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RESEARCH ARTICLE

The degree of commutativity of wreath products with infinite cyclic top group

Iker de las Heras[1,2] · Benjamin Klopsch[1] · Andoni Zozaya[2]

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Abstract
The degree of commutativity of a finite group is the probability that two uniformly and randomly chosen elements commute. This notion extends naturally to finitely generated groups G: the degree of commutativity $\mathrm{dc}_S(G)$, with respect to a given finite generating set S, results from considering the fractions of commuting pairs of elements in increasing balls around 1_G in the Cayley graph $\mathcal{C}(G, S)$. We focus on restricted wreath products of the form $G = H \wr \langle t \rangle$, where $H \neq 1$ is finitely generated and the top group $\langle t \rangle$ is infinite cyclic. In accordance with a more general conjecture, we show that $\mathrm{dc}_S(G) = 0$ for such groups G, regardless of the choice of S. This extends results of Cox who considered lamplighter groups with respect to certain kinds of generating sets. We also derive a generalisation of Cox's main auxiliary result: in 'reasonably large' homomorphic images of wreath products G as above, the image of the base group has density zero, with respect to certain types of generating sets.

Keywords Degree of commutativity · Wreath products · Density · Word growth

Mathematics Subject Classification 20P05 · 20E22 · 20F65 · 20F69 · 20F05

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✉ Benjamin Klopsch
klopsch@math.uni-duesseldorf.de

Iker de las Heras
iker.delasheras@ehu.eus

Andoni Zozaya
andoni.zozaya@ehu.eus

1 Department of Mathematics, Heinrich-Heine-Universität Düsseldorf, Mathematisch-Naturwissenschaftliche Fakultät, 40225 Düsseldorf, Germany

2 Department of Mathematics, University of the Basque Country UPV/EHU, 48940 Leioa, Spain

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1 Introduction

Let G be a finitely generated group, with finite generating set S. For $n \in \mathbb{N}_0$, let $B_S(n) = B_{G,S}(n)$ denote the ball of radius n in the Cayley graph $\mathcal{C}(G, S)$ of G with respect to S. Following Antolín, Martino and Ventura [1], we define the *degree of commutativity* of G with respect to S as

$$\mathrm{dc}_S(G) = \limsup_{n\to\infty} \frac{|\{(g, h) \in B_S(n) \times B_S(n) \mid gh = hg\}|}{|B_S(n)|^2}.$$

We remark that this notion can be viewed as a special instance of a more general concept, where the degree of commutativity is defined with respect to 'reasonable' sequences of probability measures on G, as discussed in a preliminary `arXiv`-version of [1] or, in more detail, by Tointon in [13].

If G is finite, the invariant $\mathrm{dc}_S(G)$ does not depend on the particular choice of S, as the balls stabilise and $\mathrm{dc}(G) = \mathrm{dc}_S(G)$ simply gives the probability that two uniformly and randomly chosen elements of G commute. This situation was studied already by Erdős and Turán [4], and further results were obtained by various authors over the years; for example, see [5, 6, 8, 9, 11]. For infinite groups G, it is generally not known whether $\mathrm{dc}_S(G)$ is independent of the particular choice of S.

The degree of commutativity is naturally linked to the following concept of density, which is employed, for instance, in [2]. The *density* of a subset $X \subseteq G$ with respect to S is

$$\delta_S(X) = \delta_{G,S}(X) = \limsup_{n\to\infty} \frac{|X \cap B_S(n)|}{|B_S(n)|}.$$

If the group G has sub-exponential word growth, then the density function δ_S is bi-invariant; compare with [2, Proposition 2.3]. Based on this fact, the following can be proved, initially for residually finite groups and then without this additional restriction, even in the more general context of suitable sequences of probability measures; see [1, Theorem 1.3] and [13, Theorems 1.6 and 1.17].

Theorem 1.1 (Antolín, Martino and Ventura [1]; Tointon [13]) *Let G be a finitely generated group of sub-exponential word growth, and let S be a finite generating set of G. Then $\mathrm{dc}_S(G) > 0$ if and only if G is virtually abelian. Moreover, $\mathrm{dc}_S(G)$ does not depend on the particular choice of S.*

The situation is far less clear for groups of exponential word growth. In this context, the following conjecture was raised in [1].

Conjecture 1.2 (Antolín, Martino and Ventura [1]) *Let G be a finitely generated group of exponential word growth and let S be a finite generating set of G. Then, $\mathrm{dc}_S(G) = 0$, irrespective of the choice of S.*

In [1] the conjecture was already confirmed for non-elementary hyperbolic groups, and Valiunas [14] confirmed it for right-angled Artin groups (and more general graph products of groups) with respect to certain generating sets. Furthermore, Cox [3] showed that the conjecture holds, with respect to *selected* generating sets, for (generalised) lamplighter groups, that is for restricted standard wreath products of the form $G = F \wr \langle t \rangle$, where $F \neq 1$ is finite and $\langle t \rangle$ is an infinite cyclic group. Wreath products of such a shape are basic examples of groups of exponential word growth; in Sect. 2 we briefly recall the wreath product construction, here we recall that $G = N \rtimes \langle t \rangle$ with base group $N = \bigoplus_{i \in \mathbb{Z}} F^{t^i}$. In the present paper, we make a significant step forward in two directions, by confirming Conjecture 1.2 for an even wider class of restricted standard wreath products and with respect to *arbitrary* generating sets.

Theorem A *Let $G = H \wr \langle t \rangle$ be the restricted wreath product of a finitely generated group $H \neq 1$ and an infinite cyclic group $\langle t \rangle \cong C_\infty$. Then G has degree of commutativity* $\mathrm{dc}_S(G) = 0$, *for every finite generating set S of G.*

One of the key ideas in [3] is to reduce the desired conclusion $\mathrm{dc}_S(G) = 0$, for the wreath products $G = N \rtimes \langle t \rangle$ under consideration, to the claim that the base group N has density $\delta_S(N) = 0$ in G. We proceed in a similar way and derive Theorem A from the following density result, which constitutes our main contribution.

Theorem B *Let $G = H \wr \langle t \rangle$ be the restricted wreath product of a finitely generated group H and an infinite cyclic group $\langle t \rangle \cong C_\infty$. Then the base group $N = \bigoplus_{i \in \mathbb{Z}} H^{t^i}$ has density $\delta_S(N) = 0$ in G, for every finite generating set S of G.*

The limitation in [3] to special generating sets S of lamplighter groups G is due to the fact that the arguments used there rely on explicit minimal length expressions for elements $g \in G$ with respect to S. If one restricts to generating sets which allow control over minimal length expressions in a similar, but somewhat weaker way, it is, in fact, possible to simplify and generalise the analysis considerably. In this way we arrive at the following improvement of the results in [3, Section 2.2], for homomorphic images of wreath products.

Theorem C *Let G be a finitely generated group of exponential word growth of the form $G = N \rtimes \langle t \rangle$, where*

(a) *the subgroup $\langle t \rangle$ is infinite cyclic;*
(b) *the normal subgroup $N = \left\langle \bigcup \{H^{t^i} \mid i \in \mathbb{Z}\} \right\rangle$ is generated by the $\langle t \rangle$-conjugates of a finitely generated subgroup H of N;*
(c) *the $\langle t \rangle$-conjugates of this group H commute elementwise: $[H^{t^i}, H^{t^j}] = 1$ for all $i, j \in \mathbb{Z}$ with $H^{t^i} \neq H^{t^j}$.*

Suppose further that S_0 is a finite generating set for H and that the exponential growth rates of H with respect to S_0 and of G with respect to $S = S_0 \cup \{t\}$ satisfy

$$\lim_{n\to\infty} \sqrt[n]{|B_{H,S_0}(n)|} < \lim_{n\to\infty} \sqrt[n]{|B_{G,S}(n)|}. \tag{1.1}$$

Then N has density $\delta_S(N) = 0$ in G with respect to S.

For finitely generated groups G of sub-exponential word growth, the density of a subgroup of infinite index, such as N in $G = N \rtimes \langle t \rangle$ with $\langle t \rangle \cong C_\infty$, is always 0; see [2]. Thus Theorem C has the following consequence.

Corollary 1.3 *Let $G = A \rtimes \langle t \rangle$ be a finitely generated group, where A is abelian and $\langle t \rangle \cong C_\infty$. Then A has density $\delta_S(A) = 0$ in G, with respect to any finite generating set of G that takes the form $S = S_0 \cup \{t\}$ with $S_0 \subseteq A$.*

Next we give a very simple concrete example to illustrate that the technical condition (1.1) in Theorem C is not redundant: the situation truly differs from the one for groups of sub-exponential word growth. It is not difficult to craft more complex examples.

Example 1.4 Let $G = F \times \langle t \rangle$, where $F = \langle x, y \rangle$ is the free group on two generators and $\langle t \rangle \cong C_\infty$. Then F has density $\delta_S(F) = 1/2 > 0$ in G for the 'obvious' generating set $S = \{x, y, t\}$.

Indeed, for every $i \in \mathbb{Z}$ we have

$$B_{G,S}(n) \cap Ft^i = \begin{cases} B_{F,\{x,y\}}(n - |i|)\, t^i & \text{if } n \in \mathbb{N} \text{ with } n \geqslant |i|, \\ \varnothing & \text{otherwise,} \end{cases}$$

and hence, for all $n \in \mathbb{N}$,

$$|B_{G,S}(n) \cap F| = |B_{F,\{x,y\}}(n)|$$

and

$$|B_{G,S}(n)| = |B_{F,\{x,y\}}(n)| + 2 \sum_{i=1}^{n} |B_{F,\{x,y\}}(n-i)|.$$

This yields

$$\begin{aligned} \frac{|B_{G,S}(n) \cap F|}{|B_{G,S}(n)|} &= \frac{2 \cdot 3^n - 1}{2 \cdot 3^n - 1 + 2 \sum_{i=1}^{n} (2 \cdot 3^{n-i} - 1)} \\ &= \frac{2 \cdot 3^n - 1}{4 \cdot 3^n - 2n - 3} \to \frac{1}{2} \quad \text{as } n \to \infty. \end{aligned}$$

We remark that in this example F and G have the same exponential growth rates:

$$\lim_{n \to \infty} \sqrt[n]{|B_{F,\{x,y\}}(n)|} = \lim_{n \to \infty} \sqrt[n]{|B_{G,S}(G)|} = 3.$$

Furthermore, the argument carries through without any obstacles with any finite generating set S_0 of F in place of $\{x, y\}$.

Finally, we record an open question that suggests itself rather naturally.

Question 1.5 Let G be a finitely generated group such that $\mathrm{dc}_S(G) > 0$ with respect to a finite generating set S. Does it follow that there exists an abelian subgroup $A \leqslant G$ such that $\delta_S(A) > 0$?

