\log x + i(t_2 + t_3))|^5}{(|t_1|+1)(|t_2|+2)(|t_3|+1)} dt_3, \\ H_{24}(T) &:= \int_{-T_1}^{T_1} dt_1 \int_{-T_2}^{T_2} dt_2 \int_{-T_3}^{T_3} \frac{|\zeta(11/20 + 11/\log x + i(t_1 + t_2 + t_3))|^5}{(|t_1|+1)(|t_2|+2)(|t_3|+1)} dt_3.\end{aligned}$$

By Lemma 2.7 and partial summation we get

$$H_{21}(T) = \int_{-T_1}^{T_1} \frac{dt_1}{|t_1|+1} \int_{-T_2}^{T_2} \frac{dt_2}{|t_2|+2} \int_{-T_3}^{T_3} \frac{|\zeta(11/20 + it_3)|^5}{|t_3|+1} dt_3 \ll T^\varepsilon. \quad (5.10)$$

For $H_{24}(T)$, we have

$$\begin{aligned} H_{24}(T) &\leq \int_{-T_3}^{T_3} dt_1 \int_{-T_3}^{T_3} dt_2 \int_{-T_3}^{T_3} \frac{|\zeta(11/20 + 11/\log x + i(t_1 + t_2 + t_3))|^5}{(|t_1|+1)(|t_2|+2)(|t_3|+1)} dt_3 \\ &\ll \int_{0 \leq |t_1| \leq |t_2| \leq |t_3| \leq T_3} \frac{|\zeta(11/20 + 11/\log x + i(t_1 + t_2 + t_3))|^5}{(|t_1|+1)(|t_2|+2)(|t_3|+1)} dt_3 dt_2 dt_1. \end{aligned} \quad (5.11)$$

From the condition $|t_1| \leq |t_2| \leq |t_3|$ we get

$$|t_1 + t_2 + t_3| + 1 \leq |t_1| + |t_2| + |t_3| + 1 < 3(|t_3| + 1).$$

Thus we get

$$\begin{aligned} H_{24}(T) &= \int_{0 \leq |t_1| \leq |t_2| \leq |t_3| \leq T_3} \frac{|\zeta(11/20 + 11/\log x + i(t_1 + t_2 + t_3))|^5}{(|t_1|+1)(|t_2|+2)(|t_1 + t_2 + t_3|+1)} \\ &\quad \times \frac{|t_1 + t_2 + t_3| + 1}{|t_3| + 1} dt_3 dt_2 dt_1 \\ &\ll \int_{0 \leq |t_1| \leq |t_2| \leq |t_3| \leq T_3} \frac{|\zeta(11/20 + 11/\log x + i(t_1 + t_2 + t_3))|^5}{(|t_1|+1)(|t_2|+2)(|t_1 + t_2 + t_3|+1)} dt_3 dt_2 dt_1 \\ &\leq \int_{-T_3}^{T_3} \frac{dt_1}{|t_1|+1} \int_{-T_3}^{T_3} \frac{dt_2}{|t_2|+1} \int_{-T_3}^{T_3} \frac{|\zeta(11/20 + 11/\log x + i(t_1 + t_2 + t_3))|^5}{(|t_1 + t_2 + t_3|+1)} dt_3 \\ &\leq \int_{-T_3}^{T_3} \frac{dt_1}{|t_1|+1} \int_{-T_3}^{T_3} \frac{dt_2}{|t_2|+1} \int_{-3T_3}^{3T_3} \frac{|\zeta(11/20 + 11/\log x + iv)|^5}{(|v|+1)} dv, \end{aligned}$$

which combining Lemma 2.7 gives

$$H_{24}(T) \ll T^\varepsilon. \quad (5.12)$$

Similarly, we have

$$H_{22}(T) \ll T^\varepsilon, \quad H_{23}(T) \ll T^\varepsilon. \quad (5.13)$$

Combining (5.8)–(5.13) we get

$$H_2(x, T) \ll x^{\frac{51}{20} + \varepsilon}. \quad (5.14)$$

5.1.2 Evaluation of $I_3(x, T)$

Now we evaluate $I_3(x, T)$. Since $s_3^{(3)} = 3 - s_1 - s_2$ is a simple pole of $g(s_1, s_2, s_3)$, we have

$$\begin{aligned} \text{Res}_{s_3=s_3^{(3)}} g(s_1, s_2, s_3) &= x^3 \zeta^2(s_1) \zeta^2(s_2) \zeta^2(3 - s_1 - s_2) \zeta(s_1 + s_2 - 1) \zeta(2 - s_2) \\ &\quad \times \zeta(2 - s_1) H(s_1, s_2, 3 - s_1 - s_2) \frac{1}{s_1 s_2 (3 - s_1 - s_2)}. \end{aligned}$$

Define

$$\begin{aligned} f_1(t; \sigma_1) &= \frac{\zeta^2(\sigma_1 + it) \zeta(2 - \sigma_1 - it)}{\sigma_1 + it}, \\ f_2(t; \sigma_2) &= \frac{\zeta^2(\sigma_2 + it) \zeta(2 - \sigma_2 - it)}{\sigma_2 + it}, \end{aligned}$$

$$f_3(t; \sigma_1, \sigma_2) = \frac{\zeta^2(3 - \sigma_1 - \sigma_2 - it)\zeta(\sigma_1 + \sigma_2 - 1 + it)}{3 - \sigma_1 - \sigma_2 - it},$$

$$f(t_1, t_2; \sigma_1, \sigma_2) = H(\sigma_1 + it_1, \sigma_2 + it_2, 3 - \sigma_1 - \sigma_2 - i(t_1 + t_2)).$$

From Lemma 2.4 we have

$$f_1(t; b_1) \ll \frac{\log^3(|t| + 2)}{|t| + 2}, f_2(t; b_2) \ll \frac{\log^3(|t| + 2)}{|t| + 2}, f_3(t; b_1, b_2) \ll \frac{\log^3(|t| + 2)}{|t| + 2}.$$

From Proposition 3.1 we have

$$f(t_1, t_2; b_1, b_2) \ll 1.$$

So from Lemma 2.8 with $\alpha_j = 1$, $\beta_j = 3$ ($j = 1, 2, 3$) and $U = T$, we get

$$I_3(x, T) = G(b_1, b_2)x^3 + O(x^3 T^{-1} \log^9 x), \quad (5.15)$$

where

$$G(\sigma_1, \sigma_2) = \frac{1}{(2\pi i)^2} \int_{\sigma_1 - \infty}^{\sigma_1 + \infty} \int_{\sigma_2 - \infty}^{\sigma_2 + \infty} \zeta^2(s_1) \zeta^2(s_2) \zeta^2(3 - s_1 - s_2) \zeta(s_1 + s_2 - 1) \\ \times \zeta(2 - s_2) \zeta(2 - s_1) H(s_1, s_2, 3 - s_1 - s_2) \frac{1}{s_1 s_2 (3 - s_1 - s_2)} ds_2 ds_1.$$

From the condition (3.3) in Proposition 3.1 we see that $H(s_1, s_2, 3 - s_1 - s_2)$ can be written as a Dirichlet series, which is absolutely convergent when s_1 and s_2 satisfy the following conditions:

$$\begin{aligned} 1/3 < \sigma_1 < 2, \quad 1/3 < \sigma_2 < 2, \quad 1 < \sigma_1 + \sigma_2 < 8/3, \\ 2 < 2\sigma_1 + \sigma_2 < 4, \quad 2 < 2\sigma_2 + \sigma_1 < 4, \quad |\sigma_1 - \sigma_2| < 1. \end{aligned} \quad (5.16)$$

