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RESEARCH ARTICLE

On some categories of structured sets

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Abstract

Given an arbitrary set Ω, we consider the collections $\mathrm{SS}(\Omega)$, $\mathrm{SR}(\Omega)$ and $\mathrm{SO}(\Omega)$ of all the set systems, the binary set relations and the set operators on Ω. We introduce the notion of *linking map* on Ω as any map whose domain and codomain may be chosen between the above collections. After providing a descriptive overview useful for framing the notion of linking map in a broad non-specialized context, we explain how linking maps occur in a very natural way in two specific results. The first of these results concerns the classic identification between the subfamily $\mathrm{EQ}(\Omega)$ of all the equivalence relations on Ω and the subfamily $\mathrm{SP}(\Omega)$ of all the set partitions on Ω. Starting from it, we introduce a new subfamily $\mathrm{ESO}(\Omega)$ of closure operators on Ω and four linking maps whose restrictions to the subfamilies $\mathrm{EQ}(\Omega)$, $\mathrm{SP}(\Omega)$ and $\mathrm{ESO}(\Omega)$ are bijections. The second result concerns the identification between the subfamily $\mathrm{CSO}(\Omega)$ of all closure set operators on Ω and the subfamily $\mathrm{CSS}(\Omega)$ of all closure set systems on Ω. Starting from it, we introduce a new subfamily $\mathrm{DSR}(\Omega)$ of binary set relations and four linking maps whose restrictions to the subfamilies $\mathrm{CSO}(\Omega)$, $\mathrm{CSS}(\Omega)$ and $\mathrm{DSR}(\Omega)$ are again bijections. In an attempt to extend in a natural way the above linking maps to categorical isomorphisms, after fixing a nonnegative integer k, we introduce three categories $\mathbf{SS^k}$, $\mathbf{SR^k}$ and $\mathbf{SO^k}$, whose detailed study mainly occupies the first part of the present work. Objects and arrows of these three categories are obtained by means of k-iterations of the powerset functor $\wp\colon \mathbf{Set} \longrightarrow \mathbf{Set}$, and they generalize the notions of set systems, set relations and set operators, respectively. In the second part of the paper, we extend the linking maps previously described at a categorical level in terms of isomorphisms between specific categories of set systems, binary set relations and set operators generalizing the occurring collections introduced

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before, and also prove numerous other results concerning the main properties of all these categories, such as completeness, cocompleteness and Cartesian closedness.

Keywords Set systems · Set operators · Dependence relations · Closure operators · Categories · Functors · Complete and Cartesian closed categories · Linking maps

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1 Introduction

1.1 Motivation and framing of the paper

In many contexts from both mathematics and theoretical computer science three types of families associated with an arbitrary fixed set Ω assume a substantial relevance, namely the collection $\mathrm{SS}(\Omega) := \wp(\wp(\Omega))$ of all *set systems* on Ω (in the whole paper $\wp(\Omega)$ denotes the powerset of Ω), the collection $\mathrm{SR}(\Omega) := \wp(\wp(\Omega) \times \wp(\Omega))$ of all *set relations* on Ω, and the collection $\mathrm{SO}(\Omega)$ of all the maps $\sigma : \wp(\Omega) \longrightarrow \wp(\Omega)$, that are usually called *set operators* on Ω.

More in detail, in the previous contexts, many notions and corresponding results may be reformulated and refined in terms of set systems, set relations, set operators and through appropriate maps between such collections. We will call *linking maps* on Ω these kinds of maps, and *set relation geometry* on Ω any fixed family of linking maps on Ω.

The main purpose of the present work is twofold:

- to provide an accurate study of some types of linking maps, which permits to reinterpret classical results for equivalence relations and closure operators in much more general and categorical context (see Theorems 5.4 and 6.6);
- to frame in a suitable way the notion of linking map in a categorical context, analyzing its functoriality after introducing new *concretizable* categories and subcategories [25] of set systems, set relations and set operators, by means of iterations of the usual powerset functor of the set category **Set**.

There are many studies that fit within such a framework, and to this regard we refer the reader to a (not exhaustive) part of these works, some of which on pure mathematics [2, 6, 10, 17, 20, 21], and others on an intermediate scope between mathematics and theoretical computer science [22–24, 28, 29, 35]. However, two of the main sources of inspiration for the basic ideas developed in this work can be traced back to the coarse- and weak-shape theories, mainly dealt with in topological contexts (for details on these links, we refer the reader to [31]).

Two very relevant and common cases in large part of both mathematics and theoretical computer science concern the classic notions of equivalence relation and closure operator (recall that a *set closure operator* is a set operator which is extensive, increasing and idempotent).

Clearly, any element of $\mathrm{EQ}(\Omega)$ (i.e. the collection of all equivalence relations on Ω) may be also uniquely identified with a corresponding element of $\mathrm{SR}(\Omega)$, simply by identifying any element $x \in \Omega$ with the singleton $\{x\} \in \wp(\Omega)$. Therefore we can see $\mathrm{EQ}(\Omega)$ as a subfamily of $\mathrm{SR}(\Omega)$. Let moreover $\mathrm{SP}(\Omega) \subseteq \mathrm{SS}(\Omega)$ be the collection of all set partitions on Ω and $\mathrm{ESO}(\Omega)$ be the collection of a new type of set operators on Ω, that we call *equivalence set operator*. These are extensive set operators $\sigma \in \mathrm{SO}(\Omega)$ for which $\sigma(X) = \bigcup\{\sigma(\{x\}) \mid x \in X\}$ for each $X \in \wp(\Omega)$, and such that the condition $\sigma(\{x\}) = \sigma(\{y\})$ is equivalent to $\sigma(\{x\}) \cap \sigma(\{y\}) \neq \emptyset$ for any $x, y \in \Omega$ (see Definition 5.1).

At a first elementary level, it is well known how to identify to each other the elements of $\mathrm{EQ}(\Omega)$ and $\mathrm{SP}(\Omega)$. However, going further the aforementioned level, we can see that two more general maps $\mathrm{Pa}\colon \mathrm{SR}(\Omega) \longrightarrow \mathrm{SS}(\Omega)$ and $\mathrm{Eq}\colon \mathrm{SS}(\Omega) \longrightarrow \mathrm{SR}(\Omega)$, respectively defined by $\mathrm{Pa}_{\mathcal{R}} := \{N_{\mathcal{R}}(x) \mid x \in \Omega\}$ for any $\mathcal{R} \in \mathrm{SR}(\Omega)$, where $N_{\mathcal{R}}(x) := \{y \in \Omega \mid (\{x\}, \{y\}) \in \mathcal{R}$ or $(\{y\}, \{x\}) \in \mathcal{R}\}$, and by $\mathrm{Eq}_{\mathcal{F}} := \{(\{x\}, \{y\}) \in \wp(\Omega) \times \wp(\Omega) \mid \exists\, Z \in \mathcal{F}\ [\{x, y\} \subseteq Z]\}$, for each $\mathcal{F} \in \mathrm{SS}(\Omega)$, induce such an identification.

It is clear that the two above maps Pa and Eq are not inverses of each other. Nevertheless, their corresponding restrictions to the subfamilies $\mathrm{EQ}(\Omega)$ and $\mathrm{SP}(\Omega)$, respectively, are inverses of each other, providing the usual identification between equivalence relations and set partitions.

We now analyze the link between set partitions and equivalence set operators by means of maps between $\mathrm{SS}(\Omega)$ and $\mathrm{SO}(\Omega)$. To this regard let $\mathrm{Up}\colon \mathrm{SS}(\Omega) \longrightarrow \mathrm{SO}(\Omega)$, defined by $\mathrm{Up}_{\mathcal{F}}(X) := \bigcup\{Y \in \mathcal{F} \mid X \cap Y \neq \emptyset\}$, for each $\mathcal{F} \in \mathrm{SS}(\Omega)$ and any $X \in \wp(\Omega)$, and $\mathrm{Qa}\colon \mathrm{SO}(\Omega) \longrightarrow \mathrm{SS}(\Omega)$ defined by $\mathrm{Qa}(\sigma) := \{\sigma(\{x\}) \mid x \in \Omega\}$, for each $\sigma \in \mathrm{SO}(\Omega)$. Then, even in this case, the maps Up and Qa are not inverses of each other, but they become such after restricting them to the subfamilies $\mathrm{SP}(\Omega)$ and $\mathrm{ESO}(\Omega)$, respectively.

Another similar situation occurs when we consider the subfamilies $\mathrm{CSO}(\Omega)$ of all closure set operators on Ω, $\mathrm{CSS}(\Omega)$ of all *closure set systems* on Ω (they are set systems containing Ω and closed with respect to arbitrary intersections) and $\mathrm{DSR}(\Omega)$ of all *dependence set relations* on Ω [14] (see Definition 6.1). In this case, let $\eta\colon \mathrm{SR}(\Omega) \longrightarrow \mathrm{SO}(\Omega)$, $\mathrm{Cl}\colon \mathrm{SS}(\Omega) \longrightarrow \mathrm{SO}(\Omega)$, $\Phi\colon \mathrm{SO}(\Omega) \longrightarrow \mathrm{SR}(\Omega)$ and $\mathrm{Fix}\colon \mathrm{SO}(\Omega) \longrightarrow \mathrm{SS}(\Omega)$ be the maps defined respectively by $\eta_{\mathcal{R}}(X) := \bigcup\{Y \in \wp(\Omega) \mid (Y, X) \in \mathcal{R}\}$, $\mathrm{Cl}_{\mathcal{F}}(X) := \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}$, $\Phi(\sigma) := \{(Z, W) \in \wp(\Omega) \times \wp(\Omega) \mid Z \subseteq \sigma(W)\}$ and $\mathrm{Fix}(\sigma) := \{X \in \wp(\Omega) \mid \sigma(X) = X\}$, for any $\mathcal{R} \in \mathrm{SR}(\Omega)$, $\mathcal{F} \in \mathrm{SS}(\Omega)$, $\sigma \in \mathrm{SO}(\Omega)$ and $X \in \wp(\Omega)$.

Then it results that the restriction of the map Cl to the subfamily $\mathrm{CSS}(\Omega)$ is the inverse of the restriction of the map Fix to the subfamily $\mathrm{CSO}(\Omega)$. Moreover, the restriction of the map Φ to the subfamily $\mathrm{CSO}(\Omega)$ is the inverse of the map η to the subfamily $\mathrm{DSR}(\Omega)$ (see Proposition 5.3).

The previous bijections between equivalence relations, set partitions, equivalence set operators, and between closure set operators, closure set systems, dependence set relations, provide two relevant examples of the possibility to describe classical results in terms of specific maps whose domain and codomain may be chosen between $\mathrm{SS}(\Omega)$, $\mathrm{SR}(\Omega)$ and $\mathrm{SO}(\Omega)$. Because of their importance, we name these maps that we consider as the starting point of the present work.

Definition 1.1 Let $\mathfrak{B}, \mathfrak{C} \in \{\mathrm{SO}(\Omega), \mathrm{SS}(\Omega), \mathrm{SR}(\Omega)\}$. We call any map $\beta : \mathfrak{B} \longrightarrow \mathfrak{C}$ a *linking map* on Ω. We also call any collection of linking maps on Ω a *set relation geometry* on Ω.

A particularly interesting situation occurs when one endows the ground set Ω with some algebraic, topological, combinatorial or order structure, and investigates a fixed set relation geometry on such a mathematical structure. In fact, such a type of study can help to find new and non-trivial classifications of families of substructures which are naturally induced by the mutual interrelations of the linking maps that constitute the assigned relation geometry on Ω. Some recent studies in this perspective concern the cases where Ω is an abelian variety over a finite field [26], a special type of DG-module [27], a particular space of polynomial automorphisms [4, 5], an integral domain [15, 19], a left-act of monoid [12], a module on a unitary ring [13], a finite lattice [11]. However, in the present paper we do not take any additional structure on the ground set Ω.

At this point it is convenient to frame the two specific aforementioned examples in a more formal way, introducing the next notion of *sub-bijection*, that will be a starting point in order to determine new corresponding categorial isomorphisms.

Definition 1.2 Let $\mathfrak{B}, \mathfrak{C} \in \{\mathrm{SO}(\Omega), \mathrm{SS}(\Omega), \mathrm{SR}(\Omega)\}$. We say that a formal writing of the type

$$(\mathfrak{B} \mid \mathfrak{B}') \underset{\gamma}{\overset{\beta}{\rightleftarrows}} (\mathfrak{C} \mid \mathfrak{C}')$$

is a $(\mathfrak{B}, \mathfrak{C})$-*sub-bijection* if $\mathfrak{B}' \subseteq \mathfrak{B}$, $\mathfrak{C}' \subseteq \mathfrak{C}$, and $\beta : \mathfrak{B} \longrightarrow \mathfrak{C}$, $\gamma : \mathfrak{C} \longrightarrow \mathfrak{B}$ are two linking maps such that:

(i) $\beta(\mathcal{B}) \in \mathfrak{C}'$ for all $\mathcal{B} \in \mathfrak{B}'$ and $\gamma(\mathcal{C}) \in \mathfrak{B}'$ for all $\mathcal{C} \in \mathfrak{C}'$;
(ii) $\gamma(\beta(\mathcal{X})) = \mathcal{X}$ for all $\mathcal{X} \in \mathfrak{B}'$ and $\beta(\gamma(\mathcal{Y})) = \mathcal{Y}$ for all $\mathcal{Y} \in \mathfrak{C}'$.

Moreover, if $n \geqslant 2$ and $\mathfrak{B}_1, \ldots, \mathfrak{B}_n \in \{\mathrm{SO}(\Omega), \mathrm{SS}(\Omega), \mathrm{SR}(\Omega)\}$, we say that the formal writing

$$(\mathfrak{B}_1 \mid \mathfrak{B}_1') \underset{\gamma_1}{\overset{\beta_1}{\rightleftarrows}} (\mathfrak{B}_2 \mid \mathfrak{B}_2') \underset{\gamma_2}{\overset{\beta_2}{\rightleftarrows}} \cdots \underset{\gamma_{n-2}}{\overset{\beta_{n-2}}{\rightleftarrows}} (\mathfrak{B}_{n-1} \mid \mathfrak{B}_{n-1}') \underset{\gamma_{n-1}}{\overset{\beta_{n-1}}{\rightleftarrows}} (\mathfrak{B}_n \mid \mathfrak{B}_n')$$

is a $(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$-*sub-bijection* if

$$(\mathfrak{B}_k \mid \mathfrak{B}_k') \underset{\gamma_k}{\overset{\beta_k}{\rightleftarrows}} (\mathfrak{B}_{k+1} \mid \mathfrak{B}_{k+1}')$$

is a $(\mathfrak{B}_k, \mathfrak{B}_{k+1})$-sub-bijection for each $k = 1, \ldots, n-1$.

By means of the above terminology and notations, we may express the bijection between equivalence relations, set partitions and equivalence set operators in terms of the following $(\mathrm{SR}(\Omega), \mathrm{SS}(\Omega), \mathrm{SO}(\Omega))$-sub-bijection (see Proposition 5.3):

$$(\mathrm{SR}(\Omega) \mid \mathrm{EQ}(\Omega)) \underset{\mathrm{Eq}}{\overset{\mathrm{Pa}}{\rightleftarrows}} (\mathrm{SS}(\Omega) \mid \mathrm{SP}(\Omega)) \underset{\mathrm{Qa}}{\overset{\mathrm{Up}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{ESO}(\Omega)),$$

and, similarly, we may express the bijection between closure systems, closure operators and dependence relations in terms of the following $(\mathrm{SS}(\Omega), \mathrm{SO}(\Omega), \mathrm{SR}(\Omega))$-sub-bijection (see Proposition 6.4):

$$(\mathrm{SS}(\Omega) \mid \mathrm{CSS}(\Omega)) \underset{\mathrm{Fix}}{\overset{\mathrm{Cl}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{CSO}(\Omega)) \underset{\eta}{\overset{\Phi}{\rightleftarrows}} (\mathrm{SR}(\Omega) \mid \mathrm{DSR}(\Omega)). \tag{1}$$

Clearly, when we have a set relation geometry whose linking maps are sub-bijections it is possible to carry out the results concerning specific subfamilies of set systems, and to express them in terms of set relations or set operators.

Note that in Definition 1.2 the bijections induced by the linking maps β and γ are their respective restrictions on $\mathfrak{B}'$ and $\mathfrak{C}'$. Therefore, one could question the real need for the introduction of the maps β and γ on the ambient families $\mathfrak{B}$ and $\mathfrak{C}$. The answer to such a legitimate question comes from our explicit desire to highlight that the bijections between $\mathfrak{B}'$ and $\mathfrak{C}'$ derive from maps defined on more general domains. To this regard, in fact, it can happen that there are two bijections between two pairs of distinct subfamilies $(\mathfrak{B}', \mathfrak{C}')$ and $(\mathfrak{B}'', \mathfrak{C}'')$ of $\mathfrak{B}$ and $\mathfrak{C}$ which derive from a same pair (β, γ) of linking maps defined between $\mathfrak{B}$ and $\mathfrak{C}$. For instance, if $\mathfrak{B} = \mathrm{SS}(\Omega)$, $\mathfrak{C} = \mathrm{SO}(\Omega)$, $\mathrm{ATOP}(\Omega) := \{\mathcal{F} \in \mathrm{SS}(\Omega) \mid \mathcal{F}$ is the family of closed sets of an Alexandroff topology on $\Omega\}$ and $\mathrm{ACSO}(\Omega) := \{\sigma \in \mathrm{SO}(\Omega) \mid \sigma$ is the Kuratowski closure operator of an Alexandroff topology on $\Omega\}$, then we get the further $(\mathrm{SS}(\Omega), \mathrm{SO}(\Omega))$-sub-bijection compared to the first of the sub-bijections explicited in (1)

$$(\mathrm{SS}(\Omega) \mid \mathrm{ATOP}(\Omega)) \underset{\mathrm{Fix}}{\overset{\mathrm{Cl}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{ACSO}(\Omega)).$$

Similarly, when Ω is a finite arbitrary set, the same maps give rise to the $(\mathrm{SS}(\Omega), \mathrm{SO}(\Omega))$-sub-bijection

$$(\mathrm{SS}(\Omega) \mid \mathrm{FLAT}(\Omega)) \underset{\mathrm{Fix}}{\overset{\mathrm{Cl}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{MLS}(\Omega)),$$

where $\mathrm{FLAT}(\Omega) := \{\mathcal{F} \in \mathrm{SS}(\Omega) \mid \mathcal{F}$ is the family of flats of a matroid on $\Omega\}$ and $\mathrm{MLS}(\Omega) := \{\sigma \in \mathrm{SO}(\Omega) \mid \sigma$ is the Mac Lane–Steinitz closure operator of a matroid on $\Omega\}$.Hence, the bijections between different subfamilies of $\mathfrak{B}$ and $\mathfrak{C}$ might or might not derive from a same pair (β, γ) of linking maps. In other terms, from a conceptual point of view, two distinct situations occur. In our work, we want to highlight the case where the sub-bijections come from a given fixed pair (β, γ). The reason for that is related to the observation that various *cryptomorphisms* [16], beyond those previously described, that occur in matroid theory or topology arise starting from maps defined on domains that are more general than those between which the cryptomorphism is established. Clearly, when proceeding towards a categorical extension, it is not guaranteed that the behavior of the maps β and γ is functorial. Nevertheless, though such an obstruction may occur, in Definition 1.2 we emphasized the fact that the bijections among subfamilies of $\mathfrak{B}$ and $\mathfrak{C}$ arise from two fixed maps defined on a more general ambient.

Set relation geometries with their corresponding linking maps occur, more or less explicitly, in great part of both mathematical and theoretical computer science literature. To this regard, many works have been cited in the first part of the present subsection. Here we can further mention the monographs [32, 33], and again matroid theory, with regard to its classical results [34].

It is worthwhile noticing that all the previous notions rely on the hypothesis that the ground set Ω is given. Thus it is natural to ask what happens to both the above collections and corresponding linking maps if the set Ω is no longer fixed. In this case, we are clearly led to an analysis of the aforementioned collections and maps from a categorical outlook. In fact, it is natural to think of each element of the collections $\mathrm{SS}(\Omega)$, $\mathrm{SR}(\Omega)$ and $\mathrm{SO}(\Omega)$ as a particular object of specific corresponding categories, and, possibly, each linking map as a particular type of functor among these categories or their subcategories. The introduction and the study of these categories, of some of their subcategories and of several functors defined between them is the primary purpose of the present paper, whose detailed content will be described in the next subsection.

1.2 Content of the paper

As mentioned in the previous subsection, our primary goal is to extend some classic linking maps to a categorical level. To this end, we first need to remove the fixed parameter given by the ground set Ω. Secondly, we need to determine specific families of arrows, allowing us to transform the aforementioned linking maps into suitable functors between these categories or between some of their subcategories. Our constructions rely in natural way on the k-th iterations of the usual powerset functor $\wp\colon \mathbf{Set} \longrightarrow \mathbf{Set}$, where k is a fixed nonnegative integer. The particular case which includes the study of the aforementioned linking maps corresponds to the value $k = 1$, and the case that includes the equivalence relations to the value $k = 0$.

Let us highlight that the categorical extension of linking maps passes from the suitable definition of morphisms. To this regard, as we will work with structures, i.e. sets endowed with set systems, set relations or set operators, a natural choice consists of working with structure-preserving functions.

More in detail, for any fixed integer $k \geqslant 0$, let $\wp^k$ be the composition of k times the powerset functor $\wp$ (the case $k = 0$ corresponds to the identity functor). We first consider the category $\mathbf{SS^k}$, whose objects are pairs of the form $(\Omega, \mathcal{F})$, where Ω is an arbitrary set (no longer fixed) and $\mathcal{F}$ is a set system on $\wp^{k-1}(\Omega)$ when $k \geqslant 1$, or simply a subset of Ω if $k = 0$. As arrows between two given objects $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ of $\mathbf{SS^k}$, we take the maps between sets $f\colon \Omega \longrightarrow \Omega'$ for which $\wp^k f(X) := (\wp^k f)(X) \in \mathcal{F}'$, whenever $X \in \mathcal{F}$. We call $\mathbf{SS^k}$ the category of the *k-set systems*.

Next we introduce the category $\mathbf{SR^k}$ of the *k-set relations*, whose objects are pairs of the form $(\Omega, \mathcal{R})$, where Ω is an arbitrary set and $\mathcal{R} \in \wp(\wp^k(\Omega) \times \wp^k(\Omega))$. The arrows of $\mathbf{SR^k}$ from $(\Omega, \mathcal{R})$ to $(\Omega', \mathcal{R}')$ are the set maps $f\colon \Omega \longrightarrow \Omega'$ such that $(\wp^k f(X), \wp^k f(Y)) \in \mathcal{R}'$, whenever $(X, Y) \in \mathcal{R}$.

Finally, for any $k \geqslant 1$ we introduce three categories of set operators, denoted by $\mathbf{SO^k}$, $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$, and called the *k-set operator*, $(k, \sqsubseteq)$-*set operator* and $(k, \sqsupseteq)$-*set operator* category, respectively. The objects of all these three categories are the pairs (Ω, σ), where Ω is an arbitrary set, and σ is a set operator on $\wp^{k-1}(\Omega)$. However, they differ with respect to the arrows. More specifically, given two objects (Ω, σ) and (Ω, σ'), the corresponding arrows in the above three categories $\mathbf{SO^k}$, $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$ are ordered pairs (f, f') of set maps $f, f' \colon \Omega \longrightarrow \Omega'$ such that $\wp^k f'(\sigma(X)) = \sigma'(\wp^k f(X))$, $\wp^k f'(\sigma(X)) \subseteq \sigma'(\wp^k f(X))$ and $\wp^k f'(\sigma(X)) \supseteq \sigma'(\wp^k f(X))$, for any $X \in \wp^k(\Omega)$, respectively.