For groups G of sub-exponential word growth the answer is "yes", as one can see by an easy argument from Theorem 1.1. An affirmative answer for groups of exponential word growth could be a step towards establishing Conjecture 1.2 or provide a pathway to a possible alternative outcome. At a heuristic level, an affirmative answer to Question 1.5 would fit well with the results in [12, 13].

Notation Our notation is mostly standard. For a set X, we denote by $\mathcal{P}(X)$ its power set. For elements g, h of a group G, we write $g^h = h^{-1}gh$ and $[g, h] = g^{-1}g^h$ in line with our preferred use of right actions. For a finite generating set S of G, we denote by $l_S(g)$ the length of g with respect to S, i.e., the distance between g and 1 in the corresponding Cayley graph $\mathcal{C}(G, S)$ so that

$$B_S(n) = B_{G,S}(n) = \{g \in G \mid l_S(g) \leqslant n\} \quad \text{for} \quad n \in \mathbb{N}_0.$$

Given $a, b \in \mathbb{R}$ and $T \subseteq \mathbb{R}$, we write $[a, b]_T = \{x \in T \mid a \leqslant x \leqslant b\}$; for instance, $[-2, \sqrt{2}]_{\mathbb{Z}} = \{-2, -1, 0, 1\}$. We repeatedly compare the limiting behaviour of real-valued functions defined on cofinite subsets of $\mathbb{N}_0$ which are eventually non-decreasing and take positive values. For this purpose we employ the conventional Landau symbols; specifically we write, for functions f, g of the described type,

$$f \in o(g), \quad \text{or} \quad g \in \omega(f), \quad \text{if} \ \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0, \ \text{equivalently} \ \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty.$$

As customary, we use suggestive short notation such as, for instance, $f \in o(\log n)$ in place of $f \in o(g)$ for $g \colon \mathbb{N}_{\geqslant 2} \to \mathbb{R}, n \mapsto \log n$.

2 Preliminaries

In this section, we collect preliminary and auxiliary results. Furthermore, we briefly recall the wreath product construction and establish basic notation.

2.1 Groups of exponential word growth

We concern ourselves with groups of exponential word growth. These are finitely generated groups G such that for any finite generating set S of G, the *exponential growth rate*

$$\lambda_S(G) = \lim_{n\to\infty} \sqrt[n]{|B_S(n)|} = \inf\left\{\sqrt[n]{|B_S(n)|} \mid n \in \mathbb{N}_0\right\} \tag{2.1}$$

of G with respect to S satisfies $\lambda_S(G) > 1$. Since the word growth sequence $|B_S(n)|$, $n \in \mathbb{N}$, is submultiplicative, i.e.,

$$|B_S(n+m)| \leqslant |B_S(n)||B_S(m)| \quad \text{for all} \quad n, m \in \mathbb{N},$$

the limit in (2.1) exists and is equal to the infimum as stated, by Fekete's lemma [7, Corollary VI.57]. We will use the following basic estimates:

$$\lambda_S(G)^n \leqslant |B_S(n)| \quad \text{for all} \quad n \in \mathbb{N}_0,$$

and, for each $\varepsilon \in \mathbb{R}_{>0}$,

$$|B_S(n)| \leqslant (\lambda_S(G) + \varepsilon)^n \quad \text{for all sufficiently large} \quad n \in \mathbb{N}.$$

In the proof of Theorem C, the following two auxiliary results are used.

Lemma 2.1 *For each $\alpha \in [0, 1]_{\mathbb{R}}$, the sequences $\sqrt[n]{\binom{n+\lceil \alpha n \rceil}{\lceil \alpha n \rceil}}$ and $\sqrt[n]{\binom{n}{\lceil \alpha n \rceil}}$, $n \in \mathbb{N}$, converge, and furthermore*

$$\lim_{\alpha \to 0^+} \left(\lim_{n \to \infty} \sqrt[n]{\binom{n + \lceil \alpha n \rceil}{\lceil \alpha n \rceil}} \right) = \lim_{\alpha \to 0^+} \left(\lim_{n \to \infty} \sqrt[n]{\binom{n}{\lceil \alpha n \rceil}} \right) = 1.$$

Consequently, if $f : \mathbb{N} \to \mathbb{R}_{>0}$ satisfies $f \in o(n)$, then the sequence $\binom{n+\lceil f(n) \rceil}{\lceil f(n) \rceil}$, $n \in \mathbb{N}$, grows sub-exponentially in n, viz. $\sqrt[n]{\binom{n+\lceil f(n) \rceil}{\lceil f(n) \rceil}} \to 1$ as $n \to \infty$.

Proof For each $\alpha \in [0, 1]_{\mathbb{R}}$, Stirling's approximation for factorials yields

$$\begin{aligned}\binom{n + \lceil \alpha n \rceil}{\lceil \alpha n \rceil} &\sim \frac{\sqrt{2\pi(n + \lceil \alpha n \rceil)}\,((n + \lceil \alpha n \rceil)/e)^{(n+\lceil \alpha n \rceil)}}{\sqrt{2\pi \lceil \alpha n \rceil}\,(\lceil \alpha n \rceil/e)^{\lceil \alpha n \rceil}\sqrt{2\pi n}\,(n/e)^n} \\ &= \frac{\sqrt{n + \lceil \alpha n \rceil}}{\sqrt{2\pi n \lceil \alpha n \rceil}} \cdot \frac{\lceil n + \alpha n \rceil^{\lceil n+\alpha n \rceil}}{\lceil \alpha n \rceil^{\lceil \alpha n \rceil} n^n}, \quad \text{as} \quad n \to \infty,\end{aligned}$$

i.e., the ratio of the left-hand term to the right-hand term tends to 1 as n tends to infinity. Moreover, for all $n \in \mathbb{N}$,

$$\frac{\lceil n + \alpha n \rceil^{\lceil n+\alpha n \rceil}}{\lceil \alpha n \rceil^{\lceil \alpha n \rceil}\, n^n} \geqslant \frac{(n + \alpha n)^{n+\alpha n}}{(\alpha n + 1)^{\alpha n+1}\, n^n} = n^{-1} \left(\frac{(1 + \alpha)^{1+\alpha}}{(\alpha + 1/n)^{(\alpha+1/n)}} \right)^n$$

and similarly

$$\frac{\lceil n + \alpha n \rceil^{\lceil n+\alpha n \rceil}}{\lceil \alpha n \rceil^{\lceil \alpha n \rceil}\, n^n} \leqslant \frac{(n + \alpha n + 1)^{n+\alpha n+1}}{(\alpha n)^{\alpha n}\, n^n} = n \left(\frac{(1 + \alpha + 1/n)^{(1+\alpha+1/n)}}{\alpha^\alpha} \right)^n.$$

This shows that

$$\lim_{n\to\infty} \sqrt[n]{\binom{n+\lceil \alpha n\rceil}{\lceil \alpha n\rceil}} = \frac{(1+\alpha)^{1+\alpha}}{\alpha^{\alpha}}.$$

Since $\lim_{\alpha\to 0^+} \alpha^\alpha = 1$, we conclude that

$$\lim_{\alpha\to 0^+} \left(\lim_{n\to\infty} \sqrt[n]{\binom{n+\lceil \alpha n\rceil}{\lceil \alpha n\rceil}} \right) = 1.$$

A similar computation yields that the second sequence $\sqrt[n]{\binom{n}{\lceil \alpha n\rceil}}$, $n \in \mathbb{N}$, converges. Again directly, or by virtue of

$$1 \leqslant \sqrt[n]{\binom{n}{\lceil \alpha n\rceil}} \leqslant \sqrt[n]{\binom{n+\lceil \alpha n\rceil}{\lceil \alpha n\rceil}},$$

we conclude that also the second limit, for $\alpha \to 0^+$, is equal to 1. □

Proposition 2.2 *Let G be a finitely generated group of exponential word growth, with finite generating set S. Then there exists a non-decreasing unbounded function $q\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ such that $q \in o(n)$ and*

$$\frac{|B_S(n)|}{|B_S(n-q(n))|} \to \infty \quad as \quad n \to \infty.$$

Proof We put $\lambda = \lambda_S(G) > 1$ and write $|B_S(n)| = \lambda^{\sum_{i=1}^n b_i}$, with $b_i \in \mathbb{R}_{\geqslant 0}$ for $i \in \mathbb{N}$, so that the sequence $\sum_{i=1}^n b_i$, $n \in \mathbb{N}$, is subadditive and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i = 1.$$

In this notation, we seek a non-decreasing unbounded function $q\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ such that, simultaneously,

$$\frac{q(n)}{n} \to 0 \quad \text{and} \quad \sum_{i=n-\lfloor q(n)\rfloor+1}^{n} b_i \to \infty \quad \text{as} \quad n \to \infty. \tag{2.2}$$

We show below that for every $m \in \mathbb{N}$,

$$\sum_{i=n-\lfloor n/m\rfloor+1}^{n} b_i \to \infty \quad \text{as} \quad n \to \infty. \tag{2.3}$$

From this we see that there is an increasing sequence of non-positive integers $c(m)$, $m \in \mathbb{N}$, such that, for each m,

$$c(m) \geqslant m^2 \quad \text{and} \quad \text{for all} \;\; n \in \mathbb{N}_{\geqslant c(m)} : \quad \sum_{i=n-\lfloor n/m \rfloor +1}^{n} b_i \geqslant m.$$

Setting $q_1(n) = \lfloor n/m \rfloor$ for $n \in \mathbb{N}$ with $c(m) \leqslant n < c(m+1)$ and

$$q(n) = \max \left\{ q_1(k) \mid k \in [1, n]_{\mathbb{Z}} \right\},$$

we arrive at a function $q\colon \mathbb{N} \to \mathbb{R}_{\geqslant 1}$ meeting the requirements (2.2).

