Arithmetic and topology of classical structures associated with plane quartics

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Abstract

We consider moduli spaces of plane quartics marked with various structures such as Cayley octads, Aronhold heptads, Steiner complexes and Göpel subsets and determine their cohomology. This answers a series of questions of Jesse Wolfson. We also count points of these moduli spaces over finite fields of odd characteristic.

1 Introduction

A plane quartic is a smooth curve C given by the vanishing of a homogeneous polynomial of degree 4 in three variables. The most famous result about plane quartics is the following.

Theorem 1.1 (Plücker [42], Jacobi [33]) *Every plane quartic has* 28 *bitangents.*

There are many structures beyond bitangents which one can attach to a plane quartic. Many of these structures were discovered already in the 19th century and have somewhat mysterious names such as Cayley octads, Aronhold heptads, Steiner complexes and Göpel subsets.

Given such a structure, there is a corresponding moduli space of plane quartics marked with this structure. For instance, there is a moduli space $\mathcal{Q}_{\mathrm{btg}}$ of plane quartics marked with a bitangent line and a moduli space $\mathcal{Q}_{\mathrm{CO}}$ of plane quartics marked with a Cayley octad. Jesse Wolfson posed the following question.

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Question 1.2 (Wolfson [32]) Given a structure associated with plane quartics (such as bitangents, Cayley octads, Aronhold heptads or Steiner complexes) what is the cohomology of the corresponding moduli space?

From a modern point of view, the above question is best understood in terms of curves with level 2 structure. More precisely, given a plane quartic C with level 2 structure, the group $Sp(6, \mathbb{F}_2)$ acts on the bitangents of C via changing the level 2 structure. The classical structures, such as Cayley octads and Steiner complexes, can then be understood as sets of bitangents which are fixed by some subgroup of $Sp(6, \mathbb{F}_2)$. In fact, in most cases the corresponding subgroup is maximal (the exception being Aronhold heptads). Moreover, fixed structures of maximal subgroups of $Sp(6, \mathbb{F}_2)$ have been studied classically.

The cohomology groups of the moduli space $\mathcal{Q}[2]$ of plane quartics with level 2 structure have been determined as representations of $Sp(6, \mathbb{F}_2)$ in earlier work by the author [6–9] (see also [5, 10, 11, 13]). The purpose of the present paper is the following:

- To explain how the answer to Wolfsons question can be derived from our previous work.
- To recall definitions of various structures associated to plane quartics and provide references to the classical literature.
- To recall descriptions of the various subgroups stabilizing the above structures.
- To provide point counts over finite fields (of odd characteristic) of the above moduli spaces.

Plane quartics are closely related to both Del Pezzo surfaces of degree 2 and maximally nodal double Veronese cones, see [1]. The above structures all have counterparts in these settings. Thus, most results in this note have counterparts in terms of Del Pezzo surfaces and Veronese cones (e.g. refinements of results in [4, 38] in the sense of [12, 16]). However, the precise nature of these structures and results are more subtle than one might naively expect. We have therefore not pursued these results in this paper. Another direction for future study is to investigate how the corresponding results look like for the various compactifications of $\mathcal{Q}[2]$ that have been constructed; e.g. in [31, 43, 44]. It would also be very interesting to investigate the relationship between the approach of the present paper with other recent developments in the theory of plane quartics and their bitangents; e.g. the tropical counts of Baker, Len, Morrison, Pflueger and Ren [3] and the signed counts of Larson and Vogt [35].

The paper contains more than a few moduli spaces. For convenience, we provide a table listing the most important ones below. We also tabulate the most central results about these moduli spaces.

List of moduli spaces of plane quartics

Notation	Structure
$\mathcal{Q}[2]$	Level 2 structure
$\mathcal{Q}_{btg}$	Bitangent
$\mathcal{Q}_{CO}$	Cayley octad
$\mathcal{Q}_{AH}$	Aronhold heptad
$\mathcal{Q}_{SC}$	Steiner complex
$\mathcal{Q}_{GS}$	Göpel subset
$\mathcal{Q}_{syz}$	Syzygetic tetrad
$\mathcal{Q}_{azy}$	Azygetic triad
$\mathcal{Q}_{enn}$	Ennead
$\mathcal{Q}_{OS}$	Octonionic structure