For set operators, the situation is more complex than for the categories of set systems and set relations. In this case, indeed, it is necessary to introduce the above three specific types of arrows.

In Sect. 3, after introducing all the above categories, we establish the first basic properties of their arrows. To this regard, we characterize monomorphisms, epimorphisms and isomorphisms of the categories $\mathbf{SS^k}$, $\mathbf{SR^k}$, $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$ (see Propositions 3.7, 3.8 and 3.9, respectively). With regard to the category $\mathbf{SO^k}$, we characterize its isomorphisms (see Proposition 3.10), whereas the complete characterization of its monomorphisms and epimorphisms is left as an open question.

Section 4 contains three subsections. More in detail, in Sect. 4.1 we prove that both the categories $\mathbf{SS^k}$ and $\mathbf{SR^k}$ are bicomplete (see Theorems 4.1 and 4.2), while in Sect. 4.2 we show that both the categories $\mathbf{SS^k}$ and $\mathbf{SR^k}$ are Cartesian closed (see Theorem 4.6).

Finally, in Sect. 4.3 we prove some fundamental properties of the categories $\mathbf{SO^k}$, $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$. We will see that the two categories $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$ are substantially different, though at a first glance their properties might seem to be in a certain sense dual.

In particular, in Theorem 4.7 we show that the category $\mathbf{SO^k}$ is neither complete nor cocomplete since it does not admit neither the initial nor the terminal object. With regard to the remaining two categories, using the fact $\mathbf{SO^{k,\sqsubseteq}}$ has both the initial and terminal object, whereas $\mathbf{SO^{k,\sqsupseteq}}$ admits no initial object, we prove in Theorem 4.10 that the two categories $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$ cannot be neither equivalent nor dually equivalent. Indeed, the main result of the present section (see Theorem 4.12) ensures the completeness of the category $\mathbf{SO^{k,\sqsubseteq}}$, whereas in Proposition 4.15 we show that, in general, $\mathbf{SO^{k,\sqsupseteq}}$ admits no equalizers, so that it turns out to be not complete.

In Sect. 5 we develop the considerations exposed in Sect. 1.1, concerning the extension at categorical level of the sub-bijection obtained in Proposition 5.3. To this regard, we first consider the full subcategory $\mathbf{EQ}$ of $\mathbf{SR^0}$ whose objects are pairs $(\Omega, \mathcal{R})$, where Ω is an arbitrary set and $\mathcal{R} \in \mathrm{EQ}(\Omega)$. Next, we introduce the subcategory $\mathbf{ESO^{1,\sqsubseteq,=}}$, whose objects are pairs (Ω, σ), where Ω is an arbitrary set and $\sigma \in \mathrm{ESO}(\Omega)$, and whose arrows are those of $\mathbf{SO^{1,\sqsupseteq}}$ of the form (f, f).

Finally we consider the category $\mathbf{SP}$ whose objects are pairs $(\Omega, \mathcal{F})$, where Ω is an arbitrary set, $\mathcal{F} \in \mathrm{SP}(\Omega)$ and, for every two of its objects $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$, the corresponding arrows are the set maps $f \colon \Omega \longrightarrow \Omega'$ such that $\wp^2 f(\mathcal{F}) \preceq \mathcal{F}'$, where $\preceq$ denotes the usual refining partial order on $\mathrm{SP}(\Omega')$.

Starting from of the above three categories with their corresponding arrows, we extend the linking maps given in the sub-bijection obtained in Proposition 5.3 to four corresponding functors, through which we prove in Theorem 5.4 the existence of isomorphisms between the categories **EQ**, $\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}$ and **SP**.

In Sect. 6 we deal with categories of closure operators, closure systems and dependence relations. Regarding closure set systems, we introduce the category $\mathbf{CSS}^{\mathbf{1}}$ as the full subcategory of $\mathbf{SS}^{\mathbf{1}}$ whose objects consists of the pairs $(\Omega, \mathcal{F})$, where Ω is an arbitrary set and $\mathcal{F} \in \mathrm{CSS}(\Omega)$. Analogously, we consider the full subcategory $\mathbf{DSR}^{\mathbf{1}}$ of $\mathbf{SR}^{\mathbf{1}}$ whose objects consists of the pairs $(\Omega, \mathcal{R})$, where Ω is an arbitrary set and /. Concerning closure operators, we need to introduce two distinct categories, namely $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$. Their objects agree and are the pairs (Ω, σ), where Ω is an arbitrary set and $\sigma \in \mathrm{CSO}(\Omega)$, while the arrows of $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ are those of $\mathbf{SO}^{\mathbf{1},\sqsubseteq}$ of the form (f, f) (and, similarly, for the arrows of the category $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$).

In Theorem 6.5 we show that the category $\mathbf{DSR}^{\mathbf{1}}$ is complete. Next, we again develop the considerations discussed in Sect. 1.1 about the extension at a categorical level of the sub-bijection obtained in Proposition 6.4. To this regard, such a functorial extension has been provided in Theorem 6.6.

At this point it is appropriate to make a brief consideration regarding a comparison between the statements of Theorems 5.4 and 6.6. In fact, in Theorem 5.4, each of the three factors of the sub-bijection of Proposition 5.3 “extends to a unique category”, while, in Theorem 6.6, closure operators cannot be extended to a unique category in order to obtain the desired categorical isomorphisms. Actually, to achieve such an aim, we need to consider the two distinct categories $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ which, despite having the same objects, require different types of arrows.

Therefore, although Theorem 6.6 seems to yield a weaker kind of result than that of Theorem 5.4, it might lead to the development of future research perspectives, where one deals with the possible cases occurring when the determination of categorical isomorphisms (or equivalences) needs the introduction of categories, extending specific linking maps, and having the same class of objects but different arrows.

In Proposition 6.7 we prove some further basic results concerning the categories $\mathbf{CSS}^{\mathbf{1}}$, $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$. Finally, in Theorem 6.11 we prove the commutativity condition $\Psi \circ \mathrm{Cl}_{\mathrm{DSR}(\Omega)} = \Psi = \mathrm{Fix} \circ \Gamma$, where Ψ and Γ are two linking maps frequently used in theoretical computer science.

2 Notations and brief reviews on categories, set systems and set operators

In this brief preliminary section we introduce the main notations concerning categories and functors that we will use in the whole paper. Next, we also recall some fundamental notions on set systems, set operators and set relations.

If I is an index set and $f_i : \Omega_i \longrightarrow \Omega_i'$ is a map between sets for any $i \in I$, we denote by $\prod\{f_i \mid i \in I\}$ their Cartesian product and we set $f_1 \times \cdots \times f_k := \prod\{f_i \mid i \in I\}$ when $I = \{1, \ldots, k\}$. Moreover, if $\Omega_i = \Omega$ for any $i \in I$, we denote by $\dot{\prod}\{f_i \mid i \in I\}$ the map from Ω to $\prod\{\Omega_i' \mid i \in I\}$ associating the element $(f_i(z))_{i \in I}$ with every $z \in \Omega$.

For all the main notions and results on categories, we refer the reader to [7–9], while, for the basics on classes and sets, we refer the reader to [30]. In this paper we only deal with covariant functors, so we use the term *functor* as equivalent to *covariant* *functor*. We denote by **Set** the usual category whose objects are sets and whose arrows are maps between sets.

Let $\mathfrak{C}$ be a given category. We denote by $\mathrm{Obj}(\mathfrak{C})$ and $\mathrm{Arr}(\mathfrak{C})$ the classes of the objects and arrows (or equivalently morphisms) of $\mathfrak{C}$, respectively. Moreover, if $A, B \in \mathrm{Obj}(\mathfrak{C})$, we denote by $\mathrm{Arr}_{\mathfrak{C}}(A, B)$ the arrows $f \in \mathrm{Arr}(\mathfrak{C})$ with A as domain and B as codomain.

If I is an index set and $\{X_i \mid i \in I\}$ is a family of sets, we denote by $\coprod\{X_i \mid i \in I\}$ the coproduct of $\{X_i \mid i \in I\}$ in the category of sets. This is the disjoint union $\biguplus \{X_i \times \{i\} \mid i \in I\}$. However, in this work, we always identify any set X_j with the subset $X_j \times \{j\}$ of $\coprod\{X_i \mid i \in I\}$. Notice that if $\xi_j : X_j \hookrightarrow X$ denotes the inclusion map and $Y \in \wp(X_j)$, then $\wp\xi_j(Y) = Y \in \wp\big(\coprod\{X_i \mid i \in I\}\big)$.

Definition 2.1 A category $\mathfrak{C}$ is called *concretizable* if there exists a faithful functor $F : \mathfrak{C} \longrightarrow \mathbf{Set}$.

Let $\wp$ be the powerset functor on the category **Set**. For any nonnegative integer k, we denote by $\wp^k$ the k-th iteration of the functor $\wp$ with respect to the composition. In particular, $\wp^0$ agrees with the identity functor of **Set** and $\wp^1 = \wp$. When working in a set-theoretical context, we usually write $\wp$ instead of $\wp^1$. For any $\Omega \in \mathrm{Obj}(\mathbf{Set})$ and $h \geqslant 1$, we denote by Ω^h the Cartesian product of h copies of Ω. In particular, we often write $\wp^k(\Omega)^2$ instead of $\wp^k(\Omega) \times \wp^k(\Omega)$, and so on. If $f \in \mathrm{Arr}(\mathbf{Set})$, we usually write $^{\langle 2\rangle}$ instead of $f \times f$. In particular, we write $(\wp^k f)^{\langle 2\rangle}$ instead of $\wp^k f \times \wp^k f$.

Let Ω be an arbitrary fixed set. We set $\mathrm{SS}(\Omega) := \wp(\wp(\Omega))$ and $\mathrm{SR}(\Omega) := \wp(\wp(\Omega) \times \wp(\Omega))$. We call *set system* on Ω any element $\mathcal{F} \in \mathrm{SS}(\Omega)$ and *set relation* on Ω any element $\mathcal{R} \in \mathrm{SR}(\Omega)$.

We say that a non-empty set system $\mathcal{F} \in \mathrm{SS}(\Omega)$ is:

- a *closure set system* on Ω if $\Omega \in \mathcal{F}$ and whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\bigcap \mathcal{F}' \in \mathcal{F}$. We denote by $\mathrm{CSS}(\Omega)$ the collection of all closure set systems on Ω;
- a *set partition* on Ω if $\bigcup \mathcal{F} = \Omega$ and whenever $X, Y \in \mathcal{F}$ it results that $X \cap Y = \varnothing$. We denote by $\mathrm{SP}(\Omega)$ the collection of all set partitions on Ω.

A *set operator* on Ω is any map $\sigma : \wp(\Omega) \longrightarrow \wp(\Omega)$, and we denote by $\mathrm{SO}(\Omega)$ the family of all set operators on Ω. The binary relation $\sqsubseteq$ on $\mathrm{SO}(\Omega)$ defined by

$$\sigma \sqsubseteq \sigma' \;:\Longleftrightarrow\; \forall X \in \wp(\Omega)\ [\sigma(X) \subseteq \sigma'(X)],$$

for any $\sigma, \sigma' \in \mathrm{SO}(\Omega)$ is clearly a partial order on $\mathrm{SO}(\Omega)$.

We say that a set operator $\sigma \in \mathrm{SO}(\Omega)$ is:

- *extensive* if $X \subseteq \sigma(X)$, for any $X \in \wp(\Omega)$;
- *increasing* if whenever $X, Y \in \wp(\Omega)$ and $X \subseteq Y$, then $\sigma(X) \subseteq \sigma(Y)$;
- *idempotent* if $\sigma(\sigma(X)) = \sigma(X)$, for any $X \in \wp(\Omega)$;
- a *closure set operator* on Ω if it is extensive, increasing and idempotent, and we denote by $\mathrm{CSO}(\Omega)$ the set of all the closure set operators on Ω.

If $\sigma \in \mathrm{SO}(\Omega)$, we set $\sigma^0 := \mathrm{Id}_{\wp(\Omega)}$ and $\sigma^k := \sigma^{k-1} \circ \sigma$, for any integer $k \geqslant 1$.

3 Categories of set systems, set relations and set operators

Let k be a given nonnegative integer. In this section we introduce and analyze the basic properties of the arrows of three categories $\mathbf{SS^k}$, $\mathbf{SR^k}$ and $\mathbf{SO^k}$, and some of their corresponding subcategories, which will be the main topic of the present paper. The objects of such categories are pairs constituted by a ground set and a corresponding set system, set relation or a set operator respectively. The exponent k determines the iteration of the powerset functor $\wp$ occurring in the definition of the corresponding arrows.

When we fix the ground set Ω and an associated set system $\mathcal{F} \in \wp^k(\Omega)$, we can consider the pair $(\Omega, \mathcal{F})$ as an object of a category. We may take as arrows the maps $f : \Omega \longrightarrow \Omega'$ for which the k-th iteration $\wp^k$ sends elements of $\mathcal{F}$ to elements of $\mathcal{F}'$. In such a way, we are led to consider the category $\mathbf{SS^k}$ for which

$$\mathrm{Obj}(\mathbf{SS^k}) := \{(\Omega, \mathcal{F}) \mid \Omega \in \mathrm{Obj}(\mathbf{Set}),\ \mathcal{F} \in \wp^{k+1}(\Omega)\},$$

and such that for any pair of its objects $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$, the corresponding arrow set is defined by

$$\mathrm{Arr}_{\mathbf{SS^k}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}')) := \{f \in \mathrm{Arr}_{\mathbf{Set}}(\Omega, \Omega') \mid \forall X \in \mathcal{F}\ [\wp^k f(X) \in \mathcal{F}']\},$$

where the composition law is induced by that of **Set**. Clearly, the intrinsic nature of the arrows in the category $\mathbf{SS^k}$ is local, because it is carried on $\wp^k(\Omega)$ by means of the application of the k-th iteration of $\wp$ on any function defined on the ground set Ω.

Definition 3.1 We call $\mathbf{SS^k}$ the *category of the k-set systems*.

For every k, the category $\mathbf{SS}^k$ has a reflective subcategory isomorphic to **Set**. Moreover, as one intuitively expects, the category $\mathbf{SS^k}$ embeds in a quite natural way into the category $\mathbf{SS^{k+1}}$. In fact, consider the correspondence $S^k : \mathbf{SS^k} \longrightarrow \mathbf{SS^{k+1}}$, where $S^k(\Omega, \mathcal{F}) := (\Omega, \{\{X\} \mid X \in \mathcal{F}\})$ for any $(\Omega, \mathcal{F}) \in \mathrm{Obj}(\mathbf{SS^k})$, and $S^k f := f$ for any $f \in \mathrm{Arr}(\mathbf{SS^k})$. Then the following result holds.

Proposition 3.2 *S^k is a full embedding.*

Proof It is immediate to check that S^k is an embedding by its definition. It remains to prove that S^k is a full functor. To this end, let $f \in \mathrm{Arr}_{\mathbf{SS^{k+1}}}(S^k(\Omega, \mathcal{F}), S^k(\Omega', \mathcal{F}'))$ and $X \in \mathcal{F}$. Now, since

$$\wp^{k+1} f(\{X\}) \in \{\{X'\} \mid X' \in \mathcal{F}'\} \quad \text{and} \quad \wp^{k+1} f(\{X\}) = \{\wp^k f(X)\},$$

we deduce that $\wp^k f(X) \in \mathcal{F}'$ and, hence, $f \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}'))$, so that S^k is full. □

As for set systems, when we fix Ω and a binary relation $\mathcal{R} \subseteq \wp^k(\Omega) \times \wp^k(\Omega)$, we can consider the pair $(\Omega, \mathcal{R})$ as an object of a category of binary relations. In such a way, we consider the category $\mathbf{SR^k}$, whose object class is

$$\mathrm{Obj}(\mathbf{SR^k}) := \{(\Omega, \mathcal{R}) \mid \Omega \in \mathrm{Obj}(\mathbf{Set}),\ \mathcal{R} \in \wp(\wp^k(\Omega) \times \wp^k(\Omega))\},$$

and having arrow set given by

$$\mathrm{Arr}_{\mathbf{SR^k}}((\Omega,\mathcal{R}),(\Omega',\mathcal{R}')) := \left\{ f \in \mathrm{Arr}_{\mathbf{Set}}(\Omega,\Omega') \;\middle|\; \begin{array}{l} \forall (X,Y)\in\mathcal{R} \\ [(\wp^k f(X),\wp^k f(Y))\in\mathcal{R}'] \end{array} \right\},$$

for any pair of objects $(\Omega,\mathcal{R}),(\Omega',\mathcal{R}')\in \mathrm{Obj}(\mathbf{SR^k})$. Also in this case the composition law is that induced by the corresponding composition law in **Set**.

Definition 3.3 We call $\mathbf{SR^k}$ the *category of the k-set relations*.

Also for the category $\mathbf{SR^k}$, we can check in a natural way that $\mathbf{SR^k}$ embeds into $\mathbf{SR^{k+1}}$. In fact, consider the correspondence $R^k\colon \mathbf{SR^k}\longrightarrow \mathbf{SR^{k+1}}$, where $R^k(\Omega,\mathcal{R}) := (\Omega, \{(\{X\},\{Y\}) \mid (X,Y)\in\mathcal{R}\})$ for any $(\Omega,\mathcal{R})\in\mathrm{Obj}(\mathbf{SR^k})$, and $R^k f := f$ for any $f\in\mathrm{Arr}(\mathbf{SR^k})$. Then the following result holds.

Proposition 3.4 *R^k is a full embedding.*

Proof It is immediate to verity that R^k is an embedding. We now show that R^k is also a full functor.

Let $((\Omega,\mathcal{R}),(\Omega',\mathcal{R}'))\in\mathrm{Obj}(\mathbf{SR^k})$ and $f\in\mathrm{Arr}_{\mathbf{SR^{k+1}}}(R^k(\Omega,\mathcal{R}),R^k(\Omega',\mathcal{R}'))$. By the definition of R^k, in order to obtain the conclusion of the present part (ii) we must prove that $f\in\mathrm{Arr}_{\mathbf{SR^k}}((\Omega,\mathcal{R}),(\Omega',\mathcal{R}'))$. To this regard, let $(X,Y)\in\mathcal{R}$. Then $(\{X\},\{Y\})\in R^k(\Omega,\mathcal{R})$, and therefore

$$\begin{aligned}(\{\wp^k f(X)\},\{\wp^k f(Y)\}) &= (\wp^{k+1}f(\{X\}),\wp^{k+1}f(\{Y\}))\in R^k(\Omega',\mathcal{R}')\\ &= \{(\{X'\},\{Y'\}) \mid (X',Y')\in\mathcal{R}'\},\end{aligned}$$

whence $(\wp^k f(X),\wp^k f(Y))\in\mathcal{R}'$. Hence $f\in\mathrm{Arr}_{\mathbf{SR^k}}((\Omega,\mathcal{R}),(\Omega',\mathcal{R}'))$. □

Finally we introduce the categories whose objects are set operators induced by maps between sets. In this case, assume $k\geqslant 1$. We call any map $\sigma:\wp^k(\Omega)\longrightarrow\wp^k(\Omega)$ (that is, equivalently, any arrow $\sigma\in\mathrm{Arr}_{\mathbf{Set}}(\wp^k(\Omega),\wp^k(\Omega))$ a *k-set operator* on Ω. In particular, a 0-set operator σ on Ω is a map $\sigma:\Omega\longrightarrow\Omega$ and a 1-set operator σ on Ω is a map $\sigma:\wp(\Omega)\longrightarrow\wp(\Omega)$ (which in literature is usually called a *set operator* on Ω). We introduce now three categories of k-set operators, denoted by $\mathbf{SO^k}$, $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$, which are induced by means of the functor $\wp^k$. More in detail, we set

$$\begin{aligned}\mathrm{Obj}(\mathbf{SO^k}) &= \mathrm{Obj}(\mathbf{SO^{k,\sqsubseteq}}) = \mathrm{Obj}(\mathbf{SO^{k,\sqsupseteq}})\\ &:= \big\{(\Omega,\sigma) \mid \Omega\in\mathrm{Obj}(\mathbf{Set}),\ \sigma\in\mathrm{Arr}_{\mathbf{Set}}(\wp^k(\Omega),\wp^k(\Omega))\big\},\end{aligned}$$

and for any $(\Omega,\sigma),(\Omega',\sigma')\in\mathrm{Obj}(\mathbf{SO^k})$,

$$\begin{aligned}\mathrm{Arr}_{\mathbf{SO^k}}((\Omega,\sigma),(\Omega',\sigma')) &= \big\{(g,g')\in\mathrm{Arr}_{\mathbf{Set}}(\Omega,\Omega')^2 \mid \wp^k g'\circ\sigma = \sigma'\circ\wp^k g\big\},\\ \mathrm{Arr}_{\mathbf{SO^{k,\sqsubseteq}}}((\Omega,\sigma),(\Omega',\sigma')) &= \big\{(g,g')\in\mathrm{Arr}_{\mathbf{Set}}(\Omega,\Omega')^2 \mid \wp^k g'\circ\sigma \sqsubseteq \sigma'\circ\wp^k g\big\},\\ \mathrm{Arr}_{\mathbf{SO^{k,\sqsupseteq}}}((\Omega,\sigma),(\Omega',\sigma')) &= \big\{(g,g')\in\mathrm{Arr}_{\mathbf{Set}}(\Omega,\Omega')^2 \mid \wp^k g'\circ\sigma \sqsupseteq \sigma'\circ\wp^k g\big\}.\end{aligned}$$

The composition law in $\mathbf{SO^k}$ (and analogously in $\mathbf{SO^{k,\sqsubseteq}}$ and in $\mathbf{SO^{k,\sqsupseteq}}$) is naturally defined by

$$(g, g') \circ (f, f') = (g \circ f, g' \circ f'),$$

whenever $(f, f') \in \mathrm{Arr}_{\mathbf{SO^k}}((\Omega, \sigma), (\Omega', \sigma'))$ and $(g, g') \in \mathrm{Arr}_{\mathbf{SO^k}}((\Omega', \sigma'), (\Omega'', \sigma''))$.

Definition 3.5 We call $\mathbf{SO^k}$ the *category of the k-set operators*, $\mathbf{SO^{k,\sqsubseteq}}$ the *category of the* $(k, \sqsubseteq)$*-set operators*, and $\mathbf{SO^{k,\sqsupseteq}}$ the *category of the* $(k, \sqsupseteq)$*-set operators*.

Remark 3.6 At a first glance, it might seem that the categories $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$ turn out to be isomorphic. Nevertheless, this does not correspond to reality. In fact, in Theorem 4.10 we will show that they are neither equivalent nor dual.