It remains to establish (2.3). Let $m \in \mathbb{N}$ and put $\varepsilon = \varepsilon_m = (6m)^{-1} \in \mathbb{R}_{>0}$. We choose $N = N_\varepsilon \in \mathbb{N}$ minimal subject to

$$1 - \varepsilon \leqslant \frac{1}{n} \sum_{i=1}^{n} b_i \leqslant 1 + \varepsilon \quad \text{for all} \;\; n \in \mathbb{N}_{\geqslant N}.$$

In the following we deal repeatedly with sums of the form

$$\beta(k) = \sum_{i=kN+1}^{kN+N} b_i,$$

for $k \in \mathbb{N}$, and using subadditivity, we obtain

$$\beta(k) \leqslant \beta(0) \leqslant (1+\varepsilon)N \quad \text{for all} \;\; k \in \mathbb{N}.$$

We consider $n \in \mathbb{N}$ with $n \geqslant (1+\varepsilon)\varepsilon^{-1}N \geqslant N$ and write $n = lN + r$ with $l = l_n \in \mathbb{N}$ and $r = r_n \in [0, N-1]_{\mathbb{Z}}$. Furthermore, we set

$$t = t_n = \frac{|\{k \in [0, l-1]_{\mathbb{Z}} \mid \beta(k) > \varepsilon N\}|}{l} \in [0, 1]_{\mathbb{R}}.$$

From our set-up, we deduce that

$$\begin{aligned} 1 - \varepsilon \leqslant \frac{1}{n} \sum_{i=1}^{n} b_i &\leqslant \frac{1}{lN} \left(\left(\sum_{k=0}^{l-1} \beta(k) \right) + \beta(l) \right) \\ &\leqslant \big(t(1+\varepsilon) + (1-t)\varepsilon \big) + \frac{1+\varepsilon}{l} \leqslant t + 2\varepsilon, \end{aligned}$$

hence $t \geqslant 1 - 3\varepsilon$ and consequently

$$\begin{aligned} &\left| \left\{ k \in [0, l-1]_{\mathbb{Z}} \mid \beta(k) > \varepsilon N \right\} \cap \left\{ k \in [0, l-1]_{\mathbb{Z}} \mid \lceil (1-6\varepsilon) l \rceil + 1 \leqslant k \right\} \right| \\ &\qquad \geqslant tl + (l - \lceil (1-6\varepsilon) l \rceil - 1) - l \geqslant (1 - 3\varepsilon - (1-6\varepsilon)) l - 2 = 3\varepsilon l - 2. \end{aligned}$$

Since

$$n - \lfloor n/m \rfloor = \lceil (1-6\varepsilon)n \rceil \leqslant \lceil (1-6\varepsilon)(l+1) \rceil N \leqslant (\lceil (1-6\varepsilon)l \rceil + 1) N,$$

this gives

$$\sum_{i=n-\lfloor n/m \rfloor+1}^{n} b_i \geqslant \sum_{k=\lceil (1-6\varepsilon)l \rceil+1}^{l-1} \beta(k) \geqslant (3\varepsilon l - 2)\,\varepsilon N,$$

which tends to infinity as $l \to \infty$. This proves (2.3). □

In [10, Lemma 2.2], Pittet seems to claim that

$$\liminf_{n\to\infty} \frac{|B_S(n)|}{|B_S(n-1)|} > 1,$$

from which Proposition 2.2 could be derived much more easily. However, we found the explanations in [10] not fully conclusive and thus opted to work out an independent argument. Naturally, it would be interesting to establish a more effective version of Proposition 2.2, if possible.

2.2 Wreath products

We recall that a group $G = H \wr K$ is the restricted standard *wreath product* of two subgroups H and K, if it decomposes as a semidirect product $G = N \rtimes K$, where the normal closure of H is the direct sum $N = \bigoplus_{k \in K} H^k$ of the various conjugates of H by elements of K; the groups N and K are referred to as the *base group* and the *top group* of the wreath product G, respectively. Since we do not consider complete standard wreath products or more general types of permutational wreath products, we shall drop the terms "restricted" and "standard" from now on.

Throughout the rest of this section, let

$$G = H \wr \langle t \rangle = N \rtimes \langle t \rangle \quad \text{with base group} \quad N = \bigoplus_{i \in \mathbb{Z}} H^{t^i} \tag{2.4}$$

be the wreath product of a finitely generated subgroup H and an infinite cyclic subgroup $\langle t \rangle \cong C_\infty$. Every element $g \in G$ can be written uniquely in the form

$$g = \widetilde{g}\, t^{\rho(g)},$$

where $\rho(g) \in \mathbb{Z}$ and $\widetilde{g} = \prod_{i \in \mathbb{Z}} (g_{|i})^{t^i} \in N$ with 'coordinates' $g_{|i} \in H$. The support of the product decomposition of $\widetilde{g}$ is finite and we write

$$\operatorname{supp}(g) = \{ i \in \mathbb{Z} \mid g_{|i} \neq 1 \}.$$

Furthermore, it is convenient to fix a finite symmetric generating set S of G; thus $G = \langle S \rangle$, and $g \in S$ implies $g^{-1} \in S$. We put $d = |S|$ and fix an ordering of the elements of S:

$$S = \{s_1, \ldots, s_d\}, \quad \text{with} \quad s_j = \widetilde{s_j}\, t^{\rho(s_j)} \quad \text{for} \quad j \in [1, d]_{\mathbb{Z}}, \tag{2.5}$$

where $\widetilde{s_1}, \ldots, \widetilde{s_d} \in N$. We write $r_S = \max\{\rho(s_j) \mid j \in [1, d]_{\mathbb{Z}}\} \in \mathbb{N}$.

Definition 2.3 An S-*expression* of an element $g \in G$ is (induced by) a word $W = \prod_{k=1}^{l} X_{\iota(k)}$ in the free semigroup $\langle X_1, \ldots, X_d \rangle$ such that

$$g = W(s_1, \ldots, s_d) = \prod_{k=1}^{l} s_{\iota(k)}; \tag{2.6}$$

here W determines and is determined by the function $\iota = \iota_W \colon [1, l]_{\mathbb{Z}} \to [1, d]_{\mathbb{Z}}$. In this situation the standard process of collecting powers of t to the right yields

$$g = \widetilde{g}\, t^{-\sigma(l)} \quad \text{with} \quad \widetilde{g} = \prod\nolimits_{k=1}^{l} \widetilde{s_{\iota(k)}}^{\,t^{\sigma(k-1)}}, \tag{2.7}$$

where $\sigma = \sigma_{S,W}$ is short for the negative[1] cumulative exponent function

$$\sigma_{S,W} \colon [0, l]_{\mathbb{Z}} \to \mathbb{Z}, \quad k \mapsto -\sum_{j=1}^{k} \rho(s_{\iota(j)}).$$

We define the *itinerary* of g associated to the S-expression (2.6) as the pair

$$\mathrm{It}(S, W) = (\iota_W, \sigma_{S,W}),$$

and we say that $\mathrm{It}(S, W)$ has length l, viz. the length of the word W. For the purpose of concrete calculations it is helpful to depict the functions ι_W and $\sigma_{S,W}$ as finite sequences. The term 'itinerary' refers to (2.7), which indicates how g can be built stepwise from the sequences ι_W and $\sigma_{S,W}$; see Example 2.4 below. In particular, g is uniquely determined by the itinerary $\mathrm{It}(S, W) = (\iota, \sigma)$ and, accordingly, we refer to g as the element corresponding to that itinerary. But unless G is trivial and S is empty, the element g has, of course, infinitely many S-expressions which in turn give rise to infinitely many distinct itineraries of one and the same element.

For discussing features of the exponent function $\sigma_{S,W}$, we call

$$\mathrm{maxi}(\mathrm{It}(S, W)) = \max(\sigma_{S,W}) \quad \text{and} \quad \mathrm{mini}(\mathrm{It}(S, W)) = \min(\sigma_{S,W})$$

the *maximal* and *minimal itinerary points* of $\mathrm{It}(S, W)$. Later we fix a representative function $\mathcal{W} \colon G \to \langle X_1, \ldots, X_d \rangle$, $g \mapsto W_g$ which yields for each element of G an

[1] At this stage the sign change is a price we pay for not introducing notation for left-conjugation; Example 2.4 illustrates that σ plays a convenient role in the concept of itinerary.

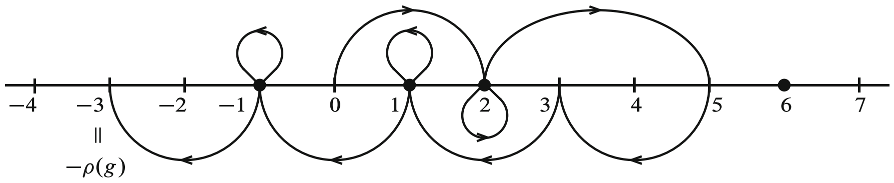

Fig. 1 An illustration of the itinerary of g in (2.9) associated to the S-expression in (2.8); the support of $\tilde{g}$ is also indicated

S-expression of shortest possible length. In that situation we suppress the reference to S and refer to

$$\mathrm{It}_{\mathcal{W}}(g) = \mathrm{It}(S, W_g), \quad \mathrm{maxi}_{\mathcal{W}}(g) = \mathrm{maxi}(\mathrm{It}_{\mathcal{W}}(g)), \quad \mathrm{mini}_{\mathcal{W}}(g) = \mathrm{mini}(\mathrm{It}_{\mathcal{W}}(g))$$

as the $\mathcal{W}$-*itinerary*, the *maximal* $\mathcal{W}$-*itinerary point* and the *minimal* $\mathcal{W}$-*itinerary point* of any given element g.

To illustrate the terminology we discuss a concrete example.

Example 2.4 A typical example of the wreath products that we consider is the lamplighter group

$$G = \left\langle a, t \mid a^2 = 1, [a, a^{t^i}] = 1 \text{ for } i \in \mathbb{Z} \right\rangle = \bigoplus_{i \in \mathbb{Z}} \langle a_i \rangle \rtimes \langle t \rangle \cong C_2 \wr C_\infty,$$

where $a_i = t^{-i} a t^i$ for $i \in \mathbb{Z}$. We consider the finite symmetric generating set

$$S = \{s_1, \dots, s_5\}$$

with

$$s_1 = a_4 t^{-3}, \quad s_2 = t^{-2}, \quad s_3 = s_1^{-1} = a_1 t^3, \quad s_4 = s_2^{-1} = t^2, \quad s_5 = a_0 = s_5^{-1}.$$

Let $g = \widetilde{g}\, t^3$ be such that $g_{|i} = a$ for $i \in \{-1, 1, 2, 6\}$ and $g_{|i} = 1$ otherwise. Then we have $\rho(g) = 3$, $\mathrm{supp}(g) = \{-1, 1, 2, 6\}$, and

$$g = t^{-2} \cdot a_0 \cdot a_4 t^{-3} \cdot (t^2)^2 \cdot a_0 \cdot t^2 \cdot a_0 \cdot t^2 = s_2\, s_5\, s_1\, s_4^2\, s_5\, s_4\, s_5\, s_4 \tag{2.8}$$

is an S-expression for g of length 9, based on $W = X_2 X_5 X_1 X_4^2 X_5 X_4 X_5 X_4$. The itinerary $I = \mathrm{It}(S, W)$ associated to this S-expression for g is

$$I = (\iota, \sigma) = \big((2, 5, 1, 4, 4, 5, 4, 5, 4), (0, 2, 2, 5, 3, 1, 1, -1, -1, -3)\big), \tag{2.9}$$

where we have written the maps ι and σ in sequence notation. Furthermore, we see that $\mathrm{maxi}(I) = 5$ and $\mathrm{mini}(I) = -3$. Figure 1 gives a pictorial description of part of the information contained in I.

