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RESEARCH ARTICLE

Finite $\mathcal{W}$-algebras of $\mathfrak{osp}_{1|2n}$ and ghost centers

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Abstract
We prove that the finite $\mathcal{W}$-algebra $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ associated to $\mathfrak{osp}_{1|2n}$ and its principal nilpotent element f_{prin} is isomorphic to Gorelik's ghost center of $\mathfrak{osp}_{1|2n}$. It is an analogue for $\mathfrak{osp}_{1|2n}$ of a theorem of Kostant (Invent Math 48(2):101–184, 1978).

Keywords Finite $\mathcal{W}$-algebras · Vertex algebras · Affine $\mathcal{W}$-algebras · Lie superalgebras · Ghost centers

Mathematics Subject Classification 17B69

1 Introduction

The Lie superalgebra $\mathfrak{osp}_{1|2n}$ is the finite-dimensional simple Lie superalgebra whose Dynkin diagram is the same as the one of type B_n except for a unique simple short root, which is replaced by a non-isotropic odd simple root in $\mathfrak{osp}_{1|2n}$. The Lie superalgebra $\mathfrak{osp}_{1|2n}$ is not a Lie algebra but it has properties similar to simple Lie algebras. For example, the category of finite-dimensional $\mathfrak{osp}_{1|2n}$-modules is semisimple and the Harish–Chandra isomorphism $Z(\mathfrak{osp}_{1|2n}) \simeq \mathbb{C}[\mathfrak{h}]^W$ holds, where $Z(\mathfrak{g})$ denotes the center of the universal enveloping algebra $U(\mathfrak{g})$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{osp}_{1|2n}$ and W is the Weyl group. However, an analogue of Duflo's theorem [7] does not hold for $\mathfrak{osp}_{1|2n}$, that is the annihilating ideals of Verma modules in $U(\mathfrak{osp}_{1|2n})$ are not generated by their intersections with the center $Z(\mathfrak{osp}_{1|2n})$. This problem was noticed by Musson [25] and solved by Gorelik and Lantzmann [17] by replacing $Z(\mathfrak{osp}_{1|2n})$ with a larger algebra, called the ghost center $\widetilde{Z}(\mathfrak{osp}_{1|2n})$.

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\mathfrak{g}_{\bar{1}} \neq 0$, the ghost center $\widetilde{Z}(\mathfrak{g})$ was introduced by Gorelik in [14] as the direct sum $Z(\mathfrak{g}) \oplus \mathcal{A}(\mathfrak{g})$, where $\mathcal{A}(\mathfrak{g})$ is the anticenter

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defined by $\mathcal{A}(\mathfrak{g}) = \{a \in U(\mathfrak{g}) \,|\, .ua - (-1)^{p(u)(p(a)+\bar{1})}au = 0 \text{ for all } u \in \mathfrak{g}\}$, where $p(\cdot)$ denotes the parity. If $\mathfrak{g}$ is a finite-dimensional simple basic classical Lie superalgebra, it is known [14] that $\widetilde{Z}(\mathfrak{g})$ coincides with the center of $U(\mathfrak{g})_{\bar{0}}$ and thus is a purely even subalgebra of $U(\mathfrak{g})$. Moreover, in the case $\mathfrak{g} = \mathfrak{osp}_{1|2n}$, there exists $T \in U(\mathfrak{g})_{\bar{0}}$ such that $\mathcal{A}(\mathfrak{osp}_{1|2n}) = Z(\mathfrak{osp}_{1|2n})T$ by [2, 17, 25]. The element T is called the Casimir's ghost [2] since $T^2 \in Z(\mathfrak{osp}_{1|2n})$. When $n = 1$, in particular, T can be expressed as $4Q - 4C + 1/2$ by using [27] and then $T^2 = 4C + 1/4$, where C is the Casimir element in $U(\mathfrak{osp}_{1|2})$ and Q is the one in $U(\mathfrak{sl}_2)$.

The finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$ is an associative superalgebra over $\mathbb{C}$ defined from a simple finite-dimensional basic classical Lie superalgebra $\mathfrak{g}$ and an even nilpotent element f [3, 11, 23, 24, 30–32]. In the case when $\mathfrak{g}$ is a simple Lie algebra and f is a principal nilpotent element f_{prin}, it was proven by Kostant [23] that the corresponding finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f_{\text{prin}})$ is isomorphic to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is a vertex superalgebra defined by the Drinfeld–Sokolov reduction associated to $\mathfrak{g}$, f and a complex number $k \in \mathbb{C}$, called the level [9, 20]. In general, (Ramond-twisted) positive-energy simple modules of a $\frac{1}{2}\mathbb{Z}$-graded vertex superalgebra V with a Hamiltonian operator H are classified in terms of an associated superalgebra called the (H-twisted) Zhu algebra of V. See Sect. 2 for the definition of Ramond-twisted modules. It was proven by De Sole and Kac [6] that the Zhu algebra of $\mathcal{W}^k(\mathfrak{g}, f)$ is isomorphic to the finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$. In particular, there exists a one-to-one correspondence between simple modules of $U(\mathfrak{g}, f)$ and Ramond-twisted positive-energy simple modules of $\mathcal{W}^k(\mathfrak{g}, f)$. The $\mathcal{W}$-algebra associated to a principal nilpotent element $f = f_{\text{prin}}$ is called the principal $\mathcal{W}$-algebra of $\mathfrak{g}$, which we denote by $\mathcal{W}^k(\mathfrak{g}) = \mathcal{W}^k(\mathfrak{g}, f_{\text{prin}})$.

Theorem A (Theorem 6.5) *$U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ is isomorphic to $\widetilde{Z}(\mathfrak{osp}_{1|2n})$ as associative algebras.*

The finite $\mathcal{W}$-algebra $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ associated to $\mathfrak{osp}_{1|2n}$ and its principal nilpotent element f_{prin} is an associative superalgebra with its non-trivial odd part, while the ghost center $\widetilde{Z}(\mathfrak{osp}_{1|2n})$ is not. However, we prove an isomorphism between them.

To prove Theorem A, we use the Miura map μ and its injectivity and relationship with the Harish–Chandra homomorphism of $\mathfrak{osp}_{1|2n}$. See Sect. 4 for the definition of μ. The map μ was originally introduced in [24]. The injectivity of μ was only known for non-super cases, but has been recently proved by [26] for super cases. As a corollary of Theorem A, it follows that Ramond-twisted positive-energy simple modules of principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$ are classified by simple modules of the ghost center $\widetilde{Z}(\mathfrak{osp}_{1|2n})$ (Corollary 6.7). We note that the definition of $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ in the paper comes from the theory of vertex superalgebras (Remark 4.4).

We will prove in the next paper that the untwisted Zhu algebra of $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$ is isomorphic to the center of $U(\mathfrak{sp}_{2n})$. This is only known in the case $n = 1$ due to [22]. Thus, by Theorem A, the untwisted Zhu algebra is isomorphic to the even part of $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$. It is also proven in (6.1) that the Zhu functor is compatible with the Miura map, and hence with the Harish–Chandra homomorphism. Since simple modules of $Z(\mathfrak{sp}_{2n})$ can be described in terms of the central characters and the Harish–Chandra homomorphism, we may apply this to construct simple modules of $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$ inside tensor products of Fock modules and the free fermion F. This allows us to analyze the coset construction of $\mathcal{W}^k(\mathfrak{osp}_{1|2n}) \simeq \mathrm{Com}(V^\ell(\mathfrak{so}_{2n+1}), V^{\ell-1}(\mathfrak{so}_{2n+1}) \otimes F^{2n+1})$, where $V^\ell(\mathfrak{so}_{2n+1})$ is the affine vertex algebra of $\mathfrak{so}_{2n+1}$ at some level ℓ. We intend to address this issue in sequels to this paper in our joint work with Thomas Creutzig.

Let us remark that a super analog of the Soergel Struktursatz for a suitable Whittaker functor from the integral BGG category $\mathcal{O}$ of any basic classical simple Lie superalgebra $\mathfrak{g}$ to the category of finite-dimensional modules of $U(\mathfrak{g}, f_{\mathrm{prin}})$ has been established in [4]. We also hope to clarify the relationship between the ghost center of $\mathfrak{g}$ and $U(\mathfrak{g}, f_{\mathrm{prin}})$ in general $\mathfrak{g}$ to apply to the Soergel Struktursatz in our future works.