The forgetful functor $U : \mathbf{SS^k} \longrightarrow \mathbf{Set}$ such that $U((\Omega, \mathcal{F})) = \Omega$ and $U(f) = f$ is clearly faithful. A similar functor may be defined on category $\mathbf{SR^k}$. Furthermore, the categories $\mathbf{SO^k}$, $\mathbf{SO^{k,\sqsubseteq}}$, $\mathbf{SO^{k,\sqsupseteq}}$ are also concretizable. In fact, just take the functor $U : \mathbf{SO^{k,\sqsubseteq}} \longrightarrow \mathbf{Set}$, such that $U((\Omega, \sigma)) = \Omega \times \Omega$ and $U((f, f')) = (f, f')$ whenever $(\Omega, \sigma), (\Omega', \sigma') \in \mathrm{Obj}(\mathbf{SO^{k,\sqsubseteq}})$ and $(f, f') \in \mathrm{Arr}_{\mathbf{SO^{k,\sqsubseteq}}}((\Omega, \sigma), (\Omega', \sigma'))$ (the other cases are similar).

In the next result we determine monomorphisms, epimorphisms and isomorphisms for the category $\mathbf{SS^k}$.

Proposition 3.7 *Let* $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}') \in \mathrm{Obj}(\mathbf{SS^k})$ *and* $f \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}'))$. *Then:*

(i) f *is a monomorphism of* $\mathbf{SS^k}$ $\iff$ f *is an injective map between* Ω *and* Ω'*;*
(ii) f *is an epimorphism of* $\mathbf{SS^k}$ $\iff$ f *is a surjective map between* Ω *and* Ω'*;*
(iii) f *is an isomorphism of* $\mathbf{SS^k}$ $\iff$ f *is bijective and* $\mathcal{F}' = \{\wp^k f(X) \mid X \in \mathcal{F}\}$.

Proof (i): Clearly, if f is injective then it is a monomorphism. Conversely, assume that f is a monomorphism and let $x, y \in \Omega$ be such that $f(x) = f(y)$. Taking the arrows $\overline{x}, \overline{y} \in \mathrm{Arr}_{\mathbf{SS^k}}((\{*\}, \varnothing), (\Omega, \mathcal{F}))$ sending $*$ to x and y respectively, we get $f \circ \overline{x} = f \circ \overline{y}$, whence $x = y$, i.e. f is injective.

(ii): Clearly, if f is surjective then it is an epimorphism. Conversely, let f be an epimorphism and take $(\{0, 1\}, \wp^k(\{0, 1\})) \in \mathrm{Obj}(\mathbf{SS^k})$ and the arrows $g, h \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega', \mathcal{F}'), (\{0, 1\}, \wp^k(\{0, 1\})))$ defined as follows:

$$\forall x' \in \Omega' \left[g(x') = 1 \ \text{ and } \ h(x') = \begin{cases} 1 & \text{if } x' \in f(\Omega), \\ 0 & \text{otherwise.} \end{cases} \right].$$

Hence $g \circ f = h \circ f$, whence $g = h$, i.e. $f(\Omega) = \Omega'$.

(iii): Assume first that f is an isomorphism of $\mathbf{SS^k}$, so that there exists $g \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega', \mathcal{F}'), (\Omega, \mathcal{F}))$ such that $f \circ g = \mathrm{Id}_{(\Omega', \mathcal{F}')}$ and $g \circ f = \mathrm{Id}_{(\Omega, \mathcal{F})}$. Clearly, f is bijective with g as its inverse. Therefore f is an isomorphism in $\mathbf{Set}$, and consequently $\wp^k f$ is also bijective with inverse $\wp^k g$. Fix now $X' \in \mathcal{F}'$ and $X \in \wp^k(\Omega)$ such that $\wp^k f(X) = X'$. Since $g \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega', \mathcal{F}'), (\Omega, \mathcal{F}))$, we easily deduce that

$X = \wp^k(g \circ f)(X) = \wp^k g(X') \in \mathcal{F}$. Thus $\mathcal{F}' \subseteq \{\wp^k f(X) \mid X \in \mathcal{F}\}$, and the reverse inclusion is a direct consequence of the choice of f.

Conversely, assume that f is a bijective map and $\mathcal{F}' = \{\wp^k f(X) \mid X \in \mathcal{F}\}$. Let $g := f^{-1}$. Then we get $\wp^k g(X') = \wp^k g(\wp^k f(X)) = \wp^k(g \circ f)(X) = \wp^k \mathrm{Id}(X) = X$ for each $X' \in \mathcal{F}'$, whence $g \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega', \mathcal{F}'), (\Omega, \mathcal{F}))$. □

The case of $\mathbf{SR^k}$ can be treated in a similar way to $\mathbf{SS^k}$, and we leave the details to the following result to the reader.

Proposition 3.8 *Let* $(\Omega, \mathcal{R}), (\Omega', \mathcal{R}') \in \mathrm{Obj}(\mathbf{SR^k})$ *and* $f \in \mathrm{Arr}_{\mathbf{SR^k}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$. *Then:*

(i) *f is a monomorphism of* $\mathbf{SR^k}$ $\iff$ *f is an injective map between Ω and Ω';*
(ii) *f is an epimorphism of* $\mathbf{SR^k}$ $\iff$ *f is a surjective map between Ω and Ω';*
(iii) *f is an isomorphism of* $\mathbf{SR^k}$ $\iff$ *f is bijective and*

$$\mathcal{R}' = \{(\wp^k f(X), \wp^k f(Y)) \mid (X, Y) \in \mathcal{R}\}.$$

The situation concerning monomorphisms, epimorphisms and isomorphisms in the previous categories of k-th set operators is more articulated. With regard to the categories $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$ we have the following results.

Proposition 3.9 *Let $\mathfrak{O}$ be one of the categories* $\mathbf{SO^{k,\sqsubseteq}}$ *or* $\mathbf{SO^{k,\sqsupseteq}}$. *Let* $(\Omega, \sigma), (\Omega', \sigma') \in \mathrm{Obj}(\mathfrak{O})$ *and* $(f, f') \in \mathrm{Arr}_{\mathfrak{O}}((\Omega, \sigma), (\Omega', \sigma'))$. *Then:*

(i) *(f, f') is a monomorphism of $\mathfrak{O}$ $\iff$ f and f' are both injective maps between Ω and Ω';*
(ii) *(f, f') is an epimorphism of $\mathfrak{O}$ $\iff$ f and f' are both surjective maps between Ω and Ω';*
(iii) *(f, f') is an isomorphism of $\mathfrak{O}$ $\iff$ f and f' are both bijective maps between Ω and Ω' and $\wp^k f' \circ \sigma = \sigma' \circ \wp^k f$.*

Proof We provide the proof in the case in which $\mathfrak{O} = \mathbf{SO^{k,\sqsubseteq}}$. The other case can be treated in a similar way, therefore we leave the proof to the reader.

(i): Using the characterization of monomorphisms of **Set**, the implication "$\Leftarrow$" is immediate.

Conversely, assume that (f, f') is a monomorphism. We first associate with any pair $(u, v) \in \Omega \times \Omega$ an arrow $(\overline{u}, \overline{v}) \in \mathrm{Arr}_{\mathbf{SO^{k,\sqsubseteq}}}((\{u, v\}, \tau), (\Omega, \sigma))$ such that $\overline{u}(u) = \overline{u}(v) = u$, $\overline{v}(u) = \overline{v}(v) = v$, and $\tau(X) = \varnothing$ for each $X \in \wp^k(\{u, v\})$. Let now $x, z, y, w \in \Omega$ be such that $(f(x), f'(z)) = (f(y), f'(w))$. In order to get the conclusion, we must show that $x = y$ and $z = w$. Clearly, the condition $(f(x), f'(z)) = (f(y), f'(w))$ can be equivalently written in the form $(f, f') \circ (\overline{x}, \overline{z}) = (f, f') \circ (\overline{y}, \overline{w})$. Therefore, since (f, f') is a monomorphism of the category $\mathbf{SO^{k,\sqsubseteq}}$, we deduce that $(\overline{x}, \overline{z}) = (\overline{y}, \overline{w})$, whence $x = y$ and $z = w$.

(ii): Let $(f, f') \in \mathrm{Arr}_{\mathbf{SO^{k,\sqsubseteq}}}((\Omega, \sigma), (\Omega', \sigma'))$. Using the characterization of epimorphisms of **Set**, if f and f' are surjective we may easily deduce that (f, f') is an epimorphism.

Conversely, assume that (f, f') is an epimorphism and let $(\{0,1\}, \tau) \in \mathrm{Obj}(\mathbf{SO}^{\mathbf{k},\sqsubseteq})$, where $\tau(X) = \wp^{k-1}(\{0,1\})$ for any $X \in \wp^k(\{0,1\})$. Let now $(g, g'), (h, h') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega', \sigma'), (\{0,1\}, \tau)))$ be defined as follows:

$$\forall x' \in \Omega' \left[g(x') = g'(x') = 1,\ h(x') = \begin{cases} 1 & \text{if } x' \in f(\Omega), \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h'(x') = \begin{cases} 1 & \text{if } x' \in f'(\Omega), \\ 0 & \text{otherwise} \end{cases} \right].$$

Hence $(g, g') \circ (f, f') = (h, h') \circ (f, f')$, so that $(g, g') = (h, h')$ since (f, f') is an epimorphism. This proves that $f(\Omega) = f'(\Omega) = \Omega'$, i.e. f and f' are surjective.

(iii): Assume first that (f, f') is an isomorphism. Then there exists $(g, g') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega', \sigma'), (\Omega, \sigma))$ such that $(g, g') \circ (f, f') = (\mathrm{Id}_\Omega, \mathrm{Id}_\Omega)$ and $(f, f') \circ (g, g') = (\mathrm{Id}_{\Omega'}, \mathrm{Id}_{\Omega'})$, and this implies that f and f' are both bijective.

Furthermore, as $(g, g') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega', \sigma'), (\Omega, \sigma))$, it results that

$$\wp^k g' \circ \sigma' \sqsubseteq \sigma \circ \wp^k g.$$

On the other hand, since $(f, f') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega, \sigma), (\Omega', \sigma'))$ and $\wp^k f$ and $\wp^k f'$ are invertible with inverses $\wp^k g$ and $\wp^k g'$, respectively, we easily get

$$\wp^k g' \circ \sigma' \sqsupseteq \sigma \circ \wp^k g.$$

Combining together the above conditions we deduce that $\wp^k g' \circ \sigma' = \sigma \circ \wp^k g$, whence, again by the invertibility of both $\wp^k g$ and $\wp^k g'$, we conclude that $\wp^k f' \circ \sigma = \sigma' \circ \wp^k f$.

Conversely, assume that f and f' are invertible, with inverses denoted by g and g' respectively, and such that $\wp^k f' \circ \sigma = \sigma' \circ \wp^k f$. Composing the last equality by $\wp^k g'$ at left and by $\wp^k g$ at right, we clearly get $\wp^k g \circ \sigma' = \sigma \circ \wp^k g$ and, so, we conclude that $(g, g') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega', \sigma'), (\Omega, \sigma))$. □

In the case of the category $\mathbf{SO}^{\mathbf{k}}$, the determination of its monomorphisms and epimorphisms is more complex than the previous cases (see Remark 3.11). Now, we provide the following result concerning the characterization of isomorphisms of $\mathbf{SO}^{\mathbf{k}}$.

Proposition 3.10 *Let* $(\Omega, \sigma), (\Omega', \sigma') \in \mathrm{Obj}(\mathbf{SO}^{\mathbf{k}})$ *and* $(f, f') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k}}}((\Omega, \sigma), ((\Omega', \sigma'))$. *Then*

$$(f, f') \in \mathrm{Iso}(\mathbf{SO}^{\mathbf{k}}) \iff f \text{ and } f' \text{ are both bijective maps between } \Omega \text{ and } \Omega'.$$

Proof Let $(f, f') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k}}}((\Omega, \sigma), (\Omega', \sigma'))$. Assume first that (f, f') is an isomorphism. Then there exists $(g, g') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k}}}((\Omega', \sigma'), (\Omega, \sigma))$ such that $(g, g') \circ (f, f') = (\mathrm{Id}_\Omega, \mathrm{Id}_\Omega)$ and $(f, f') \circ (g, g') = (\mathrm{Id}_{\Omega'}, \mathrm{Id}_{\Omega'})$, and this implies that both f and f' are bijective.

Conversely, assume that f and f' are invertible, with inverses denoted by g and g' respectively. We claim that $(g, g') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k}}}((\Omega', \sigma'), (\Omega, \sigma))$. To this regard, first note that $\wp^k f$ and $\wp^k f'$ are invertible with inverses $\wp^k g$ and $\wp^k g'$. So, as $\wp^k g \circ \sigma(X) = \sigma' \circ \wp^k f(X)$ for any $X \in \wp^k(\Omega)$, composing by $\wp^k f'$ we easily conclude that $\wp^k g' \circ \sigma'(X') = \sigma \circ \wp^k f'(X')$ for any $X' \in \wp^k(\Omega')$, i.e. $\wp^k g' \circ \sigma' = \sigma \circ \wp^k f'$. □

We leave the characterization of monomorphisms and epimorphisms as an exercise.

Remark 3.11 As we mentioned above, a complete and exhaustive characterization of the monomorphisms and epimorphisms of the category $\mathbf{SO^k}$ is more complex to be obtained with respect than the two categories $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$ in Proposition 3.9.

More in detail, the condition of equality occurring in the definition of the arrows of $\mathbf{SO^k}$ is quite strict and this reduces the possibility to construct an arrow between two objects (Ω, σ) and (Λ, τ). In fact, when an object $(\Omega, \sigma) \in \mathrm{Obj}(\mathbf{SO^k})$ is given, the k-set operator $\sigma : \wp^k(\Omega) \longrightarrow \wp^k(\Omega)$ maps any $X \in \wp^k(\Omega)$ to a $\sigma(X) \in \wp^k(\Omega)$, whose correlation with X is too general without establishing additional conditions on the k-set operator σ, such as for instance extensiveness, monotonicity or idempotency (see Sect. 6 where we deal with closure set operators).

Thus, since it is difficult to check that the equality $\wp^k f'(\sigma(X)) = \tau(\wp^k f(X))$ holds for any $X \in \wp^k(\Omega)$, some work is needed to characterize an arrow $(f, f') \in \mathrm{Arr}_{\mathbf{SO^k}}((\Omega, \sigma), (\Lambda, \tau))$.

We denote by $\mathbf{SO^{k,\sqsubseteq,=}}$, $\mathbf{SO^{k,\sqsupseteq,=}}$ and $\mathbf{SO^{k,=}}$ the wide subcategories of $\mathbf{SO^{k,\sqsubseteq}}$, $\mathbf{SO^{k,\sqsupseteq}}$ and $\mathbf{SO^k}$, respectively, whose arrows have the form (f, f). In the successive sections, we deal with some specific subcategories of the previous categories $\mathbf{SO^{k,\sqsubseteq,=}}$, $\mathbf{SO^{k,\sqsupseteq,=}}$ and $\mathbf{SO^{k,=}}$.

Remark 3.12 The analogue of Proposition 3.9 holds for both the categories $\mathbf{SO^{k,\sqsubseteq,=}}$ and $\mathbf{SO^{k,\sqsupseteq,=}}$. Similarly, the analogue of Proposition 3.10 holds for $\mathbf{SO^{k,=}}$.

4 Fundamental properties of the categories $\mathbf{SS^k}$, $\mathbf{SR^k}$ and $\mathbf{SO^k}$

Let $k \in \mathbb{N}$ be fixed. In the present section we provide two fundamental properties of the categories $\mathbf{SS^k}$ and $\mathbf{SR^k}$, i.e. that they are bicomplete (see Theorem 4.1) and Cartesian closed (see Theorem 4.6).

4.1 $\mathbf{SS^k}$ and $\mathbf{SR^k}$ are bicomplete categories

In this subsection we prove the first fundamental property of the category $\mathbf{SS^k}$, namely that it is a bicomplete category.

Theorem 4.1 $\mathbf{SS^k}$ *is a bicomplete category.*

Proof We first determine the product in $\mathbf{SS^k}$. To this regard fix $\{(X_i, \mathcal{X}_i) \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{SS^k})$. Set

$$X := \prod \{X_i \mid i \in I\} \quad \text{and} \quad \mathcal{X} := \left\{Z \in \wp^k(X) \mid \forall i \in I \; [\wp^k \pi_i(Z) \in \mathcal{X}_i]\right\},$$

where $\pi_i : X \longrightarrow X_i$ is the usual projection for any $i \in I$ (notice that $\mathcal{X} \in \wp^{k+1}(X)$). We show that

$$((X, \mathcal{X}), \{\pi_i \mid i \in I\}) \text{ is the product of the object family } \{(X_i, \mathcal{X}_i) \mid i \in I\} \text{ in } \mathbf{SS^k}. \quad (2)$$

By the definition of the arrows of $\mathbf{SS^k}$ and of the set system $\mathcal{X}$, we get $\pi_i \in \mathrm{Arr}_{\mathbf{SS^k}}((X, \mathcal{X}), (X_i, \mathcal{X}_i))$, for any $i \in I$. Let now $(Y, \mathcal{Y}) \in \mathrm{Obj}(\mathbf{SS^k})$ and $h_i \in \mathrm{Arr}_{\mathbf{SS^k}}((Y, \mathcal{Y}), (X_i, \mathcal{X}_i))$, for any $i \in I$. Let $\tilde{h} \colon Y \longrightarrow X$ be the usual universal arrow induced by the family of maps $\{h_i \mid i \in I\}$ in **Set** and $Z \in \mathcal{Y}$. Since $\pi_i \circ \tilde{h} = h_i$, it follows that

$$\wp^k \pi_i(\wp^k \tilde{h}(Z)) = (\wp^k \pi_i \circ \wp^k \tilde{h})(Z) = \wp^k(\pi_i \circ \tilde{h})(Z) = \wp^k h_i(Z) \in \mathcal{X},$$

for each $i \in I$, whence $\wp^k \tilde{h}(Z) \in \mathcal{X}$, and thus $\tilde{h} \in \mathrm{Arr}_{\mathbf{SS^k}}((Y, \mathcal{Y}), (X, \mathcal{X}))$. Therefore, in view of the universal property of the map $\tilde{h}$ in **Set**, it follows that $\tilde{h}$ is also the unique arrow in $\mathrm{Arr}_{\mathbf{SS^k}}((Y, \mathcal{Y}), (X, \mathcal{X}))$ for which $h_i = \pi_i \circ \tilde{h}$ for any $i \in I$. This proves (2).

We determine now the equalizer of any two arrows of $\mathbf{SS^k}$. Take $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}') \in \mathrm{Obj}(\mathbf{SS^k})$ and $f, g \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}'))$. Let $\Omega_* := \{x \in \Omega \mid f(x) = g(x)\}$, $\iota_* \colon \Omega_* \hookrightarrow \Omega$ be the inclusion map, and $\mathcal{F}_* := \mathcal{F} \cap \wp^k(\Omega_*)$. Now we prove that

$$((\Omega_*, \mathcal{F}_*), \iota_*) \text{ is the equalizer of } f \text{ and } g \text{ in } \mathbf{SS^k}. \tag{3}$$

We clearly have $(\Omega_*, \mathcal{F}_*) \in \mathrm{Obj}(\mathbf{SS^k})$, $\iota_* \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega_*, \mathcal{F}_*), (\Omega, \mathcal{F}))$ and $f \circ \iota_* = g \circ \iota_*$. Take $(\Omega'', \mathcal{F}'') \in \mathrm{Obj}(\mathbf{SS^k})$ and $h \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega'', \mathcal{F}''), (\Omega, \mathcal{F}))$ such that $f \circ h = g \circ h$. We show that there exists a unique arrow $h_* \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega'', \mathcal{F}''), (\Omega_*, \mathcal{F}_*))$ such that $\iota_* \circ h_* = h$.

To this regard, since $f \circ h = g \circ h$ it follows that $h(z) \in \Omega_*$. Define then the map $h' \colon \Omega'' \longrightarrow \Omega_*$ setting $h'(z) := h(z)$ for any $z \in \Lambda$. We easily deduce that h_* is the unique map such that $h = \iota_* \circ h_*$. Fix now $Z \in \mathcal{G}$. It is immediate that $\wp^k h'(Z) = \wp^k h(Z) \in \mathcal{F} \cap \wp^k(\Omega_*) = \mathcal{F}_*$, whence $h_* \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega'', \mathcal{F}''), (\Omega_*, \mathcal{F}_*))$. This proves (3), and consequently the category $\mathbf{SS^k}$ is complete in view of [7, Theorem 2.8.1].

We consider now the construction of the coproduct in $\mathbf{SS^k}$. Let $\{(\Omega_i, \mathcal{F}_i) \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{SS^k})$. We set $\widehat{\Omega} := \bigsqcup\{\Omega_i \mid i \in I\}$ and let $\xi_i : \Omega_i \longrightarrow \widehat{\Omega}$ be the inclusion map, for any $i \in I$. Set moreover $\widehat{\mathcal{F}} := \bigsqcup\{\mathcal{F}_i \mid i \in I\}$. We prove now that

$$((\widehat{\Omega}, \widehat{\mathcal{F}}), \{\xi_i \mid i \in I\}) \text{ is the coproduct of the object family } \{(\Omega_i, \mathcal{F}_i) \mid i \in I\} \text{ in } \mathbf{SS^k}. \tag{4}$$

Also in this case, by the definition of $\widehat{\mathcal{F}}$ and of $\mathrm{Arr}(\mathbf{SS^k})$, we clearly get $\xi_i \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega_i, \mathcal{F}_i), (\widehat{\Omega}, \widehat{\mathcal{F}}))$, for any $i \in I$. Let now $(\Lambda, \mathcal{G}) \in \mathrm{Obj}(\mathbf{SS^k})$ and $s_i \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega_i, \mathcal{F}_i), (\Lambda, \mathcal{G}))$ for any $i \in I$. Let $t \colon \widehat{\Omega} \longrightarrow \Lambda$ be the universal map induced by $\{s_i \mid i \in I\}$ in **Set**, i.e. $t((x, i)) := s_i(x)$ for any $(x, i) \in \widehat{\Omega}$. Take now $Z \in \widehat{\mathcal{F}}$. Then there exists $j \in I$ such that $Z \in \mathcal{F}_j$ and $\wp^k \xi_j(Z) = Z$. Thus, since $s_i = t \circ \xi_i$ for any $i \in I$, it follows that $\wp^k t(Z) = \wp^k t(\wp^k \xi_j(Z)) = (\wp^k t \circ \wp^k \xi_j)(Z) = \wp^k(t \circ \xi_j)(Z) = \wp^k s_j(Z) \in \mathcal{G}$, whence we get $\wp^k t(Z) \in \mathcal{G}$. Hence $t \in \mathrm{Arr}_{\mathbf{SS^k}}((\widehat{\Omega}, \widehat{\mathcal{F}}), (\Lambda, \mathcal{G}))$. At this point, in view of the universal property of the map t in **Set**, it follows that t is also the only arrow in $\mathrm{Arr}_{\mathbf{SS^k}}((\widehat{\Omega}, \widehat{\mathcal{F}}), (\Lambda, \mathcal{G}))$ such that $t \circ \xi_i = s_i$ for any $i \in I$. This proves (4).

Finally, we determine the coequalizer of any two arrows of $\mathbf{SS^k}$. Let again $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}') \in \mathrm{Obj}(\mathbf{SS^k})$ and $f, g \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}'))$. We set $\Omega_\diamond := \Omega'/\sim_\diamond$, where $\sim_\diamond$ is the equivalence relation on Ω' generated by the binary relation

$\{(f(x), g(x)) \mid x \in \Omega\} \subseteq \Omega' \times \Omega'$. Let moreover by $\pi_\diamond \colon \Omega' \longrightarrow \Omega'/\sim_\diamond$ be the projection on the quotient and $\mathcal{F}_\diamond := \{\wp^k \pi_\diamond(X) \mid X \in \mathcal{F}'\}$. We show then that

$$((\Omega_\diamond, \mathcal{F}_\diamond), \pi_\diamond) \text{ is the coequalizer of } f \text{ and } g \text{ in } \mathbf{SS^k}. \tag{5}$$

We have that $(\Omega_\diamond, \mathcal{F}_\diamond) \in \mathrm{Obj}(\mathbf{SS^k})$, $\pi_\diamond \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega_\diamond, \mathcal{F}_\diamond), (\Omega, \mathcal{F}))$ and $\pi_\diamond \circ f = \pi_\diamond \circ g$.