An alternative S-expression for the same element g is

$$\begin{aligned} g &= a_4 t^{-3} \cdot (t^2)^2 \cdot a_0 \cdot a_1 t^3 \cdot (t^{-2})^3 \cdot a_0 \cdot t^{-2} \cdot a_0 \cdot t^{-2} \cdot a_0 \cdot (t^2)^3 \cdot a_0 \cdot a_1 t^3 \\ &= s_1\, s_4^2\, s_5\, s_3\, s_2^3\, s_5\, s_2\, s_5\, s_2\, s_5\, s_4^3\, s_5\, s_3. \end{aligned} \tag{2.10}$$

It has length 18 and is based on the semigroup word

$$W' = X_1\, X_4^2\, X_5\, X_3\, X_2^3\, X_5\, X_2\, X_5\, X_2\, X_5\, X_4^3\, X_5\, X_3.$$

In this case, the itinerary associated to the S-expression (2.10) is

$$\begin{aligned} I' = (\iota', \sigma') = \big(&(1, 4, 4, 5, 3, 2, 2, 2, 5, 2, 5, 2, 5, 4, 4, 4, 5, 3), \\ &(0, 3, 1, -1, -1, -4, -2, 0, 2, 2, 4, 4, 6, 6, 4, 2, 0, 0, -3)\big), \end{aligned}$$

and we observe that $\operatorname{maxi}(I') = 6$ and $\operatorname{mini}(I') = -4$.

There is a natural notion of a product of two itineraries, and it has the expected properties. We collect the precise details in a lemma.

Lemma and Definition 2.5 *In the general set-up described above, suppose that $I_1 = (\iota_1, \sigma_1)$ and $I_2 = (\iota_2, \sigma_2)$ are itineraries, of lengths l_1 and l_2, associated to S-expressions W_1, W_2 for elements $g_1, g_2 \in G$. Then $W = W_1 W_2$ is an S-expression for $g = g_1 g_2$; the associated itinerary*

$$I = \operatorname{It}(S, W) = (\iota, \sigma)$$

has length $l = l_1 + l_2$ and its components are given by

$$\iota(k) = \begin{cases} \iota_1(k) & \text{if } k \in [1, l_1]_{\mathbb{Z}}, \\ \iota_2(k - l_1) & \text{if } k \in [l_1 + 1, l]_{\mathbb{Z}}, \end{cases}$$

$$\sigma(k) = \begin{cases} \sigma_1(k) & \text{if } k \in [0, l_1]_{\mathbb{Z}}, \\ \sigma_1(l_1) + \sigma_2(k - l_1) & \text{if } k \in [l_1 + 1, l]_{\mathbb{Z}}. \end{cases}$$

We refer to I as the product itinerary *and write $I = I_1 * I_2$.*

Conversely, if I is the itinerary of some element $g \in G$ associated to some S-expression of length l and if $l_1 \in [0, l]_{\mathbb{Z}}$, there is a unique decomposition $I = I_1 * I_2$ for itineraries I_1 of length l_1 and I_2 of length $l_2 = l - l_1$; the corresponding elements $g_1, g_2 \in G$ satisfy $g = g_1 g_2$.

Proof The assertions in the first paragraph are easy to verify from

$$W = W_1 W_2 = \prod_{k=1}^{l_1} X_{\iota_1(k)} \prod_{k=1}^{l_2} X_{\iota_2(k)} = \prod_{k=1}^{l_1} X_{\iota_1(k)} \prod_{k=l_1+1}^{l_1+l_2} X_{\iota_2(k-l_1)}$$

and the observation that, for $k \in [0, l]_{\mathbb{Z}}$,

$$\begin{aligned} \sigma(k) &= -\sum\nolimits_{j=1}^{k} \rho(s_{\iota(k)}) \\ &= \begin{cases} -\sum_{j=1}^{k} \rho(s_{\iota_1(k)}) = \sigma_1(k) & \text{if } k \leqslant l_1, \\ -\sum_{j=1}^{l_1} \rho(s_{\iota_1(k)}) - \sum_{j=l_1+1}^{k} \rho(s_{\iota_2(k-l_1)}) = \sigma_1(l_1) + \sigma_2(k-l_1) & \text{if } k > l_1. \end{cases} \end{aligned}$$

Conversely, let I be the itinerary of an element g, associated to some S-expression $W = \prod_{k=1}^{l} X_{\iota(k)}$ of length l, and let $l_1 \in [0, l]_{\mathbb{Z}}$. Then W decomposes uniquely as a product $W_1 W_2$ of semigroup words of lengths l_1 and $l - l_2$, namely for $W_1 = \prod_{k=1}^{l_1} X_{\iota(k)}$ and $W_2 = \prod_{k=l_1+1}^{l} X_{\iota(k)}$. These are S-expressions for elements g_1, g_2 and $g = g_1 g_2$. Moreover, W_1 and W_2 give rise to itineraries I_1, I_2 such that $I = I_1 * I_2$. Since W_1 and I_1, respectively W_2 and I_2, determine one another uniquely, the decomposition $I = I_1 * I_2$ is unique. □

For a representative function $\mathcal{W}\colon G \to \langle X_1, \ldots, X_d \rangle$, $g \mapsto W_g$, as discussed in Definition 2.3, it is typically not the case that $W_{gh} = W_g W_h$ for $g, h \in G$. Consequently, it is typically not true that $\mathrm{It}_{\mathcal{W}}(gh) = \mathrm{It}_{\mathcal{W}}(g) * \mathrm{It}_{\mathcal{W}}(h)$.

Lemma 2.6 *Let $G = H \wr \langle t \rangle$ be a wreath product as in (2.4), with generating set S as in (2.5). Put*

$$C = C(S) = 1 + \max\left\{ |i| \mid i \in \mathrm{supp}(s) \textit{ for some } s \in S \right\} \in \mathbb{N}.$$

Then the following hold:

(i) *For every $g \in G$ with itinerary I,*

$$\mathrm{mini}(I) - C < \min(\mathrm{supp}(g)) \quad \textit{and} \quad \max(\mathrm{supp}(g)) < \mathrm{maxi}(I) + C.$$

(ii) *Let $u \in H$. Put $m_S = \max\{C, r_S\} \in \mathbb{N}$ and*

$$D = D(S, u) = l_S(u) + 2\max\left\{ l_S(t^j) \mid 0 \leqslant j \leqslant m_S + r_S \right\} \in \mathbb{N}.$$

Let $g \in G$ with itinerary I, associated to an S-expression of length $l_S(g)$. Then, for every $j \in \mathbb{Z}$ with $\mathrm{mini}(I) - m_S \leqslant j \leqslant \mathrm{maxi}(I) + m_S$, the elements $h = gu^{t^{j+\rho(g)}}$, $\hbar = u^{t^j} g \in G$ satisfy $\rho(h) = \rho(\hbar) = \rho(g)$ and the 'coordinates' of h, $\hbar$ are given by

$$h_{|i} = \begin{cases} g_{|i} & \textit{if } i \neq j, \\ g_{|j}\, u & \textit{if } i = j, \end{cases} \qquad \hbar_{|i} = \begin{cases} g_{|i} & \textit{if } i \neq j, \\ u\, g_{|j} & \textit{if } i = j \end{cases} \qquad \textit{for } i \in \mathbb{Z}.$$

Furthermore,

$$l_S(h) \leqslant l_S(g) + D \quad \textit{and} \quad l_S(\hbar) \leqslant l_S(g) + D.$$

Proof We write $I = (\iota, \sigma)$ for the given itinerary of g, and l denotes the length of I.

(i) From (2.7) it follows that

$$\begin{aligned}\operatorname{supp}(g) &\subseteq \bigcup_{1 \leqslant k \leqslant l} \big(\sigma(k-1) + \operatorname{supp}(s_{\iota(k)})\big) \\ &\subseteq \bigcup_{1 \leqslant k \leqslant l} \big[\sigma(k-1) - C + 1, \sigma(k-1) + C - 1\big]_{\mathbb{Z}};\end{aligned}$$

from this inclusion the two inequalities follow readily.

(ii) In addition we now have $l = l_S(g)$. The first assertions are very easy to verify. We justify the upper bound for $l_S(h)$; the bound for $l_S(\hbar)$ is derived similarly.

Since $\operatorname{mini}(I) - m_S \leqslant j \leqslant \operatorname{maxi}(I) + m_S$ and since itineraries proceed, in the second coordinate, by steps of length at most $r_S \leqslant m_S$, there exists $k \in [0, l]_{\mathbb{Z}}$ such that $|j - \sigma(k)| \leqslant m_S$. If $|j - \sigma(l)| \leqslant m_S$ we put $k = l - 1$; otherwise we choose k maximal with $|j - \sigma(k)| \leqslant m_S$. Next we decompose the itinerary I as the product $I = I_1 * I_2$ of itineraries I_1 of length $l_1 = k + 1$ and I_2 of length $l_2 = l - k - 1$; compare with Lemma 2.5.

We denote by $g_1 = \widetilde{g}_1 t^{-\sigma(k+1)}$ and $g_2 = \widetilde{g}_2 t^{\sigma(k+1)+\rho(g)}$ the elements corresponding to I_1 and I_2 so that $g = g_1 g_2 = \widetilde{g}_1 \widetilde{g}_2^{\,t^{\sigma(k+1)}} t^{\rho(g)}$. Moreover, we observe from $|j - \sigma(k+1)| \leqslant m_S + r_S$ that

$$g_3 = u^{t^{j-\sigma(k+1)}} = t^{-j+\sigma(k+1)}\, u\, t^{j-\sigma(k+1)}$$

has length $l_3 \leqslant l_S(u) + 2\, l_S(t^{j-\sigma(k+1)}) \leqslant D$. Our choice of k guarantees that the support of $\widetilde{g}_2^{\,t^{\sigma(k+1)}}$ does not overlap with $\{j\} = \operatorname{supp}(u^{t^j})$; compare with (i). Thus $\widetilde{g}_2^{\,t^{\sigma(k+1)}}$ and u^{t^j}, both in the base group, commute with one another. This gives

$$\begin{aligned} h = g u^{t^{j+\rho(g)}} &= \widetilde{g}_1 \widetilde{g}_2^{\,t^{\sigma(k+1)}} u^{t^j} t^{\rho(g)} = \widetilde{g}_1 u^{t^j} \widetilde{g}_2^{\,t^{\sigma(k+1)}} t^{\rho(g)} \\ &= g_1 t^{-j+\sigma(k+1)} u t^{j-\sigma(k+1)} g_2 = g_1 g_3 g_2, \end{aligned}$$

and we conclude that $l_S(h) \leqslant l_1 + l_2 + l_3 \leqslant l + D = l_S(g) + D$. □

3 Proofs of Theorems A and B

First we explain how Theorem A follows from Theorem B. The first ingredient is the following general lemma.