Poincaré polynomials and point counts

Moduli space	Poincaré polynomial	Point count
$\mathcal{Q}_{btg}$	$1 + t^5 + 2t^6$	$q^6 - q + 2$
$\mathcal{Q}_{CO}$	$1 + t + t^5 + 4t^6$	$q^6 - q^5 - q + 4$
$\mathcal{Q}_{AH}$	$1 + t + t^3 + 4t^4 + 6t^5 + 6t^6$	$q^6 - q^5 - q^3 + 4q^2 - 6q + 6$
$\mathcal{Q}_{SC}$	$1 + t + 2t^5 + 5t^6$	$q^6 - q^5 - 2q + 5$
$\mathcal{Q}_{GS}$	$1 + t + 2t^5 + 11t^6$	$q^6 - q^5 - 2q + 11$
$\mathcal{Q}_{syz}$	$1 + t + t^4 + 7t^5 + 13t^6$	$q^6 - q^5 + q^2 - 7q + 13$
$\mathcal{Q}_{azy}$	$1 + t + t^3 + 3t^4 + 8t^5 + 9t^6$	$q^6 - q^5 - q^3 + 3q^2 - 8q + 9$
$\mathcal{Q}_{enn}$	$1 + 3t^3 + 11t^4 + 13t^5 + 11t^6$	$q^6 - 3q^3 + 11q^2 - 13q + 11$
$\mathcal{Q}_{OS}$	$1 + 2t^5 + 7t^6$	$q^6 - 2q + 7$

2 Background

We work over an alebraically closed field k of characteristic different from 2.

2.1 Symplectic level 2 structures

Let C be a smooth, irreducible and projective curve of genus g over k and let $\mathrm{Jac}(C)$ be its Jacobian. We will only consider group theoretic properties of $\mathrm{Jac}(C)$ so we make the identifications

$$\mathrm{Jac}(C) = \mathrm{Pic}^0(C) = \mathrm{Cl}^0(C).$$

If $D \in \mathrm{Cl}(C)$, we denote the corresponding line bundle by $\mathcal{L}(D)$ and we use the notation $h^n(D)$ for the dimension of $H^n(C, \mathcal{L}(D))$.

Let $V = \mathrm{Jac}(C)[2]$ denote the 2-torsion subgroup of $\mathrm{Jac}(C)$. This group is a vector space over $\mathbb{F}_2$ of dimension $2g$. The Weil pairing b is a symplectic bilinear form on V.

Definition 2.1 A *symplectic level* 2 *structure* on a curve C of genus g is an isometry ϕ from the standard symplectic vector space of dimension $2g$ to (V, b).

The group of isometries from (V, b) to itself is called the *symplectic group* of V and is denoted $\mathrm{Sp}(V)$. The symplectic group of the standard symplectic vector space of dimension $2g$ over $\mathbb{F}_2$ is denoted $\mathrm{Sp}(2g, \mathbb{F}_2)$.

Equivalently, a symplectic level 2 structure is a choice of an (ordered) symplectic basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ of V. We will write (C, ϕ) and $(C, x_1, \ldots, x_g, y_1, \ldots, y_g)$ interchangeably. Two curves with level 2 structures $(C, x_1, \ldots, y_g)$ and $(C', x_1', \ldots, y_g')$ are isomorphic if there is an isomorphism of curves $\varphi \colon C \to C'$ such that the induced morphism $\widetilde{\varphi} \colon V' \to V$ takes one symplectic basis to the other in the sense that

$$\widetilde{\varphi}(x_i') = x_i, \quad i = 1, \ldots, g,$$
$$\widetilde{\varphi}(y_i') = y_i, \quad i = 1, \ldots, g.$$

Let K_C denote the canonical class of C. A *theta characteristic* on C is a divisor class θ such that $2\theta = K_C$. The *Arf invariant* of a theta characteristic θ is defined as

$$\mathrm{Arf}(\theta) = h^0(\theta) \mod 2.$$

A theta characteristic θ is called *odd* (resp. *even*) if $\mathrm{Arf}(\theta) = 1$ (resp. 0). A curve C of genus g has precisely 2^{2g} theta characteristics. Of these, $2^{g-1}(2^g - 1)$ are odd and $2^{g-1}(2^g + 1)$ are even. The set Θ of theta characteristics of C can be identified with the set $Q(V)$ of quadratic forms on the symplectic vector space V via

$$\theta(v) = \mathrm{Arf}(\theta + v) + \mathrm{Arf}(\theta).$$

We denote the subset of odd theta characteristics by Θ^- and the subset of even theta characteristics by Θ^+.

Given two theta characteristics θ and θ', the difference $\theta - \theta'$ is 2-torsion. This gives Θ the structure of a V-torsor. The union $U = V \cup \Theta$ is thus a vector space over $\mathbb{F}_2$ of dimension $2g + 1$. The vector space U can also be identified with the 2-torsion subgroup of $\mathrm{Pic}(C)/\mathbb{Z}K_C$.

Let $B = (\theta_1, \ldots, \theta_{2g+1})$ be an ordered basis for U consisting of elements of Θ. For a vector $u \in U$, let $n_B(u)$ denote the number of nonzero coordinates of u in the basis B. If θ is an element of Θ, then $n_B(\theta)$ is odd. If the basis B has the property that $\mathrm{Arf}(\theta)$ only depends on the residue class of $n_B(\theta) \mod 4$ for any $\theta \in \Theta$ we say that B is an ordered *Aronhold basis*. The set $\{\theta_1, \ldots, \theta_{2g+1}\}$ is then called an *Aronhold set*. It follows directly from the definition that if B is an Aronhold basis, then $\mathrm{Arf}(\theta_1) = \cdots = \mathrm{Arf}(\theta_{2g+1})$. By [29, Proposition 2.1], Aronhold bases exist and their elements satisfy $\mathrm{Arf}(\theta_i) = 0$ if $g \equiv 0$ or $1 \mod 4$ and $\mathrm{Arf}(\theta_i) = 1$ if $g \equiv 2$ or $3 \mod 4$. Furthermore, given an Aronhold basis B, the elements