The paper is organized as follows. In Sect. 2, we introduce H-twisted Zhu algebras. In Sect. 3, we recall the definitions of $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$. In Sect. 4, we give two definitions $U(\mathfrak{g}, f)_{\mathrm{I}}$ and $U(\mathfrak{g}, f)_{\mathrm{II}}$ of finite $\mathcal{W}$-algebras and show the equivalence of the definitions, that is, $U(\mathfrak{g}, f)_{\mathrm{I}} \simeq U(\mathfrak{g}, f)_{\mathrm{II}}$. The proof is similar to [5]. In Sect. 5, we recall the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$ of $\mathfrak{osp}_{1|2n}$. In Sect. 6, we prove Theorem A.

2 *H*-twisted Zhu algebras

Let V be a vertex superalgebra. Denote by $|0\rangle$ the vacuum vector, by ∂ the translation operator, by $p(A)$ the parity of $A \in V$, and by $Y(A, z) = A(z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$ the field on V corresponding to $A \in V$. Let

$$[A_\lambda B] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} A_{(n)} B \in \mathbb{C}[\lambda] \otimes V$$

be the λ-bracket of A and B for $A, B \in V$. A Hamiltonian operator H on V is a semisimple operator on V satisfying that $[H, Y(A, z)] = z\partial_z Y(A, z) + Y(H(A), z)$ for all $A \in V$. The eigenvalue of H is called the conformal weight. If V is conformal and $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is the field corresponding to the conformal vector of V, we may choose $H = L_0$ as the Hamiltonian operator.

Suppose that V is a $\frac{1}{2}\mathbb{Z}$-graded vertex superalgebra with respect to a Hamiltonian operator H. Denote by Δ_A the conformal weight of $A \in V$. Define the $*$-product and $\circ$-product of V by

$$A * B = \sum_{j=0}^{\infty} \binom{\Delta_A}{j} A_{(j-1)} B, \quad A \circ B = \sum_{j=0}^{\infty} \binom{\Delta_A}{j} A_{(j-2)} B, \quad A, B \in V.$$

Then the quotient space

$$\mathrm{Zhu}_H V = V / V \circ V$$

has a structure of associative superalgebra with respect to the product induced from $*$, and is called the H-twisted Zhu algebra of V. Here $V \circ V = \mathrm{Span}_{\mathbb{C}}\{A \circ B \mid A, B \in V\}$.

The vacuum vector $|0\rangle$ defines a unit of $\mathrm{Zhu}_H V$. A superspace M is called a Ramond-twisted V-module if M is equipped with a parity-preserving linear map

$$Y_M\colon M \ni A \to Y_M(A,z) = \sum_{n\in\mathbb{Z}+\Delta_A} A^M_{(n)} z^{-n-1} \in (\mathrm{End}\, M)[\![z^{1/2}, z^{-1/2}]\!]$$

such that (1) for each $C \in M$, $A^M_{(n)}C = 0$ if $n \gg 0$, (2) $Y_M(|0\rangle, z) = \mathrm{id}_M$ and (3) for any $A, B \in V$, $C \in M$, $n \in \mathbb{Z}$, $m \in \mathbb{Z} + \Delta_A$ and $\ell \in \mathbb{Z} + \Delta_B$,

$$\sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \Big(A^M_{(m+n-j)} B^M_{(\ell+j)} - (-1)^{p(A)p(B)} B^M_{(\ell+n-j)} A^M_{(m+j)} \Big) C = \sum_{j=0}^{\infty} \binom{m}{j} \big(A_{(n+j)} B \big)^M_{(m+\ell-j)} C.$$

Hence the Ramond-twisted module is a twisted module of V for the automorphism $\mathrm{e}^{2\pi i H}$. In particular, M is just a V-module if V is $\mathbb{Z}$-graded. Define A^M_n by $Y_M(A,z) = \sum_{n\in\mathbb{Z}} A^M_n z^{-n-\Delta_A}$ for $A \in V$. A Ramond-twisted V-module M is called positive-energy if M has an $\mathbb{R}$-grading $M = \bigoplus_{j\in\mathbb{R}} M_j$ with $M_0 \neq 0$ such that $A^M_n M_j \subset M_{j+n}$ for all $A \in V$, $n \in \mathbb{Z}$ and $j \in \mathbb{R}$. Then M_0 is called the top space. By [6, Lemma 2.22], a linear map $V \ni A \mapsto A^M_0|_{M_0} \in \mathrm{End}\, M_0$ induces a homomorphism $\mathrm{Zhu}_H V \to \mathrm{End}\, M_0$. Thus we have a functor $M \mapsto M_0$ from the category of positive-energy Ramond-twisted V-modules to the category of $\mathbb{Z}_2$-graded $\mathrm{Zhu}_H V$-modules. By [6, Theorem 2.30], these functors establish a bijection (up to isomorphisms) between simple positive-energy Ramond-twisted V-modules and simple $\mathbb{Z}_2$-graded $\mathrm{Zhu}_H V$-modules.

3 $\mathcal{W}$-algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra with the normalized even supersymmetric invariant bilinear form $(\cdot|\cdot)$ and f be a nilpotent element in the even part of $\mathfrak{g}$. Then there exists a $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$ that is good for f. See [20] for the definitions of good gradings and [8, 18] for the classifications. Let $\mathfrak{g}_j$ be the homogeneous subspace of $\mathfrak{g}$ with degree j. The good grading $\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ for f on $\mathfrak{g}$ satisfies the following properties:

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$,
- $f \in \mathfrak{g}_{-1}$,
- $\mathrm{ad}(f)\colon \mathfrak{g}_j \to \mathfrak{g}_{j-1}$ is injective for $j \geqslant 1/2$ and surjective for $j \leqslant 1/2$,
- $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ if $i + j \neq 0$,
- $\dim \mathfrak{g}^f = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{1/2}$, where $\mathfrak{g}^f$ is the centralizer of f in $\mathfrak{g}$.

Then we can choose a set of simple roots Π of $\mathfrak{g}$ for a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ such that all positive root vectors lie in $\mathfrak{g}_{\geqslant 0}$. Denote $\Delta_j = \{\alpha \in \Delta \,|\, \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}$ and $\Pi_j = \Pi \cap \Delta_j$ for $j \in \frac{1}{2}\mathbb{Z}$. We have $\Pi = \Pi_0 \sqcup \Pi_{1/2} \sqcup \Pi_1$. Let $\chi : \mathfrak{g} \to \mathbb{C}$ be a linear map defined by $\chi(u) = (f|u)$. Since $\mathrm{ad}(f)\colon \mathfrak{g}_{1/2} \to \mathfrak{g}_{-1/2}$ is an isomorphism of vector spaces, the super skew-symmetric bilinear form $\mathfrak{g}_{1/2} \times \mathfrak{g}_{1/2} \ni (u, v) \mapsto \chi([u, v]) \in \mathbb{C}$ is non-degenerate. We fix a root vector u_α and denote by $p(\alpha)$ the parity of u_α for $\alpha \in \Delta$.

Let $V^k(\mathfrak{g})$ be the affine vertex superalgebra associated to $\mathfrak{g}$ at level $k \in \mathbb{C}$, which is generated by $u(z)$ $(u \in \mathfrak{g})$ whose parity is the same as u, satisfying that

$$[u_\lambda v] = [u, v] + k(u|v)\lambda, \quad u, v \in \mathfrak{g}.$$

Let $F(\mathfrak{g}_{1/2})$ be the neutral vertex superalgebra associated to $\mathfrak{g}_{1/2}$, which is strongly generated by $\phi_\alpha(z)$ $(\alpha \in \Delta_{1/2})$ whose parity is equal to $p(\alpha)$, satisfying that

$$[\phi_{\alpha\,\lambda}\phi_\beta] = \chi(u_\alpha, u_\beta), \quad \alpha, \beta \in \Delta_{1/2}.$$

Let $F^{\mathrm{ch}}(\mathfrak{g}_{>0})$ be the charged fermion vertex superalgebra associated to $\mathfrak{g}_{>0}$, which is strongly generated by $\varphi_\alpha(z), \varphi_\alpha^*(z)$ $(\alpha \in \Delta_{>0})$ whose parities are equal to $p(\alpha) + \bar{1}$, satisfying that

$$[\varphi_{\alpha\,\lambda}\varphi_\beta^*] = \delta_{\alpha,\beta}, \quad [\varphi_{\alpha\,\lambda}\varphi_\beta] = [\varphi^*_{\alpha\,\lambda}\varphi^*_\beta] = 0, \quad \alpha, \beta \in \Delta_{>0}.$$