Lemma 3.1 (Antolín, Martino and Ventura [1, Lemma 3.1]) *Let $G = \langle S \rangle$ be a group, with finite generating set S. Suppose that there exists a subset $X \subseteq G$ satisfying*

(a) $\delta_S(X) = 0$*;*
(b) $\sup\left\{ \frac{|C_G(g) \cap B_S(n)|}{|B_S(n)|} \mid g \in G \setminus X \right\} \to 0$ *as* $n \to \infty$*.*

Then G has degree of commutativity $\operatorname{dc}_S(G) = 0$.

The second ingredient comes from [3, Section 2.1], where Cox shows the following. If $G = H \wr \langle t \rangle$ is the wreath product of a finitely generated group $H \neq 1$ and an infinite cyclic group $\langle t \rangle$, with base group N, and if S is any finite generating set for G, then

$$\sup \left\{ \frac{|C_G(g) \cap B_S(n)|}{|B_S(n)|} \;\middle|\; g \in G \smallsetminus N \right\} \to 0 \qquad \text{as } n \to \infty.$$

The idea behind Cox' proof is as follows. For $g \in G \smallsetminus N$, the centraliser $C_G(g)$ is cyclic and the 'translation length' of g with respect to S is uniformly bounded away from 0. The latter means that there exists $\tau_S > 0$ such that

$$\inf_{n \in \mathbb{N}} \left\{ \frac{l_S(g^n)}{n} \;\middle|\; g \in G \smallsetminus N \right\} \geqslant \tau_S.$$

Consequently, for $g \in G \smallsetminus N$ the function $n \mapsto |C_G(g) \cap B_S(n)|$ is bounded uniformly by a linear function, while G has exponential word growth.

Thus, Theorem B implies Theorem A, and it remains to establish Theorem B. Throughout the rest of this section, let

$$G = H \wr \langle t \rangle = N \rtimes \langle t \rangle \quad \text{with base group} \quad N = \bigoplus_{i \in \mathbb{Z}} H^{t^i}$$

be the wreath product of a finitely generated subgroup H and an infinite cyclic subgroup $\langle t \rangle$, just as in (2.4). The exceptional case $H = 1$ poses no obstacle, hence we assume $H \neq 1$. We fix a finite symmetric generating set $S = \{s_1, \ldots, s_d\}$ for G and employ the notation established around (2.5). Finally, we recall that G has exponential word growth and we write

$$\lambda = \lambda_S(G) > 1$$

for the exponential growth rate of G with respect to S; see (2.1).

We start by showing that the subset of N consisting of all elements with suitably bounded support is negligible in the computation of the density of N.

Proposition 3.2 *Fix a representative function $\mathcal{W}$ which yields for each element of G an S-expression of shortest possible length and let $q \colon \mathbb{N} \to \mathbb{R}_{\geqslant 1}$ be a non-decreasing unbounded function such that $q \in o(\log n)$.*

Then the sequence of sets

$$R_q(n) = R_{\mathcal{W},q}(n) = \left\{ g \in N \cap B_S(n) \mid \mathrm{maxi}_{\mathcal{W}}(g) - \mathrm{mini}_{\mathcal{W}}(g) \leqslant q(n) \right\},$$

indexed by $n \in \mathbb{N}$, satisfies

$$\lim_{n \to \infty} \frac{|R_q(n)|}{|B_S(n)|} = 0.$$

The proof of Proposition 3.2 is preceded by some preparations and two auxiliary lemmata. We keep in place the set-up from Proposition 3.2. For $i \in \mathbb{Z}$, we write $H_i = H^{t^i}$. Using the notation established in Sect. 2.2, we accumulate the 'coordinates' of elements of S in a set

$$S_0 = \{s_{|i} \mid s \in S, i \in \mathbb{Z}\} = \left\{(s_j)_{|i} \mid 1 \leqslant j \leqslant d \text{ and } i \in \mathbb{Z}\right\} \subseteq H = H_0,$$

we set $S_i = S_0^{t^i} \subseteq H_i$ for $i \in \mathbb{Z}$. Then S_i is a finite symmetric generating set of H_i for each $i \in \mathbb{Z}$. Indeed, every element $h \in H$ satisfies $h = \widetilde{h} = h_{|0}$ and can thus be written in the form

$$h = \left(\prod_{k=1}^{l} \widetilde{s_{\iota(k)}}^{t^{\sigma(k-1)}}\right)\Bigg|_0 = \prod_{k=1}^{l} \left(\widetilde{s_{\iota(k)}}_{|-\sigma(k-1)}\right),$$

based upon a suitable itinerary $I = (\iota, \sigma)$ of length l. We conclude that $H = \langle S_0 \rangle$ and consequently $H_i = \langle S_i \rangle$ for $i \in \mathbb{Z}$; the generating systems inherit from S the property of being symmetric.

Moreover, we have $|B_{H_i,S_i}(n)| = |B_{H,S_0}(n)|$ for all $n \in \mathbb{N}$; consequently,

$$\lambda_{S_0}(H) = \lambda_{S_i}(H_i) \quad \text{for all} \quad i \in \mathbb{Z}.$$

It is convenient to split the analysis of the set $R_q(n)$ from Proposition 3.2 into two parts. First we take care of elements whose 'coordinates' fall within sufficiently small balls around 1 in H, with respect to the generating set S_0.

Lemma 3.3 *In addition to the set-up above, let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a non-decreasing unbounded function such that $f \in o(n/q(n))$.*

Then the sequence of subsets

$$R_q^f(n) = R_{\mathcal{W},q}^f(n) = \left\{g \in R_q(n) \mid g_{|i} \in B_{H,S_0}(f(n)) \text{ for all } i \in \mathbb{Z}\right\} \subseteq R_q(n),$$

indexed by $n \in \mathbb{N}$, satisfies

$$\lim_{n\to\infty} \frac{|R_q^f(n)|}{|B_S(n)|} = 0.$$

Proof Let $C = C(S) \in \mathbb{N}$ be as is in Lemma 2.6 (i) and choose $C' \in \mathbb{N}$ such that $\lambda^{C'} > \lambda_{S_0}(H)$. Then we have, for all sufficiently large $n \in \mathbb{N}$,

$$\left|R_q^f(n)\right| \leqslant \left|B_{H,S_0}(f(n))\right|^{2q(n)+2C} \leqslant \lambda^{2C'q(n)f(n)+2C'Cf(n)} \leqslant \lambda^{4C'Cq(n)f(n)}.$$

From $f \in o(n/q(n))$ we obtain

$$4C'Cq(n)f(n) - n \to -\infty \quad \text{as} \quad n \to \infty$$

and hence

$$\frac{|R_q^f(n)|}{|B_S(n)|} \leqslant \lambda^{4C'Cq(n)f(n)-n} \to 0 \quad \text{as} \quad n \to \infty \qquad \square$$

Next we consider $R_q(n) \smallsetminus R_q^f(n)$, for a function f as in Lemma 3.3 and $n \in \mathbb{N}$. For every $g \in R_q(n) \smallsetminus R_q^f(n)$, we pick $i(g) \in \mathbb{Z}$ with

$$\operatorname{mini}_{\mathcal{W}}(g) - C < i(g) < \operatorname{maxi}_{\mathcal{W}}(g) + C \quad \text{and} \quad g_{|i(g)} \notin B_{S_0}(f(n)),$$

where $C = C(S) \in \mathbb{N}$ continues to denote the constant from Lemma 2.6(i). Let $I = (\iota, \sigma)$, viz. $I_g = (\iota_g, \sigma_g)$, denote the $\mathcal{W}$-itinerary of g. Then

$$g_{|i(g)} = \prod_{k=1}^{l_S(g)} (s_{\iota(k)})_{|i(g)-\sigma(k-1)}.$$

By successively cancelling sub-products of adjacent factors that evaluate to 1 and have maximal length with this property (in an orderly fashion, from left to right, say), we arrive at a 'reduced' product expression

$$g_{|i(g)} = \prod_{j=1}^{\ell} (s_{\iota(\kappa(j))})_{|i(g)-\sigma(\kappa(j)-1)}, \tag{3.1}$$

for some $\ell = \ell_g \in [1, l_S(g)]_{\mathbb{Z}}$ and an increasing function $\kappa = \kappa_g \colon [1,\ell]_{\mathbb{Z}} \to [1, l_S(g)]_{\mathbb{Z}}$ that picks out a subsequence of factors. In particular, this means that, for $j_1, j_2 \in [1,\ell]_{\mathbb{Z}}$ with $j_1 < j_2$,

$$\begin{aligned} \prod_{k=\kappa(j_1)+1}^{\kappa(j_2)} (s_{\iota(k)})_{|i(g)-\sigma(k-1)} &= \prod_{j=j_1+1}^{j_2} \prod_{k=\kappa(j-1)+1}^{\kappa(j)} (s_{\iota(k)})_{|i(g)-\sigma(k-1)} \\ &= \prod_{j=j_1+1}^{j_2} (s_{\iota(\kappa(j))})_{|i(g)-\sigma(\kappa(j)-1)} \neq 1, \end{aligned} \tag{3.2}$$

and moreover we have $l_S(g) \geqslant \ell \geqslant l_{S_0}(g_{|i(g)}) \geqslant f(n)$.

By means of suitable perturbations, we aim to produce from g a collection of ℓ distinct elements $\dot{g}(1), \dots, \dot{g}(\ell)$ which each carry sufficient information to 'recover' the initial element g. We proceed as follows. For each choice of $j \in [1,\ell]_{\mathbb{Z}}$ we decompose the itinerary I for g into a product $I = I_{j,1} * I_{j,2}$ of itineraries of length $\kappa(j)$ and $l_S(g) - \kappa(j)$; compare with Lemma 2.5. Then $g = g_{j,1}g_{j,2}$, where $g_{j,1}, g_{j,2}$ denote the elements of G corresponding to $I_{j,1}, I_{j,2}$. From $g \in R_q(n)$ it follows that $\operatorname{maxi}(I_{j,1}) - \operatorname{mini}(I_{j,1})$ and $\operatorname{maxi}(I_{j,2}) - \operatorname{mini}(I_{j,2})$ are bounded by $q(n)$; in particular, $\rho(g_{j,1}) \in [-q(n), q(n)]_{\mathbb{Z}}$.

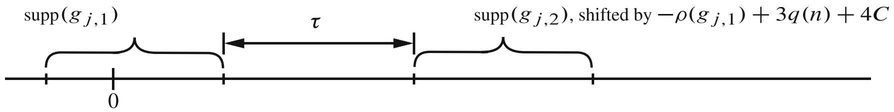

Fig. 2 An illustration of the factorisation $\dot{g}(j) = g_{j,1}\, t^{-3q(n)-4C}\, g_{j,2}$

We define

$$\dot{g}(j) = g_{j,1}\, t^{-3q(n)-4C}\, g_{j,2} \tag{3.3}$$

with $C = C(S)$ as above; see Fig. 2 for a pictorial illustration, which features an additional parameter τ that we introduce in the proof of Lemma 3.4.