From Lemma 2.3 and Lemma 2.4 we see that if $1/2 \leq \sigma_1 \leq b_1$, $1/2 \leq \sigma_2 \leq b_2$, $\sigma_1 + \sigma_2 \geq 3/2$, then

$$\begin{aligned} f_1(t; \sigma_1) &\ll \frac{(|t| + 2)^{\frac{2(1-\sigma_1)}{3}} \log^3(|t| + 2)}{|t| + 2} \ll \frac{\log^3(|t| + 2)}{(|t| + 2)^{\frac{1+2\sigma_1}{3}}}, \\ f_2(t; \sigma_2) &\ll \frac{(|t| + 2)^{\frac{2(1-\sigma_2)}{3}} \log^3(|t| + 2)}{|t| + 2} \ll \frac{\log^3(|t| + 2)}{(|t| + 2)^{\frac{1+2\sigma_2}{3}}}, \\ f_3(t; \sigma_1, \sigma_2) &\ll \frac{(|t| + 2)^{\frac{1-(\sigma_1+\sigma_2-1)}{3}} \log^3(|t| + 2)}{|t| + 2} \ll \frac{\log^3(|t| + 2)}{(|t| + 2)^{\frac{1+\sigma_1+\sigma_2}{3}}}. \end{aligned} \quad (5.17)$$

From (5.16), (5.17) and Lemma 2.8 we see that the integral $G(\sigma_1, \sigma_2)$ is absolutely convergent if $2/3 \leq \sigma_1 \leq b_1$, $2/3 \leq \sigma_2 \leq b_2$, $\sigma_1 + \sigma_2 \geq 3/2$. Certainly we can show that the integral $G(\sigma_1, \sigma_2)$ is absolutely convergent in a much wider range. However, the above range is more than enough for our proof.

Now we show that $G(b_1, b_2)$ is a constant. We write

$$\begin{aligned} G(b_1, b_2) &= \frac{1}{2\pi i} \int_{b_1 - \infty}^{b_1 + \infty} \zeta^2(s_1) \zeta(2 - s_1) \frac{ds_1}{s_1} \\ &\quad \times \frac{1}{2\pi i} \int_{b_2 - \infty}^{b_2 + \infty} \zeta^2(s_2) \zeta^2(3 - s_1 - s_2) \zeta(s_1 + s_2 - 1) \\ &\quad \times \zeta(2 - s_2) H(s_1, s_2, 3 - s_1 - s_2) \frac{ds_2}{s_2(3 - s_1 - s_2)} \\ &= \frac{1}{2\pi i} \int_{b_1 - \infty}^{b_1 + \infty} \zeta^2(s_1) \zeta(2 - s_1) \frac{ds_1}{s_1} \frac{1}{2\pi i} \int_{b_2 - \infty}^{b_2 + \infty} Y(s_1, s_2) ds_2, \end{aligned} \quad (5.18)$$

say. We move the integral line of the inner integral from $\Re s_2 = b_2$ to $\Re s_2 = 3/4$. The function $Y(s_1, s_2)$ has two poles in this range, which are $s_2 = 1$ (degree 3) and $s_2 = 2 - s_1$ (degree 3), respectively. We get

$$G(b_1, b_2) = \frac{1}{2\pi i} \int_{b_1-\infty}^{b_1+\infty} \zeta^2(s_1) \zeta(2-s_1) G_0(s_1) \frac{ds_1}{s_1} + \frac{1}{2\pi i} \int_{b_1-\infty}^{b_1+\infty} \zeta^2(s_1) \zeta(2-s_1) \frac{G_1(s_1) + G_2(s_1)}{s_1} ds_1, \quad (5.19)$$

where

$$G_0(s_1) = \frac{1}{2\pi i} \int_{3/4-\infty}^{3/4+\infty} Y(s_1, s_2) ds_2, \\ G_1(s_1) = \text{Res}_{s_2=1} Y(s_1, s_2), \quad G_2(s_1) = \text{Res}_{s_2=2-s_1} Y(s_1, s_2).$$

Both $G_1(s_1)$ and $G_2(s_1)$ can be computed by Lemmas 2.9 and 2.10, respectively. But the explicit expressions of them are not important here. So we omit the details.

We move both integral lines in (5.19) from $\Re s_1 = b_1$ to $\Re s_1 = 3/4$, we get

$$G(b_1, b_2) = \frac{1}{2\pi i} \int_{3/4-\infty}^{3/4+\infty} \zeta^2(s_1) \zeta(2-s_1) G_0(s_1) \frac{ds_1}{s_1} + \frac{1}{2\pi i} \int_{3/4-\infty}^{3/4+\infty} \zeta^2(s_1) \zeta(2-s_1) \frac{G_1(s_1) + G_2(s_1)}{s_1} ds_1 + c_1 + c_2, \quad (5.20)$$

where

$$c_1 = \text{Res}_{s_1=1} \zeta^2(s_1) \zeta(2-s_1) \frac{G_0(s_1)}{s_1}, \quad c_2 = \text{Res}_{s_1=1} \zeta^2(s_1) \zeta(2-s_1) \frac{G_1(s_1) + G_2(s_1)}{s_1}.$$

From (5.20) we see that $G(b_1, b_2)$ is an absolute constant. We denote it by C . Hence we have

$$I_3(x, T) = Cx^3 + O\left(\frac{x^3 \log^9 x}{T}\right). \quad (5.21)$$

5.2 Evaluation of $I_0(x, T)$

In this subsection, we shall evaluate $I_0(x, T)$. By Lemma 2.9 we get that

$$\text{Res}_{s_3=1} g(s_1, s_2, s_3) = \sum_{j=1}^7 K_j(s_1, s_2), \quad (5.22)$$

where (γ is Euler constant)

$$K_1(s_1, s_2) := 2\gamma \zeta^3(s_1) \zeta^3(s_2) \zeta^2(s_1 + s_2 - 1) H(s_1, s_2, 1) \frac{x^{s_1+s_2+1}}{s_1 s_2}, \\ K_2(s_1, s_2) := \zeta^2(s_1) \zeta'(s_1) \zeta^3(s_2) \zeta^2(s_1 + s_2 - 1) H(s_1, s_2, 1) \frac{x^{s_1+s_2+1}}{s_1 s_2}, \\ K_3(s_1, s_2) := \zeta^3(s_1) \zeta^2(s_2) \zeta'(s_2) \zeta^2(s_1 + s_2 - 1) H(s_1, s_2, 1) \frac{x^{s_1+s_2+1}}{s_1 s_2}, \\ K_4(s_1, s_2) := \zeta^3(s_1) \zeta^3(s_2) \zeta(s_1 + s_2 - 1) \zeta'(s_1 + s_2 - 1) H(s_1, s_2, 1) \frac{x^{s_1+s_2+1}}{s_1 s_2}, \\ K_5(s_1, s_2) := \zeta^3(s_1) \zeta^3(s_2) \zeta^2(s_1 + s_2 - 1) \frac{\partial H(s_1, s_2, s_3)}{\partial s_3} \Big|_{s_3=1} \frac{x^{s_1+s_2+1}}{s_1 s_2}, \\ K_6(s_1, s_2) := -\zeta^3(s_1) \zeta^3(s_2) \zeta^2(s_1 + s_2 - 1) H(s_1, s_2, 1) \frac{x^{s_1+s_2+1}}{s_1 s_2},$$

$$K_7(s_1, s_2) := \zeta^3(s_1)\zeta^3(s_2)\zeta^2(s_1 + s_2 - 1)H(s_1, s_2, 1)\frac{x^{s_1+s_2+1}}{s_1s_2}\log x.$$