Let $C^k(\mathfrak{g}, f) = V^k(\mathfrak{g}) \otimes F(\mathfrak{g}_{1/2}) \otimes F^{\mathrm{ch}}(\mathfrak{g}_{>0})$ and d be an odd element in $C^k(\mathfrak{g}, f)$ defined by

$$d = \sum_{\alpha\in\Delta_{>0}} (-1)^{p(\alpha)} u_\alpha \varphi_\alpha^* - \frac{1}{2} \sum_{\alpha,\beta,\gamma\in\Delta_{>0}} (-1)^{p(\alpha)p(\gamma)} c_{\alpha,\beta}^{\gamma} :\varphi_\gamma \varphi_\alpha^* \varphi_\beta^*: \\ + \sum_{\alpha\in\Delta_{1/2}} \phi_\alpha \varphi_\alpha^* + \sum_{\alpha\in\Delta_{>0}} \chi(u_\alpha)\varphi_\alpha^*.$$

Then $(C^k(\mathfrak{g}, f), d_{(0)})$ defines a cochain complex with respect to the charged degree: $\operatorname{charge}\varphi_\alpha = -\operatorname{charge}\varphi_\alpha^* = 1$ $(\alpha \in \Delta_{>0})$ and $\operatorname{charge} A = 0$ for all $A \in V^k(\mathfrak{g}) \otimes F(\mathfrak{g}_{1/2})$. The (affine) $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ associated to $\mathfrak{g}$, f at level k is defined by

$$\mathcal{W}^k(\mathfrak{g}, f) = H(C^k(\mathfrak{g}, f), d_{(0)}).$$

Let $C^k(\mathfrak{g}, f)_+$ be a subcomplex generated by $\phi_\alpha(z)$ $(\alpha \in \Delta_{1/2})$, $\varphi_\alpha^*(z)$ $(\alpha \in \Delta_{>0})$ and

$$J^u(z) = u(z) + \sum_{\alpha,\beta\in\Delta_{>0}} c_{\beta,u}^{\alpha} :\varphi_\beta^*(z)\varphi_\alpha(z):, \quad u \in \mathfrak{g}_{\leqslant 0}.$$

Then we have [21]

$$\mathcal{W}^k(\mathfrak{g}, f) = H(C^k(\mathfrak{g}, f), d_{(0)}) = H^0(C^k(\mathfrak{g}, f)_+, d_{(0)}).$$

Thus, $\mathcal{W}^k(\mathfrak{g}, f)$ is a vertex subalgebra of $C^k(\mathfrak{g}, f)_+$. Using the fact that

$$[J^u{}_\lambda J^v] = J^{[u,v]} + \tau(u|v)\lambda, \quad u, v \in \mathfrak{g}_{\leqslant 0},$$
$$\tau(u|v) = k(u|v) + \frac{1}{2}\kappa_{\mathfrak{g}}(u|v) - \frac{1}{2}\kappa_{\mathfrak{g}_0}(u|v), \quad u, v \in \mathfrak{g}_{\leqslant 0},$$

where $\kappa_{\mathfrak{g}}$ denotes the Killing form on $\mathfrak{g}$, it follows that the vertex algebra generated by $J^u(z)$ ($u \in \mathfrak{g}_{\leqslant 0}$) is isomorphic to the affine vertex superalgebra associated to $\mathfrak{g}_{\leqslant 0}$ and τ, which we denote by $V^\tau(\mathfrak{g}_{\leqslant 0})$. Therefore the homogeneous subspace of $C^k(\mathfrak{g}, f)_+$ with charged degree 0 is isomorphic to $V^\tau(\mathfrak{g}_{\leqslant 0}) \otimes F(\mathfrak{g}_{1/2})$. The projection $\mathfrak{g}_{\leqslant 0} \twoheadrightarrow \mathfrak{g}_0$ induces a vertex superalgebra surjective homomorphism $V^\tau(\mathfrak{g}_{\leqslant 0}) \otimes F(\mathfrak{g}_{1/2}) \twoheadrightarrow V^\tau(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$ so that we have

$$\Upsilon : \mathcal{W}^k(\mathfrak{g}, f) \to V^\tau(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$$

by the restriction. The map Υ is called the Miura map and is injective thanks to [1, 10, 26].

4 Finite $\mathcal{W}$-algebras

Recall the definitions of finite $\mathcal{W}$-algebras $U(\mathfrak{g}, f)$, following [5]. We give two definitions in (4.1), (4.2) denoted by $U(\mathfrak{g}, f)_{\mathrm{I}}$, $U(\mathfrak{g}, f)_{\mathrm{II}}$, respectively, and prove the isomorphism $U(\mathfrak{g}, f)_{\mathrm{I}} \simeq U(\mathfrak{g}, f)_{\mathrm{II}}$ in Theorem 4.2.

Let Φ be an associative $\mathbb{C}$-superalgebra generated by Φ_α ($\alpha \in \Delta_{1/2}$) that has the same parity as u_α, satisfying that

$$[\Phi_\alpha, \Phi_\beta] = \chi([u_\alpha, u_\beta]), \quad \alpha, \beta \in \Delta_{1/2}.$$

Here $[A, B]$ denotes $AB - (-1)^{p(A)\,p(B)} BA$. We extend the definition of Φ_α for all $\alpha \in \Delta_{>0}$ by $\Phi_\alpha = 0$ for $\alpha \in \Delta_{\geqslant 1}$. Let $\Lambda(\mathfrak{g}_{>0})$ be the Clifford superalgebra associated to $\mathfrak{g}_{>0}$, which is an associative $\mathbb{C}$-superalgebra generated by ψ_α, ψ_α^* ($\alpha \in \Delta_{>0}$) with the opposite parity to that of u_α, satisfying that

$$[\psi_\alpha, \psi_\beta^*] = \delta_{\alpha,\beta}, \quad [\psi_\alpha, \psi_\beta] = [\psi_\alpha^*, \psi_\beta^*] = 0, \quad \alpha, \beta \in \Delta_{>0}.$$

The Clifford superalgebra $\Lambda(\mathfrak{g}_{>0})$ has the charged degree defined by $\deg(\psi_\alpha) = 1 = -\deg(\psi_\alpha^*)$ for all $\alpha \in \Delta_{>0}$. Set

$$C_{\mathrm{I}} = U(\mathfrak{g}) \otimes \Phi \otimes \Lambda(\mathfrak{g}_{>0}), \quad d_{\mathrm{I}} = \mathrm{ad}(Q),$$
$$Q = \sum_{\alpha \in \Delta_{>0}} (-1)^{p(\alpha)} X_\alpha \psi_\alpha - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \Delta_{>0}} (-1)^{p(\alpha)p(\gamma)} c_{\alpha,\beta}^\gamma \psi_\gamma \psi_\alpha^* \psi_\beta^*,$$
$$X_\alpha = u_\alpha + (-1)^{p(\alpha)}(\Phi_\alpha + \chi(u_\alpha)), \quad \alpha \in \Delta_{>0},$$

where $c_{\alpha,\beta}^{\gamma}$ is the structure constant defined by $[u_\alpha, u_\beta] = \sum_{\gamma\in\Delta_{>0}} c_{\alpha,\beta}^{\gamma} u_\gamma$. Then a pair $(C_{\rm I}, d_{\rm I})$ forms a cochain complex with respect to the charged degree on $\Lambda(\mathfrak{g}_{>0})$ and the cohomology

$$U(\mathfrak{g}, f)_{\rm I} = H^{\bullet}(C_{\rm I}, d_{\rm I}) \tag{4.1}$$

has a structure of an associative $\mathbb{C}$-superalgebra inherited from that of $C_{\rm I}$. Let

$$j^u = u + \sum_{\alpha,\beta\in\Delta_{>0}} c_{\beta,u}^{\alpha} \psi_\beta^* \psi_\alpha, \quad u \in \mathfrak{g}.$$

Then

$$\mathrm{ad}(Q)\cdot\psi_\alpha = j^{u_\alpha} + (-1)^{p(\alpha)}(\Phi_\alpha + \chi(u_\alpha)) = X_\alpha + \sum_{\alpha,\beta\in\Delta_{>0}} c_{\beta,u}^{\alpha} \psi_\beta^* \psi_\alpha, \quad \alpha \in \Delta_{>0}.$$

Let C_- be the subalgebra of $C_{\rm I}$ generated by ψ_α, $\mathrm{ad}(Q)\cdot\psi_\alpha$ $(\alpha \in \Delta_{>0})$ and C_+ be the subalgebra of $C_{\rm I}$ generated by j^u $(u \in \mathfrak{g}_{\leqslant 0})$, Φ_α $(\alpha \in \Delta_{1/2})$ and ψ_α^* $(\alpha \in \Delta_{>0})$. Then $(C_\pm, d_{\rm I})$ form subcomplexes and $C_{\rm I} \simeq C_- \otimes C_+$ as vector superspaces. Since $H(C_-, d_{\rm I}) = \mathbb{C}$, we have

$$H(C_{\rm I}, d_{\rm I}) \simeq H(C_-, d_{\rm I}) \otimes H(C_+, d_{\rm I}) = H(C_+, d_{\rm I}).$$