Take now $(\Lambda, \mathcal{G}) \in \mathrm{Obj}(\mathbf{SS^k})$ and $h \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega', \mathcal{F}'), (\Lambda, \mathcal{G}))$ such that $h \circ f = h \circ g$. Then there exists a unique map $h' \colon \Omega_\diamond \longrightarrow \Lambda$ such that $h' \circ \pi_\diamond = h$.

We finally claim that $h' \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega_\diamond, \mathcal{F}_\diamond), (\Lambda, \mathcal{G}))$. In fact, let $Z \in \mathcal{F}_\diamond$. In view of the definition of $\mathcal{F}_\diamond$, there exists $X \in \mathcal{F}'$ such that $Z = \wp^k \pi_\diamond(X)$. Thus, since $h \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega', \mathcal{F}'), (\Lambda, \mathcal{G}))$, we get $\wp^k h'(Z) = \wp^k h'(\wp^k \pi_\diamond(X)) = \wp^k h'(X) \in \mathcal{G}$. Hence $h' \in \mathrm{Arr}_{\mathbf{SS^k}}((\Omega_\diamond, \mathcal{F}_\diamond), (\Lambda, \mathcal{G}))$, and therefore (5) holds. Consequently, the category $\mathbf{SS^k}$ is cocomplete in view of [7, Theorem 2.8.1]. □

In the next result we show that $\mathbf{SR^k}$ is also a bicomplete category. The proof is similar to that of the category $\mathbf{SS^k}$ and, hence, we only sketch it, leaving to the reader to fix the details.

Theorem 4.2 $\mathbf{SR^k}$ *is a bicomplete category.*

Proof We first determine the product in $\mathbf{SR^k}$. To this regard, let $\{(\Omega_i, \mathcal{R}_i) \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{SR^k})$, and we take Ω and π_i (where $i \in I$) as in (2). We set now $\mathcal{B} := \{(Z, W) \in \wp^k(\Omega) \times \wp^k(\Omega) \mid \forall i \in I\ [(\wp^k \pi_i(Z), \wp^k \pi_i(W)) \in \mathcal{R}_i]\}$. We leave to the reader the proof of the fact that $((\Omega, \mathcal{B}), \{\pi_i \mid i \in I\})$ is the product of $\{(\Omega_i, \mathcal{R}_i) \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{SR^k})$ in $\mathbf{SR^k}$.

Now determine the equalizer of any two arrows of $\mathbf{SR^k}$. Let $(\Omega, \mathcal{R}), (\Omega', \mathcal{R}') \in \mathrm{Obj}(\mathbf{SR^k})$ and $f, g \in \mathrm{Arr}_{\mathbf{SR^k}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$. Take $\Omega_* := \{x \in \Omega \mid f(x) = g(x)\}$ and $\iota_* \colon \Omega_* \hookrightarrow \Omega$ be as in (3) and also set $\mathcal{R}^*_{f,g} := \mathcal{R} \cap (\wp^k(\Omega_*) \times \wp^k(\Omega_*))$. Also in this case, we leave to the reader the proof of the fact that $((\Omega_*, \mathcal{R}^*_{f,g}), \iota_*)$ is the equalizer of f and g in $\mathbf{SR^k}$.

We consider now the construction of the coproduct in $\mathbf{SR^k}$. Take $\widehat{\Omega}, \xi_i$ (where $i \in I$) as in (4) and set $\mathcal{R} := \bigsqcup \{\mathcal{R}_i \mid i \in I\}$. We leave to the reader the proof of the fact that $((\Omega, \mathcal{R}), \{\xi_i \mid i \in I\})$ is the coproduct of $\{(\Omega_i, \mathcal{R}_i) \mid i \in I\}$ in $\mathbf{SR^k}$.

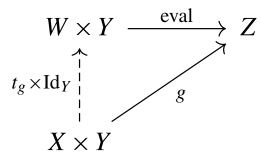
Finally, we determine the coequalizer of any two arrows of $\mathbf{SR^k}$. Let again $(\Omega, \mathcal{R})$, $(\Omega', \mathcal{R}') \in \mathrm{Obj}(\mathbf{SR^k})$ and $f, g \in \mathrm{Arr}_{\mathbf{SR^k}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$. Take $\Omega_\diamond := \Omega'/\sim_\diamond$ and $\pi_\diamond \colon \Omega' \longrightarrow \Omega'/\sim_\diamond$ as in (5) and set $\mathcal{R}^\diamond_{f,g} := \{(\wp^k \pi_\diamond(X), (\wp^k \pi_\diamond(Y)) \mid (X, Y) \in \mathcal{R}'\}$. We leave to the reader the proof of the fact that $((\Omega_\diamond, \mathcal{R}^\diamond_{f,g}), \pi_\diamond)$ is the coequalizer of f and g in $\mathbf{SR^k}$. □

4.2 The categories $\mathbf{SS^k}$ and $\mathbf{SR^k}$ are Cartesian closed

The main result of the present subsection is Theorem 4.6, where we determine the exponential objects of both the categories $\mathbf{SS^k}$ and $\mathbf{SR^k}$ and deduce that such categories are Cartesian closed.

We now recall two classical basic notions of category theory in the next two definitions.

Definition 4.3 Let $\mathfrak{C}$ be a category with binary products and $Z, Y \in \mathrm{Obj}(\mathfrak{C})$. We say that a pair (W, eval) is a (Z, Y)-*exponential object* in $\mathfrak{C}$ if $W \in \mathrm{Obj}(\mathfrak{C})$, $\mathrm{eval} \in \mathrm{Arr}_{\mathfrak{C}}(W \times Y, Z)$ and for any $X \in \mathrm{Obj}(\mathfrak{C})$, $g \in \mathrm{Arr}_{\mathfrak{C}}(X \times Y, Z)$ there is a unique morphism $t_g : X \longrightarrow W$ (usually called the *transpose* of g) such that the diagram



commutes.

Definition 4.4 A category $\mathfrak{C}$ is said *Cartesian closed* if it admits finite products and exponential objects exist for any pair of objects.

In the next Theorem 4.6 we exhibit the explicit characterization of the exponential objects for both the categories $\mathbf{SS^k}$ and $\mathbf{SR^k}$, proving hence that they are Cartesian closed. To this regard, for any $X, Y \in \mathrm{Obj}(\mathbf{Set})$, we consider the map $\mathrm{ev}_{X,Y} : X^Y \times Y \longrightarrow X$ defined by

$$\mathrm{ev}_{X,Y} : (f, y) \in X^Y \times Y \mapsto f(y) \in X.$$

Let now $\Omega, \Lambda, \Theta \in \mathrm{Obj}(\mathbf{Set})$, $g \in \mathrm{Arr}(\Theta \times \Lambda, \Omega)$ and $(\mathcal{X}, \mathcal{Y}) \in \wp^k(\Theta) \times \wp^k(\Theta)$ be arbitrary and fixed. Let moreover $\pi_1 : \mathrm{Arr}(\Lambda, \Omega) \times \Lambda \longrightarrow \mathrm{Arr}(\Lambda, \Omega)$, $\pi_2 : \mathrm{Arr}(\Lambda, \Omega) \times \Lambda \longrightarrow \Lambda$, $p_1 : \Theta \times \Lambda \longrightarrow \Theta$ and $p_2 : \Theta \times \Lambda \longrightarrow \Lambda$ denote usual Cartesian projections. Consequently, we define the map $t_g : \Theta \longrightarrow \mathrm{Arr}(\Lambda, \Omega)$ such that $t_g(\theta) := \widetilde{t}_{g,\theta}$ for any $\theta \in \Theta$, where $\widetilde{t}_{g,\theta} : \lambda \in \Lambda \mapsto \widetilde{t}_{g,\theta}(\lambda) := g(\theta, \lambda) \in \Omega$. Clearly, in view of the definition of the exponential object in $\mathbf{Set}$, t_g is the only map for which

$$\mathrm{ev}_{\Omega,\Lambda} \circ (t_g \times \mathrm{Id}_\Lambda) = g. \tag{6}$$

We set now

$$\mathcal{C}_{(\mathcal{X},\mathcal{Y})} := \left\{ (\mathcal{Z}, \mathcal{W}) \in \wp^k(\mathrm{Arr}(\Lambda, \Omega) \times \Lambda)^2 \mid (\wp^k \pi_1)^{\langle 2 \rangle}(\mathcal{Z}, \mathcal{W}) = (\wp^k t_g)^{\langle 2 \rangle}(\mathcal{X}, \mathcal{Y}) \right\}$$

and, for any $(\mathcal{Z}, \mathcal{W}) \in \mathcal{C}_{(\mathcal{X},\mathcal{Y})}$,

$$\mathcal{U}_{\mathcal{X},\mathcal{Z}} := \left\{ U \in \wp^{k-1}(\Theta \times \Lambda) \mid \wp^{k-1} p_1(U) \in \mathcal{X} \text{ and } \wp^{k-1}(t_g \times \mathrm{Id}_\Lambda)(U) \in \mathcal{Z} \right\},$$
$$\mathcal{U}_{\mathcal{Y},\mathcal{W}} := \left\{ U \in \wp^{k-1}(\Theta \times \Lambda) \mid \wp^{k-1} p_1(U) \in \mathcal{Y} \text{ and } \wp^{k-1}(t_g \times \mathrm{Id}_\Lambda)(U) \in \mathcal{W} \right\}.$$

At this point we are ready to exhibit a preliminary technical result before providing a complete proof of Theorem 4.6 in the case of $\mathbf{SR^k}$.

Lemma 4.5 *Let* $k \geqslant 1$, $\Omega, \Lambda, \Theta \in \mathrm{Obj}(\mathbf{Set})$, $g \in \mathrm{Arr}(\Theta \times \Lambda, \Omega)$ *and* $(\mathcal{X}, \mathcal{Y}) \in \wp^k(\Theta)^2$. *Then, with the previous notations, for every* $(\mathcal{Z}, \mathcal{W}) \in \mathcal{C}_{(\mathcal{X},\mathcal{Y})}$ *we have that:*

(a) $(\wp^k(t_g \times \mathrm{Id}_\Lambda))^{\langle 2\rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) = (\mathcal{Z}, \mathcal{W})$;
(b) $(\wp^k p_1)^{\langle 2\rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) = (\mathcal{X}, \mathcal{Y})$;
(c) $(\wp^k p_2)^{\langle 2\rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) = (\wp^k \pi_2)^{\langle 2\rangle}(\mathcal{Z}, \mathcal{W})$.

Proof We proceed inductively on k. Let first $k = 1$. Then we respectively have

$$\mathcal{U}_{\mathcal{X},\mathcal{Z}} = \left\{(\theta, \lambda) \in \Theta \times \Lambda \mid \theta \in \mathcal{X} \text{ and } (t_g \times \mathrm{Id}_\Lambda)(\theta, \lambda) = (\tilde{t}_{g,\theta}, \lambda) \in \mathcal{Z}\right\}$$

and

$$\mathcal{U}_{\mathcal{Y},\mathcal{W}} = \left\{(\theta, \lambda) \in \Theta \times \Lambda \mid \theta \in \mathcal{Y} \text{ and } (t_g \times \mathrm{Id}_\Lambda)(\theta, \lambda) = (\tilde{t}_{g,\theta}, \lambda) \in \mathcal{W}\right\}.$$

Let us prove Condition (a). The inclusions $\wp(t_g \times \mathrm{Id}_\Lambda)(\mathcal{U}_{\mathcal{X},\mathcal{Z}}) \subseteq \mathcal{Z}$ and $\wp(t_g \times \mathrm{Id}_\Lambda)(\mathcal{U}_{\mathcal{Y},\mathcal{W}}) \subseteq \mathcal{W}$ respectively hold by the definition of $\mathcal{U}_{\mathcal{X},\mathcal{Z}}$ and $\mathcal{U}_{\mathcal{Y},\mathcal{W}}$. Conversely, fix $((z, \mu), (w, \nu)) \in (\mathcal{Z}, \mathcal{W}) \in \wp(\mathrm{Arr}(\Lambda, \Omega) \times \Lambda)^2$. By our choice of $(\mathcal{Z}, \mathcal{W})$, there exist $\theta \in \mathcal{X}$ and $\delta \in \mathcal{Y}$ such that $t_g^{\langle 2\rangle}(\theta, \delta) = (z, w)$. Hence, we clearly have $((\theta, \mu), (\delta, \nu)) \in (\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}})$, proving that $\mathcal{Z} \subseteq \wp(t_g \times \mathrm{Id}_\Lambda)(\mathcal{U}_{\mathcal{X},\mathcal{Z}})$ and $\mathcal{W} \subseteq \wp(t_g \times \mathrm{Id}_\Lambda)(\mathcal{U}_{\mathcal{Y},\mathcal{W}})$.

Let us prove Condition (b). By the definitions of $\mathcal{U}_{\mathcal{X},\mathcal{Z}}$ and $\mathcal{U}_{\mathcal{Y},\mathcal{W}}$, the inclusions $\wp p_1(\mathcal{U}_{\mathcal{X},\mathcal{Z}}) \subseteq \mathcal{X}$ and $\wp p_1(\mathcal{U}_{\mathcal{Y},\mathcal{W}}) \subseteq \mathcal{Y}$ hold. Vice versa, let $\theta' \in \mathcal{X}$. By our choice of $\mathcal{Z}$, we can find an element $(t_g(\theta'), \lambda') \in \mathcal{Z} \cap \pi_1^{-1}(t_g(\theta'))$. Then $(\theta', \lambda') \in \mathcal{U}_{\mathcal{X},\mathcal{Z}}$ and $\theta' = p_1(\theta', \lambda')$, with $(\theta', \lambda') \in \mathcal{U}_{\mathcal{X},\mathcal{Z}}$. In such a way, we showed that $\mathcal{X} \subseteq \wp p_1(\mathcal{U}_{\mathcal{X},\mathcal{Z}})$. Analogously, we can demonstrate the inclusion $\mathcal{Y} \subseteq \wp p_1(\mathcal{U}_{\mathcal{Y},\mathcal{W}})$.

Let us prove Condition (c). First observe that from $p_2 = \pi_2 \circ (t_g \times \mathrm{Id}_\Lambda)$ and our definition of $\mathcal{U}_{\mathcal{X},\mathcal{Z}}$ and of $\mathcal{U}_{\mathcal{Y},\mathcal{W}}$, we easily get the inclusions $\wp p_2(\mathcal{U}_{\mathcal{X},\mathcal{Z}}) \subseteq \wp\pi_2(\mathcal{Z})$ and $\wp p_2(\mathcal{U}_{\mathcal{Y},\mathcal{W}}) \subseteq \wp\pi_2(\mathcal{W})$. Conversely, let $\lambda'' \in \wp\pi_2(\mathcal{Z})$ and $\nu'' \in \wp\pi_2(\mathcal{W})$. By the choice of $(\mathcal{Z}, \mathcal{W})$, it is quite simple to check the existence of $x'' \in \mathcal{X}$ and $y'' \in \mathcal{Y}$ such that $(x'', \lambda'') \in \mathcal{U}_{\mathcal{X},\mathcal{Z}}$ and $(y'', \lambda'') \in \mathcal{U}_{\mathcal{Y},\mathcal{Z}}$. This proves that $\wp\pi_2(\mathcal{Z}) \subseteq \wp p_2(\mathcal{U}_{\mathcal{X},\mathcal{Z}})$ and $\wp\pi_2(\mathcal{W}) \subseteq \wp p_2(\mathcal{U}_{\mathcal{Y},\mathcal{W}})$. This shows Condition (c) for $k = 0$.

Fix now an integer $k \geqslant 2$ and suppose that our conclusion holds for every $j \in \{1, \ldots, k-1\}$. Let then $(\mathcal{X}, \mathcal{Y}) \in \wp^k(\Theta)^2$ and $(\mathcal{Z}, \mathcal{W}) \in \mathcal{C}_{(\mathcal{X},\mathcal{Y})}$. First note that the inclusions

$$\wp^k(t_g \times \mathrm{Id}_\Lambda)(\mathcal{U}_{\mathcal{X},\mathcal{Z}}) \subseteq \mathcal{Z}, \quad \wp^k p_2(\mathcal{U}_{\mathcal{X},\mathcal{Z}}) \subseteq \wp^k\pi_2(\mathcal{Z}), \quad \wp^k p_1(\mathcal{U}_{\mathcal{X},\mathcal{Z}}) \subseteq \mathcal{X},$$

and

$$\wp^k(t_g \times \mathrm{Id}_\Lambda)(\mathcal{U}_{\mathcal{Y},\mathcal{W}}) \subseteq \mathcal{W}, \quad \wp^k p_1(\mathcal{U}_{\mathcal{Y},\mathcal{W}}) \subseteq \mathcal{Y} \quad \wp^k p_2(\mathcal{U}_{\mathcal{Y},\mathcal{W}}) \subseteq \wp^k\pi_2(\mathcal{W})$$

follow from the definition of $\mathcal{U}_{\mathcal{X},\mathcal{Z}}$, $\mathcal{U}_{\mathcal{Y},\mathcal{W}}$ and of the equality $\pi_2 \circ (t_g \times \mathrm{Id}_\Lambda) = p_2$.

We must prove the reverse inclusions to get our conclusions. We only show the three first inclusions, as the others may be obtained in the same way. Take then $Z_1 \in \mathcal{Z}$ and set $Z := \wp^{k-1}\pi_1(Z_1) \in \wp^k\pi_1(\mathcal{Z})$. As $(\mathcal{Z}, \mathcal{W}) \in \mathcal{C}_{(\mathcal{X},\mathcal{Y})}$, we can find some $X \in \mathcal{X}$ such that $\wp^{k-1}t_g(X) = Z$. Set

$$U_{X,Z_1} := \left\{u \in \wp^{k-2}\,(\Theta \times \Lambda) \mid \wp^{k-2} p_1(u) \in X \text{ and } \wp^{k-2}(t_g \times \mathrm{Id}_\Lambda)(u) \in Z_1\right\}$$
$$\in \wp^{k-1}(\Theta \times \Lambda).$$

Note that $U_{X,Z_1} \in \mathcal{U}_{\mathcal{X},\mathcal{Z}}$ because, by the inductive hypothesis, it clearly results that $\wp^{k-1} p_1(U_{X,Z_1}) = X \in \mathcal{X}$ and $\wp^{k-1}(t_g \times \mathrm{Id}_\Lambda)(U_{X,Z_1}) = Z_1 \in \mathcal{Z}$. Thus $\mathcal{Z} \subseteq \wp^k(t_g \times \mathrm{Id}_\Lambda)(\mathcal{U}_{\mathcal{X},\mathcal{Z}})$ and (a) holds.

Let us now prove the reverse inclusion needed for obtaining Condition (b). Take $X \in \mathcal{X}$ and set $Z := \wp^{k-1} t_g(X) \in \mathcal{Z}$. Let moreover $Z_1 \in \mathcal{Z}$ be such that $\wp^{k-1}\pi_1(Z_1) = Z$ and, as before, set

$$U_{X,Z_1} := \left\{u \in \wp^{k-2}\,(\Theta \times \Lambda) \mid \wp^{k-2} p_1(u) \in X \text{ and } \wp^{k-2}(t_g \times \mathrm{Id}_\Lambda)(u) \in Z_1\right\}.$$

By the inductive hypothesis, we get $\wp^{k-1} p_1(U) = X$ and $\wp^{k-1}(t_g \times \mathrm{Id}_\Lambda)(U) = Z_1$, so that $U \in \mathcal{U}_{\mathcal{X},\mathcal{Z}}$ and $\mathcal{X} \subseteq \wp^k p_1(\mathcal{U}_{\mathcal{X},\mathcal{Z}})$. So (b) holds.

Finally, by our choice of $(\mathcal{Z}, \mathcal{W})$, if $Y \in \wp^k \pi_2(\mathcal{Z})$, then we can easily find $U \in \mathcal{U}_{\mathcal{X},\mathcal{Z}}$ such that $\wp^{k-1} p_2(U) = Y$, so we get $\wp^k \pi_2(\mathcal{Z}) \subseteq \wp^k p_1(\mathcal{U}_{\mathcal{X},\mathcal{Z}})$ and Condition (c) holds. □

Let now $(\Omega, \mathcal{F}), (\Lambda, \mathcal{G}) \in \mathbf{SS^k}$ and $(\Omega, \mathcal{R}), (\Lambda, \mathcal{S}) \in \mathbf{SR^k}$ be arbitrary and fixed. We consider the k-set system $\mathcal{L}_{(\Omega,\mathcal{F}),(\Lambda,\mathcal{G})}$ on $\mathrm{Arr}(\Lambda, \Omega)$ and the k-set relation $\mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})}$ on $\mathrm{Arr}(\Lambda, \Omega)$ defined respectively by:

$$\mathcal{L}_{(\Omega,\mathcal{F}),(\Lambda,\mathcal{G})} := \left\{ Z \in \wp^k(\mathrm{Arr}(\Lambda, \Omega)) \;\middle|\; \begin{array}{l} \wp^k \mathrm{ev}_{\Omega,\Lambda}(W) \in \mathcal{F}\ \forall W \in \wp^k(\mathrm{Arr}(\Lambda, \Omega) \times \Lambda) \\ \text{such that } \wp^k \pi_1(W) = Z \text{ and } \wp^k \pi_2(W) \in \mathcal{G} \end{array} \right\},$$

$$\mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})} := \left\{ (X, Y) \in \wp^k(\mathrm{Arr}(\Lambda, \Omega))^2 \;\middle|\; \begin{array}{l} (\wp^k \mathrm{ev}_{\Omega,\Lambda})^{\langle 2\rangle}(W, Z) \in \mathcal{R} \\ \forall (W, Z) \in \wp^k(\mathrm{Arr}(\Lambda, \Omega) \times \Lambda)^2 \\ \text{such that } (\wp^k \pi_1)^{\langle 2\rangle}(W, Z) = (X, Y), \\ (\wp^k \pi_2)^{\langle 2\rangle}(W, Z) \in \mathcal{S} \end{array} \right\}. \tag{7}$$

So we get the following characterization that also provides an alternative proof for the Cartesian closedness of $\mathbf{SS^k}$ and $\mathbf{SR^k}$. We give the proof only for $\mathbf{SR^k}$ (part (ii) of Theorem 4.6) leaving as an exercise for readers the case of $\mathbf{SS^k}$ (part (i) of Theorem 4.6).

Theorem 4.6 *Let* $(\Omega, \mathcal{F}), (\Lambda, \mathcal{G}) \in \mathbf{SS^k}$ *and* $(\Omega, \mathcal{R}), (\Lambda, \mathcal{S}) \in \mathbf{SR^k}$. *The following conditions hold:*

(i) $((\mathrm{Arr}(\Lambda, \Omega), \mathcal{L}_{(\Omega,\mathcal{F}),(\Lambda,\mathcal{G})}), \mathrm{ev}_{\Omega,\Lambda})$ *is the exponential object of* $(\Omega, \mathcal{F})$ *and* $(\Lambda, \mathcal{G})$*;*
(ii) $((\mathrm{Arr}(\Lambda, \Omega), \mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})}), \mathrm{ev}_{\Omega,\Lambda})$ *is the exponential object of* $(\Omega, \mathcal{R})$ *and* $(\Lambda, \mathcal{S})$.