So we have

$$I_0(x, T) = \sum_{j=1}^7 \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} \int_{b_2-iT_2}^{b_2+iT_2} K_j(s_1, s_2) ds_2 ds_1. \quad (5.23)$$

5.2.1 Integration over s_2

We consider the domain formed by the four points $s_2 = b_2 \pm iT_2$, $s_2 = 11/20 \pm iT_2$. It is easy to see that for each $1 \leq j \leq 7$, the function $K_j(s_1, s_2)$ has two poles, which are $s_2^{(1)} = 1$ and $s_2^{(2)} = 2 - s_1$, respectively. By the residue theorem, we have

$$I_0(x, T) = \sum_{j=1}^7 \sum_{\ell=1}^2 I_0^{(j,\ell)}(x, T) + \sum_{j=1}^7 (H_j^{(1)}(x, T) + H_j^{(2)}(x, T) - H_j^{(3)}(x, T)), \quad (5.24)$$

where ($1 \leq j \leq 7$)

$$\begin{aligned} I_0^{(j,1)}(x, T) &:= \frac{1}{2\pi i} \int_{b_1-iT_1}^{b_1+iT_1} \operatorname{Res}_{s_2=1} K_j(s_1, s_2) ds_1, \\ I_0^{(j,2)}(x, T) &:= \frac{1}{2\pi i} \int_{b_1-iT_1}^{b_1+iT_1} \operatorname{Res}_{s_2=2-s_1} K_j(s_1, s_2) ds_1, \\ H_j^{(1)}(x, T) &:= \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} \int_{11/20-iT_2}^{b_2+iT_2} K_j(s_1, s_2) ds_2 ds_1, \\ H_j^{(2)}(x, T) &:= \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} \int_{11/20-iT_2}^{11/20+iT_2} K_j(s_1, s_2) ds_2 ds_1, \\ H_j^{(3)}(x, T) &:= \frac{1}{(2\pi i)^2} \int_{b_1-iT_1}^{b_1+iT_1} \int_{11/20-iT_2}^{b_2-iT_2} K_j(s_1, s_2) ds_2 ds_1. \end{aligned}$$

Similar to the estimate of $H_1(x, T)$ we have

$$H_j^{(1)}(x, T) \ll \frac{x^3 \log^{10} x}{T}, \quad H_j^{(3)}(x, T) \ll \frac{x^3 \log^{10} x}{T}, \quad (j = 1, \dots, 7). \quad (5.25)$$

Similar to the estimate of $H_2(x, T)$ we have

$$H_j^{(2)}(x, T) \ll x^{51/20+\varepsilon}, \quad (j = 1, \dots, 7). \quad (5.26)$$

5.2.2 Evaluations of $I_0^{(j,1)}(x, T)$

Now we evaluate $I_0^{(j,\ell)}(x, T)$ ($1 \leq j \leq 7, \ell = 1, 2$). We consider $j = 7$. The proof of other cases are the same.

First consider the pole $s_2 = 1$, which is of degree 3 in $K_7(s_1, s_2)$. Define

$$\begin{aligned} A &:= \zeta^3(s_1) \frac{x^{s_1+1}}{s_1} \log x, \quad M_1(s_2) := \zeta^3(s_2), \\ M_2(s_2) &:= \zeta^2(s_1 + s_2 - 1)H^*(s_1, s_2)x^{s_2}, \quad H^*(s_1, s_2) = H(s_1, s_2, 1)s_2^{-1}. \end{aligned} \quad (5.27)$$

Write

$$M_1(s_2) = \sum_{m=-3}^{\infty} a_m (s_2 - 1)^m, \quad a_{-3} = 1.$$

So by Lemmas 2.9 and 2.10 we get

$$\begin{aligned} \operatorname{Res}_{s_2=1} K_7(s_1, s_2) &= A \sum_{m=0}^2 \frac{a_{-m-1}}{m!} M_2^{(m)}(1) \\ &= A \sum_{m=0}^2 \frac{a_{-m-1}}{m!} \sum_{\substack{n_1+n_2+n_3=m \\ n_1, n_2, n_3 \geq 0}} (\zeta^2(s_1 + s_2 - 1))^{(n_1)} \frac{\partial^{n_2} H^*(s_1, s_2)}{\partial s_2^{n_2}} (x^{s_2})^{(n_3)}|_{s_2=1}. \end{aligned} \quad (5.28)$$

We have

$$\begin{aligned} (\zeta^2(s_1 + s_2 - 1))^{(0)}|_{s_2=1} &= \zeta^2(s_1), \\ (\zeta^2(s_1 + s_2 - 1))^{(1)}|_{s_2=1} &= 2\zeta(s_1)\zeta'(s_1), \\ (\zeta^2(s_1 + s_2 - 1))^{(2)}|_{s_2=1} &= 2\zeta(s_1)\zeta''(s_1) + 2(\zeta'(s_1))^2. \end{aligned} \quad (5.29)$$

Suppose s_1, s_2, s_3 satisfy the condition (3.3). Then we can write

$$H(s_1, s_2, s_3) = \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3)}{n_1^{s_1} n_2^{s_2} n_3^{s_3}},$$

which is absolutely convergent for the variables s_1, s_2, s_3 . So for any $(\ell_1, \ell_2, \ell_3) \in \mathbb{N}_0^3$, we have

$$\frac{\partial^{\ell_1+\ell_2+\ell_3} H(s_1, s_2, s_3)}{\partial s_1^{\ell_1} \partial s_2^{\ell_2} \partial s_3^{\ell_3}} = (-1)^{\ell_1+\ell_2+\ell_3} \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3) \prod_{j=1}^3 (\log n_j)^{\ell_j}}{n_1^{s_1} n_2^{s_2} n_3^{s_3}}. \quad (5.30)$$

Define

$$\begin{aligned} h_0(s_1) &:= H^*(s_1, 1) = H(s_1, 1, 1), \\ h_1(s_1) &:= \frac{\partial H^*(s_1, s_2)}{\partial s_2}|_{s_2=1}, \quad h_2(s_1) := \frac{\partial^2 H^*(s_1, s_2)}{\partial s_2^2}|_{s_2=1}. \end{aligned} \quad (5.31)$$

From (5.30) it is easy to see that

$$\begin{aligned} h_1(s_1) &= - \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3)(\log n_2 - 1)}{n_1^{s_1} n_2 n_3}, \\ h_2(s_1) &= \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3)(\log^2 n_2 + 2 \log n_2 + 2)}{n_1^{s_1} n_2 n_3}. \end{aligned}$$

From the condition (3.3) it is easy to see that $h_0(s_1), h_1(s_1)$ and $h_2(s_1)$ are all analytic when $\Re s_1 > 1/2$.