Using the same argument as in [21], it follows that $H^n(C_+, d_{\rm I}) = 0$ for $n \neq 0$. Therefore $U(\mathfrak{g}, f)_{\rm I}$ is a subalgebra of C_+^0, which is generated by j^u $(u \in \mathfrak{g}_{\leqslant 0})$ and Φ_α $(\alpha \in \Delta_{1/2})$. Since $[j^u, j^v] = j^{[u,v]}$ for $u, v \in \mathfrak{g}_{\leqslant 0}$, there exists an isomorphism $C_+^0 \simeq U(\mathfrak{g}_{\leqslant 0}) \otimes \Phi$ as associative $\mathbb{C}$-superalgebras. The projection $\mathfrak{g}_{\leqslant 0} \twoheadrightarrow \mathfrak{g}_0$ induces an associative $\mathbb{C}$-superalgebra surjective homomorphism $U(\mathfrak{g}_{\leqslant 0}) \otimes \Phi \twoheadrightarrow U(\mathfrak{g}_0) \otimes \Phi$ so that we have

$$\mu\colon U(\mathfrak{g}, f)_{\rm I} \to U(\mathfrak{g}_0) \otimes \Phi$$

by the restriction. The map μ is called the Miura map for the finite $\mathcal{W}$-algebras and it is injective by [13, 24, 26]. Let $\mathbb{C}_{-\chi}$ be the one-dimensional $\mathfrak{g}_{\geqslant 1}$-module defined by $\mathfrak{g}_{\geqslant 1} \ni u \mapsto -\chi(u) \in \mathbb{C}$ and $M_{\rm II}$ be the induced left $\mathfrak{g}$-module

$$M_{\rm II} = \mathrm{Ind}_{\mathfrak{g}_{\geqslant 1}}^{\mathfrak{g}} \mathbb{C}_{-\chi} = U(\mathfrak{g}) \underset{U(\mathfrak{g}_{\geqslant 1})}{\otimes} \mathbb{C}_{-\chi} \simeq U(\mathfrak{g})/I_{-\chi},$$

where $I_{-\chi}$ is a left $U(\mathfrak{g})$-module generated by $u+\chi(u)$ for all $u \in \mathfrak{g}_{\geqslant 1}$. Then $M_{\rm II}$ has a structure of the $\mathrm{ad}(\mathfrak{g}_{>0})$-module inherited from that of $U(\mathfrak{g})$. Set the $\mathrm{ad}(\mathfrak{g}_{>0})$-invariant subspace

$$U(\mathfrak{g}, f)_{\rm II} = (M_{\rm II})^{\mathrm{ad}(\mathfrak{g}_{>0})}. \tag{4.2}$$

Then $U(\mathfrak{g}, f)_{\mathrm{II}}$ also has a structure of an associative $\mathbb{C}$-superalgebra inherited from that of $U(\mathfrak{g})$. We may also define $U(\mathfrak{g}, f)_{\mathrm{II}}$ as the Chevalley cohomology $H(\mathfrak{g}_{>0}, M_{\mathrm{II}})$ of the left $\mathfrak{g}_{>0}$-module M_{II}:

Lemma 4.1 ([11, 26])

$$H(\mathfrak{g}_{>0}, M_{\mathrm{II}}) = H^0(\mathfrak{g}_{>0}, M_{\mathrm{II}}) = (M_{\mathrm{II}})^{\mathrm{ad}(\mathfrak{g}_{>0})}.$$

Proof Though the assertion is proved in [11] for Lie algebras $\mathfrak{g}$, the same proof together with [26, Corollary 2.6] applies. □

Theorem 4.2 ([5, Theorem A.6]) *There exists an isomorphism $U(\mathfrak{g}, f)_{\mathrm{I}} \simeq U(\mathfrak{g}, f)_{\mathrm{II}}$ as associative $\mathbb{C}$-superalgebras.*

Proof Though the assertion is proved in [5] for Lie algebras $\mathfrak{g}$, the same proof applies as follows. Let $C_{\mathrm{II}} = \Lambda(\mathfrak{g}_{>0})_c \otimes M_{\mathrm{II}}$ be the Chevalley cohomology complex of the left $\mathfrak{g}_{>0}$-module M_{II}, where $\Lambda(\mathfrak{g}_{>0})_c$ is the subalgebra of $\Lambda(\mathfrak{g}_{>0})$ generated by ψ_α^* for all $\alpha \in \Delta_{>0}$, and d_{II} be the derivation of the cochain complex C_{II}. Let $U(\mathfrak{g}_{>0})_{-\chi} = U(\mathfrak{g}_{>0}) \otimes \mathbb{C}_{-\chi}$ be a left $\mathfrak{g}_{\geqslant 1}$-module defined by the diagonal action, where $U(\mathfrak{g}_{>0})$ is considered as a left $\mathfrak{g}_{\geqslant 1}$-module by the left multiplication, and M_{III} be the induced left $\mathfrak{g}$-module

$$M_{\mathrm{III}} = \mathrm{Ind}_{\mathfrak{g}_{\geqslant 1}}^{\mathfrak{g}} U(\mathfrak{g}_{>0})_{-\chi} = U(\mathfrak{g}) \underset{U(\mathfrak{g}_{\geqslant 1})}{\otimes} U(\mathfrak{g}_{>0})_{-\chi}.$$

Let $\mathbb{C}_\chi$ be the one-dimensional $\mathfrak{g}_{\geqslant 1}$-module defined by $\mathfrak{g}_{\geqslant 1} \ni u \mapsto \chi(u) \in \mathbb{C}$ and $U(\mathfrak{g})_\chi = U(\mathfrak{g}) \otimes \mathbb{C}_\chi$ be a right $\mathfrak{g}_{\geqslant 1}$-module defined by the diagonal action, where $U(\mathfrak{g})$ is considered as a right $\mathfrak{g}_{\geqslant 1}$-module by the right multiplication. Then we have

$$M_{\mathrm{III}} \simeq U(\mathfrak{g})_\chi \underset{U(\mathfrak{g}_{\geqslant 1})}{\otimes} U(\mathfrak{g}_{>0})$$

so that M_{III} is a left $\mathfrak{g}$- right $\mathfrak{g}_{>0}$-bimodule. Note that there is an isomorphism $\Lambda(\mathfrak{g}_{>0}) \simeq \Lambda(\mathfrak{g}_{>0})_h \otimes \Lambda(\mathfrak{g}_{>0})_c$ of vector superspaces, where $\Lambda(\mathfrak{g}_{>0})_h$ is the subalgebra of $\Lambda(\mathfrak{g}_{>0})$ generated by ψ_α for all $\alpha \in \Delta_{>0}$. Let d_h be the derivation of the Chevalley homology complex $M_{\mathrm{III}} \otimes \Lambda(\mathfrak{g}_{>0})_h$ of the right $\mathfrak{g}_{>0}$-module M_{III}. Then $M_{\mathrm{III}} \otimes \Lambda(\mathfrak{g}_{>0})_h$ is clearly a left $\mathfrak{g}_{>0}$-module with respect to the adjoint $\mathfrak{g}_{>0}$-action. Now, let $\overline{d}_c$ be the derivation of the Chevalley cohomology complex $\Lambda(\mathfrak{g}_{>0})_c \otimes M_{\mathrm{III}} \otimes \Lambda(\mathfrak{g}_{>0})_h$ of the left $\mathfrak{g}_{>0}$-module $M_{\mathrm{III}} \otimes \Lambda(\mathfrak{g}_{>0})_h$. Then, as in [5], we get a new cochain complex $(C_{\mathrm{III}}, d_{\mathrm{III}})$ defined by

$$C_{\mathrm{III}} = \Lambda(\mathfrak{g}_{>0})_c \otimes M_{\mathrm{III}} \otimes \Lambda(\mathfrak{g}_{>0})_h, \quad d_{\mathrm{III}} = d_c + (-1)^{\delta-1} \otimes d_h,$$

where δ denotes the parity of the part of elements in $\Lambda(\mathfrak{g}_{>0})_c$. Then it is easy to check that the following linear map

$$\begin{aligned} i_{\mathrm{III}} \colon C_{\mathrm{III}} \ni\ & \psi_{\beta_1}^* \cdots \psi_{\beta_i}^* \otimes \Big(v_1 \cdots v_s \underset{U(\mathfrak{g}_{\geqslant 1})}{\otimes} u_{\alpha_1} \cdots u_{\alpha_t} \Big) \otimes \psi_{\gamma_1} \cdots \psi_{\gamma_j} \\ & \mapsto\ \psi_{\beta_1}^* \cdots \psi_{\beta_i}^* \cdot v_1 \cdots v_s \cdot X_{\alpha_1} \cdots X_{\alpha_t} \cdot \psi_{\gamma_1} \cdots \psi_{\gamma_j} \in \overline{C}_{\mathrm{I}} \end{aligned}$$