Lemma 3.4 *In the set-up above, the elements $\dot{g}(1), \ldots, \dot{g}(\ell)$ defined in (3.3) satisfy the following:*

(i) *for each $j \in [1,\ell]_{\mathbb{Z}}$ the element $\dot{g}(j)$ lies in $B_S(n + (3q(n) + 4C)\, l_S(t))$;*
(ii) *for each $j \in [1,\ell]_{\mathbb{Z}}$ the original element g can be recovered from $\dot{g}(j)$;*
(iii) *the elements $\dot{g}(1), \ldots, \dot{g}(\ell)$ are pairwise distinct.*

Proof (i) Lemma 2.5 gives $l_S(g_{j,1}) + l_S(g_{j,2}) = \ell \leqslant l_S(g) \leqslant n$, and it is clear that $l_S(t^{-3q(n)-4C}) \leqslant (3q(n) + 4C)\, l_S(t)$.

(ii) Let $j \in [1,\ell]_{\mathbb{Z}}$, and write $\mathcal{G}_1 = \mathrm{supp}(g_{j,1})$, $\mathcal{G}_2 = \mathrm{supp}(g_{j,2})$. Lemma 2.6(i) implies that the sets $\mathcal{G}_1$ and $\mathcal{G}_2 - \rho(g_{j,1}) = \mathrm{supp}(t^{\rho(g_{j,1})} g_{j,2})$ lie wholly within the interval $[-q(n) - C, q(n) + C]_{\mathbb{Z}}$, hence

$$\mathrm{supp}(\dot{g}(j)) = \mathcal{G}_1 \cup \big(\mathcal{G}_2 - \rho(g_{j,1}) + 3q(n) + 4C\big) \tag{3.4}$$

with a gap

$$\tau = \underbrace{\min\big(\mathcal{G}_2 - \rho(g_{j,1}) + 3q(n) + 4C\big)}_{\geqslant -q(n)-C+3q(n)+4C = 2q(n)+3C} - \underbrace{\max(\mathcal{G}_1)}_{\leqslant q(n)+C} \geqslant q(n) + 2C,$$

subject to the standard conventions $\min \varnothing = +\infty$ and $\max \varnothing = -\infty$ in special circumstances; see Fig. 2 for a pictorial illustration.

In contrast, gaps between two elements in $\mathcal{G}_1$ or two elements in $\mathcal{G}_2$ are strictly less than $q(n) + 2C \leqslant \tau$. Consequently, we can identify the two components in (3.4) and thus $\mathcal{G}_1$ and $\mathcal{G}_2 - \rho(g_{j,1})$, without any prior knowledge of j or $g_{j,1}, g_{j,2}$. Therefore, for each $i \in \mathbb{Z}$ the ith coordinate of g satisfies

$$g_{|i} = \begin{cases} \dot{g}(j)_{|i}\, \dot{g}(j)_{|i+3q(n)+4C} & \text{if } i \in [-q(n) - C, q(n) + C], \\ 1 & \text{otherwise,} \end{cases}$$

and hence g can be recovered from $\dot{g}(j)$.

(iii) For $j_1, j_2 \in [1, \ell]_{\mathbb{Z}}$ with $j_1 < j_2$ we conclude from our choice of the 'reduced' product expression (3.1) and its consequence (3.2) that

$$\begin{aligned}\dot{g}(j_1)_{|i(g)} = (g_{j_1,1})_{|i(g)} &= \prod_{k=1}^{\kappa(j_1)} (s_{\iota(k)})_{|i(g)-\sigma(k-1)} \\ &\neq \prod_{k=1}^{\kappa(j_2)} (s_{\iota(k)})_{|i(g)-\sigma(k-1)} = (g_{j_2,1})_{|i(g)} = \dot{g}(j_2)_{|i(g)}\end{aligned}$$

and hence $\dot{g}(j_1) \neq \dot{g}(j_2)$. □

For the proof of Proposition 3.2 we now make a more careful choice of the non-decreasing unbounded function $f : \mathbb{N} \to \mathbb{R}_{>0}$, which entered the stage in Lemma 3.3: we arrange that

$$f \in o(n/q(n)) \quad \text{and} \quad f \in \omega\big((\lambda+1)^{m(n)}\big) \quad \text{for} \quad m(n) = (3q(n)+4C)\, l_S(t),$$

with $C = C(S)$ as in Lemma 2.6 (i). For instance, we can take $f = f_\alpha$ for any real parameter α with $0 < \alpha < 1$, where $f_\alpha(n) = \max\{k^\alpha/q(k) \mid k \in [1,n]_{\mathbb{Z}}\}$ for $n \in \mathbb{N}$. Indeed, since $q(n) \in o(\log n)$ and $q(n) \geqslant 1$ for all $n \in \mathbb{N}$, each of these functions satisfies

$$\lim_{n\to\infty} \frac{f_\alpha(n) q(n)}{n} \leqslant \lim_{n\to\infty} \frac{n^\alpha q(n)}{n} = 0.$$

Furthermore, $q(n) \in o(\log n)$ implies $q(n)\, a^{q(n)} \in o(n^\beta)$ for all $a \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{R}_{>0}$ so that

$$\begin{aligned}\lim_{n\to\infty} \frac{(\lambda+1)^{m(n)}}{f_\alpha(n)} &\leqslant \lim_{n\to\infty} \frac{q(n)(\lambda+1)^{m(n)}}{n^\alpha} \\ &= (\lambda+1)^{4C\, l_S(t)} \lim_{n\to\infty} \frac{q(n)(\lambda+1)^{3 l_S(t) q(n)}}{n^\alpha} = 0.\end{aligned}$$

Proof of Proposition 3.2 We continue with the set-up established above; in particular, we make use of the refined choice of f. In view of Lemma 3.3 it remains to show that

$$\frac{|R_q(n) \smallsetminus R_q^f(n)|}{|B_S(n)|} \to 0 \quad \text{as} \quad n \to \infty.$$

We define a map

$$\begin{aligned} F_n \colon R_q(n) \smallsetminus R_q^f(n) &\to \mathcal{P}(B_S(n+m(n))) \\ g &\mapsto \{\dot{g}(j) \mid 1 \leqslant j \leqslant \ell_g\};\end{aligned}$$

see (3.3) and Lemma 3.4 (i). From Lemma 3.4 (ii) we deduce that $F_n(g_1) \cap F_n(g_2) = \varnothing$ for all $g_1, g_2 \in R_q(n) \smallsetminus R_q^f(n)$ with $g_1 \neq g_2$. In addition, from $\ell_g \geqslant f(n)$ and

Lemma 3.4(iii) we deduce that $|F_n(g)| \geqslant f(n)$ for all $g \in R_q(n) \smallsetminus R_q^f(n)$. This yields

$$|B_S(n + m(n))| \geqslant f(n)\,|R_q(n) \smallsetminus R_q^f(n)|,$$

and hence, by submultiplicativity,

$$\begin{aligned}\frac{|R_q(n) \smallsetminus R_q^f(n)|}{|B_S(n)|} \leqslant \frac{|B_S(n+m(n))|}{f(n)\,|B_S(n)|} &\leqslant \frac{|B_S(m(n))|}{f(n)} \\ &\leqslant \frac{(\lambda+1)^{m(n)}}{f(n)} \to 0 \quad \text{as } n \to \infty. \qquad \square\end{aligned}$$

Remark 3.5 Proposition 3.2 can be established much more easily under the extra assumption that H has sub-exponential word growth. Indeed, in this case, one can prove that

$$\lim_{n\to\infty} \frac{|R_q(n)|}{|B_S(n)|} = 0$$

for any non-decreasing unbounded function $q\colon \mathbb{N} \to \mathbb{R}_{>1}$ such that $q \in o(n)$; the proof is similar to the one of Lemma 4.1 below.

If we assume that H is finite, it is easy to see that there exists $\alpha \in \mathbb{R}_{>0}$ such that

$$\lim_{n\to\infty} \frac{|R_q(n)|}{|B_S(n)|} = 0 \quad \text{for} \quad q\colon \mathbb{N} \to \mathbb{R}_{>1}, \quad n \mapsto 1 + \alpha n.$$

Next we establish Theorem B, using ideas that are similar to those in the proof of Proposition 3.2: again we work with perturbations of a given element g in such a manner that the original element can be retrieved easily. We begin with some preparations to establish an auxiliary lemma.

Fix a representative function $\mathcal{W}$ which yields for each element of G an S-expression of shortest possible length, and fix an element $u \in H \smallsetminus \{1\}$. Consider $g \in N$ with $\mathcal{W}$-itinerary $I = (\iota, \sigma)$, viz. $I_g = (\iota_g, \sigma_g)$. We put

$$\sigma^+ = \sigma_g^+ = \mathrm{maxi}_{\mathcal{W}}(g) \quad \text{and} \quad \sigma^- = \sigma_g^- = \mathrm{mini}_{\mathcal{W}}(g).$$

For the time being, we suppose that

$$\begin{aligned} k^+ &= k^+_{\mathcal{W},g} = \min\left\{k \mid 0 \leqslant k \leqslant l_S(g) \text{ and } \sigma(k) = \sigma^+\right\}, \\ k^- &= k^-_{\mathcal{W},g} = \min\left\{k \mid 0 \leqslant k \leqslant l_S(g) \text{ and } \sigma(k) = \sigma^-\right\} \end{aligned}$$

satisfy $k^+ \leqslant k^-$. We decompose the itinerary for g as $I = I_1 * I_2 * I_3$, where I_1, I_2, I_3 have lengths $k^+, k^- - k^+, l_S(g) - k^-$; compare with Lemma 2.5.

If $x = x_{\mathcal{W},g}$, $y = y_{\mathcal{W},g}$, $z = z_{\mathcal{W},g}$ denote the elements corresponding to I_1, I_2, I_3 then $g = xyz$; observe that the lengths of I_1, I_2, I_3 are automatically minimal, i.e,

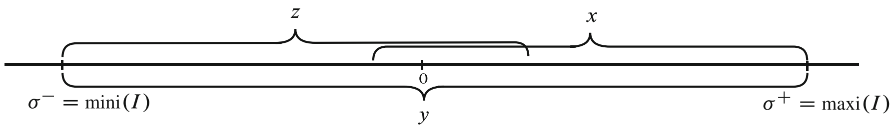

Fig. 3 A schematic illustration of the decomposition $g = xyz$

equal to $l_S(x), l_S(y), l_S(z)$. All this is illustrated schematically in Fig. 3. Observe that I_1, associated to x, 'starts' at 0 and 'ends' at σ^+, the shifted I_2, associated to y, 'starts' at σ^+ and 'ends' at σ^-, and the shifted I_3, associated to z, 'starts' at σ^- and 'ends' at 0.