From (5.27) to (5.31) it is easy to see that

$$\operatorname{Res}_{s_2=1} K_7(s_1, s_2) = S_1(s_1) + S_2(s_1) + S_3(s_1) + S_4(s_1), \quad (5.32)$$

where

$$\begin{aligned} S_1(s_1) &:= \zeta^5(s_1) V_1(s_1, x), \\ S_2(s_1) &:= \zeta^4(s_1) \zeta'(s_1) V_2(s_1, x), \\ S_3(s_1) &:= \zeta^4(s_1) \zeta''(s_1) \frac{x^{2+s_1} \log x}{s_1} h_0(s_1), \\ S_4(s_1) &:= \zeta^3(s_1) (\zeta'(s_1))^2 \frac{x^{2+s_1} \log x}{s_1} h_0(s_1), \end{aligned} \quad (5.33)$$

$$\begin{aligned}
V_1(s_1, x) &= \frac{x^{2+s_1} \log x}{s_1} \left(a_{-1}h_0(s_1) + a_{-2}h_1(s_1) + \frac{1}{2}h_2(s_1) \right) \\
&\quad + \frac{x^{2+s_1} \log x}{s_1} \left(\frac{1}{2}h_0(s_1) \log^2 x + \frac{1}{2}h_1(s_1) \log x + a_{-2}h_0(s_1) \log x \right), \\
V_2(s_1, x) &= \frac{x^{2+s_1} \log x}{s_1} (2a_{-2}h_0(s_1) + h_1(s_1) + h_0(s_1) \log x).
\end{aligned}$$

Hence we have

$$I_0^{(7,1)}(x, T) = \sum_{j=1}^4 \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} S_j(s_1) ds_1. \quad (5.34)$$

We consider the domain formed by the four points $s_1 = b_1 \pm iT$, $s_1 = 11/20 \pm iT$. By the residue theorem, we get

$$I_0^{(7,1)}(x, T) = \sum_{j=1}^4 \operatorname{Res}_{s_1=1} S_j(s_1) + \sum_{j=1}^4 (L_{1,j}(x, T) + L_{2,j}(x, T) - L_{3,j}(x, T)), \quad (5.35)$$

where

$$\begin{aligned}
L_{1,j}(x, T) &= \frac{1}{2\pi i} \int_{11/20+iT}^{b_1+iT} S_j(s_1) ds_1, \\
L_{2,j}(x, T) &= \frac{1}{2\pi i} \int_{11/20-iT}^{11/20+iT} S_j(s_1) ds_1, \\
L_{3,j}(x, T) &= \frac{1}{2\pi i} \int_{11/20-iT}^{b_1-iT} S_j(s_1) ds_1.
\end{aligned}$$

Similar to (5.5), by Lemmas 2.3 and 2.4 we have for $1 \leq j \leq 4$ that

$$\begin{aligned}
L_{1,j}(x, T) &\ll \frac{x^3 \log^{10} x}{T} + x^{\frac{51}{20}} T^{-\frac{1}{4}} \log^{12} x, \\
L_{3,j}(x, T) &\ll \frac{x^3 \log^{10} x}{T} + x^{\frac{51}{20}} T^{-\frac{1}{4}} \log^{12} x.
\end{aligned} \quad (5.36)$$

Similar to the estimate of $H_2(x, T)$, by Lemma 2.7 we have

$$L_{2,j}(x, T) \ll x^{51/20+\varepsilon}, \quad (j = 1, 2, 3, 4). \quad (5.37)$$

By (5.33) and Lemma 2.9 we get

$$\operatorname{Res}_{s_1=1} S_j(s_1) = x^3 Q_j(\log x), \quad (j = 1, 2, 3, 4), \quad (5.38)$$

where $Q_j(u)$ ($j = 1, 2, 3, 4$) are polynomials in u of degree 7.

From (5.35) to (5.38), we get

$$I_0^{(7,1)}(x, T) = x^3 Q_{71}(\log x) + O\left(x^{51/20+\varepsilon} + \frac{x^3 \log^{10} x}{T}\right), \quad (5.39)$$

where $Q_{71}(u)$ is a polynomial in u of degree 7. Similarly for each $1 \leq j \leq 6$, we have

$$I_0^{(j,1)}(x, T) = x^3 Q_{j1}(\log x) + O\left(x^{51/20+\varepsilon} + \frac{x^3 \log^{10} x}{T}\right), \quad (5.40)$$

where $Q_{j1}(u)$ is a polynomial in u of degree 7.

Now we consider the pole $s_2 = 2 - s_1$, which is of degree 2 in $K_7(s_1, s_2)$. Write

$$M_1^*(s_2) = \zeta^2(s_1 + s_2 - 1), \quad M_2^*(s_2) = \zeta^3(s_2)H(s_1, s_2, 1)\frac{x^{s_2}}{s_2}, \quad (5.41)$$

where

$$M_1^*(s_2) = \frac{1}{(s_2 - (2 - s_1))^2} + \sum_{n=-1}^{\infty} b_n (s_2 - (2 - s_1))^n,$$

By Lemma 2.9 we have

$$\text{Res}_{s_2=2-s_1} K_7(s_1, s_2) = Ab_{-1}M_2^*(2 - s_1) + AM_2^{*'}(2 - s_1) = \sum_{j=5}^9 S_j(s_1), \quad (5.42)$$

where $A = \zeta^3(s_1)\frac{x^{s_1+1}}{s_1} \log x$ and

$$\begin{aligned} S_5(s_1) &:= b_{-1} \frac{\zeta^3(s_1)\zeta^3(2-s_1)}{s_1(2-s_1)} h_3(s_1) \times x^3 \log x, \\ S_6(s_1) &:= 3 \frac{\zeta^3(s_1)\zeta^2(2-s_1)\zeta'(2-s_1)}{s_1(2-s_1)} h_3(s_1) \times x^3 \log x, \\ S_7(s_1) &:= \frac{\zeta^3(s_1)\zeta^3(2-s_1)}{s_1(2-s_1)} h_4(s_1) \times x^3 \log x, \\ S_8(s_1) &:= -\frac{\zeta^3(s_1)\zeta^3(2-s_1)}{s_1(2-s_1)^2} h_3(s_1) \times x^3 \log x, \\ S_9(s_1) &:= \frac{\zeta^3(s_1)\zeta^3(2-s_1)}{s_1(2-s_1)} h_3(s_1) \times x^3 \log^2 x, \\ h_3(s_1) &:= H(s_1, 2 - s_1, 1) = \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3)}{n_1^{s_1} n_2^{2-s_1} n_3}, \\ h_4(s_1) &:= -\sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3) \log n_2}{n_1^{s_1} n_2^{2-s_1} n_3}. \end{aligned} \quad (5.43)$$

By the condition (3.3) it is easy to see that both $h_3(s_1)$ and $h_4(s_1)$ are analytic when $1/2 < \Re s_1 < 3/2$. Similar to (5.21), we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} S_j(s_1) ds_1 &= C_j x^3 \log x + O\left(\frac{x^3 \log^5 x}{T}\right), \quad (j = 5, 6, 7, 8), \\ \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} S_9(s_1) ds_1 &= C_9 x^3 \log^2 x + O\left(\frac{x^3 \log^6 x}{T}\right). \end{aligned} \quad (5.44)$$

where C_j ($j = 5, 6, 7, 8, 9$) are absolute constants.

Thus we get

$$I_0^{(7,2)}(x, T) = x^3 Q_{72}(\log x) + O\left(\frac{x^3 \log^6 x}{T}\right), \quad (5.45)$$

where $Q_{72}(u)$ is a polynomial in u of degree 2. Similarly for each $1 \leq j \leq 6$, we have

$$I_0^{(j,2)}(x, T) = x^3 Q_{j2}(\log x) + O\left(\frac{x^3 \log^6 x}{T}\right), \quad (5.46)$$

where $Q_{j2}(u)$ is a polynomial in u of degree not exceeding 2.