with $v_1, \ldots, v_s \in \mathfrak{g}$, $\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_i, \gamma_1, \ldots, \gamma_j \in \Delta_{>0}$ is well defined and induces an isomorphism of complexes $(C_{\mathrm{III}}, d_{\mathrm{III}}) \to (C_{\mathrm{I}}, d_{\mathrm{I}})$ since $i_{\mathrm{III}\to\mathrm{I}} \circ d_{\mathrm{III}} = d_{\mathrm{I}} \circ i_{\mathrm{III}\to\mathrm{I}}$. Now

$$\begin{aligned} H_n(C_{\mathrm{III}}, d_h) &= \Lambda(\mathfrak{g}_{>0})_c \otimes H_n\big(M_{\mathrm{III}} \otimes \Lambda(\mathfrak{g}_{>0})_h, d_h\big) \\ &= \Lambda(\mathfrak{g}_{>0})_c \otimes U(\mathfrak{g})_\chi \underset{U(\mathfrak{g}_{\geqslant 1})}{\otimes} H_n(\mathfrak{g}_{>0}, U(\mathfrak{g}_{>0})) \\ &= \delta_{n,0}\, \Lambda(\mathfrak{g}_{>0})_c \otimes U(\mathfrak{g})_\chi \underset{U(\mathfrak{g}_{\geqslant 1})}{\otimes} \mathbb{C} \simeq \delta_{n,0}\, C_{\mathrm{II}}. \end{aligned}$$

Thus, since d_c and $(-1)^{\delta-1} \otimes d_h$ commute, we have

$$H(C_{\mathrm{III}}, d_{\mathrm{III}}) \simeq H(H(C_{\mathrm{III}}, d_h), d_c) \simeq H(C_{\mathrm{II}}, d_{\mathrm{II}}).$$

The above argument together with the isomorphism $i_{\mathrm{III}\to\mathrm{I}}$ of complexes shows that $(C_{\mathrm{I}}, d_{\mathrm{I}})$ and $(C_{\mathrm{II}}, d_{\mathrm{II}})$ are quasi-isomorphic via the following quasi-isomorphism

$$\begin{aligned} i_{\mathrm{I}\to\mathrm{II}} \colon C_{\mathrm{I}} \ni\ & \psi^*_{\beta_1} \cdots \psi^*_{\beta_i} \cdot v_1 \cdots v_s \cdot X_{\alpha_1} \cdots X_{\alpha_t} \cdot \psi_{\gamma_1} \cdots \psi_{\gamma_j} \\ & \mapsto\ \delta_{t,0}\, \delta_{j,0}\, \psi^*_{\beta_1} \cdots \psi^*_{\beta_i} \cdot v_1 \cdots v_s \in C_{\mathrm{II}}, \end{aligned} \tag{4.3}$$

which preserves the associative superalgebra structures on the cohomologies. □

Definition 4.3 The finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$ associated to $\mathfrak{g}, f$ is defined to be the superalgebra $U(\mathfrak{g}, f)_{\mathrm{I}}$, which is isomorphic to $U(\mathfrak{g}, f)_{\mathrm{II}}$ due to Theorem 4.2.

Remark 4.4 The same result as Theorem 4.2 for Poisson superalgebra versions has been studied in [33]. Also remark that our definitions of the finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$ are not necessarily equivalent to the definitions in some literature [28, 29, 34]. In fact, in case that $\mathfrak{g} = \mathfrak{osp}_{1|2n}$ and $f = f_{\text{prin}}$ its principal nilpotent element, we have $\dim \mathfrak{g}_{1/2} = \dim \mathfrak{g}_{1/2,\bar{1}} = 1$ and thus $\mathfrak{g}_{\geqslant 1} \subsetneq \mathfrak{g}_{>0}$. Then $U(\mathfrak{g}, f) \simeq U(\mathfrak{g}, f)_{\mathrm{II}} = (U(\mathfrak{g})/I_{-\chi})^{\mathrm{ad}(\mathfrak{g}_{>0})}$ is a proper subalgebra of $(U(\mathfrak{g})/I_{-\chi})^{\mathrm{ad}(\mathfrak{g}_{\geqslant 1})} = \mathrm{End}_{U(\mathfrak{g})}\, U(\mathfrak{g})/I_{-\chi}$.

The vertex superalgebra $C^k(\mathfrak{g}, f)$ has a conformal vector ω if $k \neq -h^\vee$, which defines the conformal weights on $C^k(\mathfrak{g}, f)$ by L_0, where $\omega(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. See [20] for the details. Then $H = L_0$ defines a Hamiltonian operator on $C^k(\mathfrak{g}, f)$, the vertex subalgebra $C^k(\mathfrak{g}, f)_+$, and the corresponding $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$. Moreover the Hamiltonian operator L_0 is well defined for all $k \in \mathbb{C}$. Recall that $\mathrm{Zhu}_H V$ is the H-twisted Zhu algebra of V, see Sect. 2. Let $x \in \mathfrak{h}$ be such that $[x, u] = ju$ for $u \in \mathfrak{g}_j$. Then by [1, 6],

$$\mathrm{Zhu}_H C^k(\mathfrak{g}, f)_+ \simeq C_+, \quad J^u \mapsto j^u + \tau(x\,|\,u), \quad \phi_\alpha \mapsto \Phi_\alpha, \quad \varphi^*_\alpha \mapsto \psi^*_\alpha \tag{4.4}$$

for $u \in \mathfrak{g}_{\leqslant 0}$, $\alpha \in \Delta_{>0}$ and $\mathrm{Zhu}_H H^0(C^k(\mathfrak{g}, f)_+, d_{(0)}) \simeq H^0(C_+, d_{\mathrm{I}})$ so that

$$\mathrm{Zhu}_H \mathcal{W}^k(\mathfrak{g}, f) \simeq U(\mathfrak{g}, f). \tag{4.5}$$

Let V_1, V_2 be any $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$-graded vertex superalgebras with the Hamiltonian operators and $g\colon V_1 \to V_2$ any vertex superalgebra homomorphism preserving the conformal weights. Since $g(V_1 \circ V_1) = g(V_1) \circ g(V_1) \subset V_2 \circ V_2$, the map g induces an algebra homomorphism

$$\mathrm{Zhu}_H(g)\colon \mathrm{Zhu}_H V_1 \to \mathrm{Zhu}_H V_2.$$

Apply for $g = \Upsilon$. Then we get

$$\mathrm{Zhu}_H(\Upsilon) = \mu$$

by construction.

5 Principal $\mathcal{W}$-algebras of $\mathfrak{osp}_{1|2n}$

Consider the case that

$$\mathfrak{g} = \mathfrak{osp}_{1|2n} = \left\{ u = \left(\begin{array}{c|cc} 0 & {}^t y & -{}^t x \\ \hline x & a & b \\ y & c & -{}^t a \end{array}\right) \in \mathfrak{gl}_{1|2n} \;\middle|\; \begin{array}{l} a, b, c \in \mathrm{Mat}_{\mathbb{C}}(n \times n), \\ x, y \in \mathrm{Mat}_{\mathbb{C}}(n \times 1), \\ b = {}^t b,\ c = {}^t c \end{array} \right\},$$

where ${}^t A$ denotes the transpose of A. Let $\{e_{i,j}\}_{i,j \in I}$ be the standard basis of $\mathfrak{gl}_{1|2n}$ with the index set $I = \{0, 1, \ldots, n, -1, \ldots, -n\}$ and $h_i = e_{i,i} - e_{-i,-i}$ $(i = 1, \ldots, n)$. Then $\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{h_i\}_{i=1}^n$ is a Cartan subalgebra of $\mathfrak{osp}_{1|2n}$. Define $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(h_j) = \delta_{i,j}$. Then $\Delta_+ = \{\epsilon_i, 2\epsilon_i\}_{i=1}^n \sqcup \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j\}_{1 \leqslant i < j \leqslant n}$ forms a set of positive roots with simple roots $\Pi = \{\alpha_i\}_{i=1}$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(i = 1, \ldots, n-1)$ and $\alpha_n = \epsilon_n$, and $\epsilon_1, \ldots, \epsilon_n$ are the (non-isotropic) odd roots in Δ_+. Set $\Delta_- = -\Delta_+$ and $(u \,|\, v) = -\,\mathrm{str}(uv)$ for $u, v \in \mathfrak{osp}_{1|2n}$. We may identify $\mathfrak{h}^*$ with $\mathfrak{h}$ through $\nu\colon \mathfrak{h}^* \ni \lambda \mapsto \nu(\lambda) \in \mathfrak{h}$ defined by $\lambda(h) = (h \,|\, \nu(\lambda))$ for $h \in \mathfrak{h}$, which induces a non-degenerate bilinear form on $\mathfrak{h}^*$ by $(\lambda \,|\, \mu) = (\nu(\lambda) \,|\, \nu(\mu))$ so that $(\epsilon_i \,|\, \epsilon_j) = \delta_{i,j}/2$. Then h_i corresponds to $2\epsilon_i = 2\sum_{j=i}^n \alpha_j$ by ν. We have

$$(\alpha_i \,|\, \alpha_i) = 1, \quad (\alpha_i \,|\, \alpha_{i+1}) = -\frac{1}{2}, \quad i = 1, \ldots, n-1; \quad (\alpha_n \,|\, \alpha_n) = \frac{1}{2}.$$