$$
\begin{aligned}
&x_1 = \theta_1 + \theta_2 \quad y_1 = \theta_1 + \theta_{2g+1}, \\
&x_2 = \theta_3 + \theta_4 \quad y_2 = \theta_1 + \theta_2 + \theta_3\theta_{2g+1}, \\
&\qquad \cdots \\
&x_g = \theta_{2g-1} + \theta_{2g} \quad y_g = \theta_1 + \theta_2 + \cdots + \theta_{2g-1} + \theta_{2g+1}
\end{aligned}
$$

constitute a level 2 structure, see [29, Section 2] (recall that the addition is taken modulo K_C). This establishes a bijection between the set of Aronhold bases of C and the set of level 2 structures.

2.2 Bitangents and level 2 structures

A *plane quartic* is a smooth curve C in the projective plane given by a polynomial of degree 4. By the genus-degree formula, such a curve has genus 3. Conversely, every non-hyperelliptic curve of genus 3 can be embedded as a plane quartic via its canonical linear system. A *bitangent* to a plane quartic C is a line L which is tangent to C at two points or has contact order 4 at one point. If L intersects C at two distinct points we say that L is a *genuine bitangent*. If L only intersects C at one point we say that it is a *hyperflex line*. Every plane quartic has precisely 28 bitangents.

A representative of the canonical class of a plane quartic C is obtained by intersecting C with a line L, i.e. $K_C \sim L \cdot C$. Suppose that L is a bitangent to C. Then $L \cdot C = 2P + 2Q$ for some points $P, Q \in C$ (where $P = Q$ if L is a hyperflex line). Therefore, the divisor $D = P + Q$ satisfies $2D = 2P + 2Q \sim K_C$. In other words, D is a theta characteristic of C. Since there are 28 bitangents of C we obtain 28 theta characteristics in this way. These are exactly the $2^{3-1}(2^3 - 1) = 28$ odd theta characteristics. We combine this identification with the bijection between Aronhold bases and level 2 structures. In this, we may interpret level 2 structures in terms of (special) collections of seven bitangents and vice versa.

2.3 Plane quartics and Del Pezzo surfaces

A *Del Pezzo surface* is a smooth and projective surface S such that the anticanonical class $-K_S$ is ample. The number K_S^2 is called the *degree* of S.

Let C be a smooth plane quartic curve. The double cover $\pi : S \to \mathbb{P}^2$ branched over C is a Del Pezzo surface of degree 2. Every Del Pezzo surface of degree 2 is obtained in this way. The surface S has 56 exceptional lines. The covering map π maps pairs of these lines to the 28 bitangents of C.

Let $P_1, \ldots, P_7 \in \mathbb{P}^2$ be seven points in general position. In other words

- no three of the points $P_1, \ldots, P_7$ lie on a line, and
- no six of the points $P_1, \ldots, P_7$ lie on a conic.

By blowing up the points $P_1, \ldots, P_7$ we obtain a Del Pezzo surface S of degree 2. All Del Pezzo surfaces of degree 2 can be obtained in this way. The seven exceptional curves $E_1, \ldots, E_7$ together with the strict transform E_0 of a line in $\mathbb{P}^2$ constitute a basis for $\mathrm{Pic}(S)$

$$\mathrm{Pic}(S) = \mathbb{Z}E_0 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_7.$$

Such a basis (coming from a blow up) is called a *geometric marking* of S. The intersection theory of S is given by

$$E_0^2 = 1, \quad E_i^2 = -1, \ i \neq 0, \quad E_i \cdot E_j = 0, \ i \neq j.$$

The group permuting the geometric markings of a given surface is isomorphic to the Weyl group $W(E_7)$. This group is isomorphic to the direct product of $\mathrm{Sp}(6, \mathbb{F}_2)$ with a group with two elements.

We thus have two ways of obtaining a Del Pezzo surface S: as a cover $\pi : S \to \mathbb{P}^2$ branched over a plane quartic curve C and as the blow-up $S = \mathrm{Bl}_{P_1, \ldots, P_7}$. The images $\pi(E_1), \ldots, \pi(E_7)$ of the exceptional curves are seven bitangents of the plane quartic C which constitute an Aronhold set.

Denote the moduli space of geometrically marked Del Pezzo surfaces of degree 2 by $\mathcal{DP}_2^{\mathrm{gm}}$, denote the moduli space of seven ordered points in $\mathbb{P}^2$ in general position by $\mathcal{P}_7^2$ and denote the moduli space of plane quartics with level 2 structure by $\mathcal{Q}[2]$. The moduli spaces $\mathcal{P}_7^2$ and $\mathcal{Q}[2]$ are fine moduli spaces. Every geometrically marked Del Pezzo surface of degree 2 has precisely one non-trivial automorphism (namely the involution changing the two sheets of the cover $\pi : S \to \mathbb{P}^2$) so the moduli space $\mathcal{DP}_2^{\mathrm{gm}}$ is not fine (as a scheme). On the level of coarse moduli spaces we have the following result due to van Geemen.