Combining (5.24)–(5.26), (5.39), (5.40), (5.45) and (5.46) we get

$$I_0(x, T) = x^3 P_0(\log x) + O\left(x^{51/20+\varepsilon} + \frac{x^3 \log^{10} x}{T}\right), \quad (5.47)$$

where $P_0(u)$ is a polynomial in u of degree 7.

5.3 Evaluation of $I_1(x, T)$

In this subsection we evaluate $I_1(x, T)$. Since $s_3^{(1)} = 2 - s_1$ is a simple pole of $g(s_1, s_2, s_3)$, we have

$$\begin{aligned} \operatorname{Res}_{s_3=2-s_1} g(s_1, s_2, s_3) &= \zeta^2(s_1) \zeta^3(s_2) \zeta^2(2-s_1) \zeta(s_1+s_2-1) \\ &\quad \times \zeta(s_2-s_1+1) H(s_1, s_2, 2-s_1) \frac{x^{s_2+2}}{s_1 s_2 (2-s_1)} \\ &=: g_2(s_1, s_2), \end{aligned} \quad (5.48)$$

say. Consider the domain formed by the four points $s_2 = b_2 \pm iT_2$, $s_2 = 11/20 \pm iT_2$. The function $g_2(s_1, s_2)$ has three poles in this domain, which are $s_2^{(1)} = 1$ (of degree 3), $s_2^{(2)} = s_1$ (simple) and $s_2^{(3)} = 2 - s_1$ (simple), respectively. By the residue theorem, we get

$$I_1(x, T) = \sum_{j=1}^3 I_{1j}(x, T) + O\left(x^{51/20+\varepsilon} + \frac{x^3 \log^{10} x}{T}\right), \quad (5.49)$$

where

$$I_{1j}(x, T) = \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} \operatorname{Res}_{s_2=s_2^{(j)}} g(s_1, s_2) ds_1, \quad (j = 1, 2, 3). \quad (5.50)$$

We omit the details of the proof of (5.49), since it is similar to Sect. 5.1.

5.3.1 Evaluation of $I_{11}(x, T)$

Since $s_2 = 1$ is a pole (of order 3) of $g_2(s_1, s_2)$, by Lemmas 2.9 and 2.10 we have

$$\begin{aligned} \operatorname{Res}_{s_2=1} g_2(s_1, s_2) &= \sum_{m=0}^2 \frac{a_{-m-1}}{m!} \sum_{\substack{n_1+n_2+n_3+n_4=m \\ n_j \geq 0}} x^3 \log^{n_4} x \\ &\quad \times \frac{\zeta^2(s_1) \zeta^2(2-s_1) \zeta^{(n_1)}(s_1) \zeta^{(n_2)}(2-s_1)}{s_1(2-s_1)} h_{n_3}(s_1), \end{aligned} \quad (5.51)$$

where

$$h_{n_3}(s_1) = \frac{\partial^{n_3} H^{**}(s_1, s_2)}{\partial s_2^{n_3}} \Big|_{s_2=1}, \quad H^{**}(s_1, s_2) := H(s_1, s_2, 2-s_1) s_2^{-1}.$$

From the condition (3.3) of Proposition 3.1 we see easily that $H(s_1, 1, 2-s_1)$ is analytic when $1/2 < \Re s_1 < 3/2$. Hence we have

$$h_{n_3}(s_1) \ll_{\varepsilon} 1, \quad (1/2 + \varepsilon < \Re s_1 < 3/2 - \varepsilon).$$

Thus similar to (5.21) we can get

$$I_{11}(x, T) = x^3 P_{11}(\log x) + O\left(\frac{x^3 \log^{10} x}{T}\right), \quad (5.52)$$

where $P_{11}(u)$ is a polynomial in u of degree 2.

5.3.2 Evaluation of $I_{12}(x, T)$

Since $s_2 = s_1$ is a simple pole of $g_2(s_1, s_2)$, we have

$$\text{Res}_{s_2=s_1} g_2(s_1, s_2) = \zeta^5(s_1) \zeta^2(2-s_1) \zeta(2s_1-1) H(s_1, s_1, 2-s_1) \frac{x^{s_1+2}}{s_1^2(2-s_1)}, \quad (5.53)$$

which implies that

$$\begin{aligned} I_{12}(x, T) &= \frac{1}{2\pi i} \int_{b_1-i\infty}^{b_1+i\infty} \zeta^5(s_1) \zeta^2(2-s_1) \zeta(2s_1-1) H(s_1, s_1, 2-s_1) \frac{x^{s_1+2}}{s_1^2(2-s_1)} ds_1 \\ &\quad + O\left(\frac{x^3 \log^8 x}{T^2}\right). \end{aligned} \quad (5.54)$$

By the condition (3.3) we see easily that $H(s_1, s_1, 2-s_1)$ is analytic if $2/3 < \Re s_1 < 5/3$. Hence we have $H(s_1, s_1, 2-s_1) \ll_\varepsilon 1$ if $2/3 + \varepsilon < \Re s_1 < 5/3 - \varepsilon$. The function $\zeta^5(s_1) \zeta^2(2-s_1) \zeta(2s_1-1) H(s_1, s_1, 2-s_1) \frac{x^{s_1+2}}{s_1^2(2-s_1)}$ has a pole $s_1 = 1$ in the region $2/3 + \varepsilon < \Re s_1 \leq b_1$, which is of order 8. So moving the integration line in (5.54) from $\Re s_1 = b_1$ to $\Re s_1 = 2/3 + \varepsilon$, we get

$$I_{12}(x, T) = x^3 P_{12}(\log x) + O\left(x^{\frac{8}{3}+\varepsilon} + \frac{x^3 \log^8 x}{T^2}\right), \quad (5.55)$$

where $P_{12}(u)$ is a polynomial in u of degree 7, and where we used the estimate

$$\int_{2/3+\varepsilon-i\infty}^{2/3+\varepsilon+i\infty} \left| \zeta^5(s_1) \zeta^2(2-s_1) \zeta(2s_1-1) H(s_1, s_1, 2-s_1) \frac{1}{s_1^2(2-s_1)} \right| dt_1 \ll 1.$$

The above estimate follows easily from Lemma 2.4.

5.3.3 Evaluation of $I_{13}(x, T)$

Now we consider $I_{13}(x, T)$. Since $s_2 = 2-s_1$ is a simple pole of $g_2(s_1, s_2)$, we have

$$\text{Res}_{s_2=2-s_1} g_2(s_1, s_2) = \zeta^2(s_1) \zeta^5(2-s_1) \zeta(3-2s_1) h_5(s_1) \frac{x^{4-s_1}}{s_1(2-s_1)^2}, \quad (5.56)$$

where $h_5(s_1) := H(s_1, 2-s_1, 2-s_1)$. So we get

$$\begin{aligned} I_{13}(x, T) &= \frac{1}{2\pi i} \int_{b_1-i\infty}^{b_1+i\infty} \zeta^2(s_1) \zeta^5(2-s_1) \zeta(3-2s_1) h_5(s_1) \frac{x^{4-s_1}}{s_1(2-s_1)^2} ds_1 \\ &\quad + O\left(\frac{x^3 \log^8 x}{T}\right). \end{aligned} \quad (5.57)$$