Note that the dual Coxeter number of $\mathfrak{osp}_{1|2n}$ is equal to $n + 1/2$. Let

$$f_{\mathrm{prin}} = \sum_{i=1}^{n-1} u_{-\alpha_i} + u_{-2\alpha_n}$$

be a principal nilpotent element in the even part of $\mathfrak{osp}_{1|2n}$, where u_α denotes some root vector for $\alpha \in \Delta$. Then there exists a unique good grading on $\mathfrak{osp}_{1|2n}$ such that

$\Pi_1 = \{\alpha_i\}_{i=1}^{n-1}$ and $\Pi_{1/2} = \{\alpha_n\}$. Thus

$$\mathfrak{g}_0 = \mathfrak{h}, \quad \mathfrak{g}_{>0} = \mathfrak{n} := \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{<0} = \mathfrak{n}_- := \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha.$$

Let

$$\mathcal{W}^k(\mathfrak{osp}_{1|2n}) := \mathcal{W}^k(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$$

be the principal $\mathcal{W}$-algebra of $\mathfrak{osp}_{1|2n}$ at level k. The Miura map for $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$ is

$$\Upsilon\colon \mathcal{W}^k(\mathfrak{osp}_{1|2n}) \to \pi \otimes F,$$

where π is the Heisenberg vertex algebra generated by even fields $\alpha_i(z)$, $i = 1, \ldots, n$, satisfying that

$$[\alpha_i{}_\lambda \alpha_j] = \left(k + n + \frac{1}{2}\right)(\alpha_i|\alpha_j)\lambda, \quad i, j = 1, \ldots, n,$$

and F is the free fermion vertex superalgebra generated by an odd field $\phi(z)$ satisfying that

$$[\phi_\lambda \phi] = 1.$$

By [12, Theorem 6.4], $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$ is strongly generated by $G, W_2, W_4, \ldots, W_{2n}$ for odd G and even $W_2, W_4, \ldots, W_{2n}$ elements of conformal weights $n + 1/2$ and $2, 4, \ldots, 2n$ such that

$$\begin{aligned} &\Upsilon(G)(z) = :(2(k+n)\partial + h_1(z)) \cdots (2(k+n)\partial + h_n(z))\phi(z):, \\ &\Upsilon(W_{2i})(z) \equiv \sum_{1\leqslant j_1<\cdots<j_i\leqslant n} :h_{j_1}^2(z)\cdots h_{j_i}^2(z): \pmod{C_2(\pi\otimes F)}, \\ &C_2(\pi\otimes F) = \{A_{(-2)}B \mid A, B \in \pi\otimes F\}, \end{aligned} \tag{5.1}$$

and

$$[G_\lambda G] = W_{2n} + \sum_{i=1}^{n-1} \gamma_i \left(\frac{\lambda^{2i-1}}{(2i-1)!} W_{2n-2i+1} + \frac{\lambda^{2i}}{(2i)!} W_{2n-2i} \right) + \gamma_n \frac{\lambda^{2n}}{(2n)!} \tag{5.2}$$

for some $W_{2j+1} \in \mathcal{W}^k(\mathfrak{osp}_{1|2n})$, where

$$h_i(z) = 2\sum_{j=i}^{n} \alpha_j(z), \quad \gamma_i = (-1)^i \prod_{j=1}^{i} (2(2j-1)(k+n)-1)(4j(k+n)+1),$$

which satisfy that

$$[h_{i\,\lambda} h_j] = (2k + 2n + 1)\,\delta_{i,j}\lambda, \quad i, j = 1, \ldots, n.$$

If $k + n + 1/2 \neq 0$,

$$L = \frac{W_2}{2(2k + 2n + 1)}$$

is a unique conformal vector of $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$ with the central charge

$$c(k) = -\frac{(2n + 1)(2(2n - 1)(k + n) - 1)(4n(k + n) + 1)}{2(2k + 2n + 1)}.$$

6 Zhu algebras of $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$

By (4.5), we have an isomorphism

$$\iota_1 \colon \mathrm{Zhu}_H \mathcal{W}^k(\mathfrak{osp}_{1|2n}) \xrightarrow{\simeq} U(\mathfrak{osp}_{1|2n}, f_{\mathrm{prin}}).$$

Then ι_1 is induced by (4.4):

$$\begin{aligned} &\mathrm{Zhu}_H C^k(\mathfrak{osp}_{1|2n}, f_{\mathrm{prin}}) \xrightarrow{\simeq} C_+, \\ &J^u \mapsto j^u + (2k + 2n + 1)(\rho_{\mathfrak{osp}} \,|\, u), \quad \phi_\alpha \mapsto \Phi_\alpha, \quad \varphi_\alpha^* \mapsto \psi_\alpha^*, \end{aligned}$$

where

$$\rho_{\mathfrak{osp}} = \frac{1}{2} \sum_{\alpha \in \Delta_+} (-1)^{p(\alpha)} \alpha.$$

Let $\mathbb{C}[\mathfrak{h}^*] = U(\mathfrak{h})$ and set an isomorphism

$$\begin{aligned} &\iota_2 \colon \mathrm{Zhu}_H \pi \otimes \mathrm{Zhu}_H F \xrightarrow{\simeq} \mathbb{C}[\mathfrak{h}^*] \otimes \Phi, \\ &h_i \mapsto h_i + (2n - 2i + 1)\left(k + n + \frac{1}{2}\right), \quad \phi_{\alpha_n} \mapsto \Phi_{\alpha_n}. \end{aligned}$$

Then we have a commutative diagram of Miura maps

$$\begin{array}{ccc} \mathrm{Zhu}_H \mathcal{W}^k(\mathfrak{osp}_{1|2n}) & \xrightarrow{\mathrm{Zhu}_H(\Upsilon)} & \mathrm{Zhu}_H \pi \otimes \mathrm{Zhu}_H F \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} \\ U(\mathfrak{osp}_{1|2n}, f_{\mathrm{prin}}) & \xrightarrow{\mu} & \mathbb{C}[\mathfrak{h}^*] \otimes \Phi. \end{array} \tag{6.1}$$

By [6], $\mathrm{Zhu}_H \mathcal{W}^k(\mathfrak{osp}_{1|2n})$ has a PBW basis generated by $G, W_2, W_4, \ldots, W_{2n}$. By abuse of notation, we shall use the same notation for the generators of $U(\mathfrak{osp}_{1|2n}, f_{\mathrm{prin}})$ corresponding to $G, W_2, W_4, \ldots, W_{2n}$ by ι_1.

Lemma 6.1 $\mu(G) = (h_1 + \rho_{\mathfrak{osp}}(h_1))(h_2 + \rho_{\mathfrak{osp}}(h_2)) \cdots (h_n + \rho_{\mathfrak{osp}}(h_n)) \otimes \Phi_{\alpha_n}$.