Theorem 2.2 ([19, Chapter IX, Theorem 1]) *There are* $\mathrm{Sp}(6, \mathbb{F}_2)$-*equivariant isomorphisms of coarse moduli spaces*

$$\mathcal{DP}_2^{\mathrm{gm}} \cong \mathcal{P}_7^2 \cong \mathcal{Q}[2].$$

2.4 The cohomology of the moduli space of quartics with level structure

Let ℓ be a prime number different from the characteristic of the field k. Let $\mathbb{Q}_\ell$ denote the field of ℓ-adic numbers. For a variety X, let $H^i(X, \mathbb{Q})$ denote its ith de Rham cohomology group, let $H_c^i(X, \mathbb{Q})$ denote its ith de Rham cohomology group with compact support, let $H_{\mathrm{ét}}^i(X, \mathbb{Q}_\ell)$ denote its ith étale cohomology group and let $H_{\mathrm{ét,c}}^i(X, \mathbb{Q}_\ell)$ denote its ith étale cohomology group with compact support.

Theorem 2.3 ([8, Section 6]) *The cohomology groups* $H^i(\mathcal{Q}[2], \mathbb{Q})$ *are pure of Tate type* (i, i). *Their structure as representations of* $\mathrm{Sp}(6, \mathbb{F}_2)$ *are as given in Table* 1.

By Poincaré duality, the cohomology groups $H_c^i(\mathcal{Q}[2])$ are pure of weight $2i - 2\dim(\mathcal{Q}[2])$.

Definition 2.4 (Dimca and Lehrer [17]) Let X be an irreducible and separated scheme of finite type over k. If, for all i, the cohomology groups $H_c^i(\mathcal{Q}[2])$ are pure of weight $2i - 2\dim(\mathcal{Q}[2])$ we say that X is minimally pure.

There is also an étale version of minimal purity.

Definition 2.5 (Dimca and Lehrer [17]) An irreducible and separated scheme X of finite type over $\overline{\mathbb{F}_q}$ is called *minimally pure* if the Frobenius endomorphism F acts on $H_{\mathrm{ét,c}}^i(X, \mathbb{Q}_\ell)$ with all eigenvalues equal to $q^{i - \dim(X)}$.

Corollary 2.6 *For odd* q, *the moduli space* $\mathcal{Q}[2]$ *is minimally pure also in the étale sense and the cohomology groups* $H_{\mathrm{ét}}^i(X, \mathbb{Q}_\ell)$ *are as given in Table* 1.

Proof By Theorem 2.2, $\mathcal{Q}[2]$ can be identified with the complement of a finite number of hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. In particular, $\mathcal{Q}[2]$ is smooth. Sekiguchi has constructed a compactification $\overline{\mathcal{Q}[2]}$ of $\mathcal{Q}[2]$ with many nice properties, see [43, 44]. Hacking, Keel and Tevelev have shown that over $\mathbb{Z}[1/2]$ the compactification $\overline{\mathcal{Q}[2]}$ is smooth and the boundary $\overline{\mathcal{Q}[2]} \setminus \mathcal{Q}[2]$ is a union of smooth divisors with normal crossings, see [31, Theorm 1.2]. By [34, Corollary 1.3] we may now identify the de Rham cohomology of $\mathcal{Q}[2]$ over $\mathbb{C}$ with the étale cohomology of $\mathcal{Q}[2]$ over $\overline{\mathbb{F}}_q$ (but only for odd q). $\qquad\square$

Proposition 2.7 ([8, Section 5]) *Let* $\mathcal{Q}_{\mathrm{btg}}[2]$ *denote the moduli space of plane quartics with level* 2 *structure and one marked bitangent line. The cohomology groups* $H^i(\mathcal{Q}_{\mathrm{btg}}[2], \mathbb{Q})$ *are pure of Tate type* (i, i). *Their structure as representations of* $\mathrm{Sp}(6, \mathbb{F}_2)$ *are as given in Table* 2.

2.5 Cohomology of quotients

Let X be a variety over the field k. Let G be a finite group of rational automorphisms of X such that the morphism $f : X \to X/G$ is Galois. In what follows, we will typically have cohomological information about X and we will want to use this information to obtain cohomological information about X/G. The following facts allow us to achieve this.

Proposition 2.8 $H^i(X/G, \mathbb{Q}) \cong H^i(X, \mathbb{Q})^G$. *Moreover, if X is minimally pure (in the sense of mixed Hodge theory) then so is X/G.*

The first part of result can be found e.g. in [30], the corollary following Proposition 5.2.4. The second part follows from Peters–Steenbrink's generalization of the Leray–Hirsch theorem, see [41, Theorem 2].