By the condition (3.3) of Proposition 3.1 it is easy to see that $h_5(s_1)$ is analytic in the region $1/3 < \Re s_1 < 4/3$. So we have $h_5(s_1) \ll_\varepsilon 1$ when $1 \leq \Re s_1 \leq 4/3 - \varepsilon$. Also, the integral function in (5.57) has no poles in the range $1 < \Re s_1 \leq 4/3 - \varepsilon$. Moving the integral

line in (5.57) from $\Re s_1 = b_1$ to $\Re s_1 = 4/3 - \varepsilon$, by the residue theorem and Lemma 2.4 we get

$$\begin{aligned} I_{13}(x, T) &= O\left(\frac{x^3 \log^6 x}{T}\right) \\ &+ \frac{1}{2\pi i} \int_{4/3-\varepsilon-i\infty}^{4/3-\varepsilon+i\infty} \zeta^2(s_1) \zeta^5(2-s_1) \zeta(3-2s_1) h_5(s_1) \frac{x^{4-s_1}}{s_1(2-s_1)^2} ds_1 \\ &\ll \frac{x^3 \log^8 x}{T} + x^{\frac{8}{3}+\varepsilon} \left| \int_{4/3-\varepsilon-i\infty}^{4/3-\varepsilon+i\infty} \frac{\zeta^2(s_1) \zeta^5(2-s_1) \zeta(3-2s_1)}{s_1(2-s_1)^2} ds_1 \right| \\ &\ll \frac{x^3 \log^8 x}{T} + x^{\frac{8}{3}+\varepsilon}. \end{aligned} \quad (5.58)$$

From (5.49), (5.52), (5.55), (5.58) we get

$$I_1(x, T) = x^3 P_1(\log x) + O\left(\frac{x^3 \log^8 x}{T} + x^{\frac{8}{3}+\varepsilon}\right), \quad (5.59)$$

where $P_1(u)$ is a polynomial in u of degree 7.

5.4 Evaluation of $I_2(x, T)$

In this subsection we shall evaluate $I_2(x, T)$. Since $s_3^{(2)} = 2 - s_2$ is a simple pole of $g(s_1, s_2, s_3)$, we have

$$\begin{aligned} \operatorname{Res}_{s_3=2-s_2} g(s_1, s_2, s_3) &= \zeta^3(s_1) \zeta^2(s_2) \zeta^2(2-s_2) \zeta(s_1+s_2-1) \\ &\quad \times \zeta(1+s_1-s_2) H(s_1, s_2, 2-s_2) \frac{x^{s_1+2}}{s_1 s_2 (2-s_2)} \\ &=: \frac{\zeta^3(s_1) x^{s_1+2}}{s_1} \times g_3(s_1, s_2), \end{aligned} \quad (5.60)$$

where

$$g_3(s_1, s_2) := \zeta^2(s_2) \zeta^2(2-s_2) \zeta(s_1+s_2-1) \zeta(1+s_1-s_2) \frac{H(s_1, s_2, 2-s_2)}{s_2(2-s_2)}.$$

5.4.1 Integration over s_2

By the condition (3.3) of Proposition 3.1 it is easy to see that $H(s_1, s_2, 2-s_2)$ is analytic when σ_1 and σ_2 satisfy

$$\sigma_1 > 1/2, \quad \max(1/3, 2-2\sigma_1, 1-\sigma_1/2) < \sigma_2 < \min(5/3, 2\sigma_1, 1+\sigma_1/2). \quad (5.61)$$

We write

$$I_2(x, T) = \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} \frac{\zeta^3(s_1) x^{s_1+2}}{s_1} \left(\frac{1}{2\pi i} \int_{b_2-iT_2}^{b_2+iT_2} g_3(s_1, s_2) ds_2 \right) ds_1.$$

From (5.60) and (5.61) it is easy to check that

$$I_2(x, T) = \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} \frac{\zeta^3(s_1) x^{s_1+2}}{s_1} \left(\frac{1}{2\pi i} \int_{b_2-i\infty}^{b_2+i\infty} g_3(s_1, s_2) ds_2 \right) ds_1 + O\left(\frac{x^3 \log^9 x}{T}\right). \quad (5.62)$$

We move the integral line of s_2 from $\Re s_2 = b_2$ to $\Re s_2 = 1 - \varepsilon$. In the range $1 - \varepsilon \leq \Re s_2 \leq b_2$, the function $g_3(s_1, s_2)$ has three poles, which are $s_2 = 1$ (of order 4), $s_2 = s_1$ (simple) and $s_2 = 2 - s_1$ (simple), respectively. By the residue theorem we have

$$\frac{1}{2\pi i} \int_{b_2-i\infty}^{b_2+i\infty} g_3(s_1, s_2) ds_2 = \sum_{j=3}^6 G_j(s_1), \quad (5.63)$$

where

$$\begin{aligned} G_3(s_1) &= \operatorname{Res}_{s_2=s_1} g_3(s_1, s_2), \quad G_4(s_1) = \operatorname{Res}_{s_2=2-s_1} g_3(s_1, s_2), \\ G_5(s_1) &= \operatorname{Res}_{s_2=1} g_3(s_1, s_2), \quad G_6(s_1) = \frac{1}{2\pi i} \int_{1-\varepsilon-i\infty}^{1-\varepsilon+i\infty} g_3(s_1, s_2) ds_2. \end{aligned} \quad (5.64)$$

From (5.62)–(5.64) we have

$$I_2(x, T) = \sum_{j=3}^6 I_{2j}(x, T) + O\left(\frac{x^3 \log^9 x}{T}\right), \quad (5.65)$$

where

$$I_{2j}(x, T) := \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} \frac{\zeta^3(s_1) x^{s_1+2}}{s_1} G_j(s_1) ds_1 \quad (j = 3, 4, 5, 6). \quad (5.66)$$

5.4.2 Integration over s_1

We first consider $I_{23}(x, T)$. Since $s_2 = s_1$ is a simple pole of $g_3(s_1, s_2)$, we have

$$G_3(s_1) = -\zeta^2(s_1) \zeta^2(2-s_1) \zeta(2s_1-1) \frac{H(s_1, s_1, 2-s_1)}{s_1(2-s_1)}. \quad (5.67)$$

From (3.3) we see that $H(s_1, s_1, 2-s_1)$ can be written as a Dirichlet series, which is absolutely convergent for $2/3 < \Re s_1 < 5/3$. Hence we have $H(s_1, s_1, 2-s_1) \ll_\varepsilon 1$ if $2/3 + \varepsilon < \Re s_1 < 5/3 - \varepsilon$. Note that $s_1 = 1$ is the pole of $\frac{\zeta^3(s_1) x^{s_1+2}}{s_1} G_3(s_1)$ of order 8. By the residue theorem, Lemmas 2.3, 2.4 and 2.9 we have

$$\begin{aligned} I_{23}(x, T) &= \frac{1}{2\pi i} \int_{b_1-i\infty}^{b_1+i\infty} \frac{\zeta^3(s_1) x^{s_1+2}}{s_1} G_3(s_1) ds_1 + O\left(\frac{x^3 \log^8 x}{T^2}\right) \\ &= x^3 P_{23}(\log x) + \frac{1}{2\pi i} \int_{b_1-i\infty}^{b_1+i\infty} \frac{\zeta^3(s_1) x^{s_1+2}}{s_1} G_3(s_1) ds_1 + O\left(\frac{x^3 \log^8 x}{T^2}\right) \\ &= x^3 P_{23}(\log x) + O\left(x^{\frac{8}{3}+\varepsilon} + \frac{x^3 \log^8 x}{T^2}\right), \end{aligned} \quad (5.68)$$

where $P_{23}(u)$ is a polynomial in u of degree 7. Similarly we have

$$I_{24}(x, T) = x^3 P_{24}(\log x) + O\left(x^{\frac{8}{3}+\varepsilon} + \frac{x^3 \log^8 x}{T^2}\right) \quad (5.69)$$

if noting that $G_4(s_1) = \zeta^2(s_1) \zeta^2(2-s_1) \zeta(2s_1-1) \frac{H(s_1, 2-s_1, s_1)}{s_1(2-s_1)}$, where $P_{24}(u)$ is also a polynomial in u of degree 7.