Next, we put to use the element $u \in H \smallsetminus \{1\}$ that was fixed and define, for any given $J \subseteq [\sigma^-, \sigma^+]_{\mathbb{Z}}$, perturbations

$$\dot{x}(J) = \dot{x}_{\mathcal{W},g}(J, u), \quad \dot{y}(J) = \dot{y}_{\mathcal{W},g}(J, u), \quad \dot{z}(J) = \dot{z}_{\mathcal{W},g}(J, u)$$

of the elements x, y, z that are specified by

$$\begin{aligned} &\rho(\dot{x}(J)) = \rho(x) = -\sigma^+, \quad \rho(\dot{y}(J)) = \rho(y) = -\sigma^- + \sigma^+, \\ &\rho(\dot{z}(J)) = \rho(z) = \sigma^- \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \dot{x}(J)_{|i} &= \begin{cases} x_{|i}\, u & \text{for } i \in J_{\geqslant 0}, \\ x_{|i} & \text{otherwise}, \end{cases} \\ \dot{y}(J)_{|i} &= \begin{cases} u^{-1}\, y_{|i} & \text{for } i \in \mathbb{Z} \text{ such that } i + \sigma^+ \in J_{\geqslant 0}, \\ y_{|i}\, u^{-1} & \text{for } i \in \mathbb{Z} \text{ such that } i + \sigma^+ \in J_{<0}, \\ y_{|i} & \text{otherwise}, \end{cases} \\ \dot{z}(J)_{|i} &= \begin{cases} u\, z_{|i} & \text{for } i \in \mathbb{Z} \text{ such that } i + \sigma^- \in J_{<0}, \\ z_{|i} & \text{otherwise}, \end{cases} \end{aligned} \tag{3.6}$$

where we suggestively write $J_{\geqslant 0} = \{j \in J \mid j \geqslant 0\}$ and $J_{<0} = \{j \in J \mid j < 0\}$. We observe that

$$g = \dot{x}(J)\, \dot{y}(J)\, \dot{z}(J). \tag{3.7}$$

Let $C = C(S) \in \mathbb{N}$ be as in Lemma 2.6 (i). We call

$$\ddot{g}(J) = \dot{x}(J)\, t^{-2C}\, \dot{y}(J)^{-1}\, t^{-2C} \dot{z}(J)$$

the J-*variant of* g; see Fig. 4 for a schematic illustration.

Observe that

$$\ddot{g}(J) \in N t^{\rho(\ddot{g}(J))}, \quad \text{where } \rho(\ddot{g}(J)) = 2(\sigma_g^- - \sigma_g^+) - 4C \leqslant -4. \tag{3.8}$$

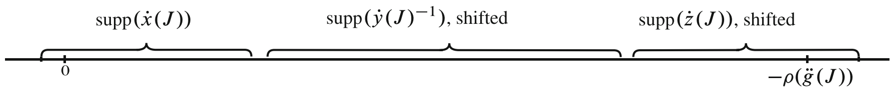

Fig. 4 A schematic illustration of the support components of $\ddot{g}(J)$

Up to now we assumed that $k^+ \leqslant k^-$. If instead $k^- < k^+$, a similar construction at this stage yields elements

$$\ddot{g}(J) \in Nt^{\rho(\ddot{g}(J))}, \quad \text{where} \;\; \rho(\ddot{g}(J)) = 2(\sigma_g^+ - \sigma_g^-) + 4C \geqslant 4; \tag{3.9}$$

in particular, there is no overlap between elements $\ddot{g}(J)$ arising from these two different cases.

For our purposes, it suffices to work with subsets $J \subseteq [\sigma^-, \sigma^+]_{\mathbb{Z}}$ of size $|J| = 2$ and we streamline the discussion to this situation.

Lemma 3.6 *In the set-up described above, suppose that $J \subseteq [\sigma^-, \sigma^+]_{\mathbb{Z}}$ with $|J| = 2$. Let $D = D(S, u) \in \mathbb{N}$ be as in Lemma 2.6* (ii). *Then*

(i) *$l_S(\ddot{g}(J)) \leqslant l_S(g) + D'$ for $D' = 6D + 2\,l_S(t^{2C})$;*
(ii) *the element g can be recovered from $\ddot{g}(J)$ and any one of σ^+, σ^-;*
(iii) *the resulting variants of g are pairwise distinct, i.e., $\ddot{g}(J) \neq \ddot{g}(J')$ for all $J' \subseteq [\sigma^-, \sigma^+]_{\mathbb{Z}}$ with $|J'| = 2$ and $J \neq J'$.*

Proof (i) Since

$$\begin{aligned} J_{\geqslant 0} &\subseteq [0, \sigma^+]_{\mathbb{Z}} \subseteq [\mathrm{mini}(I_1), \mathrm{maxi}(I_1)]_{\mathbb{Z}}, \\ J - \sigma^+ &\subseteq [\sigma^- - \sigma^+, 0]_{\mathbb{Z}} = [\mathrm{mini}(I_2), \mathrm{maxi}(I_2)]_{\mathbb{Z}}, \\ J_{<0} - \sigma^- &\subseteq [0, -\sigma^-]_{\mathbb{Z}} \subseteq [\mathrm{mini}(I_3), \mathrm{maxi}(I_3)]_{\mathbb{Z}} \end{aligned}$$

we can apply Lemma 2.6 (ii), if necessary twice, to deduce that

$$l_S(\dot{x}(J)) \leqslant l_S(x) + 2D, \quad l_S(\dot{y}(J)) \leqslant l_S(y) + 2D, \quad l_S(\dot{z}(J)) \leqslant l_S(z) + 2D.$$

Since $l_S(x) + l_S(y) + l_S(z) = l_S(g)$, this gives

$$l_S(\ddot{g}(J)) \leqslant l_S(g) + D' \quad \text{for} \;\; D' = 6D + 2\,l_S(t^{2C}).$$

(ii) As in the discussion above suppose that $k^+ = k^+_{\mathcal{W},g}$ and $k^- = k^-_{\mathcal{W},g}$ satisfy $k^+ \leqslant k^-$; the other case $k^- < k^+$ can be dealt with similarly. We have to check that g can be recovered from $\ddot{g}(J)$, if we are allowed to use one of the parameters σ^+, σ^-. Indeed, from $-\rho(\ddot{g}(J)) = 2(\sigma^+ - \sigma^-) + 4C$ we deduce that in such a case both, σ^+ and σ^- are available to us. Furthermore, Lemma 2.6 (i) gives

$$\begin{aligned} \mathrm{supp}(\dot{x}(J)) &\subseteq [\sigma^- - C + 1, \sigma^+ + C - 1]_{\mathbb{Z}}, \\ \mathrm{supp}(\dot{y}(J)^{-1}) &\subseteq [-C + 1, \sigma^+ - \sigma^- + C - 1]_{\mathbb{Z}}, \\ \mathrm{supp}(\dot{z}(J)) &\subseteq [-C + 1, \sigma^+ - \sigma^- + C - 1]_{\mathbb{Z}}, \end{aligned}$$

and thus

$$\operatorname{supp}(\ddot{g}(J)) = \operatorname{supp}(\dot{x}(J)) \cup \big(\operatorname{supp}(\dot{y}(J)^{-1}) + \sigma^{+} + 2C\big) \\ \cup \big(\operatorname{supp}(\dot{z}(J)) + 2\sigma^{+} - \sigma^{-} + 4C\big)$$

allows us to recover $\dot{x}(J)$, $\dot{y}(J)$ and $\dot{z}(J)$ via (3.5) and

$$\dot{x}(J)_{|i} = \begin{cases} \ddot{g}(J)_{|i} & \text{for } i \in [\sigma^{-} - C, \sigma^{+} + C]_{\mathbb{Z}}, \\ 1 & \text{for } i \in \mathbb{Z} \smallsetminus [\sigma^{-} - C, \sigma^{+} + C]_{\mathbb{Z}}, \end{cases}$$
$$(\dot{y}(J)^{-1})_{|i} = \begin{cases} \ddot{g}(J)_{|i+\sigma^{+}+2C} & \text{for } i \in [-C, \sigma^{+} - \sigma^{-} + C]_{\mathbb{Z}}, \\ 1 & \text{for } i \in \mathbb{Z} \smallsetminus [-C, \sigma^{+} - \sigma^{-} + C]_{\mathbb{Z}}, \end{cases}$$
$$\dot{z}(J)_{|i} = \begin{cases} \ddot{g}(J)_{|i+2\sigma^{+}-\sigma^{-}+4C} & \text{for } i \in [-C, \sigma^{+} - \sigma^{-} + C]_{\mathbb{Z}}, \\ 1 & \text{for } i \in \mathbb{Z} \smallsetminus [-C, \sigma^{+} - \sigma^{-} + C]_{\mathbb{Z}}. \end{cases}$$

Using (3.7), we recover $g = \dot{x}(J)\,\dot{y}(J)\,\dot{z}(J)$.

(iii) Again we suppose that $k^{+} = k^{+}_{\mathcal{W},g}$ and $k^{-} = k^{-}_{\mathcal{W},g}$ satisfy $k^{+} \leqslant k^{-}$; the other case $k^{-} < k^{+}$ can be dealt with similarly. Let $J' \subseteq [\sigma^{-}, \sigma^{+}]_{\mathbb{Z}}$ with $|J'| = 2$ such that $\ddot{g}(J) = \ddot{g}(J')$. As explained above, we can not only recover g but even $\dot{x}(J) = \dot{x}(J')$, $\dot{y}(J) = \dot{y}(J')$ and $\dot{z}(J) = \dot{z}(J')$ from $\ddot{g}(J) = \ddot{g}(J')$ and σ^{+}, say. Since $u \neq 1$ we deduce from (3.6) that $J = J'$. □

Proof of Theorem B We continue within the set-up established above; in particular, we employ the J-variants $\ddot{g}(J)$ of elements $g \in N$ for two-element subsets $J \subseteq [\sigma_g^{-}, \sigma_g^{+}]_{\mathbb{Z}}$, with respect to a fixed representative function $\mathcal{W}$ and a chosen element $u \in H \smallsetminus \{1\}$.

Let $q : \mathbb{N} \to \mathbb{R}_{\geqslant 1}$ be a non-decreasing unbounded function such that $q \in o(\log n)$. We make use of the decomposition

$$N \cap B_S(n) = R_q(n) \cup R_q^{\flat}(n), \quad \text{for } n \in \mathbb{N}, \tag{3.10}$$

where $R_q(n) = R_{\mathcal{W},q}(n)$ is defined as in Proposition 3.2 and $R_q^{\flat}(n) = R_{\mathcal{W},q}^{\flat}(n)$ denotes the corresponding complement in $N \cap B_S(n)$. Let $D' \in \mathbb{N}$ be as in Lemma 3.6 (i). Below we show that

$$|B_S(n + D')| > \frac{q(n)}{2} \, |R_q^{\flat}(n)| \qquad \text{for } n \in \mathbb{N}. \tag{3.11}$$

This bound and submultiplicativity yield

$$\frac{|R_q^{\flat}(n)|}{|B_S(n)|} < \frac{2|B_S(n + D')|}{q(n)|B_S(n)|} \leqslant \frac{2|B_S(D')|}{q(n)} \to 0 \quad \text{as } n \to \infty.$$

Together with Proposition 3.2 we deduce from (3.10) that N has density zero:

$$\delta_S(N) = \lim_{n\to\infty} \frac{|N \cap B_S(n)|}{|B_S(n)|} = 0,$$

properly as a limit.