Proposition 2.9 $H^i_{\text{ét}}(X/G, \mathbb{Q}) \cong H^i_{\text{ét}}(X, \mathbb{Q})^G$. *Moreover, if X is minimally pure (in the étale sense) then so is X/G.*

This follows from the Hochschild–Serre spectral sequence, see e.g. [40, Theorem 2.20].

3 Constructions of classical structures

In this section we recall some classical results and constructions related to plane quartics and their bitangents.

Each of these structures can be understood as the structure stabilized by some subgroup of $\mathrm{Sp}(6, \mathbb{F}_2)$. A small computation in GAP shows that there are 1369 conjugacy classes of subgroups of $\mathrm{Sp}(6, \mathbb{F}_2)$ so we could, in principle, investigate this many structures. However, most of the structures occurring in the classical literature correspond to maximal subgroups of $\mathrm{Sp}(6, \mathbb{F}_2)$. The exception to this rule is Aronhold heptads—these are stabilized by a subgroup isomorphic to S_7 which is not quite maximal (it sits as a maximal subgroup inside an S_8 which is in turn maximal in $\mathrm{Sp}(6, \mathbb{F}_2)$). Also, out of the eight maximal subgroups of $\mathrm{Sp}(6, \mathbb{F}_2)$, only seven occur as stabilizers

Table 1 The cohomology groups of the moduli space of plane quartics with level 2 structure as representations of $\mathrm{Sp}(6, \mathbb{F}_2)$

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	.	.	.	.	.	.	.	.	.
H^1	.	.	.	.	.	.	.	1	.	.
H^2	.	.	.	.	.	.	.	.	.	.
H^3	.	.	.	1	.	.	.	.	.	.
H^4	.	.	.	.	.	.	.	.	.	1
H^5	.	.	.	.	.	1	1	1	.	.
H^6	1	.	2	.	1	1	1	3	.	.

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	.	.	.	.	.	.	.	.	.	.
H^1	.	.	.	.	.	.	.	.	.	.
H^2	.	.	.	.	.	.	.	.	.	1
H^3	.	.	1	.	.	.	1	.	.	2
H^4	.	.	2	.	2	1	2	1	.	3
H^5	1	2	2	1	2	4	3	3	3	4
H^6	5	1	1	4	.	3	2	2	5	3

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	.	.	.	.	.	.	.	.	.	.
H^1	.	.	.	.	.	.	.	.	.	.
H^2	.	.	.	1	.	.	.	.	.	.
H^3	1	.	.	.	.	.	1	2	2	1
H^4	4	.	3	1	3	2	3	6	5	4
H^5	4	4	4	6	5	6	6	6	8	9
H^6	1	6	3	6	1	6	4	2	6	6

The characters are ordered as in [15, p. 47]. The indices are the degrees of the representations followed by a letter to distinguish characters of the same degree, see [15, Section 5.1]

of some structure in the classical literature (the group $G_2(2)$ of automorphisms of the integral octonions is missing).

In this section, we also describe the stabilizers in $\mathrm{Sp}(6, \mathbb{F}_2)$ of each structure and identify the structure corresponding to $G_2(2)$. For a different approach, via root subsystems of E_7, to find most of these structures (all except those in Sects. 3.8 and 3.9), see [39, Section 4].

3.1 Bitangents and odd theta characteristics

3.1.1 Constructions

See Sect. 2.2 for a discussion about bitangents to plane quartic curves and their relation to odd theta characteristics.

Table 2 The cohomology groups of the moduli space of plane quartics with a marked bitangent line and level two structure as representations of $\mathrm{Sp}(6, \mathbb{F}_2)$

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	.	.	.	.	1	.	.	.	.
H^1	.	.	.	.	.	1	.	2	.	.
H^2	.	.	.	1	.	1	.	2	.	.
H^3	.	.	.	5	.	3	1	3	.	3
H^4	.	.	1	7	1	8	7	9	8	17
H^5	1	2	7	10	8	16	17	21	22	31
H^6	2	2	10	6	10	13	14	20	16	20

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	.	.	.	.	.	.	.	.	.	.
H^1	.	.	1	.	1	1	.	.	.	1
H^2	1	.	3	2	4	4	3	.	.	7
H^3	5	1	11	8	15	16	19	3	5	25
H^4	18	16	34	24	41	50	54	33	33	65
H^5	43	46	54	50	62	89	92	83	86	106
H^6	42	37	35	46	39	65	65	66	77	76

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	.	.	.	.	.	.	.	.	.	.
H^1	.	.	.	1	.	.	.	.	.	.
H^2	2	1	.	7	.	2	1	6	4	4
H^3	16	8	11	23	13	19	21	38	33	34
H^4	60	42	64	74	73	79	89	122	114	130
H^5	103	103	129	143	145	160	176	198	205	247
H^6	68	90	95	112	100	126	131	129	151	181

The characters are ordered as in [15, p. 47]. The indices are the degrees of the representations followed by a letter to distinguish characters of the same degree, see [15, Section 5.1]

3.1.2 Stabilizer subgroup

The stabilizer of a bitangent (or, if one prefers, an odd theta characteristic) is a maximal subgroup of order 51840. It is isomorphic to the Weyl group $W(E_6)$ of the root system E_6. It has index 28 in $\mathrm{Sp}(6, \mathbb{F}_2)$.