Therefore $\mathbf{SS^k}$ *and* $\mathbf{SR^k}$ *are both Cartesian closed categories.*

Proof Clearly, $(\mathrm{Arr}(\Lambda, \Omega), \mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})}) \in \mathbf{SR^k}$. Consider now the product $(\mathrm{Arr}(\Lambda, \Omega) \times \Lambda, \mathcal{A})$ in $\mathbf{SR^k}$ of $(\mathrm{Arr}(\Lambda, \Omega), \mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})})$ and $(\Lambda, \mathcal{S})$, for a suitable $\mathcal{A} \in \wp^k(\mathrm{Arr}(\Lambda, \Omega) \times \Lambda)^2$. By using (7), we easily get

$$\mathrm{ev}_{\Omega,\Lambda} \in \mathrm{Arr}_{\mathbf{SR^k}}((\mathrm{Arr}(\Lambda, \Omega) \times \Lambda), \mathcal{A}), (\Omega, \mathcal{R})).$$

Fix now $(\Theta, \mathcal{T}) \in \mathbf{SR^k}$ and take the product $(\Theta \times \Lambda, \mathcal{B})$ in $\mathbf{SR^k}$ of $(\Theta, \mathcal{T})$ and $(\Lambda, \mathcal{S})$, for a suitable $\mathcal{B} \in \wp^k(\Theta \times \Lambda)^2$. Take $g \in \mathrm{Arr}_{\mathbf{SR^k}}((\Theta \times \Lambda, \mathcal{B}), (\Omega, \mathcal{R}))$. We claim that

$$t_g \in \mathrm{Arr}_{\mathbf{SR^k}}((\Theta, \mathcal{T}), (\mathrm{Arr}(\Lambda, \Omega), \mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})})),$$

that is:

$$\forall (\mathcal{X}, \mathcal{Y}) \in \mathcal{T}\, [(\wp^k t_g)^{\langle 2 \rangle}(\mathcal{X}, \mathcal{Y}) \in \mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})}]. \tag{8}$$

Fix therefore $(\mathcal{X}, \mathcal{Y}) \in \mathcal{T}$. We first show our claim when $k = 0$. To this regard, note that

$$\mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})} = \left\{ (h, s) \in \mathrm{Arr}(\Lambda, \Omega)^2 \;\middle|\; \begin{array}{l} \mathrm{ev}^{\langle 2 \rangle}_{\Omega,\Lambda}((h, z), (s, w)) \in \mathcal{R} \\ \forall ((h, z), (s, w)) \in (\mathrm{Arr}(\Lambda, \Omega) \times \Lambda)^2 \\ \text{such that } (z, w) \in \mathcal{S} \end{array} \right\}.$$

Now, if $(\lambda, \mu) \in \mathcal{S}$ and $((\tilde{t}_{g,\mathcal{X}}, \lambda), (\tilde{t}_{g,\mathcal{Y}}, \mu)) \in (\mathrm{Arr}(\Lambda, \Omega) \times \Lambda)^2$, we clearly have

$$\mathrm{ev}^{\langle 2 \rangle}_{\Omega,\Lambda}((\tilde{t}_{g,\mathcal{X}}, \lambda), (\tilde{t}_{g,\mathcal{Y}}, \mu)) = (\tilde{t}_{g,\mathcal{X}}(\lambda), \tilde{t}_{g,\mathcal{Y}}(\mu)) = g^{\langle 2 \rangle}((\mathcal{X}, \lambda), (\mathcal{Y}, \mu)) \in \mathcal{R}$$

as $((\mathcal{X}, \lambda), (\mathcal{Y}, \mu)) \in \mathcal{B}$, whence $t_g^{\langle 2 \rangle}(\mathcal{X}, \mathcal{Y}) \in \mathcal{M}_{(\Omega,\mathcal{R}),(\Lambda,\mathcal{S})}$. This shows (8) when $k = 0$.

Suppose now $k \geqslant 1$ and take any $(\mathcal{Z}, \mathcal{W}) \in \mathcal{C}_{(\mathcal{X},\mathcal{Y})}$ such that $(\wp^k \pi_2)^{\langle 2 \rangle}(\mathcal{Z}, \mathcal{W}) \in \mathcal{S}$. We can then apply Lemma 4.5 to $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Z}, \mathcal{W})$, so that

(a) $(\wp^k(t_g \times \mathrm{Id}_\Lambda))^{\langle 2 \rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) = (\mathcal{Z}, \mathcal{W})$;
(b) $(\wp^k p_1)^{\langle 2 \rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) = (\mathcal{X}, \mathcal{Y})$;
(c) $(\wp^k p_2)^{\langle 2 \rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) = (\wp^k \pi_2)^{\langle 2 \rangle}(\mathcal{Z}, \mathcal{W})$.

Note that conditions (b)–(c) imply that $(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) \in \mathcal{B}$. Thus, as $g \in \mathrm{Arr}_{\mathbf{SR^k}}((\Theta \times \Lambda, \mathcal{B}), (\Omega, \mathcal{R}))$, we have that $(\wp^k g)^{\langle 2 \rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) \in \mathcal{R}$. Next, by condition (a) and by using (6), we get

$$\begin{aligned}(\wp^k \mathrm{ev}_{\Omega,\Lambda})^{\langle 2 \rangle}(\mathcal{Z}, \mathcal{W}) &= (\wp^k(\mathrm{ev}_{\Omega,\Lambda} \circ (t_g \times \mathrm{Id}_\Lambda)))^{\langle 2 \rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) \\ &= (\wp^k g)^{\langle 2 \rangle}(\mathcal{U}_{\mathcal{X},\mathcal{Z}}, \mathcal{U}_{\mathcal{Y},\mathcal{W}}) \in \mathcal{R},\end{aligned}$$

so that (8) holds. □

4.3 Some basic properties of $\mathbf{SO^k}$, $\mathbf{SO^{k,\sqsubseteq}}$ and $\mathbf{SO^{k,\sqsupseteq}}$

For the category $\mathbf{SO^k}$, the properties holding for $\mathbf{SS^k}$ and $\mathbf{SR^k}$ are not valid in general. In the next result we show that such a category is neither complete nor cocomplete.

Theorem 4.7 *The category* $\mathbf{SO^k}$ *is neither complete nor cocomplete.*

Proof In view of [7, Theorem 2.8.1], by proving the non-existence of a terminal object we will check that $\mathbf{SO^k}$ is not complete. To this regard, assume by contradiction that (Ω, σ) is the terminal object in the category $\mathbf{SO^k}$. We clearly have $\Omega \neq \emptyset$. Moreover, it results that $|\mathrm{Arr}_{\mathbf{SO^k}}((\Lambda, \tau), (\Omega, \sigma))| = 1$ for any $(\Lambda, \tau) \in \mathrm{Obj}(\mathbf{SO^k})$. Fix first $(\Lambda, \tau) \in \mathrm{Obj}(\mathbf{SO^k})$ such that $\tau(\emptyset) = \emptyset$, and let (g, h) be the only morphism in $\mathrm{Arr}_{\mathbf{SO^k}}((\Lambda, \tau), (\Omega, \sigma))$. Thus the equality $\wp^k h \circ \tau = \sigma \circ \wp^k g$ holds, and we easily deduce that $\sigma(\emptyset) = \emptyset$. On the other hand, take $(\Lambda', \tau') \in \mathrm{Obj}(\mathbf{SO^k})$ such that $\tau'(\emptyset) \neq \emptyset$, and denote by (g', h') the only morphism in $\mathrm{Arr}_{\mathbf{SO^k}}((\Lambda', \tau'), (\Omega, \sigma))$. Thus the equality $\wp^k h' \circ \tau = \sigma \circ \wp^k g'$ holds, and we easily deduce that $\sigma(\emptyset) \neq \emptyset$, which provides a contradiction.

In view of [7, Theorem 2.8.1], by proving the non-existence of an initial object we will check that $\mathbf{SO^k}$ is not bicomplete. To this end, assume by contradiction that (Ω, σ) is the initial object in $\mathbf{SO^k}$. Hence $|\mathrm{Arr}_{\mathbf{SO^k}}((\Omega, \sigma), (\Lambda, \tau))| = 1$ for any $(\Lambda, \tau) \in \mathrm{Obj}(\mathbf{SO^k})$. Fix now $(\Lambda, \tau) \in \mathrm{Obj}(\mathbf{SO^k})$, with $\tau(\emptyset) = \emptyset$ and denote by (g, h) the only morphism in $\mathrm{Arr}_{\mathbf{SO^k}}((\Omega, \sigma), (\Lambda, \tau))$. Thus, as the equality $\wp^k h \circ \sigma = \tau \circ \wp^k g$ holds, we easily get $\sigma(\emptyset) = \emptyset$.

On the other hand, $(\Lambda, \tau) \in \mathrm{Obj}(\mathbf{SO^k})$, with $\tau(\emptyset) \neq \emptyset$ and denote again by (g, h) the only morphism in $\mathrm{Arr}_{\mathbf{SO^k}}((\Omega, \sigma), (\Lambda, \tau))$. Thus, as the equality $\wp^k h \circ \sigma = \tau \circ \wp^k g$ holds, we easily get $\sigma(\emptyset) \neq \emptyset$, which provides a contradiction. □

Remark 4.8 The following conditions hold:

(i) $(\{a\}, \sigma)$, where $\sigma : \wp^k(\{a\}) \longrightarrow \wp^k(\{a\})$ is defined by $\sigma(X) = \wp^{k-1}(\{a\})$ for each $X \in \wp^k(\{a\})$, is the terminal object in the category $\mathbf{SO^{k,\sqsubseteq}}$;
(ii) $(\{a\}, \sigma)$, where $\sigma : \wp^k(\{a\}) \longrightarrow \wp^k(\{a\})$ is defined by $\sigma(X) = \emptyset$ for each $X \in \wp^k(\{a\})$, is the terminal object in the category $\mathbf{SO^{k,\sqsupseteq}}$;
(iii) $(\emptyset, \overline{\sigma})$, where $\overline{\sigma} : \wp^k(\emptyset) \longrightarrow \wp^k(\emptyset)$ is defined by $\overline{\sigma}(X) = \emptyset$ for each $X \in \wp^k(\emptyset)$, is the initial object in the category $\mathbf{SO^{k,\sqsubseteq}}$.

Before proving the next Theorem 4.10, we need the following preliminary result that we leave as an exercise to the reader.

Proposition 4.9 *Let* $f \in \mathrm{Arr}_{\mathbf{Set}}(\Omega, \Omega')$ *and* $k \geqslant 1$ *be a given integer. If* $\wp^k f$ *is surjective, then* $\wp^{k-1} f$ *is also surjective.*

Theorem 4.10 *The following conditions hold:*

(i) *there is no initial object in* $\mathbf{SO^{k,\sqsupseteq}}$*;*
(ii) *the category* $\mathbf{SO^{k,\sqsupseteq}}$ *is not cocomplete;*
(iii) *the categories* $\mathbf{SO^{k,\sqsubseteq}}$ *and* $\mathbf{SO^{k,\sqsupseteq}}$ *are not equivalent;*
(iv) *the categories* $\mathbf{SO^{k,\sqsubseteq}}$ *and* $\mathbf{SO^{k,\sqsupseteq}}$ *are not dually equivalent.*

Proof (i): Assume by contradiction that (Λ, τ) is the initial object in $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$. We cannot have $\Lambda \neq \varnothing$. In fact, assume that $\Lambda \neq \varnothing$ and let $(\Omega, \sigma) \in \mathrm{Obj}(\mathbf{SO}^{\mathbf{k},\sqsupseteq})$, where Ω contains at least two elements and $\sigma(X) = \wp^{k-1}(\Omega)$ for each $X \in \wp^k(\Omega)$, and let (f, f') be the only arrow in $\mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsupseteq}}((\Lambda, \tau), (\Omega, \sigma))$. As $\wp^k f' \circ \tau \sqsupseteq \sigma \circ \wp^k f$, we easily deduce that $\wp^{k-1} f'$ is surjective and, by Proposition 4.9, we also conclude that f' is surjective. So $|\Lambda| \geqslant |\Omega|$. At this point, notice that we can choose f in more than one way, contradicting the uniqueness of (f, f').

Thus $\Lambda = \varnothing$ and, moreover, if $(\Omega, \sigma) \in \mathrm{Obj}(\mathbf{SO}^{\mathbf{k},\sqsupseteq})$ and (f, f') is the only arrow in $\mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsupseteq}}((\Lambda, \tau), (\Omega, \sigma))$, we also deduce f and f' are the empty maps. Nevertheless, the inclusion $\wp^k f' \circ \tau \sqsupseteq \sigma \circ \wp^k f$ cannot hold if we take a non-empty set Ω and the k-set operator σ on Ω such that $\sigma(X) = \wp^{k-1}(\Omega)$ for each $X \in \wp^k(\Omega)$. In fact, just note that if $z \in \Omega$, then $\wp^{k-1}(\{z\}) \in \wp^k(\Omega) \setminus \wp^k(\wp^{k-1}(\varnothing))$. This shows that (Λ, τ) cannot exist.

(ii): It is an immediate consequence of the above part (i) and of [7, Theorem 2.8.1].

(iii): Use part (iii) of Remark 4.8, the above part (i) and the fact that equivalence of categories preserve co-limit objects.

(iv): Assume by contradiction that $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ and $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ were dually equivalent categories. Then $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$ would be equivalent to the dual category of $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$. Now, by part (i) of Remark 4.8 there exists the terminal object of $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$ and, hence, even the dual category of $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ admits terminal object. This implies that $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$ has an initial object, in contrast with the above part (i). $\square$

Another substantial difference between the categories $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$ and $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ concerns their completeness. In fact, with regard to the category $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$, we prove in the next Theorem 4.12 that it is complete, while for $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ we check that in general it does not admit equalizers (see Proposition 4.15).

In order to prove Theorem 4.12, we need the following construction. Let $\mathfrak{G} := \{X_i \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{Set})$ be a set-indexed family of sets. For every $n \in \mathbb{N}$, let $\Pi^{(n)}_{\mathfrak{G}} := \prod\{\wp^n(X_i) \mid i \in I\}$ be the Cartesian product of the family $\{\wp^n(X_i) \mid i \in I\}$ in $\mathbf{Set}$, and $\pi^{(n)}_{\mathfrak{G},i} : \Pi^{(n)}_{\mathfrak{G}} \longrightarrow \wp^n(X_i)$ the i-th projection, for any $i \in I$. Next, we consider:

- for any $n \geqslant 1$, the map $c^{(n)}_{\mathfrak{G}} : \Pi^{(n)}_{\mathfrak{G}} \longrightarrow \wp(\Pi^{(n-1)}_{\mathfrak{G}})$ defined by setting $c^{(n)}_{\mathfrak{G}}((\mathcal{K}_i)_{i \in I}) := \prod\{\mathcal{K}_i \mid i \in I\}$;
- for any $n \in \mathbb{N}$, the map $s^{(n)}_{\mathfrak{G}} : \Pi^{(n)}_{\mathfrak{G}} \longrightarrow \wp^n(\Pi^{(0)}_{\mathfrak{G}})$ defined by inductively setting $s^{(0)}_{\mathfrak{G}} := \mathrm{Id}_{\Pi^{(0)}_{\mathfrak{G}}}$ and $s^{(n)}_{\mathfrak{G}} := \wp(s^{(n-1)}_{\mathfrak{G}}) \circ c^{(n)}_{\mathfrak{G}}$ for every $n \geqslant 1$.

Lemma 4.11 *Let* $\mathfrak{G} := \{X_i \mid i \in I\}, \mathfrak{H} := \{Y_i \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{Set})$ *be two families of sets indexed by the same set* I*,* $f_i \in \mathrm{Arr}_{\mathbf{Set}}(X_i, Y_i)$ *for every* $i \in I$*,* $Z \in \mathrm{Obj}(\mathbf{Set})$ *and* $g_i \in \mathrm{Arr}_{\mathbf{Set}}(Z, Y_i)$ *for any* $i \in I$*. The following properties hold:*

(i) $\wp\, \pi^{(n-1)}_{\mathfrak{G},i} \circ c^{(n)}_{\mathfrak{G}} = \pi^{(n)}_{\mathfrak{G},i}$ *for any* $i \in I$ *and any* $n \geqslant 1$*;*

(ii) $\wp\big(\prod\{\wp^{n-1} f_i \mid i \in I\}\big) \circ c^{(n)}_{\mathfrak{G}} = c^{(n)}_{\mathfrak{H}} \circ \big(\prod\{\wp^n f_i \mid i \in I\}\big)$ *for any* $n \geqslant 1$*;*

(iii) $\wp\big(\dot{\prod}\{\wp^{n-1} g_i \mid i \in I\}\big)(\mathcal{K}) \subseteq \big(c^{(n)}_{\mathfrak{H}} \circ \big(\dot{\prod}\{\wp^{n-1}\, g_i \mid i \in I\}\big)\big)(\mathcal{K})$ *for any* $\mathcal{K} \in \wp^n(Z)$ *and any* $n \geqslant 1$*;*

(iv) $\wp^n \pi^{(0)}_{\mathfrak{G},i} \circ s^{(n)}_{\mathfrak{G}} = \pi^{(n)}_{\mathfrak{G},i}$ *for any* $i \in I$ *and* $n \in \mathbb{N}$*;*

(v) $\wp^n\big(\prod\{f_i \mid i \in I\}\big) \circ s_{\mathfrak{G}}^{(n)} = s_{\mathfrak{H}}^{(n)} \circ \prod\{\wp^n f_i \mid i \in I\}$ *for any* $n \in \mathbb{N}$;

(vi) $\wp^n\big(\dot{\prod}\{g_i \mid i \in I\}\big)(\mathcal{K}) \subseteq (s_{\mathfrak{H}}^{(n)} \circ \dot{\prod}\{\wp^n g_i \mid i \in I\})(\mathcal{K})$ *for any* $\mathcal{K} \in \wp^n(Z)$ *and any* $n \geqslant 1$.

Proof (i): Straightforward.

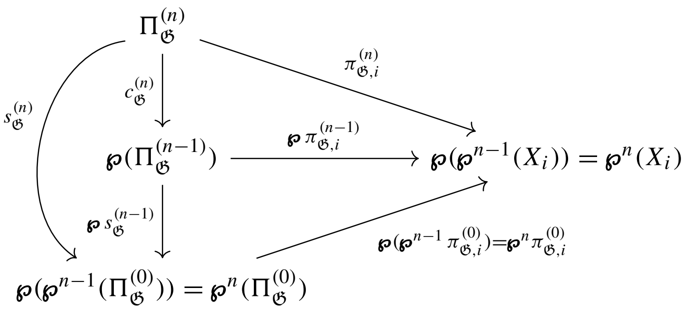
(ii): Let $(\mathcal{K}_i)_{i\in I} \in \Pi_{\mathfrak{G}}^{(n)}$. Then

$$\begin{aligned}
&\Big(\wp\Big(\prod\big\{\wp^{n-1} f_i \mid i \in I\big\}\Big) \circ c_{\mathfrak{G}}^{(n)}\Big)((\mathcal{K}_i)_{i\in I}) \\
&\quad = \wp\Big(\prod\{\wp^{n-1} f_i \mid i \in I\}\Big)\Big(\prod\{\mathcal{K}_i \mid i \in I\}\Big) \\
&\quad = \big\{(\wp^{n-1} f_i(\mathcal{X}_i))_{i\in I} \mid \mathcal{X}_i \in \mathcal{K}_i \ \forall i \in I\big\} \\
&\quad = \prod\big\{\{\wp^{n-1} f_i(\mathcal{X}_i) \mid \mathcal{X}_i \in \mathcal{K}_i\} \mid i \in I\big\} \\
&\quad = c_{\mathfrak{H}}^{(n)}\big(((\wp^n f_i)(\mathcal{K}_i))_{i\in I}\big) \\
&\quad = \Big(c_{\mathfrak{H}}^{(n)} \circ \prod\big\{\wp^n f_i \mid i \in I\big\}\Big)((\mathcal{K}_i)_{i\in I}).
\end{aligned}$$

(iii): Let $\mathcal{K} \in \wp^n(Z) = \wp(\wp^{n-1}(Z))$. Then

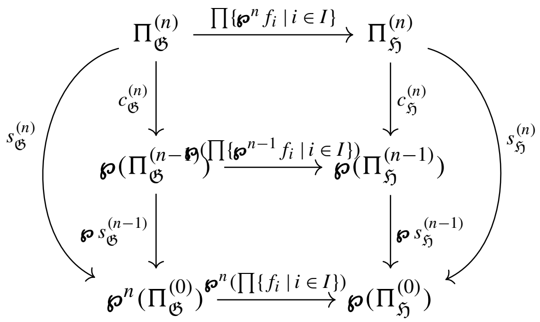
$$\begin{aligned}
\wp\Big(\dot{\prod}\{\wp^{n-1} g_i \mid i \in I\}\Big)(\mathcal{K}) &= \Big\{\dot{\prod}\{\wp^{n-1} g_i \mid i \in I\}(\mathcal{X}) \mid \mathcal{X} \in \mathcal{K}\Big\} \\
&= \big\{(\wp^{n-1} g_i(\mathcal{X}))_{i\in I} \mid \mathcal{X} \in \mathcal{K}\big\} \\
&\subseteq \prod\{\wp^n g_i(\mathcal{K}) \mid i \in I\} = c_{\mathfrak{H}}^{(n)}((\wp^n g_i(\mathcal{K}))_{i\in I}) \\
&= \Big(c_{\mathfrak{H}}^{(n)} \circ \dot{\prod}\{\wp^n g_i \mid i \in I\}\Big)(\mathcal{K}).
\end{aligned}$$

(iv): We proceed inductively on n. The case $n = 0$ is trivial. Fix now an integer $n \geqslant 1$ and assume that the claim holds for $1 \leqslant k \leqslant n-1$. Let $i \in I$ and consider the following diagram:



In view of the above part (i), we have that $\wp\pi_{\mathfrak{G},i}^{(n-1)} \circ c_{\mathfrak{G}}^{(n)} = \pi_{\mathfrak{G},i}^{(n)}$. On the other hand, by the inductive hypothesis we have $\wp^{n-1}\pi_{\mathfrak{G},i}^{(0)} \circ s_{\mathfrak{G}}^{(n-1)} = \pi_{\mathfrak{G},i}^{(n-1)}$. Hence, applying $\wp$

to the previous equality, we get the commutativity of the lower triangle of the diagram and we easily deduce the equality $\wp^n(\pi^{(0)}_{\mathfrak{G},i}) \circ s^{(n)}_{\mathfrak{G}} = \pi^{(n)}_{\mathfrak{G},i}$.
(v): We proceed inductively on n. The case $n = 0$ is trivial. Fix now an integer $n \geqslant 1$ and assume that the claim holds for $1 \leqslant k \leqslant n-1$. Consider the following diagram:



In view of the above part (ii), we have that the equality $\wp\big(\prod\{\wp^{n-1} f_i \mid i \in I\}\big) \circ c^{(n)}_{\mathfrak{G}}$ $c^{(n)}_{\mathfrak{H}} \circ \big(\prod\{\wp^n f_i \mid i \in I\}\big)$ holds. On the other hand, by the inductive hypothesis we get $\wp^{n-1}\big(\prod\{f_i \mid i \in I\}\big) \circ s^{(n-1)}_{\mathfrak{G}} = s^{(n-1)}_{\mathfrak{H}} \circ \prod\{\wp^{n-1} f_i \mid i \in I\}$. Hence, applying $\wp$ to the previous equality, we get the commutativity of the lower square of the diagram. So, we clearly deduce the equality $\wp^n\big(\prod\{f_i \mid i \in I\}\big) \circ s^{(n)}_{\mathfrak{H}} = s^{(n)}_{\mathfrak{G}} \circ \prod\{\wp^n f_i \mid i \in I\}$.
(vi): We proceed inductively on n. The case $n = 0$ is trivial. Fix now an integer $n \geqslant 1$ and assume that our assertion holds for $1 \leqslant k \leqslant n-1$. Then $\wp^{n-1}\big(\dot{\prod}\{g_i \mid i \in I\}\big)(\mathcal{K}) \subseteq \big(s^{(n-1)}_{\mathfrak{H}} \circ \dot{\prod}\{\wp^{n-1} g_i \mid i \in I\}\big)(\mathcal{K})$ for every $\mathcal{K} \in \wp^{n-1}(Z)$. We must prove our claim for n. To this regard, let $\mathcal{K} \in \wp^n(Z)$. By the above part (iii) it results that

$$\begin{aligned}\wp^n\Big(\dot{\prod}\{g_i \mid i \in I\}\Big)(\mathcal{K}) &\subseteq \wp\, s^{(n-1)}_{\mathfrak{H}} \circ \wp\Big(\dot{\prod}\{\wp^{n-1} g_i \mid i \in I\}\Big)(\mathcal{K})\\ &\subseteq \wp\, s^{(n-1)}_{\mathfrak{H}} \circ \Big(c_n \circ \dot{\prod}\{\wp^n g_i \mid i \in I\}\Big)(\mathcal{K})\\ &= s^{(n)}_{\mathfrak{H}} \circ \Big(\dot{\prod}\{\wp^n g_i \mid i \in I\}\Big)(\mathcal{K}).\end{aligned}$$ □

At this point we can show the completenesss of the category $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$.