Proof We have

$$\begin{aligned}\Upsilon(G) &= :(2(k+n)\partial + h_1)\cdots(2(k+n)\partial + h_n)\phi: \\ &\equiv (-(2n-1)(k+n) + h_1) * (-(2n-3)(k+n) + h_2) \\ &\quad \cdots * (-(k+n) + h_n) * \phi \quad \big(\mathrm{mod}\ \mathcal{W}^k(\mathfrak{osp}_{1|2n}) \circ \mathcal{W}^k(\mathfrak{osp}_{1|2n})\big).\end{aligned}$$

Thus

$$\begin{aligned}\mu(G) &= \iota_2\big((-(2n-1)(k+n) + h_1) * (-(2n-3)(k+n) + h_2) \\ &\qquad \cdots * (-(k+n) + h_n) * \phi\big) \\ &= \left(h_1 + n - 1 + \frac{1}{2}\right)\left(h_2 + n - 2 + \frac{1}{2}\right)\cdots\left(h_n + \frac{1}{2}\right) \otimes \Phi_{\alpha_n}.\end{aligned}$$

Therefore the assertion follows from the fact that $\rho_{\mathfrak{osp}}(h_i) = n - i + 1/2$. □

For a basic classical Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{g}_{\bar{1}} \neq 0$, denote by

$$\begin{aligned}Z(\mathfrak{g}) &= \big\{z \in U(\mathfrak{g}) \mid uz - (-1)^{p(u)p(z)} zu = 0 \text{ for all } u \in \mathfrak{g}\big\}, \\ \mathcal{A}(\mathfrak{g}) &= \big\{a \in U(\mathfrak{g}) \mid ua - (-1)^{p(u)(p(a)+\bar{1})} au = 0 \text{ for all } u \in \mathfrak{g}\big\}, \\ \widetilde{Z}(\mathfrak{g}) &= Z(\mathfrak{g}) \oplus \mathcal{A}(\mathfrak{g}),\end{aligned}$$

called the center, the anticenter and the ghost center of $U(\mathfrak{g})$, respectively due to [14]. Then the ghost center $\widetilde{Z}(\mathfrak{g})$ coincides with the center of $U(\mathfrak{g})_{\bar{0}}$ by [14, Corollary 4.4.4]. In case that $\mathfrak{g} = \mathfrak{osp}_{1|2n}$, there exists $T \in U(\mathfrak{g})_{\bar{0}}$ [2, 17, 25] such that

$$\mathcal{A}(\mathfrak{osp}_{1|2n}) = Z(\mathfrak{osp}_{1|2n})T, \quad (\sigma \circ \eta)(T) = h_1 h_2 \cdots h_n,$$

where

$$\eta \colon U(\mathfrak{osp}_{1|2n}) \twoheadrightarrow U(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$$

is the projection induced by the decomposition

$$U(\mathfrak{osp}_{1|2n}) \simeq \mathfrak{n}_- U(\mathfrak{osp}_{1|2n}) \oplus U(\mathfrak{h}) \oplus U(\mathfrak{osp}_{1|2n})\,\mathfrak{n}$$

and σ is an isomorphism defined by

$$\sigma : \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*], \quad f \mapsto (\sigma(f) \colon \lambda \mapsto f(\lambda - \rho_{\mathfrak{osp}})).$$

The element T is called the Casimir's ghost [2] since $T^2 \in Z(\mathfrak{osp}_{1|2n})$ is such that $(\sigma\circ\eta)(T^2) = h_1^2 \cdots h_n^2$, and is studied for general $\mathfrak{g}$ in [14]. It is well known [15, 19] that the restriction of $\sigma\circ\eta$ to $Z(\mathfrak{g})$ is injective and maps onto $\mathbb{C}[\mathfrak{h}^*]^W$, where W is the Weyl group of $\mathfrak{sp}_{2n}$, called the Harish–Chandra homomorphism of $\mathfrak{osp}_{1|2n}$. Recall that

$$U(\mathfrak{osp}_{1|2n}, f_{\text{prin}}) \simeq U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}} = (U(\mathfrak{osp}_{1|2n})/I_{-\chi})^{\operatorname{ad}\mathfrak{n}},$$

where $I_{-\chi}$ is a left $U(\mathfrak{osp}_{1|2n})$-module generated by $u_\alpha + (f_{\text{prin}}|u_\alpha)$ for all $\alpha \in \Delta_+\setminus\{\alpha_n\}$. Define the projections q_1, q_2 by

$$\begin{aligned}
&q_1\colon U(\mathfrak{osp}_{1|2n}) \twoheadrightarrow U(\mathfrak{osp}_{1|2n})/I_{-\chi},\\
&q_2\colon U(\mathfrak{osp}_{1|2n})/I_{-\chi} \simeq \mathfrak{n}_- U(\mathfrak{osp}_{1|2n})/I_{-\chi} \oplus U(\mathfrak{h}) \oplus U(\mathfrak{h})\,u_{\alpha_n} \twoheadrightarrow U(\mathfrak{h}) \oplus U(\mathfrak{h})\,u_{\alpha_n}
\end{aligned}$$

and a linear map q_3 by

$$q_3\colon U(\mathfrak{h}) \oplus U(\mathfrak{h})\,u_{\alpha_n} \to \mathbb{C}[\mathfrak{h}^*]\otimes\Phi, \quad (f_1,\ f_2\cdot u_{\alpha_n}) \mapsto f_1\otimes 1 + f_2\otimes\Phi_{\alpha_n}.$$

Then, using the quasi-isomorphism $i_{\text{I}\to\text{II}}$ in (4.3), the Miura map μ can be identified with the restriction of the composition map $q_3\circ q_2$ to $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}}$ since $u_{\alpha_n} = X_{\alpha_n} + \Phi_{\alpha_n}$.

Lemma 6.2 *$q_1(Tu_{\alpha_n})$ is the element of $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}}$ corresponding to G.*

Proof First of all, we show that $q_1(Tu_{\alpha_n}) \in U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}}$. It is enough to show that $[u_\alpha, Tu_{\alpha_n}] \equiv 0 \pmod{I_{-\chi}}$ for all $\alpha \in \Delta_+$. Let $\Delta_{+,\bar{i}} = \{\alpha \in \Delta_+ \mid p(u_\alpha) = \bar{i}\}$. Since $[u_\alpha, T] = 0$ for $\alpha \in \Delta_{+,\bar{0}}$, we have

$$[u_\alpha, Tu_{\alpha_n}] = T[u_\alpha, u_{\alpha_n}] \equiv 0 \pmod{I_{-\chi}}, \quad \alpha \in \Delta_{+,\bar{0}}.$$

Next, for $\alpha \in \Delta_{+,\bar{1}}\setminus\{\alpha_n\}$, since $u_\alpha T + Tu_\alpha = 0$, we also have

$$[u_\alpha, Tu_{\alpha_n}] = -T[u_\alpha, u_{\alpha_n}] + 2Tu_{\alpha_n}u_\alpha \equiv 0 \pmod{I_{-\chi}}, \quad \alpha \in \Delta_{+,\bar{1}}\setminus\{\alpha_n\}.$$

Finally, in case that $\alpha = \alpha_n$,

$$[u_{\alpha_n}, Tu_{\alpha_n}] = (u_{\alpha_n}T + Tu_{\alpha_n})u_{\alpha_n} = 0.$$

Therefore, $q_1(Tu_{\alpha_n})$ belongs to $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}}$. Now $\mu = q_3\circ q_2|_{U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}}}$ and by definition,

$$\begin{aligned}
((\sigma\otimes 1)\circ\mu)(q_1(Tu_{\alpha_n})) &= ((\sigma\otimes 1)\circ q_3\circ q_2\circ q_1)(Tu_{\alpha_n})\\
&= (\sigma\circ\eta)(T)\otimes\Phi_{\alpha_n} = h_1\cdots h_n\otimes\Phi_{\alpha_n}.
\end{aligned}$$

By Lemma 6.1, $((\sigma\otimes 1)\circ\mu)(G) = h_1\cdots h_n\otimes\Phi_{\alpha_n}$. Since $(\sigma\otimes 1)\circ\mu$ is injective, we have $q_1(Tu_{\alpha_n}) = G$. ◻

Theorem 6.3 $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\bar{0}} \simeq Z(\mathfrak{osp}_{1|2n})$.

Proof Since $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ has a PBW basis generated by $G, W_2, W_4, \ldots, W_{2n}$ and G is a unique odd generator, $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\bar{0}}$ has a PBW basis generated by $W_2, W_4, \ldots, W_{2n}$. Now Φ is a superalgebra generated by Φ_{α_n} with the relation $2\Phi_{\alpha_n}^2 = \chi(u_{\alpha_n}, u_{\alpha_n})$. Thus μ maps $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\bar{0}}$ to $\mathbb{C}[\mathfrak{h}^*]$. By (5.1), $\mu(W_{2i})$ for $i = 1, \ldots, n$ are algebraically independent in $\mathbb{C}[\mathfrak{h}^*]$ with degree $2i$ (but not necessary homogeneous). Now, by definition, $q_2 \circ q_1 = \eta$ on $Z(\mathfrak{osp}_{1|2n})$. Hence $q_2 \circ q_1|_{Z(\mathfrak{osp}_{1|2n})}$ is injective. In particular, $q_1|_{Z(\mathfrak{osp}_{1|2n})}$ is injective. Clearly, $q_1(Z(\mathfrak{osp}_{1|2n}))$ is $\operatorname{ad}\mathfrak{n}$-invariant. Thus, $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}}) \simeq U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}}$ contains $Z(\mathfrak{osp}_{1|2n})$ through q_1. Moreover

$$\mu(Z(\mathfrak{osp}_{1|2n})) = (q_3 \circ q_2 \circ q_1)(Z(\mathfrak{osp}_{1|2n})) = \eta(Z(\mathfrak{osp}_{1|2n})) = \sigma^{-1}(\mathbb{C}[\mathfrak{h}^*]^W).$$

Since $\mathbb{C}[\mathfrak{h}^*]^W$ is a symmetric algebra of $h_1^2, \ldots, h_n^2$, $\mu(Z(\mathfrak{osp}_{1|2n}))$ must contain all $\mu(W_{2i})$ for $i = 1, \ldots, n$. Therefore

$$U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\bar{0}} \simeq Z(\mathfrak{osp}_{1|2n}).$$

This completes the proof. □

Corollary 6.4 $(\text{Zhu}_H \mathcal{W}^k(\mathfrak{osp}_{1|2n}))_{\bar{0}} \simeq Z(\mathfrak{osp}_{1|2n})$.