It remains to establish (3.11). The set $R_q^\flat(n)$ decomposes into a disjoint union of subsets

$$R_{q,\ell}^\flat(n) = \{g \in N \cap B_S(n) \mid \sigma_g^+ - \sigma_g^- = \ell\}, \quad \ell > q(n),$$

and the map

$$\begin{aligned} F_n \colon R_q^\flat(n) &\to \mathcal{P}(B_S(n + D')), \\ g &\mapsto \{\ddot{g}(J) \mid J \subseteq [\sigma_g^-, \sigma_g^+]_{\mathbb{Z}} \text{ with } |J| = 2\} \end{aligned}$$

restricts for each $\ell \in \mathbb{N}$ with $\ell > q(n)$, to a mapping

$$F_{n,\ell} \colon R_{q,\ell}^\flat(n) \to \mathcal{P}\big((Nt^{-2\ell-4C} \cup Nt^{2\ell+4C}) \cap B_S(n + D')\big);$$

see Lemma 3.6 (i), (3.8) and (3.9).

We contend that for every $h \in (Nt^{-2\ell-4C} \cup Nt^{2\ell+4C}) \cap B_S(n + D')$, where $\ell > q(n)$, there are at most $\ell + 1$ elements $g \in R_{q,\ell}^\flat(n)$ such that $h \in F_n(g)$. Indeed, suppose that $h \in Nt^{2\ell+4C} \cap B_S(n + D')$, with $\ell > q(n)$, and suppose that $g \in R_{q,\ell}^\flat(n)$ such that $h = \ddot{g}(J)$ for some $J \subseteq [\sigma_g^-, \sigma_g^+]_{\mathbb{Z}}$ with $|J| = 2$. Then $\sigma_g^+ \in [0, \ell]_{\mathbb{Z}}$ takes one of $\ell + 1$ values, and once σ^+ is fixed, there is a way of recovering g, by Lemma 3.6 (ii). For $h \in Nt^{-2\ell-4C} \cap B_S(n + D')$ the argument is similar.

From this observation and Lemma 3.6 (ii) we conclude that

$$\begin{aligned} \big|\big(Nt^{-2\ell-4C} \cup Nt^{2\ell+4C}\big) \cap B_S(n + D')\big| &\geqslant \frac{1}{\ell+1}\binom{\ell+1}{2} \big|R_{q,\ell}^\flat(n)\big| \\ &> \frac{q(n)}{2} \big|R_{q,\ell}^\flat(n)\big|. \end{aligned}$$

Hence

$$|B_S(n + D')| > \frac{q(n)}{2} \sum_{\ell > q(n)} \big|R_{q,\ell}^\flat(n)\big| = \frac{q(n)}{2} \big|R_q^\flat(n)\big|,$$

which is the bound (3.11) we aimed for. □

4 Proof of Theorem C

Throughout this section let G denote a finitely generated group of exponential word growth of the form $G = N \rtimes \langle t \rangle$, where

(a) the subgroup $\langle t \rangle$ is infinite cyclic;
(b) the normal subgroup $N = \left\langle \bigcup \{ H^{t^i} \mid i \in \mathbb{Z} \} \right\rangle$ is generated by the $\langle t \rangle$-conjugates of a finitely generated subgroup $H \; N$;
(c) the $\langle t \rangle$-conjugates of this group H commute elementwise: $[H^{t^i}, H^{t^j}] = 1$ for all $i, j \in \mathbb{Z}$ with $H^{t^i} \neq H^{t^j}$.

Suppose further that $S_0 = \{a_1, \ldots, a_d\} \subseteq H$ is a finite symmetric generating set for H and that the exponential growth rates of H with respect to S_0 and of G with respect to $S = S_0 \cup \{t, t^{-1}\}$ satisfy

$$\lim_{n \to \infty} \sqrt[n]{|B_{H,S_0}(n)|} < \lim_{n \to \infty} \sqrt[n]{|B_{G,S}(n)|}. \tag{4.1}$$

This is essentially the setting of Theorem C; for technical reasons we prefer to work with symmetric generating sets. Our ultimate aim is to show that $\delta_S(N) = 0$.

Using the commutation rules recorded in (c), it is not difficult to see that every $g \in N$ admits S-expressions of minimal length that take the special form

$$g = t^{-\sigma^-} \cdot \left(\prod_{i=\sigma^-}^{\sigma^+ - 1} (w_i(a_1, \ldots, a_d)\, t^{-1}) \right) \cdot w_{\sigma^+}(a_1, \ldots, a_d) \cdot t^{\sigma^+}, \tag{4.2}$$

$$g = t^{-\sigma^+} \cdot \left(\prod_{j=\sigma^-}^{\sigma^+ - 1} (w_{\sigma^+ + \sigma^- - j}(a_1, \ldots, a_d)\, t) \right) \cdot w_{\sigma^-}(a_1, \ldots, a_d) \cdot t^{\sigma^-}, \tag{4.3}$$

where the parameters $\sigma^-, \sigma^+ \in \mathbb{Z}$ satisfy $\sigma^- \leqslant \sigma^+$ and, for every $i \in [\sigma^-, \sigma^+]_{\mathbb{Z}}$, we have picked a suitable semigroup word $w_i = w_i(Y_1, \ldots, Y_d)$ in d variables of length $l_{S_0}(w_i(a_1, \ldots, a_d))$. The lengths of the expressions (4.2) and (4.3) are equal to

$$l_S(g) = |\sigma^-| + (\sigma^+ - \sigma^-) + |\sigma^+| + \sum_{i=\sigma^-}^{\sigma^+} l_{S_0}(w_i(a_1, \ldots, a_d)).$$

For the following we fix, for each $g \in N$, expressions as described and we use subscripts to stress the dependency on g: we write σ_g^-, σ_g^+ and $w_{g,i}$ for $i \in [\sigma_g^-, \sigma_g^+]_{\mathbb{Z}}$, where necessary. The notation is meant to be reminiscent of the one introduced in Definition 2.3, but one needs to keep in mind that we are dealing with a larger class of groups now.

Lemma 4.1 *In addition to the general set-up described above, let $q \colon \mathbb{N} \to \mathbb{R}_{>0}$ be a non-decreasing unbounded function such that $q \in o(n)$. Then the sequence of sets*

$$R_q(n) = \left\{ g \in N \cap B_S(n) \mid -q(n) \leqslant \sigma_g^- \leqslant \sigma_g^+ \leqslant q(n) \right\},$$

indexed by $n \in \mathbb{N}$*, satisfies*

$$\lim_{n\to\infty} \frac{|R_q(n)|}{|B_S(n)|} = 0.$$

Proof For short we set $\mu = \lim_{n\to\infty} \sqrt[n]{|B_{H,S_0}(n)|}$ and $\lambda = \lim_{n\to\infty} \sqrt[n]{|B_{G,S}(n)|}$. According to (4.1) we find $\varepsilon \in \mathbb{R}_{>0}$ such that $(\mu+\varepsilon)/\lambda \leqslant 1-\varepsilon$ and $M = M_\varepsilon \in \mathbb{N}$ such that

$$|B_{H,S_0}(n)| \leqslant M(\mu+\varepsilon)^n \quad \text{for all} \;\; n \in \mathbb{N}_0.$$

This allows us to bound the number of possibilities for the elements $w_{g,i}(a_1,\ldots,a_d)$ in an S-expression of the form (4.2) for $g \in R_q(n)$ and, writing $\tilde{q}(n) = 2\lfloor q(n)\rfloor + 1$, we obtain

$$\begin{aligned} |R_q(n)| &\leqslant \sum_{\substack{m_{-\lfloor q(n)\rfloor},\ldots,m_{\lfloor q(n)\rfloor}\in\mathbb{N}_0 \text{ st} \\ m_{-\lfloor q(n)\rfloor}+\cdots+m_{\lfloor q(n)\rfloor}\leqslant n}} \; \prod_{i=-\lfloor q(n)\rfloor}^{\lfloor q(n)\rfloor} |B_{H,S_0}(m_i)| \\ &\leqslant \binom{n+\tilde{q}(n)}{\tilde{q}(n)} M^{\tilde{q}(n)} (\mu+\varepsilon)^n, \end{aligned}$$

and hence

$$\frac{|R_q(n)|}{|B_S(n)|} \leqslant \frac{|R_q(n)|}{\lambda^n} \leqslant \binom{n+\tilde{q}(n)}{\tilde{q}(n)} M^{\tilde{q}(n)} (1-\varepsilon)^n \quad \text{for} \;\; n \in \mathbb{N}. \tag{4.4}$$

We notice that $q \in o(n)$ implies $\tilde{q} \in o(n)$. Thus Lemma 2.1 implies that $\binom{n+\tilde{q}(n)}{\tilde{q}(n)} M^{\tilde{q}(n)}$ grows sub-exponentially, and the term on the right-hand side of (4.4) tends to 0 as n tends to infinity. □

Proof of Theorem C We continue to work in the notational set-up introduced above. In addition we fix a non-decreasing unbounded function $q : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ such that $q \in o(n)$ and

$$\frac{|B_S(n)|}{|B_S(n-q(n))|} \to \infty \quad \text{as} \;\; n \to \infty; \tag{4.5}$$

see Proposition 2.2. As in the proof of Theorem B, we make use of a decomposition

$$N \cap B_S(n) = R_q(n) \mathbin{\dot\cup} R_q^\flat(n), \quad \text{for} \;\; n \in \mathbb{N},$$

where $R_q(n)$ is defined as in Lemma 4.1 and $R_q^\flat(n)$ denotes the corresponding complement in $N \cap B_S(n)$.

In view of Lemma 4.1 it suffices to show that

$$\frac{|R_q^\flat(n)|}{|B_S(n)|} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.6}$$

It is enough to consider sufficiently large n so that $n > q(n)$ holds. For every such n and $g \in R_q^\flat(n)$, with chosen minimal S-expressions (4.2) and (4.3), we have $\sigma^- = \sigma_g^- < -q(n)$ or $\sigma^+ = \sigma_g^+ > q(n)$, hence

$$\left\{gt^{-q(n)}, gt^{q(n)}\right\} \cap B_S(n - q(n)) \neq \varnothing.$$

As each of the right translation maps $g \mapsto gt^{-q(n)}$ and $g \mapsto gt^{q(n)}$ is injective, we conclude that

$$|R_q^\flat(n)| \leqslant 2|B_S(n - q(n))|,$$

and thus (4.6) follows from (4.5). □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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