The action of $W(E_6)$ is probably best understood in the world of Del Pezzo surfaces. Let C be a plane quartic and let $\pi : S \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched over C. Recall from Sect. 2.3, that S is a Del Pezzo surface of degree 2 and that π identifies the 56 exceptional curves of S with the 28 bitangents of C.

Let L be a bitangent of C and let E be an exceptional curve of S such that $\pi(E) = L$. By blowing down E we obtain a Del Pezzo surface S' of degree 3 (i.e. a smooth cubic

surface). Such a surface has 27 lines. From this perspective, these 27 lines correspond to the 27 bitangents of C different from L. The group $W(E_6)$ is the group permuting the 27 lines of S'. See Sect. 3.3 for a few more details on this perspective.

We denote the moduli space of (smooth) plane quartics marked with a bitangent by $\mathcal{Q}_{btg}$.

3.2 Cayley octads and even theta characteristics

3.2.1 Constructions

There is also a close relationship between even theta characteristics of a non-hyperelliptic genus 3 curve C and bitangents of its canonical model as a plane quartic Q. We explain this relationship below (for a more complete account, see [29, Section 6]).

Let θ be an even theta characteristic of C and consider the linear system corresponding to the divisor class $K_C + \theta$. This linear system is very ample and gives an embedding of C into $\mathbb{P}^3$ as a curve of degree 6. We denote this curve by B.

To be more precise, let $V = H^0(C, K_C)^*$, let $W = H^0(C, K_C+\theta)$. We obtain a map $\varphi \colon V \to \mathrm{Sym}^2 W^*$ given in the following way. Let $s_0, \ldots, s_3$ be a basis for W. The products $s_i s_j$ are then sections of the line bundle corresponding to $2(K_C + \theta) = 3K_C$. Thus, $s_i s_j$ can be identified with a cubic polynomial f_{ij} on $\mathbb{P}V$. Construct the matrix $M = (f_{ij})$ and let N_0 be the matrix of cofactors of M. Let F be the defining polynomial for Q. One may show that N_0 is divisible by F^2. Let N be the matrix N_0/F^2. The matrix N is a matrix representation of φ.

We make the identifications $\mathbb{P}V = \mathbb{P}^2$ and $\mathbb{P}W = \mathbb{P}^3$. We think of φ as a net[1] $\mathcal{N}$ of quadrics in $\mathbb{P}^3$. With these identifications at hand, the quartic Q is the locus

$$Q := \{[v] \in \mathbb{P}V \mid \varphi(v) \text{ is singular}\}$$

and the sextic $B \subset \mathbb{P}W$ is the locus of singular points of members of $\mathcal{N}$. Moreover, for every point $P = [v]$ of Q we may consider $\varphi(P)$ to be a map $W \to W^*$. We let L denote the divisor class corresponding to the dual of the line bundle whose fiber at $P \in Q$ is $\ker(\varphi(P))$. It can then be shown that

$$\theta = L - K_C.$$

Somewhat remarkably, the above process can be abstracted in the following sense. Let V be a vector space of dimension 2 and let W be a vector space of dimension 3. Let $\varphi \colon V \to \mathrm{Sym}^2 W^*$ be a linear map such that the net $\mathcal{N}$ given by φ has eight points in general linear position as its base locus. The process described above then yields a genus 3 curve C with an even theta characteristic.

Not all 8-tuples of points in $\mathbb{P}^3$ occur as the base locus of a net of quadrics. When this is the case, the 8-tuple uniquely determines the net.

Definition 3.1 A *Cayley octad* is an unordered 8-tuple of points in $\mathbb{P}^3$ in general linear position which is the base locus of a net of quadrics.

---

[1] I.e. a linear system of dimension 2.

Cayley octads are a special case of self-associated point sets, see [19, Section III.3]. Note that a plane quartic curve with an even theta characteristic uniquely defines a Cayley octad and vice versa. In particular, we see that, up to projective equivalence, there are 36 Cayley octads associated to each plane quartic.

Let $\Omega = \{Q_1, \ldots, Q_8\}$ be a Cayley octad and let B be the corresponding sextic curve in $\mathbb{P}^3$ given by the theta characteristic θ. The 28 lines L_{ij}, $1 \leqslant i < j \leqslant 8$, are bisecants to B, i.e. they intersect B in two points each and each of the 28 lines cuts out an odd theta characteristic on B. Recall that the odd theta characteristics are naturally identified with bitangents of the canonical model of B as a plane quartic. We thus see that θ can be specified by labelling the 28 bitangents with the 28 pairs of points of a set of eight elements (in a compatible way according to the above constructions). Sometimes Cayley octads are defined as such labellings, see e.g. [15].