Now we consider $I_{25}(x, T)$. We compute $G_5(s_1)$ first. Let

$$\begin{aligned} \mathbf{M}_1(s_2) &:= \zeta^2(s_2) \zeta^2(2-s_2), \quad \mathbf{H}_{s_1}(s_2) := \frac{H(s_1, s_2, 2-s_2)}{s_2(2-s_2)}, \\ \mathbf{M}_2(s_2) &:= \zeta(s_1+s_2-1) \zeta(1+s_1-s_2) \mathbf{H}_{s_1}(s_2). \end{aligned}$$

Since $s_2 = 1$ is the pole of $M_1(s_2)$ with order 4, we can write

$$M_1(s_2) = \sum_{m=-4}^{\infty} c_m (s_2 - 1)^m, \quad c_{-4} = 1.$$

By Lemmas 2.9 and 2.10 we get

$$\begin{aligned} G_5(s_1) &= \sum_{m=0}^3 \frac{c_{-m-1}}{m!} M_2^{(m)}(1) \\ &= \sum_{m=0}^3 \frac{c_{-m-1}}{m!} \sum_{\substack{n_1+n_2+n_3=m \\ n_1, n_2, n_3 \geq 0}} (-1)^{n_2} \zeta^{(n_1)}(s_1) \zeta^{(n_2)}(s_1) H_{s_1}^{(n_3)}(1). \end{aligned}$$

So we have

$$\begin{aligned} I_{25}(x, T) &= \sum_{m=0}^3 \frac{c_{-m-1}}{m!} \sum_{\substack{n_1+n_2+n_3=m \\ n_1, n_2, n_3 \geq 0}} (-1)^{n_2} \\ &\quad \times \frac{1}{2\pi i} \int_{b_1-iT}^{b_1+iT} \frac{\zeta^3(s_1) x^{s_1+2} \zeta^{(n_1)}(s_1) \zeta^{(n_2)}(s_1)}{s_1} H_{s_1}^{(n_3)}(1) ds_1. \end{aligned} \quad (5.70)$$

Suppose $\Re s_1 = b_1$, $|\sigma_2 - 1| < 1/1000$. Then $s_1, s_2, 2 - s_2$ satisfy the condition (3.3). Thus

$$H(s_1, s_2, 2 - s_2) = \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3)}{n_1^{s_1} n_2^{s_2} n_3^{2-s_2}} = \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3)}{n_1^{s_1} n_3^2} \left(\frac{n_3}{n_2}\right)^{s_2}.$$

Hence for any $\ell \geq 0$ we have

$$\frac{\partial^\ell H(s_1, s_2, 2 - s_2)}{\partial s_2^\ell} = \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3)}{n_1^{s_1} n_3^2} \left(\frac{n_3}{n_2}\right)^{s_2} \log^\ell \frac{n_3}{n_2},$$

which implies that

$$\frac{\partial^\ell H(s_1, s_2, 2 - s_2)}{\partial s_2^\ell} \Big|_{s_2=1} = \sum_{n_1, n_2, n_3=1}^{\infty} \frac{h(n_1, n_2, n_3) (\log n_3 - \log n_2)^\ell}{n_1^{s_1} n_2 n_3}.$$

The condition (3.3) implies that the above infinite series is absolutely convergent for $\sigma_1 > 1/2$. Thus we get that for any fixed $n_3 \geq 0$

$$H_{s_1}^{(n_3)}(1) \ll_\varepsilon 1, \quad \sigma_1 \geq 1/2 + \varepsilon. \quad (5.71)$$

The integral function $\frac{\zeta^3(s_1) x^{s_1+2} \zeta^{(n_1)}(s_1) \zeta^{(n_2)}(s_1)}{s_1} H_{s_1}^{(n_3)}(1)$ has a pole $s_1 = 1$ of order $5 + n_1 + n_2$. It is easy to see the highest order is 8, which corresponds to $n_1 + n_2 = 3$. Consider the domain formed by the four points $s_1 = b_1 \pm iT$, $11/20 \pm iT$. So similar to (5.39), by the residue theorem, Lemmas 2.3, 2.4, 2.7 and 2.9 we get

$$I_{25}(x, T) = x^3 P_{25}(\log x) + O\left(x^{\frac{51}{20}+\varepsilon} + \frac{x^3 \log^8 x}{T}\right), \quad (5.72)$$

where $P_{25}(u)$ is a polynomial in u of degree 7.

Finally we consider $I_{26}(x, T)$. From the condition (3.3) we see that if $\Re s_2 = 1 - \varepsilon$, then the function $H_{s_1}(s_2)$ is analytic for $\Re s_1 > 1/2 + \varepsilon/2$ and hence is bounded if $\Re s_1 > 1/2 + \varepsilon$. Later we always suppose $\Re s_1 > 1/2 + \varepsilon$. We need two estimates of $G_6(s_1)$.

Since $\Re s_2 = 1 - \varepsilon$, we have $\zeta^2(2 - s_2) \ll 1$ and $H(s_1, s_2, 2 - s_2) \ll 1$. By Lemma 2.4 we have

$$\begin{aligned}
G_6(s_1) &\ll \int_{1-\varepsilon-i\infty}^{1-\varepsilon+i\infty} \left| \frac{\zeta^2(s_2)\zeta(s_1+s_2-1)\zeta(1+s_1-s_2)}{s_2(2-s_2)} \right| dt_2 \\
&\ll \int_{-\infty}^{\infty} \left| \frac{(|t_2|+1)^{\varepsilon}(|t_1+t_2|+1)^{\frac{1-\sigma_1}{3}+\varepsilon}(|t_1-t_2|+1)^{\frac{1-\sigma_1}{3}+\varepsilon}}{(|t_2|+1)^2} \right| dt_2 \\
&\ll (|t_1|+1)^{\frac{2(1-\sigma_1)}{3}+\varepsilon}.
\end{aligned} \tag{5.73}$$

The second estimate involving $G_6(s_1)$ is the following:

$$\int_1^U |G_6(11/20 + it_1)|^{\frac{5}{2}} dt_1 \ll U^{1+\varepsilon}. \tag{5.74}$$