Theorem 4.12 *The category* $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$ *is complete.*

Proof In view of [7, Proposition 2.8.1] it suffices to check the existence of the product of any indexed set family of objects and of equalizers. We begin with products. To this regard, let us consider $\mathfrak{S} := \{\Omega_i \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{Set})$ and $\{(\Omega_i, \sigma_i) \mid i \in I\} \subseteq \mathbf{SO}^{\mathbf{k},\sqsubseteq}$. Set moreover $\Omega := \prod \mathfrak{S}$ and let $\pi^{(0)}_{\mathfrak{S},i} \colon \Omega \longrightarrow \Omega_i$ be the usual projections for any $i \in I$ and $\sigma := s^{(k)}_{\mathfrak{S}} \circ \big(\dot{\prod}\{\sigma_i \circ \wp^k \pi^{(0)}_{\mathfrak{S},i} \mid i \in I\}\big)$. Let us check that $((\Omega, \sigma), \{(\pi^{(0)}_{\mathfrak{S},i}, \pi^{(0)}_{\mathfrak{S},i}) \mid i \in I\})$ is the product of $\{(\Omega_i, \sigma_i) \mid i \in I\}$ in $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$.

We clearly have $(\pi^{(0)}_{\mathfrak{S},i}, \pi^{(0)}_{\mathfrak{S},i}) \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega,\sigma),(\Omega_i,\sigma_i))$ for any $i \in I$ because, by Lemma 4.11 (iv) and of the definition of σ, we get

$$\wp^k \pi^{(0)}_{\mathfrak{S},i} \circ \sigma = \pi^{(k)}_{\mathfrak{S},i} \circ \Big(\dot{\prod} \{\sigma_i \circ \wp^k \pi^{(0)}_{\mathfrak{S},i} \mid i \in I\}\Big) = \sigma_i \circ \wp^k \pi^{(0)}_{\mathfrak{S},i}.$$

Let now $(\Lambda,\tau) \in \mathbf{SO}^{\mathbf{k},\sqsubseteq}$ and $(q_i, q_i') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Lambda,\tau),(\Omega_i,\sigma_i))$ for any $i \in I$. Set first $r := \dot{\prod}\{q_i \mid i \in I\}$ and $r' := \dot{\prod}\{q_i' \mid i \in I\}$.

We claim that $(r, r') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Lambda,\tau),(\Omega,\sigma))$, i.e. that $\wp^k r' \circ \tau \sqsubseteq \sigma \circ \wp^k r$. By Lemma 4.11 (iv), the definition of r and since $(q_i, q_i') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Lambda,\tau),(\Omega_i,\sigma_i))$ for any $i \in I$, we have

$$\begin{aligned}
\wp^k r' \circ \tau &\sqsubseteq s^{(k)}_{\mathfrak{S}} \circ \dot{\prod} \{\wp^k q_i' \mid i \in I\} \circ \tau = s^{(k)}_{\mathfrak{S}} \circ \dot{\prod} \{\wp^k q_i' \circ \tau \mid i \in I\} \\
&\sqsubseteq s^{(k)}_{\mathfrak{S}} \circ \dot{\prod} \{\sigma_i \circ \wp^k q_i \mid i \in I\} \\
&= s^{(k)}_{\mathfrak{S}} \circ \dot{\prod} \big\{\big(\sigma_i \circ \wp^k (\pi^{(0)}_{\mathfrak{S},i} \circ r)\big) \mid i \in I\big\} \\
&= \Big(s^{(k)}_{\mathfrak{S}} \circ \dot{\prod} \big\{\big(\sigma_i \circ \wp^k (\pi^{(0)}_{\mathfrak{S},i})\big) \mid i \in I\big\}\Big) \circ \wp^k r = \sigma \circ \wp^k r.
\end{aligned}$$

Furthermore we have $(q_i, q_i') = (\pi^{(0)}_{\mathfrak{S},i}, \pi^{(0)}_{\mathfrak{S},i}) \circ (r, r')$ for each $i \in I$.

To complete the proof, let us compute the equalizers. To this regard, consider $(\Omega_1,\sigma_1), (\Omega_2,\sigma_2) \in \mathbf{SO}^{\mathbf{k},\sqsubseteq}$ and $(f, f'), (g, g') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega_1,\sigma_1),(\Omega_2,\sigma_2))$. Respectively set

$$\begin{aligned}
&\Omega := \{x \in \Omega_1 \mid f(x) = g(x) \text{ and } f'(x) = g'(x)\} \quad \text{and} \\
&\forall X \in \wp^k(\Omega) \ \ [\sigma(X) := \sigma_1(X) \cap \wp^{k-1}(\Omega)].
\end{aligned}$$

We claim that $((\Omega,\sigma),(\xi,\xi))$, where $\xi : \Omega \longrightarrow \Omega_1$ is the usual set-theoretical inclusion, is the equalizer of (f, f') and (g, g') in $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$. Clearly, $(\xi,\xi) \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Omega,\sigma),$
$(\Omega_1,\sigma_1))$ and $(f, f') \circ (\xi,\xi) = (g, g') \circ (\xi,\xi)$.

Take now $(\Lambda,\tau) \in \mathbf{SO}^{\mathbf{k},\sqsubseteq}$ and $(m, m') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Lambda,\tau),(\Omega_1,\sigma_1))$ such that $(f, f') \circ (m, m') = (g, g') \circ (m, m')$. Define the maps $n : \Lambda \longrightarrow \Omega$ and $n' : \Lambda \longrightarrow \Omega$ respectively setting $n(\lambda) := m(\lambda)$ and $n'(\lambda) := m'(\lambda)$ for any $\lambda \in \Lambda$. We claim that $(n, n') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsubseteq}}((\Lambda,\tau),(\Omega,\sigma))$. To this end, fix $Z \in \wp^k(\Lambda)$. Hence we get the inclusion $(\wp^k n' \circ \tau)(Z) = (\wp^k m' \circ \tau)(Z) \subseteq (\sigma_1 \circ \wp^k m)(Z) \cap \wp^{k-1}(\Omega) = (\sigma \circ \wp^k m)(Z)$, whence our conclusion. Finally, it is obvious that (n, n') is the only morphism such that $(m, m') = (\xi,\xi) \circ (n, n')$. □

Exercise 4.13 Using a proof similar as that of Theorem 4.12, it may be easily shown that $\mathbf{SO}^{\mathbf{k},\sqsubseteq,=}$ is a complete category.

In order to prove the next Proposition 4.15, we need the following preliminary result.

Lemma 4.14 ([7], Proposition 2.4.3) *Let $\mathfrak{C}$ be a category, $A, B \in \mathrm{Obj}(\mathfrak{C})$ and $f, g \in \mathrm{Arr}_{\mathfrak{C}}(A, B)$. If (H, h) is the equalizer of f and g, the arrow $h \in \mathrm{Arr}_{\mathfrak{C}}(H, A)$ is a monomorphism.*

In the next result we will find two morphisms of $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ not admitting equalizers.

Proposition 4.15 *In general, the category $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ does not admit equalizers.*

Proof Let $(\Omega_1, \sigma_1), (\Omega_2, \sigma_2) \in \mathrm{Obj}(\mathbf{SO}^{\mathbf{k},\sqsupseteq})$, where $\Omega_1 = \{x\}, \Omega_2 = \{a, b\}, \sigma_1(X) = \wp^{k-1}(\Omega_1)$ for each $X \in \wp^k(\Omega_1)$ and $\sigma_2(Z) = \varnothing$ for each $Z \in \wp^k(\Omega_2)$.

Consider moreover the four maps $f, f', g, g' : \Omega_1 \longrightarrow \Omega_2$ defined as follows: $f(x) = f'(x) = g'(x) = a$ and $g(x) = b$. Notice that $(f, f'), (g, g') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsupseteq}}((\Omega_1, \sigma_1), (\Omega_2, \sigma_2))$.

Assume by contradiction that $((\Lambda, \tau), (h, h'))$ is an equalizer of (f, f') and (g, g') in the category $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$. By Proposition 4.14 it follows that (h, h') is a monomorphism and, thus, by Proposition 3.9 (i) we deduce that both h and h' are injective maps from Λ to Ω_1.

On the other hand, as $(h, h') \in \mathrm{Arr}_{\mathbf{SO}^{\mathbf{k},\sqsupseteq}}((\Lambda, \tau), (\Omega_1, \sigma_1))$, we get the inclusion $\wp^k h' \circ \tau \sqsupseteq \sigma_1 \circ \wp^k h$ and, using our choice of σ_1, this implies that $\wp^{k-1} h'$ is surjective. So, by Proposition 4.9 we deduce that h' is surjective. Thus, h' is a bijection between Λ and Ω_1. This shows that Λ contains only one element.

Furthermore, we deduce that h is also bijective. Now, since $((\Lambda, \tau), (h, h'))$ is the equalizer of (f, f') and (g, g') in the category $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$, it result that $f \circ h = g \circ h$, whence $f = g$, in contrast with our choice of f and g. This shows that there is no equalizer of (f, f') and (g, g') in the category $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$. □

As an immediate consequence of Proposition 4.15, we obtain the following result.

Corollary 4.16 *The category $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ is not complete.*

We conclude this section showing that the category $\mathbf{SO}^{\mathbf{k}}$ does not have neither products nor coproducts.

Proposition 4.17 *The following conditions hold:*

(i) $\mathbf{SO}^{\mathbf{k}}$ *does not admit products;*
(ii) $\mathbf{SO}^{\mathbf{k}}$ *does not admit coproducts.*

Proof In both proofs, we will consider the objects $(\Omega_1, \sigma_1), (\Omega_2, \sigma_2) \in \mathrm{Obj}(\mathbf{SO}^{\mathbf{k}})$, where $\Omega_1 = \Omega_2 = \{x\}, \sigma_1(X) = \wp^{k-1}(\Omega_1)$ for each $X \in \wp^k(\Omega_1)$ and $\sigma_2 = \mathrm{Id}_{\wp^k(\Omega_2)}$.
(i): Assume by contradiction that there exists a product $((\Omega, \tau), (p_1, p_1'), (p_2, p_2'))$ of (Ω_1, σ_1) and (Ω_2, σ_2). Then it follows that

$$\wp^k p_1' \circ \tau = \sigma_1 \circ \wp^k p_1 \quad \text{and} \quad \wp^k p_2' \circ \tau = \sigma_2 \circ \wp^k p_2. \tag{9}$$

Now, by (9) and the fact that $\sigma_2(\varnothing) = \varnothing$, we get $\tau(\varnothing) = \varnothing$ and $\sigma_1(\varnothing) = \varnothing$, contradicting our choice of σ_1. This proves that (Ω_1, σ_1) and (Ω_2, σ_2) do not have a product.

(ii): Assume by contradiction that there exists a coproduct $((\Omega, \tau), (i_1, i_1'), (i_2, i_2'))$ of (Ω_1, σ_1) and (Ω_2, σ_2). Then it follows that

$$\wp^k i_1' \circ \sigma_1 = \tau \circ \wp^k i_1 \quad \text{and} \quad \wp^k i_2' \circ \sigma_2 = \tau \circ \wp^k i_2. \tag{10}$$

Now, by (10) and the fact that $\sigma_2(\varnothing) = \varnothing$, we get $\tau(\varnothing) = \varnothing$ and $\sigma_1(\varnothing) = \varnothing$, contradicting our choice of σ_1. This proves that (Ω_1, σ_1) and (Ω_2, σ_2) do not have a coproduct. ◻

5 A new interpretation of the category of the equivalence relations

In any classical mathematical textbook an *equivalence relation* $\mathcal{R}$ on a given set Ω is usually studied in a direct analogy with its corresponding counterpart which is the induced *set partition* on the same ground set Ω. In the scope of such an identification, we introduce a new notion, which is a class of Alexandroff closure set operators, and that we call *equivalence closure set operators*. In this section, we first address the identification of the three notions of equivalence relations, set partitions and equivalence closure set operators within our formalism of sub-bijection (see Proposition 5.3). Next, we extend the above sub-bijection to a categorial level (see Theorem 5.4).

To this regard, for any given set Ω we denote by $\mathrm{EQ}(\Omega)$ the collection of all the equivalence relations on Ω, that is all the set relations $\mathcal{R} \in \wp(\wp^0(\Omega) \times \wp^0(\Omega)) = \wp(\Omega \times \Omega) = \mathrm{Obj}(\mathbf{SR^0})$ which are reflexive, symmetric and transitive. Notice also that we may consider $\mathrm{EQ}(\Omega)$ as a subfamily of $\mathrm{SR}(\Omega)$, by means of the identification of any pair $(x, y) \in \Omega \times \Omega$ with the pair $(\{x\}, \{y\}) \in \wp(\Omega) \times \wp(\Omega)$. Let $\mathbf{EQ}$ be the full subcategory of $\mathbf{SR^0}$ defined by

$$\mathrm{Obj}(\mathbf{EQ}) = \big\{(\Omega, \mathcal{R}) \mid \Omega \in \mathrm{Obj}(\mathbf{Set}),\ \mathcal{R} \in \mathrm{EQ}(\Omega)\big\}.$$

The main purpose of the present section is the proof of the next Theorem 5.4, where we determine two isomorphisms between the above category $\mathbf{EQ}$ and other two categories of set systems and set operators, respectively. We need first to introduce some preliminary linking maps, between binary relations and set systems, and between set systems and set operators. To this regard we define:

- $\mathrm{Pa}\colon \mathrm{SR}(\Omega) \longrightarrow \mathrm{SS}(\Omega)$ such that $\mathrm{Pa}_{\mathcal{R}} := \{N_{\mathcal{R}}(x) \mid x \in \Omega\}$ for any $\mathcal{R} \in \mathrm{SR}(\Omega)$, where

$$N_{\mathcal{R}}(x) := \big\{y \in \Omega \mid (\{x\}, \{y\}) \in \mathcal{R} \text{ or } (\{y\}, \{x\}) \in \mathcal{R}\big\};$$

- $\mathrm{Qa}\colon \mathrm{SO}(\Omega) \longrightarrow \mathrm{SS}(\Omega)$ defined by $\mathrm{Qa}(\sigma) := \{\sigma(\{x\}) \mid x \in \Omega\}$, for each $\sigma \in \mathrm{SO}(\Omega)$;
- $\mathrm{Up}\colon \mathrm{SS}(\Omega) \longrightarrow \mathrm{SO}(\Omega)$ such that $Up_{\mathcal{F}}(X) := \bigcup\{Y \in \mathcal{F} \mid X \cap Y \neq \varnothing\}$, for each $\mathcal{F} \in \mathrm{SS}(\Omega)$ and any $X \in \wp(\Omega)$;
- $\mathrm{Eq}\colon \mathrm{SS}(\Omega) \longrightarrow \mathrm{SR}(\Omega)$ defined by $\mathrm{Eq}_{\mathcal{F}} := \{(x, y) \in \Omega \times \Omega \mid \exists\, Z \in \mathcal{F}\ [\{x, y\} \subseteq Z]\}$, for each $\mathcal{F} \in \mathrm{SS}(\Omega)$.

We introduce now a new sub-collection of closure set operators, which we call *equivalence set operators* in view of the categorial isomorphism obtained in Theorem 5.4.

Definition 5.1 We say that a set operator $\sigma \in \mathrm{SO}(\Omega)$ is an *equivalence set operator* on Ω if it satisfies the following properties:

(E1) σ is extensive;
(E2) $\sigma(X) = \bigcup\{\sigma(\{x\}) \mid x \in X\}$, for any $X \in \wp(\Omega)$;
(E3) for any $x, y \in \Omega$, the condition $\sigma(\{x\}) = \sigma(\{y\})$ is equivalent to $\sigma(\{x\}) \cap \sigma(\{y\}) \neq \varnothing$.

We denote by $\mathrm{ESO}(\Omega)$ the family of all equivalence set operators on Ω.

Remark 5.2 Notice that equivalence set operators are particular types of closure set operators, that is $\mathrm{ESO}(\Omega) \subseteq \mathrm{CSO}(\Omega)$.

A direct consequence of Remark 5.2 and of property (E2) is that any equivalence set operator is also an *Alexandroff closure set operator*. We refer the interested reader to know further properties of Alexandroff closure set operators to the work [12], where such operators are investigated mainly in relation to their links with monoid actions.

The next result is preliminary to the proof of Theorem 5.4. Here, we frame the collections $\mathrm{EQ}(\Omega)$, $\mathrm{SP}(\Omega)$ and $\mathrm{ESO}(\Omega)$ within the formalism of sub-bijections, introduced in Definition 1.2.

Proposition 5.3 *We have that*

$$(\mathrm{SR}(\Omega) \mid \mathrm{EQ}(\Omega)) \underset{\mathrm{Eq}}{\overset{\mathrm{Pa}}{\rightleftarrows}} (\mathrm{SS}(\Omega) \mid \mathrm{SP}(\Omega)) \underset{\mathrm{Qa}}{\overset{\mathrm{Up}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{ESO}(\Omega))$$

is an $(\mathrm{SR}(\Omega), \mathrm{SS}(\Omega), \mathrm{SO}(\Omega))$*-sub-bijection.*

Proof The fact that Pa is a bijection between $\mathrm{EQ}(\Omega)$ and $\mathrm{SP}(\Omega)$, with inverse Eq, is a classical result. So, it suffices to show that Up is a bijection between $\mathrm{SP}(\Omega)$ and $\mathrm{ESO}(\Omega)$, with inverse Qa.

Let $\mathcal{F} \in \mathrm{SP}(\Omega)$. We claim that $\mathrm{Up}_{\mathcal{F}} \in \mathrm{ESO}(\Omega)$. To this regard, by the definition of Up we clearly have that $\mathrm{Up}_{\mathcal{F}}(\varnothing) = \varnothing$ and, moreover, it is also easy to prove that $\mathrm{Up}_{\mathcal{F}}$ is an extensive and increasing set operator. Furthermore it satisfies property (E3) because, for every $x \in \Omega$, it results that $\mathrm{Up}_{\mathcal{F}}(\{x\})$ agrees with the only member of the set partition $\mathcal{F}$ of Ω that contains the element x.

Let us now prove that $\mathrm{Up}_{\mathcal{F}}$ satisfies (E2). To this end, fix $X \in \wp(\Omega)$ and, for every $y \in \Omega$, denote by B_y the only member of $\mathcal{F}$ containing y. Then, we clearly have

$$y \in \mathrm{Up}_{\mathcal{F}}(X) \iff X \cap B_y \neq \varnothing \iff \exists\, x \in X \;\; [B_y \subseteq \mathrm{Up}_{\mathcal{F}}(\{x\})].$$

This shows (E2) and, so, $\mathrm{Up}_{\mathcal{F}} \in \mathrm{ESO}(\Omega)$.

Let $\sigma \in \mathrm{ESO}(\Omega)$. As σ is extensive and by property (E3) we easily deduce that $\mathrm{Qa}(\sigma) \in \mathrm{SP}(\Omega)$.

Now, let us prove that $\mathrm{Up}_{\mathrm{Qa}(\sigma)} = \sigma$ for each $\sigma \in \mathrm{ESO}(\Omega)$. To this regard, when $\sigma \in \mathrm{ESO}(\Omega)$, it is sufficient to observe that

$$X \cap \sigma(\{x\}) \neq \varnothing \iff \exists\, y \in X \;\; [\sigma(\{x\}) = \sigma(\{y\})].$$

On the other hand, we clearly have $\mathrm{Qa}(\mathrm{Up}_{\mathcal{F}}) = \mathcal{F}$ for each $\mathcal{F} \in \mathrm{SP}(\Omega)$. □

Now, in order to obtain a categorical extension preserving the directions of the arrows of the above sub-bijection, we consider the following $(\mathrm{SS}(\Omega), \mathrm{SR}(\Omega), \mathrm{SO}(\Omega))$-sub-bijection induced by that previously obtained in Proposition 5.3:

$$(\mathrm{SS}(\Omega) \mid \mathrm{SP}(\Omega)) \underset{\mathrm{Pa}}{\overset{\mathrm{Eq}}{\rightleftarrows}} (\mathrm{SR}(\Omega) \mid \mathrm{EQ}(\Omega)) \underset{\mathrm{Za}}{\overset{\mathrm{Ta}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{ESO}(\Omega)), \qquad (11)$$

where $\mathrm{Ta} := \mathrm{Up} \circ \mathrm{Pa}$ and $\mathrm{Za} := \mathrm{Eq} \circ \mathrm{Qa}$.

We introduce now the following category, whose objects are equivalence set operators.

- Let $\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}$ be the full subcategory of $\mathbf{SO}^{\mathbf{1},\sqsubseteq,=}$ for which $\mathrm{Obj}(\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}) = \{(\Omega, \sigma) \mid \sigma \in \mathrm{ESO}(\Omega)\}$.

Before proving Theorem 5.4, we need to introduce the category **SP** whose objects are set partitions, and the candidate correspondences to be the categorial isomorphisms established in Theorem 5.4.