3.2.2 Stabilizer subgroup

The stabilizer of a Cayley octad is a maximal subgroup of order 40320 and it is isomorphic to S_8, the symmetric group on eight elements. It has index 36 in $\mathrm{Sp}(6, \mathbb{F}_2)$. Here, the action is plainly seen as the permutation action of S_8 on the eight points of the Cayley octad.

We denote the moduli space of plane quartics marked with a Cayley octad by $\mathcal{Q}_{\mathrm{CO}}$.

3.3 Aronhold heptads

3.3.1 Constructions

Let $\Omega = \{Q_1, \ldots, Q_8\}$ be a Cayley octad, let B be the corresponding sextic curve in $\mathbb{P}^3$ and let θ be the corresponding even theta characteristic. By projecting from one of the points, say Q_8, we obtain seven points $P_1, \ldots, P_7$ in $\mathbb{P}^2$ in general position, i.e. no three of them lie on a line and no six of them lie on a conic. The image $\widetilde{B}$ of B under the projection is a plane sextic curve with double points at the seven points. The seven lines $L_{18}, \ldots, L_{78}$ define seven odd theta characteristics $\theta_1, \ldots, \theta_7$ on B. Such a 7-tuple of odd theta characteristics is called an *Aronhold heptad*. A more direct definition is the following.

Definition 3.2 An Aronhold heptad η is a 7-tuple of odd theta characteristics such that if θ_1, θ_2 and θ_3 are distinct elements of η, then

$$\theta_1 + \theta_2 + \theta_3$$

is an even theta characteristic.

Recall that the sum is taken modulo K_C. We remark that we can get the even theta characteristic back from the Aronhold heptad via

$$\theta = \sum_{i=1}^{7} \theta_i.$$

There are eight different points of the Cayley octad to project from, thus yielding eight different Aronhold heptads. We thus see that there are $8 \cdot 36 = 288$ projectively inequivalent Aronhold heptads associated to a plane quartic.

3.3.2 Stabilizer subgroup

The stabilizer of an Aronhold heptad is a subgroup isomorphic to S_7, the symmetric group on seven elements. This subgroup is not maximal (it is contained in the S_8 stabilizing the associated Cayley octad). The index of S_7 in $\mathrm{Sp}(6, \mathbb{F}_2)$ is 288. Again, the action is plainly seen as the permutation action of S_7 on the seven theta characteristics of the Aronhold heptad.

We denote the moduli space of plane quartics marked with an Aronhold heptad by $\mathcal{Q}_{\mathrm{AH}}$.

3.4 Steiner complexes

3.4.1 Constructions

Let Q be a plane quartic and let $\mathcal{P}$ be the set of unordered distinct pairs of odd theta characteristics of Q. Consider the map

$$s \colon \mathcal{P} \to \mathrm{Jac}(Q)[2] \setminus \{0\}$$

sending a pair $\{\theta_1, \theta_2\}$ to $\theta_1 + \theta_2 - K_C$.

Definition 3.3 (see [18]) Let $v \in \mathrm{Jac}(C)[2]$ be a nonzero element. The set

$$\Sigma(v) = \bigcup_{\{\theta_1, \theta_2\} \in s^{-1}(v)} \{\theta_1, \theta_2\}$$

is called the *Steiner complex associated to* v.

Thus, there is one Steiner complex for each of the $2^{2 \cdot 3} - 1 = 63$ elements of $\mathrm{Jac}(Q)[2]$. Each Steiner complex contains precisely 12 odd theta characteristics (or 12 bitangents, if one prefers this viewpoint).

A more direct definition of a Steiner complex is

$$\Sigma(v) = \{\theta \in \Theta^- \mid \theta(v) = 0\}.$$

However, from Definition 3.3 it is clear how the 12 elements of a Steiner complex naturally form six pairs.

3.4.2 Stabilizer subgroup

Permuting the six pairs stabilizes a Steiner complex and so does interchanging the two elements of a pair. This suggests that the stabilizer subgroup of a Steiner complex

should be the wreath product $\mathbb{F}_2 \wr S_6$ (i.e. the semidirect product $\mathbb{F}_2^6 \rtimes S_6$ with the canonical action of S_6 on $\mathbb{F}_2^6$). However, it turns out (see e.g. [18] for details) that a nontrivial parity condition must be satisfied by the $\mathbb{F}_2^6$-part; once five of the switches are chosen the sixth is determined. This gives the stabilizer subgroup the structure of a semidirect product $\mathbb{F}_2^5 \rtimes S_6$. It is a maximal subgroup of cardinality 23040 and index 63. It is also possible to identify the stabilizer subgroup with the Weyl group of D_6, see [19].

We denote the moduli space of plane quartics marked with a Steiner complex by $\mathcal{Q}_{SC}$.

3.5 Göpel subsets and maximal isotropic subspaces

3.5.1 Constructions

Definition 3.4 Let C be a plane quartic. A *Göpel subspace* is a maximal isotropic subspace of $\mathrm{Jac}(C)[2]$ with respect to the Weil pairing. The set of the seven nonzero elements of a Göpel subspace is called a *Göpel subset*.