In order to prove (5.74) suppose $s_1 = 11/20 + it_1$ and $s_2 = 1 - \varepsilon + it_2$. Then $s_1 + s_2 - 1 = 11/20 - \varepsilon + i(t_1 + t_2)$ and $1 + s_1 - s_2 = 11/20 + \varepsilon + i(t_1 - t_2)$. By Hölder's inequality we get

$$\begin{aligned}
G_6(s_1) &\ll \int_{-\infty}^{\infty} \frac{|\zeta^2(s_2)\zeta(s_1+s_2-1)\zeta(1+s_1-s_2)|}{(|t_2|+1)^{\frac{2}{5}+\frac{4}{5}+\frac{4}{5}}} dt_2 \\
&\ll \left(\int_{-\infty}^{\infty} \frac{|\zeta(1-\varepsilon+it_2)|^{10}}{(|t_2|+1)^2} dt_2 \right)^{\frac{1}{5}} \times \left(\int_{-\infty}^{\infty} \frac{|\zeta(11/20-\varepsilon+i(t_1+t_2))|^{\frac{5}{2}}}{(|t_2|+1)^2} dt_2 \right)^{\frac{2}{5}} \\
&\quad \times \left(\int_{-\infty}^{\infty} \frac{|\zeta(11/20+\varepsilon+i(t_1-t_2))|^{\frac{5}{2}}}{(|t_2|+1)^2} dt_2 \right)^{\frac{2}{5}} \\
&\ll K_1^{\frac{2}{5}}(t_1) K_2^{\frac{2}{5}}(t_1)
\end{aligned}$$

if noting that (by Lemma 2.5)

$$\int_{-\infty}^{\infty} \frac{|\zeta(1-\varepsilon+it_2)|^{10}}{(|t_2|+1)^2} dt_2 \ll 1,$$

where

$$\begin{aligned}
K_1(t_1) &:= \int_{-\infty}^{\infty} \frac{|\zeta(11/20-\varepsilon+i(t_1+t_2))|^{\frac{5}{2}}}{(|t_2|+1)^2} dt_2, \\
K_2(t_1) &:= \int_{-\infty}^{\infty} \frac{|\zeta(11/20+\varepsilon+i(t_1-t_2))|^{\frac{5}{2}}}{(|t_2|+1)^2} dt_2.
\end{aligned}$$

Hence we have by Cauchy's inequality that

$$\begin{aligned}
\int_1^U |G_6(11/20 + it_1)|^{\frac{5}{2}} dt_1 &\ll \int_1^U K_1(t_1) K_2(t_1) dt_1 \\
&\ll \left(\int_1^U K_1^2(t_1) dt_1 \right)^{1/2} \left(\int_1^U K_2^2(t_1) dt_1 \right)^{1/2}.
\end{aligned} \tag{5.75}$$

By Cauchy's inequality again we have

$$\begin{aligned}
K_1^2(t_1) &= \left(\int_{-\infty}^{\infty} \frac{|\zeta(11/20-\varepsilon+i(t_1+t_2))|^{\frac{5}{2}}}{(|t_2|+1)^{\frac{5}{4}+\frac{3}{4}}} dt_2 \right)^2 \\
&\leq \int_{-\infty}^{\infty} \frac{|\zeta(11/20-\varepsilon+i(t_1+t_2))|^5}{(|t_2|+1)^{\frac{5}{2}}} dt_2 \int_{-\infty}^{\infty} \frac{1}{(|t_2|+1)^{\frac{3}{2}}} dt_2 \\
&\ll \int_{-\infty}^{\infty} \frac{|\zeta(11/20-\varepsilon+i(t_1+t_2))|^5}{(|t_2|+1)^{\frac{5}{2}}} dt_2.
\end{aligned} \tag{5.76}$$

Hence by Lemma 2.7 we have

$$\begin{aligned} \int_1^U K_1^2(t_1) dt_1 &\ll \int_{-\infty}^{\infty} \frac{1}{(|t_2|+1)^{\frac{5}{2}}} dt_2 \int_1^U |\zeta(11/20 - \varepsilon + i(t_1 + t_2))|^5 dt_1 \\ &= \int_{-\infty}^{\infty} \frac{1}{(|t_2|+1)^{\frac{5}{2}}} dt_2 \int_{1+t_2}^{U+t_2} |\zeta(11/20 - \varepsilon + iu)|^5 du \\ &\ll \int_{-\infty}^{\infty} \frac{|U+t_2|^{1+\varepsilon} + |t_2+1|^{1+\varepsilon}}{(|t_2|+1)^{\frac{5}{2}}} dt_2 \\ &\ll U^{1+\varepsilon}. \end{aligned} \quad (5.77)$$

Similarly we have

$$\int_1^U K_2^2(t_1) dt_1 \ll U^{1+\varepsilon}. \quad (5.78)$$

The estimate (5.74) follows from (5.75) and (5.78).

Consider the domain formed by the four points $s_1 = b_1 \pm iT, 11/20 \pm iT$. So by the residue theorem, we get

$$\begin{aligned} I_{26}(x, T) &= \operatorname{Res}_{s_1=1} \frac{\zeta^3(s_1)x^{s_1+2}}{s_1} G_6(s_1) \\ &\quad + H_{261}(x, T) + H_{262}(x, T) - H_{263}(x, T), \end{aligned} \quad (5.79)$$

where

$$\begin{aligned} H_{261}(x, T) &:= \frac{1}{2\pi i} \int_{11/20+iT}^{b_1+iT} \frac{\zeta^3(s_1)x^{s_1+2}}{s_1} G_6(s_1) ds_1, \\ H_{262}(x, T) &:= \frac{1}{2\pi i} \int_{11/20-iT}^{11/20+iT} \frac{\zeta^3(s_1)x^{s_1+2}}{s_1} G_6(s_1) ds_1, \\ H_{263}(x, T) &:= \frac{1}{2\pi i} \int_{11/20-iT}^{b_1-iT} \frac{\zeta^3(s_1)x^{s_1+2}}{s_1} G_6(s_1) ds_1. \end{aligned}$$

From Lemma 2.4 and (5.73) we get

$$H_{261}(x, T) \ll \frac{x^{3+\varepsilon}}{T}, \quad H_{263}(x, T) \ll \frac{x^{3+\varepsilon}}{T}. \quad (5.80)$$

By Lemma 2.7 and (5.74) we get via Hölder's inequality that

$$\int_1^U |\zeta^3(11/20 + it_1) G_6(11/20 + it_1)| dt_1 \ll U^{1+\varepsilon},$$

which implies that

$$H_{262}(x, T) \ll x^{51/20+\varepsilon}. \quad (5.81)$$

From (5.79) to (5.81) and Lemma 2.9 we get that

$$I_{26}(x, T) = x^3 P_{26}(\log x) + O\left(x^{\frac{51}{20}+\varepsilon} + \frac{x^{3+\varepsilon}}{T}\right), \quad (5.82)$$

where $P_{26}(u)$ is a polynomial in u of degree 2.

From (5.65), (5.68), (5.69), (5.72) and (5.82) we get

$$I_2(x, T) = x^3 P_2(\log x) + O\left(x^{\frac{8}{3}+\varepsilon} + \frac{x^{3+\varepsilon}}{T}\right), \quad (5.83)$$

where $P_2(u)$ is a polynomial in u of degree 7.

5.5 Completion of the proof

First suppose $10 \leq x \notin \mathbb{N}$. Suppose T is a large parameter such that $5 \leq T \leq x/2$. Combining (5.1), (5.3), (5.5), (5.6), (5.14), (5.21), (5.47), (5.59), (5.83) and Proposition 4.1 we get by taking $T = x^{1/2}$,

$$\begin{aligned} C_3(x) &= x^3 P(\log x) + O\left(x^{\frac{8}{3}+\varepsilon} + \frac{x^{3+\varepsilon}}{T}\right) \\ &= x^3 P(\log x) + O\left(x^{\frac{8}{3}+\varepsilon}\right). \end{aligned} \quad (5.84)$$

where $P(u)$ is a polynomial in u of degree 7. Hence our theorem is true when $x \notin \mathbb{N}$.

Now suppose $10 \leq x \in \mathbb{N}$. Then we have by (5.84) that

$$\begin{aligned} C_3(x) &= C_3(x + 1/2) \\ &= (x + 1/2)^3 P(\log(x + 1/2)) + O\left(x^{\frac{8}{3}+\varepsilon}\right) \\ &= x^3 P(\log x) + O\left(x^{\frac{8}{3}+\varepsilon}\right). \end{aligned} \quad (5.85)$$

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