Proof The assertion is immediate from Theorem 6.3 and the fact that

$$\text{Zhu}_H \mathcal{W}^k(\mathfrak{osp}_{1|2n}) \simeq U(\mathfrak{osp}_{1|2n}, f_{\text{prin}}).$$ □

Consider a linear isomorphism

$$\xi : \widetilde{Z}(\mathfrak{osp}_{1|2n}) = Z(\mathfrak{osp}_{1|2n}) \oplus \mathcal{A}(\mathfrak{osp}_{1|2n}) \xrightarrow{\simeq} Z(\mathfrak{osp}_{1|2n}) \oplus \mathcal{A}(\mathfrak{osp}_{1|2n})\, u_{\alpha_n}$$

defined by $\xi(z, a) = (z, a\, u_{\alpha_n})$. Then by Lemma 6.2 and the fact that $\mathcal{A}(\mathfrak{osp}_{1|2n}) = Z(\mathfrak{osp}_{1|2n})T$, we have $(q_1 \circ \xi)(\widetilde{Z}(\mathfrak{osp}_{1|2n})) \subset U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})_{\text{II}}$.

Theorem 6.5 *The map* $q_1 \circ \xi : \widetilde{Z}(\mathfrak{osp}_{1|2n}) \to U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ *is an isomorphism of associative algebras.*

Proof By definition and Lemma 6.2, $(q_3 \circ q_2 \circ q_1 \circ \xi)(zT) = (q_3 \circ q_2 \circ q_1)(zT u_{\alpha_n}) = \eta(z)G$ for all $z \in Z(\mathfrak{osp}_{1|2n})$. Thus, $q_3 \circ q_2 \circ q_1 \circ \xi|_{\mathcal{A}(\mathfrak{osp}_{1|2n})}$ is injective. In particular, $q_1 \circ \xi|_{\mathcal{A}(\mathfrak{osp}_{1|2n})}$ is injective. Using the fact that $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ has a PBW basis generated by $G, W_2, W_4, \ldots, W_{2n}$ and Theorem 6.3, it follows that $q_1 \circ \xi$ is a linear isomorphism. Now, we may suppose that $\chi(u_{\alpha_n}, u_{\alpha_n}) = 2$. Then $\Phi_{\alpha_n}^2 = 1$ so that $\mu(T^2) = \sigma^{-1}(h_1^2 \cdots h_n^2) = \mu(G^2)$. Therefore $q_1 \circ \xi$ defines an isomorphism of associative algebras. □

Let $L(\lambda)$ be the simple highest weight $\mathfrak{osp}_{1|2n}$-module with the highest weight λ. Then there exists $\chi_\lambda\colon Z(\mathfrak{osp}_{1|2n}) \to \mathbb{C}$ such that z acts on $\chi_\lambda(z)$ on $L(\lambda)$ for all $z \in Z(\mathfrak{osp}_{1|2N})$. The map χ_λ is called a central character of $\mathfrak{osp}_{1|2n}$ and is induced by η and one-dimensional $\mathbb{C}[\mathfrak{h}^*]$-module $\mathbb{C}_\lambda$ defined by $f \mapsto f(\lambda)$. Using the Harish–Chandra homomorphism, it follows that $\chi_{\lambda_1} = \chi_{\lambda_2}$ if and only if $\lambda_2 = w(\lambda_1 + \rho_{\mathfrak{osp}}) - \rho_{\mathfrak{osp}}$ for some $w \in W$. Let

$$D = \left\{\lambda \in \mathfrak{h}^* \;\middle|\; \prod_{\alpha \in \Delta_{\bar{1}}} (\lambda + \rho_{\mathfrak{osp}} \,|\, \alpha) = 0\right\}.$$

Denote by $\chi_\lambda \in D$ if $\lambda \in D$. Since $w(\Delta_{\bar{1}}) \subset \Delta_{\bar{1}}$ for all $w \in W$, we have $\lambda \in D \Rightarrow w(\lambda + \rho_{\mathfrak{osp}}) - \rho_{\mathfrak{osp}} \in D$ for any $w \in W$ so that $\chi_\lambda \in D$ is well defined.

From now on, we will identify $\widetilde{Z}(\mathfrak{osp}_{1|2n})$ with $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$ by Theorem 6.5. Then $\widetilde{Z}(\mathfrak{osp}_{1|2n})$ is a superalgebra such that $\widetilde{Z}(\mathfrak{osp}_{1|2n})_{\bar{1}} = \mathcal{A}(\mathfrak{osp}_{1|2n})$. Let E be a finite-dimensional $\mathbb{Z}_2$-graded simple $\widetilde{Z}(\mathfrak{osp}_{1|2n})$-module. Then $Z(\mathfrak{osp}_{1|2n})$ acts on E as χ_λ for some $\lambda \in \mathfrak{h}^*$. For a non-zero parity-homogeneous element $v \in E$, Tv has an opposite parity to v such that $T^2 v = \chi_\lambda(T^2)v$. Recall that the set $\{h_1, \dots, h_n\}$ is identified with $2\Delta_{+,\bar{1}}$ by $\mathfrak{h} \simeq \mathfrak{h}^*$. Then, using the fact that $\eta(T^2) = \sigma^{-1}(h_1^2 \cdots h_n^2)$, it follows that

$$\chi_\lambda(T^2) = \prod_{i=1}^{n} \big((\lambda + \rho_{\mathfrak{osp}})(h_i)\big)^2 = \prod_{\alpha \in \Delta_{+,\bar{1}}} (\lambda + \rho_{\mathfrak{osp}} \,|\, 2\alpha)^2.$$

Hence $\chi_\lambda(T^2) = 0$ if and only if $\chi_\lambda \in D$. Since E is simple, $E = \mathbb{C}v$ if $\chi_\lambda \in D$ and $E = \mathbb{C}v \oplus \mathbb{C}Tv$ if $\chi_\lambda \notin D$, which we denote by E_{χ_λ}. Here we identify E_{χ_λ} with the parity change of E_{χ_λ} if $\chi_\lambda(T^2) = 0$. Therefore we obtain the following results:

Proposition 6.6 *A finite-dimensional $\mathbb{Z}_2$-graded simple $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$-module is isomorphic to E_{χ_λ} for some $\lambda \in \mathfrak{h}^*$. In particular, there exists one-to-one correspondence between isomorphism classes (up to the parity change) of finite-dimensional $\mathbb{Z}_2$-graded simple $U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$-modules and central characters of $\mathfrak{osp}_{1|2n}$.*

Corollary 6.7 *There exists a bijective correspondence between central characters of $\mathfrak{osp}_{1|2n}$ and isomorphism classes (up to the parity change) of simple positive-energy Ramond-twisted $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$-modules with finite-dimensional top spaces.*

Proof The assertion is immediate from $\text{Zhu}_H \mathcal{W}^k(\mathfrak{osp}_{1|2n}) \simeq U(\mathfrak{osp}_{1|2n}, f_{\text{prin}})$, Proposition 6.6 and [6, Theorem 2.30]. □

Corollary 6.7 implies that dimensions of the top spaces E_{χ_λ} of simple positive-energy Ramond-twisted $\mathcal{W}^k(\mathfrak{osp}_{1|2n})$-modules are equal to 2 if and only if $(\lambda + \rho_{\mathfrak{osp}} \,|\, \alpha) \neq 0$ for all $\alpha \in \Delta_{\bar{1}}$. We remark that this condition is equivalent to one that the annihilator of the Verma module $M(\lambda)$ is generated by its intersection with the center $Z(\mathfrak{osp}_{1|2n})$ by [16].

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Declarations

Conflict of interest The author declares no conflict of interest associated with this manuscript.

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