- Let **SP** be the category for which $\mathrm{Obj}(\mathbf{SP}) = \{(\Omega, \mathcal{F}) \mid \Omega \in \mathrm{Obj}(\mathbf{Set}),\ \mathcal{F} \in \mathrm{SP}(\Omega)\}$, and such that $\mathrm{Arr}_{\mathbf{SP}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}')) = \{f \in \mathrm{Arr}_{\mathbf{Set}}(\Omega, \Omega') \mid \wp^2 f(\mathcal{F}) \preceq \mathcal{F}'\}$ for each $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}') \in \mathrm{Obj}(\mathbf{SP})$, where the symbol $\preceq$ denotes the usual partial order on $\mathrm{SP}(\Omega')$ defined by

$$\forall\, \mathcal{F}, \mathcal{G} \in \mathrm{SP}(\Omega') \;\big[\mathcal{F} \preceq \mathcal{G} :\iff \forall X \in \mathcal{F}\ \exists\, Y \in \mathcal{G}\ [X \subseteq Y]\big].$$

We now extended to a functorial level the linking maps Za, Ta, Pa and Eq, respectively as follows:

- $\widehat{\mathrm{Za}}\colon \mathbf{ESO}^{\mathbf{1},\sqsubseteq,=} \longrightarrow \mathbf{EQ}$, where $\widehat{\mathrm{Za}}((\Omega, \sigma)) := (\Omega, \mathrm{Za}(\sigma))$ and $\widehat{\mathrm{Za}}((f, f)) := f$, for $(\Omega, \sigma), (\Omega', \sigma') \in \mathrm{Obj}(\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=})$ and $(f, f) \in \mathrm{Arr}_{\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}}((\Omega, \sigma), (\Omega', \sigma'))$;
- $\widehat{\mathrm{Ta}}\colon \mathbf{EQ} \longrightarrow \mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}$, where $\widehat{\mathrm{Ta}}((\Omega, \mathcal{R})) := (\Omega, \mathrm{Ta}(\mathcal{R}))$ and $\widehat{\mathrm{Ta}}(f) := (f, f)$, whenever $(\Omega, \mathcal{R}), (\Omega', \mathcal{R}') \in \mathrm{Obj}(\mathbf{EQ})$ and $f \in \mathrm{Arr}_{\mathbf{EQ}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$;
- $\widehat{\mathrm{Pa}}\colon \mathbf{EQ} \longrightarrow \mathbf{SP}$, where $\widehat{\mathrm{Pa}}((\Omega, \mathcal{R})) := (\Omega, \mathrm{Pa}_{\mathcal{R}})$ and $\widehat{\mathrm{Pa}}(f) := f$, whenever $(\Omega, \mathcal{R}), (\Omega', \mathcal{R}') \in \mathrm{Obj}(\mathbf{EQ})$ and $f \in \mathrm{Arr}_{\mathbf{EQ}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$;
- $\widehat{\mathrm{Eq}}\colon \mathbf{SP} \longrightarrow \mathbf{EQ}$, where $\widehat{\mathrm{Eq}}((\Omega, \mathcal{F})) := (\Omega, \mathrm{Eq}(\mathcal{F}))$ and $\widehat{\mathrm{Eq}}(f) := f$, whenever $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}') \in \mathrm{Obj}(\mathbf{SP})$, for any $f \in \mathrm{Arr}_{\mathbf{SP}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}'))$.

Through the categories $\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}$, **EQ** and **SP** and the previous correspondences, in the next result we extend to a categorical level the sub-bijection given in (11).

Theorem 5.4 *The arrows of the diagram*

$$\mathbf{SP} \underset{\widehat{\mathrm{Pa}}}{\overset{\widehat{\mathrm{Eq}}}{\rightleftarrows}} \mathbf{EQ} \underset{\widehat{\mathrm{Za}}}{\overset{\widehat{\mathrm{Ta}}}{\rightleftarrows}} \mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}$$

are isomorphisms of categories, where $\widehat{\mathrm{Pa}}$ *is the inverse of* $\widehat{\mathrm{Eq}}$ *and* $\widehat{\mathrm{Za}}$ *is the inverse of* $\widehat{\mathrm{Ta}}$.

Proof The claim concerning $\widehat{\mathrm{Pa}}$ and $\widehat{\mathrm{Eq}}$ is easy and straightforward. Let us prove that $\widehat{\mathrm{Za}}$ and $\widehat{\mathrm{Ta}}$ are well defined. To this regard, let us consider $(\Omega, \sigma), (\Omega', \sigma') \in \mathrm{Obj}(\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=})$ and $(f, f) \in \mathrm{Arr}_{\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}}((\Omega, \sigma), (\Omega', \sigma'))$. Let moreover $x_1, x_2 \in \Omega$ be such that $(x_1, x_2) \in \mathrm{Za}(\sigma)$. Hence we get $(\wp f \circ \sigma)(\{x_1\}) = \wp f(\sigma(\{x_1\})) = \wp f(\sigma(\{x_2\})) = (\wp f \circ \sigma)(\{x_2\})$, so that

$$\begin{aligned}&\emptyset \neq (\wp f \circ \sigma)(\{x_1\})\\&\quad\subseteq (\sigma' \circ \wp f)(\{x_1\}) \cap (\sigma' \circ \wp f)(\{x_2\}) = \sigma'(f(\{x_1\}) \cap \sigma'(f(\{x_2\}))\end{aligned}$$

and, by (E3), we conclude that $\sigma'(f(\{x_1\}) == \sigma'(f(\{x_2\}))$, i.e. $(f(\{x_1\}, f(\{x_2\}) \in \mathrm{Za}(\sigma')$. This proves that $\widehat{\mathrm{Za}}((f, f)) = f \in \mathrm{Arr}_{\mathbf{EQ}}((\Omega, \mathrm{Za}(\sigma)), (\Omega', \mathrm{Za}(\sigma')))$.

Let us analyze the correspondence $\widehat{\mathrm{Ta}}$. Let $(\Omega, \mathcal{R}), (\Omega', \mathcal{R}') \in \mathrm{Obj}(\mathbf{EQ})$ and $f \in \mathrm{Arr}_{\mathbf{EQ}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$. Take $X \in \wp(\Omega)$. Then we have

$$\begin{aligned}(\mathrm{Ta}(\mathcal{R}') \circ \wp f)(X) &= \bigcup \{[f(x)]_{\mathcal{R}'} \mid x \in X\},\\(\wp f \circ \mathrm{Ta}(\mathcal{R}))(X) &= \wp f\Big(\bigcup \{[x]_{\mathcal{R}} \mid x \in X\}\Big).\end{aligned}$$

At this point, we take an element $y' \in (\wp f \circ \mathrm{Ta}(\mathcal{R}))(X)$. Clearly there exist $x, y \in X$ such that $y \in [x]_{\mathcal{R}}$ and $y' = f(y)$. Therefore, since $f \in \mathrm{Arr}_{\mathbf{EQ}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$, it follows that $(f(x), f(y)) \in \mathcal{R}'$, whence $y' \in (\mathrm{Ta}(\mathcal{R}') \circ \wp f)(X)$. Then $\widehat{\mathrm{Ta}}(f) = (f, f) \in \mathrm{Arr}_{\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}}((\Omega, \mathrm{Ta}(\sigma)), (\Omega', \mathrm{Ta}(\sigma')))$. □

In the next result we prove that **EQ** is a reflective subcategory of $\mathbf{SR^0}$.

Proposition 5.5 **EQ** *is a reflective subcategory of* $\mathbf{SR^0}$.

Proof Let $(\Omega, \mathcal{R}) \in \mathbf{SR^0}$ and consider the transitive closure $\widehat{\mathcal{R}}$ of the binary relation

$$\mathcal{R} \cup \{(x, x) \in \Omega \times \Omega \mid x \in \Omega\} \cup \{(z, w) \in \Omega \times \Omega \mid (w, z) \in \mathcal{R}\}.$$

Then $\mathrm{Id}_\Omega \in \mathrm{Arr}((\Omega, \mathcal{R}), (\Omega, \widehat{\mathcal{R}}))$ and it is immediate to check that $((\Omega, \widehat{\mathcal{R}}), \mathrm{Id}_\Omega)$ is the wanted reflection. □

As an immediate corollary we get the following result.

Corollary 5.6 *The following conditions hold:*

(i) **EQ** *is bicomplete;*
(ii) **SP** *and* $\mathbf{ESO}^{\mathbf{1},\sqsubseteq,=}$ *are bicomplete.*

Proof (i): It follows by Theorem 4.2 and by Proposition 5.5.

(ii): It is an immediate consequence of Theorem 5.4 and of the above part (i). □

6 Closure set operators, closure set systems and dependence set relations

In this section we consider the pairs (Ω, σ), where Ω is a fixed ground set and $\sigma \in \mathrm{CSO}(\Omega)$, as the objects of two specific categories, that we denote respectively by $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$.

More in detail, we must introduce two different conditions in order to define the previous two categories. Using the two categories $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ we prove the main result (Theorem 6.6) of this section. In such a result we obtain a categorical version of the usual bijective correspondence between closure set operators and closure set systems, and, moreover, provide an isomorphism between one among the previous categories of closure set operators and a specific kind of binary relations (called *dependence relations*) introduced formally in [14].

Let therefore Ω be a fixed arbitrary set. We recall the notion of dependence relation on Ω.

Definition 6.1 ([14]) We call any set relation $\mathcal{R} \in \mathrm{SR}(\Omega)$ a dependence set relation on Ω if the following conditions hold:

(D1) if $(Y, X) \in \wp(\Omega) \times \wp(\Omega)$ and $Y \subseteq X$, then $(Y, X) \in \mathcal{R}$;
(D2) if $(Z, Y), (Y, X) \in \mathcal{R}$ then $(Z, X) \in \mathcal{R}$;
(D3) $(Y, X) \in \mathcal{R}$ if and only if $(\{y\}, X) \in \mathcal{R}$ for any $y \in Y$.

We denote by $\mathrm{DSR}(\Omega)$ the family of all dependence relations on Ω.

The proof of the next result is easy and we leave it to the reader.

Proposition 6.2 *The following conditions hold:*

(i) $\mathcal{R} \in \mathrm{DSR}(\Omega)$ *if and only if it satisfies* (D1), (D2) *and if* $\{(Y_i, X) \mid i \in I\} \subseteq \mathcal{R}$ *then* $\left(\bigcup\{Y_i \mid i \in I\}, X\right) \in \mathcal{R}$*;*
(ii) *if* $\mathcal{R} \in \mathrm{DSR}(\Omega)$ *and* $\{(Z_i, W_i) \mid i \in I\} \subseteq \mathcal{R}$ *then* $\left(\bigcup\{Z_i \mid i \in I\}, \bigcup\{W_i \mid i \in I\}\right) \in \mathcal{R}$*;*
(iii) *if* $\mathfrak{E} \subseteq \mathrm{DSR}(\Omega)$ *then* $\bigcap \mathfrak{E} \in \mathrm{DSR}(\Omega)$.

If $\mathcal{R} \in \mathrm{SR}(\Omega)$, we set

$$\mathcal{R}^+ := \bigcap \left\{\mathcal{E} \in \mathrm{DSR}(\Omega) \mid \mathcal{R} \subseteq \mathcal{E}\right\}.$$

Notice that $\mathcal{R}^+$ is the smallest dependence relation on Ω containing $\mathcal{R}$ as a subfamily. We say that $\mathcal{R}^+$ the *dependence closure* of $\mathcal{R}$.

In order to prepare the proof of Theorem 6.6 we first introduce the following new linking maps:

- $\eta \colon \mathrm{SR}(\Omega) \longrightarrow \mathrm{SO}(\Omega)$ and $\mathrm{Cl} \colon \mathrm{SS}(\Omega) \longrightarrow \mathrm{SO}(\Omega)$, defined respectively by:

$$\forall X \in \wp(\Omega) \quad \left[\eta_{\mathcal{R}}(X) := \bigcup \{Y \in \wp(\Omega) \mid (Y, X) \in \mathcal{R}\}\right], \tag{12}$$

$$\forall X \in \wp(\Omega) \quad \left[\mathrm{Cl}_{\mathcal{F}}(X) := \bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\}\right]; \tag{13}$$

- $\Phi\colon \mathrm{SO}(\Omega) \longrightarrow \mathrm{SR}(\Omega)$ and $\mathrm{Fix}\colon \mathrm{SO}(\Omega) \longrightarrow \mathrm{SS}(\Omega)$, defined respectively by:

$$\forall \sigma \in \mathrm{SO}(\Omega) \ \left[\Phi(\sigma) := \{(Z, W) \in \wp(\Omega) \times \wp(\Omega) \mid Z \subseteq \sigma(W)\}\right],$$
$$\forall \sigma \in \mathrm{SO}(\Omega) \ \left[\mathrm{Fix}(\sigma) := \{X \in \wp(\Omega) \mid \sigma(X) = X\}\right].$$

In the next result (whose simple proof is left to the reader) we establish the basic properties of the above linking maps.

Proposition 6.3 *Let* $\mathcal{R} \in \mathrm{DSR}(\Omega)$ *and* $X \in \wp(\Omega)$. *Then:*

(i) $\eta_{\mathcal{R}}(X) = \max\{W \in \wp(\Omega) \mid (W, X) \in \mathcal{R}\}$;
(ii) $\eta_{\mathcal{R}} \in \mathrm{CSO}(\Omega)$;
(iii) $\mathcal{R}^{+} = \mathrm{Cl}_{\mathrm{DSR}(\Omega)}(\mathcal{R})$.

The next result is preliminary to the proof of Theorem 6.6. Here, we frame the collections $\mathrm{DSR}(\Omega)$, $\mathrm{CSS}(\Omega)$ and $\mathrm{CSO}(\Omega)$ within the formalism of sub-bijections, introduced in Definition 1.2.

Proposition 6.4 *We have that*

$$(\mathrm{SS}(\Omega) \mid \mathrm{CSS}(\Omega)) \underset{\mathrm{Fix}}{\overset{\mathrm{Cl}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{CSO}(\Omega)) \underset{\eta}{\overset{\Phi}{\rightleftarrows}} (\mathrm{SR}(\Omega) \mid \mathrm{DSR}(\Omega))$$

is an $(\mathrm{SS}(\Omega), \mathrm{SO}(\Omega), \mathrm{SR}(\Omega))$*-sub-bijection.*

Proof Using a classical result of Birkhoff concerning closure set systems and closure set operators (see [3]) it results that

$$(\mathrm{SS}(\Omega) \mid \mathrm{CSS}(\Omega)) \underset{\mathrm{Fix}}{\overset{\mathrm{Cl}}{\rightleftarrows}} (\mathrm{SO}(\Omega) \mid \mathrm{CSO}(\Omega))$$

is an $(\mathrm{SS}(\Omega), \mathrm{SO}(\Omega))$-sub-bijection. Therefore it remains to show that also

$$(\mathrm{SO}(\Omega) \mid \mathrm{CSO}(\Omega)) \underset{\eta}{\overset{\Phi}{\rightleftarrows}} (\mathrm{SR}(\Omega) \mid \mathrm{DSR}(\Omega))) \tag{14}$$

is an $(\mathrm{SO}(\Omega), \mathrm{SR}(\Omega))$-sub-bijection.

To this regard, since $\mathcal{R} \in \mathrm{DSR}(\Omega)$, in view of Proposition 6.3 (ii) we deduce that $\eta_{\mathcal{R}} \in \mathrm{CSO}(\Omega)$. Let now $\sigma \in \mathrm{CSO}(\Omega)$. We first show that $\Phi(\sigma) \in \mathrm{DSR}(\Omega)$, namely that the set relation $\Phi(\sigma)$ satisfies the properties (D1), (D2) and (D3). With regard to both the properties (D1) and (D2), they follow respectively by the extensiveness and the monotonicity of the set operator σ.

With regard to the property (D3), let $X, Y \in \wp(\Omega)$ and $y \in \Omega$. Then we get $(Y, X) \in \Phi(\sigma)$ if and only if $(\{y\}, X) \in \Phi(\sigma)$ for each $y \in Y$. In fact, notice that $(Y, X) \in \Phi(\sigma)$ if and only if $Y \subseteq \sigma(X)$, that is equivalent to say that $y \in \sigma(X)$ for each $y \in Y$, i.e. $(\{y\}, X) \in \Phi(\sigma)$ for any $y \in Y$. Hence (D3) holds.

Let now $\mathcal{R} \in \mathrm{DSR}(\Omega)$. We claim that $\Phi(\eta_{\mathcal{R}}) = \mathcal{R}$. To this end, if $(Y, X) \in \mathcal{R}$, then $Y \subseteq \eta_{\mathcal{R}}(X)$ by (12), whence $(Y, X) \in \Phi(\eta_{\mathcal{R}})$. Conversely, if $(Y, X) \in \Phi(\eta_{\mathcal{R}})$, then $Y \subseteq \eta_{\mathcal{R}}(X)$. Therefore, for any element $y \in Y$ there exists $Z_y \in \wp(\Omega)$ for which $y \in Z_y$ and $(Z_y, X) \in \mathcal{R}$. Consequently, since $\mathcal{R} \in DSR(\Omega)$, by (D3) it follows that $(\{y\}, X) \in \mathcal{R}$ for each $y \in Y$. Thus, again by (D3) we deduce that $(Y, X) \in \mathcal{R}$.

At this point, notice that $\eta_{\Phi(\sigma)} = \sigma$. In fact, if $X \in \wp(\Omega)$, we have that

$$\begin{aligned}\eta_{\Phi(\sigma)}(X) &= \bigcup \{Y \in \wp(\Omega) \mid (Y, X) \in \Phi(\sigma)\}\\ &= \bigcup \{Y \in \wp(\Omega) \mid Y \subseteq \sigma(X)\} = \sigma(X).\end{aligned}$$

Hence (14) is an $(SO(\Omega), SR(\Omega))$-sub-bijection on Ω. □

We now introduce the following new categories of set operators (for brevity we provide explicitly only the definition concerning $\sqsubseteq$, because the corresponding definition with $\sqsupseteq$ is similar):

- the full subcategory $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ of $\mathbf{SO}^{\mathbf{1},\sqsubseteq,=}$, where $\mathrm{Obj}(\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}) := \{(\Omega, \sigma) \in \mathrm{Obj}(\mathbf{SO}^{\mathbf{1},\sqsubseteq,=}) \mid \sigma \in CSO(\Omega)\}$.

We will use closure set systems and dependence relations as objects of the following two categories:

- the full subcategory $\mathbf{CSS^1}$ of $\mathbf{SS^1}$, where $\mathrm{Obj}(\mathbf{CSS^1}) := \{(\Omega, \mathcal{F}) \in \mathrm{Obj}(\mathbf{SS^1}) \mid \mathcal{F} \in CSS(\Omega)\}$;
- the full subcategory $\mathbf{DSR^1}$ of $\mathbf{SR^1}$, where $\mathrm{Obj}(\mathbf{DSR^1}) := \{(\Omega, \mathcal{R}) \mid \mathcal{R} \in DSR(\Omega)\}$.

In the next result we prove that the category $\mathbf{DSR^1}$ is complete.

Theorem 6.5 *The category* $\mathbf{DSR^1}$ *is complete.*

Proof In view of [7, Theorem 2.8.1], we must prove the existence of products and equalizers in $\mathbf{DSR^1}$. To this regard, first take $\{(\Omega_i, \mathcal{R}_i) \mid i \in I\} \subseteq \mathrm{Obj}(\mathbf{DSR^1}) \subseteq \mathrm{Obj}(\mathbf{SR^1})$. Consider their product $(\Omega, \mathcal{B})$ in the category $\mathbf{SR^1}$, computed in the proof of Theorem 4.2. As $\mathbf{DSR^1}$ is a full subcategory of $\mathbf{SR^1}$, it suffices to check that $(\Omega, \mathcal{B}) \in \mathrm{Obj}(\mathbf{DSR^1})$.

To this end, let first $(Y, X) \in \wp(\Omega) \times \wp(\Omega)$ be such that $Y \subseteq X$. Then, as $\wp\pi_i$ is increasing and $\mathcal{R}_i \in DSR(\Omega_i)$ for any $i \in I$, we get $(\wp\pi_i(Y), \wp\pi_i(X)) \in \mathcal{R}_i$ for all $i \in I$, so that $(Y, Z) \in \mathcal{B}$.

Suppose now that $(Z, W), (W, T) \in \mathcal{B}$. We claim that $(Z, T) \in \mathcal{B}$. To this regard, just observe that our assumptions imply that $(\wp\pi_i(Z), \wp\pi_i(W)), (\wp\pi_i(W), \wp\pi_i(T)) \in \mathcal{R}_i$ for each $i \in I$. Hence $(\wp\pi_i(Z), \wp\pi_i(T)) \in \mathcal{R}_i$ for each $i \in I$ because $\mathcal{R}_i \in DSR(\Omega_i)$. This proves that $(Z, T) \in \mathcal{B}$.

Finally let us prove that $(Z, W) \in \mathcal{B}$ if and only if $(\{z\}, W) \in \mathcal{B}$ for each $z \in Z$. To this regard, using the fact that $\wp\pi_i(Z) = \bigcup\{\wp\pi_i(\{z\}) \mid z \in Z\}$, we easily get

$$\forall i \in I \ \left[(\wp\pi_i(Z), \wp\pi_i(W)) \in \mathcal{R}_i \iff \forall z \in Z\,[(\wp\pi_i(\{z\}), \wp\pi_i(W)) \in \mathcal{R}_i]\right].$$

This proves that $\mathcal{B} \in DSR(\Omega)$.

Let now $(\Omega, \mathcal{R}), (\Omega', \mathcal{R}') \in \mathrm{Obj}(\mathbf{DSR^1}) \subseteq \mathrm{Obj}(\mathbf{SR^1})$ and $f, g \in \mathrm{Arr}_{\mathbf{DSR^1}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$. As $\mathbf{DSR^1}$ is a full subcategory of $\mathbf{SR^1}$, we have $\mathrm{Arr}_{\mathbf{DSR^1}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}')) = \mathrm{Arr}_{\mathbf{SR^1}}((\Omega, \mathcal{R}), (\Omega', \mathcal{R}'))$.

Thus we can consider the equalizer $((\Omega_*, \mathcal{R}^*_{f,g}), \iota_*)$ of f and g in $\mathbf{SR^1}$ computed in the proof of Theorem 4.2. By a straightforward check, we easily deduce that $(\Omega_*, \mathcal{R}^*_{f,g}) \in \mathrm{Obj}(\mathbf{DSR^1})$. $\square$

The last tools to be introduced before proving the next Theorem 6.6 are the following correspondences:

- $\widehat{\mathrm{Fix}}: \mathbf{CSO}^{\mathbf{1},\sqsupseteq,=} \longrightarrow \mathbf{CSS^1}$, where $\widehat{\mathrm{Fix}}((\Omega,\sigma)) := (\Omega, \mathrm{Fix}(\sigma))$ and $\widehat{\mathrm{Fix}}(f,f) := f$, whenever $(\Omega,\sigma), (\Omega',\sigma') \in \mathrm{Obj}(\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=})$ and $(f,f) \in \mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}}((\Omega,\sigma), (\Omega',\sigma'))$;
- $\widehat{\mathrm{Cl}}: \mathbf{CSS^1} \longrightarrow \mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$, where $\widehat{\mathrm{Cl}}((\Omega,\mathcal{F})) := (\Omega, \mathrm{Cl}_{\mathcal{F}})$ and $\widehat{\mathrm{Cl}}(f) := (f,f)$, whenever $(\Omega,\mathcal{F}), (\Omega',\mathcal{F}') \in \mathrm{Obj}(\mathbf{CSS^1})$ and $f \in \mathrm{Arr}_{\mathbf{CSS^1}}((\Omega,\mathcal{F}), (\Omega',\mathcal{F}'))$;
- $\widehat{\eta}: \mathbf{DSR^1} \longrightarrow \mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$, where $\widehat{\eta}((\Omega,\mathcal{R})) := (\Omega, \eta_{\mathcal{R}})$ and $\widehat{\eta}(f) := (f,f)$, whenever $(\Omega,\mathcal{R}), (\Omega',\mathcal{R}') \in \mathrm{Obj}(\mathbf{DSR^1})$ and $f \in \mathrm{Arr}_{\mathbf{DSR^1}}((\Omega,\mathcal{R}), (\Omega',\mathcal{R}'))$;
- $\widehat{\Phi}: \mathbf{CSO}^{\mathbf{1},\sqsubseteq,=} \longrightarrow \mathbf{DSR^1}$, where $\widehat{\Phi}((\Omega,\sigma)) := (\Omega, \Phi(\sigma))$ and $\widehat{\Phi}(f,f) := f$, whenever $(\Omega,\sigma), (\Omega',\sigma') \in \mathrm{Obj}(\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=})$ and $(f,f) \in \mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}}((\Omega,\sigma), (\Omega',\sigma'))$.

Now we provide the proof of the main result of the present section.

Theorem 6.6 *The arrows of the diagrams*

$$\mathbf{CSS^1} \underset{\widehat{\mathrm{Fix}}}{\overset{\widehat{\mathrm{Cl}}}{\rightleftarrows}} \mathbf{CSO}^{\mathbf{1},\sqsupseteq,=} \quad \textit{and} \quad \mathbf{CSO}^{\mathbf{1},\sqsubseteq,=} \underset{\widehat{\eta}}{\overset{\widehat{\Phi}}{\rightleftarrows}} \mathbf{DSR^1}$$

are isomorphisms of categories.

Proof We must only show the four correspondences are well defined on arrows. Let $(\Omega,\sigma), (\Omega',\sigma') \in \mathrm{Obj}(\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=})$ and $(f,f) \in \mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}}((\Omega,\sigma), (\Omega',\sigma'))$. Let moreover $(X,Y) \in \Phi(\sigma)$. By the definitions of $\Phi(\sigma)$ and of the arrows in $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ it follows that $\wp f(X) \subseteq \wp f(\sigma(Y)) \subseteq \sigma'(\wp f(Y))$, whence $(\wp f(X), \wp f(Y)) \in \Phi(\sigma')$ and $\widehat{\Phi}(f,f) := f \in \mathrm{Arr}_{\mathbf{DSR^1}}((\Omega, \widehat{\Phi}(\sigma)), (\Omega', \widehat{\Phi}(\sigma')))$. Thus $\widehat{\Phi}$ is well defined.

We examine now $\widehat{\eta}$. To this end, let $(\Omega,\mathcal{R}), (\Omega',\mathcal{R}') \in \mathrm{Obj}(\mathbf{DSR^1})$ and $f \in \mathrm{Arr}_{\mathbf{DSR^1}}((\Omega,\mathcal{R}), (\Omega',\mathcal{R}'))$. Let moreover $X \in \wp(\Omega)$. Then, by Proposition 6.3 (i) and since $\mathcal{R} \in \mathrm{DSR}(\Omega)$, it follows that $(\eta_{\mathcal{R}}(X), X) \in \mathcal{R}$. Hence, in view of the definition of the arrows in the category $\mathbf{DSR^1}$, it results that $(\wp f(\eta_{\mathcal{R}}(X)), \wp f(X)) \in \mathcal{R}'$. Again, by Proposition 6.3 (i) and since $\mathcal{R}' \in \mathrm{DSR}(\Omega')$, we also obtain that $\wp f(\eta_{\mathcal{R}}(X)) \subseteq \eta_{\mathcal{R}'}(\wp f(X))$, whence $\wp f \circ \eta_{\mathcal{R}} \sqsubseteq \eta_{\mathcal{R}'} \circ \wp f$ in view of the arbitrariness of X. This proves that $\widehat{\eta}(f) = (f,f) \in \mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}}((\Omega, \eta_{\mathcal{R}}), (\Omega', \eta_{\mathcal{R}'}))$. Therefore, also the correspondence $\widehat{\eta}$ is well defined.

We now prove that $\widehat{\mathrm{Fix}}: \mathbf{CSO}^{\mathbf{1},\sqsupseteq,=} \longrightarrow \mathbf{CSS^1}$ is well defined on arrows. Let therefore $(\Omega,\sigma), (\Omega',\sigma') \in \mathrm{Obj}(\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=})$ and $(f,f) \in \mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}}((\Omega,\sigma), (\Omega',\sigma'))$. Let moreover $X \in \mathrm{Fix}(\sigma)$ and set $Y := \wp f(X)$. By the definition of the arrows of $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ and our choice of X, it follows that

$$\sigma'(Y) = \sigma'(\wp f(X)) \subseteq \wp f(\sigma(X)) = \wp f(X) = Y.$$

On the other hand, since $\sigma' \in \mathrm{CSO}(\Omega')$, we also have $Y \subseteq \sigma'(Y)$, so that $\sigma'(Y) = Y$, i.e. $Y \in \mathrm{Fix}(\sigma')$. Therefore $\widehat{\mathrm{Fix}}(f, f) = f \in \mathrm{Arr}_{\mathbf{CSS^1}}((\Omega, \widehat{\mathrm{Fix}}(\sigma)), (\Omega', \widehat{\mathrm{Fix}}(\sigma')))$, i.e. $\widehat{\mathrm{Fix}}$ is well defined.

Let us examine $\widehat{\mathrm{Cl}}$. Let $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}') \in \mathrm{Obj}(\mathbf{CSS^1})$ and $f \in \mathrm{Arr}_{\mathbf{CSS^1}}((\Omega, \mathcal{F}), (\Omega', \mathcal{F}'))$. Since $\wp f$ is increasing and $\mathrm{Cl}_{\mathcal{F}'} \in \mathrm{CSO}(\Omega')$, in view of Proposition 6.4 and the definition of the arrows of $\mathbf{CSS^1}$, it follows that

$$\mathrm{Cl}_{\mathcal{F}'}(\wp f(X)) \subseteq \mathrm{Cl}_{\mathcal{F}'}(\wp f(\mathrm{Cl}_{\mathcal{F}}(X))) = \wp f(\mathrm{Cl}_{\mathcal{F}}(X))$$

for any $X \in \wp(\Omega)$. Thus $\widehat{\mathrm{Cl}}(f) = (f, f) \in \mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}}((\Omega, \mathrm{Cl}_{\mathcal{F}}), (\Omega', \mathrm{Cl}_{\mathcal{F}'}))$ and the correspondence $\widehat{\mathrm{Cl}}$ is also well defined. □

In the next result we establish further properties of the categories $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$, $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ and $\mathbf{CSS^1}$. In particular, we will see that the two categories $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ are deeply different because they are neither equivalent nor dually equivalent.

Proposition 6.7 *The following conditions hold:*

(i) *there is no initial object in* $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$*;*
(ii) $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ *is not cocomplete;*
(iii) *the categories* $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ *and* $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ *are not equivalent;*
(iv) *the categories* $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ *and* $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ *are not dually equivalent;*
(v) *the category* $\mathbf{CSS^1}$ *is not cocomplete;*
(vi) *the categories* $\mathbf{CSS^1}$ *and* $\mathbf{DSR^1}$ *are not equivalent;*
(vii) *the category* $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ *is complete.*

Proof (i): Assume by contradiction that (Λ, τ) is the initial object in $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$. We cannot have $\Lambda \neq \varnothing$. In fact, assume that $\Lambda \neq \varnothing$ and let $(\Omega, \sigma) \in \mathrm{Obj}(\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=})$, where Ω contains at least two elements and $\sigma(X) = \Omega$ for each $X \in \wp(\Omega)$, and let (f, f) be the only arrow in $\mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}}((\Lambda, \tau), (\Omega, \sigma))$. As $\wp f \circ \tau(X) \sqsupseteq \sigma \circ \wp f(X)$, taking $X = \Lambda$, we easily deduce that f is surjective. So $|\Lambda| \geqslant |\Omega|$ and we can choose the surjective function $f : \Lambda \longrightarrow \Omega$ in a more than one arbitrary way, contradicting the uniqueness of (f, f).

Thus $\Lambda = \varnothing$ and $\tau = \mathrm{Id}_{\wp(\varnothing)}$. Moreover, if $(\Omega, \sigma) \in \mathrm{Obj}(\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=})$ and (f, f) is the only arrow in $\mathrm{Arr}_{\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}}((\Lambda, \tau), (\Omega, \sigma))$, we also deduce f is the empty map. Nevertheless, the inclusion $\wp f \circ \tau \sqsupseteq \sigma \circ \wp f$ cannot hold if we take a non-empty set Ω and the set operator σ on Ω such that $\sigma(X) = \Omega$ for each $X \in \wp(\Omega)$. This shows that (Λ, τ) cannot exist.

(ii): It follows immediately by the above part (i) and by [7, Theorem 2.8.1].

(iii): In view of Proposition 4.7 (v), the category $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ admits an initial object. Now, if the categories $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ were equivalent, then even $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ would have an initial object, in contrast with the above part (i).

(iv): In view of Proposition 4.7 (iv), the category $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ admits a terminal object. Now, if the categories $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ were dually equivalent, then $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$ would admit an initial object, in contrast with the above part (i).

(v): It follows by the above part (ii) and by Theorem 6.6.

(vi): If $\mathbf{CSS^1}$ and $\mathbf{DSR^1}$ were equivalent, then by Theorem 6.6 we would get an equivalence between the categories $\mathbf{CSO}^{\mathbf{1},\sqsubseteq,=}$ and $\mathbf{CSO}^{\mathbf{1},\sqsupseteq,=}$, contradicting the above part (iii).

(vii): It follows immediately by Theorems 6.5 and 6.6. □

We conclude the present section proving next Theorem 6.11. With regard to this result, it is convenient first provide a new interpretation of the linking map Cl when the ground set Ω is substituted with $\wp(\Omega) \times \wp(\Omega)$. In fact, with such a choice, it is clear that $\mathrm{DSR}(\Omega) \in \mathrm{SS}(\wp(\Omega) \times \wp(\Omega))$. Consequently, in view of (13) it follows that $\mathrm{Cl}_{\mathrm{DSR}(\Omega)}$ is an element of $\mathrm{SO}(\wp(\Omega) \times \wp(\Omega))$, i.e.

$$\mathrm{Cl}_{\mathrm{DSR}(\Omega)} : \wp(\wp(\Omega) \times \wp(\Omega)) = \mathrm{SR}(\Omega) \longrightarrow \wp(\wp(\Omega) \times \wp(\Omega)) = \mathrm{SR}(\Omega).$$

Now, if we interpret $\mathrm{Cl}_{\mathrm{DSR}(\Omega)}$ as a linking map between $\mathrm{SR}(\Omega)$ into itself, it is also natural to consider its relationships with a linking map Ψ, from relations to set systems, that is implicitly used (although not explicitly defined) in several scopes of theoretical computer science (see for example [35]).

Formally, we introduce the linking map $\Psi : \mathrm{SR}(\Omega) \longrightarrow \mathrm{SS}(\Omega)$ defined by

$$\Psi(\mathcal{R}) := \big\{Z \in \wp(\Omega) \mid \forall (Y, X) \in \mathcal{R}[X \subseteq Z \implies Y \subseteq Z]\big\},$$

for any $\mathcal{R} \in \mathrm{SR}(\Omega)$.

The map Ψ enables us to highlight a fundamental aspect related to the sub-bijections obtained through linking maps. To this end, we consider both the linking maps Ψ and $\mathrm{Fix} \circ \eta$. Then, we have the following result, whose simple proof is left to the reader.

Proposition 6.8 *The following conditions hold:*

(i) $\mathrm{Fix}(\eta_{\mathcal{R}}) = \Psi(\mathcal{R})$ *for any* $\mathcal{R} \in \mathrm{DSR}(\Omega)$*;*
(ii) $\Psi(\mathcal{R})$ *is a closure set system on* Ω*, for any* $\mathcal{R} \in \mathrm{SR}(\Omega)$*;*
(iii) *there exists* $\mathcal{R}^{\S} \in \mathrm{SR}(\Omega)$ *such that* $(\mathrm{Fix} \circ \eta)(\mathcal{R}^{\S}) \notin \mathrm{CSS}(\Omega)$.

In view of Proposition 6.8 we observe that the linking maps Ψ and $\mathrm{Fix} \circ \eta$ differ from each other. In fact, by Proposition 6.8 (ii) it follows that Ψ maps any set relation on Ω to a closure set system on the same ground set, while the same does not hold for the linking map $\mathrm{Fix} \circ \eta$, as one can see by Proposition 6.8 (iii).

In addition, by Proposition 6.8 (i) we can also notice that the restrictions of Ψ and $\mathrm{Fix} \circ \eta$ to the collection of all dependence relations on Ω agree. From this and Proposition 6.4, we then obtain the two following distinct $(\mathrm{SR}(\Omega), \mathrm{SS}(\Omega))$-sub-bijections:

$$(\mathrm{SR}(\Omega) \mid \mathrm{DSR}(\Omega)) \xrightarrow[\xleftarrow[\Phi \circ \mathrm{Cl}]{}]{\mathrm{Fix} \circ \eta \quad \Psi} (\mathrm{SS}(\Omega) \mid \mathrm{CSS}(\Omega)).$$

In the next Theorem 6.11 we will establish the natural commutativities between the linking maps Fix, Ψ and $\mathrm{Cl}_{\mathrm{DSR}(\Omega)}$ when Ω is a fixed finite ground set. To this regard it is first necessary to introduce two linking maps $\Theta, \Gamma : \mathrm{SR}(\Omega) \longrightarrow \mathrm{SO}(\Omega)$ defined by:

$$\forall \mathcal{R} \in \mathrm{SR}(\Omega), \forall X \in \wp(\Omega)$$
$$\Big[\Theta_{\mathcal{R}}(X) := X \cup \bigcup \{\eta_{\mathcal{R}}(Y) \mid Y \subseteq X\} \text{ and } \Gamma_{\mathcal{R}}(X) := \bigcup \{\Theta^k_{\mathcal{R}}(X) \mid k \in \mathbb{N}\}\Big].$$

With regard to the above linking maps, we get the following results.

Proposition 6.9 *Let $\mathcal{R} \in \mathrm{SR}(\Omega)$, $k \geqslant 0$, $X \in \wp(\Omega)$ and $Y \subseteq \Theta^k_{\mathcal{R}}(X)$. Then $(Y, X) \in \mathcal{R}^+$.*

Proof Use induction on k and the definition of dependence relation, $\mathcal{R}^+$ and of the set operator Θ. □

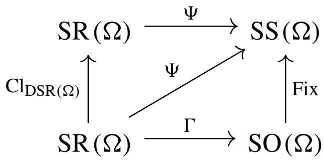
Proposition 6.10 *For any $\mathcal{R} \in \mathrm{SR}(\Omega)$, the following conditions hold:*

(i) $\eta_{\mathcal{R}} \sqsubseteq \Theta_{\mathcal{R}} \sqsubseteq \Gamma_{\mathcal{R}}$;
(ii) $\Gamma_{\mathcal{R}} \in \mathrm{CSO}(\Omega)$.

Proof Straightforward. □

At this point we are able to prove the last result of this section.

Theorem 6.11 *Let Ω be a fixed set. Then the diagram*



commutes, i.e. $\Psi \circ \mathrm{Cl}_{\mathrm{DSR}(\Omega)} = \Psi = \mathrm{Fix} \circ \Gamma$.

Proof Let $\mathcal{R} \in \mathrm{SR}(\Omega)$ be fixed. The commutativity of the above diagram is equivalent to show that $\Psi(\mathcal{R}^+) = \Psi(\mathcal{R}) = \mathrm{Fix}(\Gamma_{\mathcal{R}})$. We want first show that

$$\mathrm{Fix}(\Gamma_{\mathcal{R}}) = \Psi(\mathcal{R}^+). \tag{15}$$

To this end, we need to show that

$$\Phi \circ \Gamma = \mathrm{Cl}_{\mathrm{DSR}(\Omega)} \quad \text{and} \quad \eta \circ \mathrm{Cl}_{\mathrm{DSR}(\Omega)} = \Gamma. \tag{16}$$

Let us check that

$$\mathcal{R}^+ \subseteq \Phi(\Gamma_{\mathcal{R}}). \tag{17}$$

To this regard, take $(Y, X) \in \mathcal{R}$. First note that $Y \subseteq \Gamma_{\mathcal{R}}(X)$ by (12) and by Proposition 6.10 (i), so that $\mathcal{R} \subseteq \Phi(\Gamma_{\mathcal{R}})$. By the definition of $\mathcal{R}^+$, it is now sufficient to check that

$\Phi(\Gamma_{\mathcal{R}}) \in DSR(\Omega)$ and this is a consequence of Proposition 6.4 because $\Gamma_{\mathcal{R}} \in CSO(\Omega)$ by Proposition 6.10 (ii). So (17) holds.

We now show that $\eta_{\mathcal{R}^+} = \Gamma_{\mathcal{R}}$. To this regard, fix $X \in \wp(\Omega)$.

As $\Phi(\Gamma_{\mathcal{R}}) \in DSR(\Omega)$ and $\Gamma_{\mathcal{R}} \in CSO(\Omega)$, by Proposition 6.4 we have that $\Gamma_{\mathcal{R}} = \eta_{\Phi(\Gamma_{\mathcal{R}})}$. Therefore, by (17) and Proposition 6.3 (i), we easily get the inclusion $\eta_{\mathcal{R}^+}(X) \subseteq \Gamma_{\mathcal{R}}(X)$.

Conversely, let $y \in \Gamma_{\mathcal{R}}(X) = \bigcup\{\Theta^k_{\mathcal{R}}(X) \mid k \in \mathbb{N}\}$. Then either $y \in X$ or there exists a minimum integer $s \geqslant 1$ such that $y \in \Theta^s_{\mathcal{R}}(X)$. If $y \in X$, then we get $(\{y\}, X) \in \mathcal{R}^+$, whence $y \in \eta_{\mathcal{R}^+}(X)$.

Assume therefore the existence of a minimum integer $s \geqslant 1$ such that

$$y \in \Theta^s_{\mathcal{R}}(X) = \Theta^{s-1}_{\mathcal{R}}(X) \cup \bigcup\{\eta_{\mathcal{R}}(W) \mid W \subseteq \Theta^{s-1}_{\mathcal{R}}(X)\}.$$

In view of the minimality of s, there exists $Z \subseteq \Theta^{s-1}_{\mathcal{R}}(X)$ such that $y \in \eta_{\mathcal{R}}(Z)$. Thus, by (12), there exists $Z_y \in \wp(\Omega)$ such that $y \in Z_y$ and $(Z_y, Z) \in \mathcal{R} \subseteq \mathcal{R}^+$. Now, as $Z \subseteq \Theta^{s-1}_{\mathcal{R}}(X)$, by Proposition 6.9 it follows that $(Z, X) \in \mathcal{R}^+$. So, using (D2) on the pairs (Z_y, Z) and (Z, X), we get $(Z_y, X) \in \mathcal{R}^+$, whence $y \in \eta_{\mathcal{R}^+}(X)$. Therefore we obtain the inclusion $\Gamma_{\mathcal{R}}(X) \subseteq \eta_{\mathcal{R}^+}(X)$, so that $\eta_{\mathcal{R}^+} = \Gamma_{\mathcal{R}} = \eta_{\Phi(\Gamma_{\mathcal{R}})}$. Then we also get the equality $\mathcal{R}^+ = \Phi(\Gamma_{\mathcal{R}})$ by Proposition 6.4. This proves that (16) holds.

At this point, we can conclude the proof of (15). In fact, as $\mathcal{R}^+ \in DSR(\Omega)$, we get our conclusion using (16) and Proposition 6.8 (i).

In order to complete the proof, let us finally show that $\Psi(\mathcal{R}) = \Psi(\mathcal{R}^+)$. To this end, note first that $\Psi(\mathcal{R}^+) \subseteq \Psi(\mathcal{R})$ since $\mathcal{R} \subseteq \mathcal{R}^+$ and the map Ψ is decreasing with respect to set-theoretical inclusion. Take now $Z \in \Psi(\mathcal{R})$. We claim that $Z \in \Psi(\mathcal{R}^+)$. To this end, let us check that $Z \in Fix(\Theta_{\mathcal{R}})$. By the definition of $\Theta_{\mathcal{R}}$, we clearly have $Z \subseteq \Theta_{\mathcal{R}}(Z)$. Conversely, let $w \in \Theta_{\mathcal{R}}(Z)$. Then there exists $Y \in \wp(Z)$ such that $w \in \eta_{\mathcal{R}}(Y)$. At this point, by the definition of $\eta_{\mathcal{R}}$, we get $w \in W$, for some $W \in \wp(\Omega)$ such that $(W, Y) \in \mathcal{R}$. Since $Z \in \Psi(\mathcal{R})$ and $Y \subseteq Z$, we deduce that $W \subseteq Z$, whence $w \in Z$. Thus $\Theta_{\mathcal{R}}(Z) \subseteq Z$ and, hence, $Z \in Fix(\Theta_{\mathcal{R}})$. Now, using the fact that $Fix(\Theta_{\mathcal{R}}) = Fix(\Gamma_{\mathcal{R}})$, we deduce that $\Gamma_{\mathcal{R}}(Z) = Z$, whence $Z \in \Psi(\mathcal{R}^+)$ by virtue of (15). This proves that $\Psi(\mathcal{R}) = \Psi(\mathcal{R}^+)$. □

7 Conclusions

In this paper we introduced new categories $\mathbf{SS^k}$, $\mathbf{SR^k}$, $\mathbf{SO^{k,\sqsubseteq}}$, $\mathbf{SO^{k,\sqsupseteq}}$ and $\mathbf{SO^k}$ generalizing at a categorical level the notions of set system, set relation and set operator, respectively. These categories depend on a nonnegative parameter k, on the basis of which the objects and arrows of the above categories were defined by means of k successive compositions of the classic powerset functor.

In the first part of the paper we studied the basic properties of these categories. A particular attention has been addressed towards the analysis of the main properties of the previous categories; among the most important, we proved for instance the bi-completeness and Cartesian closedness of $\mathbf{SS^k}$ and $\mathbf{SR^k}$.

In the second part of the work we extended both the bijection between equivalence relations, set partitions and equivalence set operators, and also that between closure set operators, closure set systems and dependence set relations, determining more general categorical isomorphisms. In order to establish the aforementioned isomorphisms, we have also introduced and studied in detail suitable relevant subcategories of the main categories $\mathbf{SS^k}$, $\mathbf{SR^k}$, $\mathbf{SO}^{\mathbf{k},\sqsubseteq}$, $\mathbf{SO}^{\mathbf{k},\sqsupseteq}$ and $\mathbf{SO^k}$.

The study of these sub-categories may be seen as the starting point of the development of an articulated framework, which we believe is worthy of being investigated even in successive works.

The future research perspectives appear to be quite promising, because of the novelty of these categories and since we can study them relatively with topics typical of various sectors of both pure mathematics (e.g. commutative algebra, topology and matroid theory, to name few specific examples), and of theoretical computer science (e.g. granular and soft computing, rough set theory, or functional programming).

Furthermore, the complete investigation of such categories is far to be reached, because most of the usual properties of the aforementioned categories have not been yet analyzed. So, many possible open problems (which we believe to be not appropriate to list in this short final section) occur and this situation may constitute an interesting future research perspective.

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Declarations

Conflict of interest The authors declare no competing interests.

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