There are 135 maximal isotropic subspaces of a symplectic vector space of dimension 6 over $\mathbb{F}_2$, see [2]. Thus, there are 135 Göpel subsets. For more details and further perspectives, see [14, 19, 39].

3.5.2 Stabilizer subgroup

Stabilizers of isotropic subspaces are maximal parabolic subgroups of $\mathrm{Sp}(6, \mathbb{F}_2)$. We give a more precise description below.

A Göpel subset is naturally a Fano plane. Therefore, the automorphism group $\mathrm{PGL}(3, \mathbb{F}_2)$ of the Fano plane is naturally a subgroup of the stabilizer of a Göpel subset.

To understand the rest of the stabilizer it is convenient to once again recall the Del Pezzo picture (see Sect. 2.3). In particular, recall that if C is a plane quartic and S is the corresponding Del Pezzo surface then a level 2 structure on C naturally corresponds to a geometric marking of S. Under this correspondence, the 63 nonzero elements of $\mathrm{Jac}(C)[2]$ correspond to the 63 positive roots of the root system E_7 (defined with respect to the geometric marking). Thus, the seven elements of a Göpel subset give rise to seven positive roots. The remainder of the stabilizer of a Göpel subset comes from the operation of switching such a root to its negative. However, we may only choose the direction of six out of seven roots freely—once six directions are chosen there is a unique direction of the final root so that the seven roots constitute part of a choice of positive roots for E_7. This explains how the stabilizer subgroup of a Göpel subset is identified with $\mathrm{PGL}(3, \mathbb{F}_2) \times \mathbb{F}_2^6$. It is a maximal subgroup of cardinality 10752 and index 135.

We denote the moduli space of plane quartics marked with a Göpel subset by $\mathcal{Q}_{GS}$.

3.6 Syzygetic tetrads and isotropic planes

3.6.1 Constructions

Definition 3.5 Let $\{\theta_1, \theta_2, \theta_3\}$ be a set of three odd theta characteristics on a curve C. The set is called a *syzygetic triad* if

$$\mathrm{Arf}\,(\theta_1) + \mathrm{Arf}\,(\theta_2) + \mathrm{Arf}\,(\theta_3) + \mathrm{Arf}\,(\theta_1 + \theta_2 + \theta_3) = 0,$$

otherwise it is called an *azygetic triad*.

Note that if $\{\theta_1, \theta_2, \theta_3\}$ is a syzygetic triad of odd theta characteristics, then $\theta_{123} = \theta_1 + \theta_2 + \theta_3$ is an odd theta characteristic. Moreover, any subset of three elements of $\{\theta_1, \theta_2, \theta_3, \theta_{123}\}$ is a syzygetic triad such that the sum of the three elements is equal to the fourth.

Definition 3.6 A *syzygetic tetrad* is a set $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ of four odd theta characteristics such that any subset of three elements is a syzygetic triad and such that the sum of any three elements is equal to the fourth.

In terms of bitangents, a syzygetic tetrad is a set of four bitangents such that the intersection points of the bitangents and the quartic lie on a conic, see Fig. 1. Given a syzygetic tetrad of odd theta characteristics $\{\theta_1, \theta_2, \theta_3, \theta_4\}$, we may choose one of the four theta characteristics θ of the tetrad and construct the plane

$$V = \big\{\theta_1 - \theta, \theta_2 - \theta, \theta_3 - \theta, \theta_4 - \theta\big\} \subset \mathrm{Jac}\,(C)[2].$$

One may show that V is isotropic and independent of the choice of θ, see [18, Corollary 5.4.5]. Furthermore, the isotropic planes of $\mathrm{Jac}\,(C)[2]$ correspond bijectively to the syzygetic tetrads. Thus, there are $(2^6 - 1) \cdot (2^5 - 2)/|\mathrm{GL}(2, \mathbb{F}_2)| = 315$ syzygetic tetrads on a plane quartic.

3.6.2 Stabilizer subgroup

Again, stabilizers of isotropic subspaces are maximal parabolic subgroups of $\mathrm{Sp}(6, \mathbb{F}_2)$. We give a more precise description below.

The determination of the stabilizer of a syzygetic tetrad follows from standard results around stabilizers of isotropic subspaces in symplectic spaces. For completeness, we sketch the argument.

An element σ of $\mathrm{Sp}(6, \mathbb{F}_2)$ stabilizing the isotropic plane V will necessarily stabilize the orthogonal complement $V^\perp$ of V. Since V is isotropic we have $V \subset V^\perp$. Thus, σ preserves the flag $0 \subset V \subset V^\perp \subset \mathrm{Jac}\,(C)[2]$. The Weil pairing induces a symplectic pairing on the quotient $V^\perp/V$ so the stabilizer of V contains a copy of $\mathrm{Sp}(V^\perp/V) \cong \mathrm{Sp}(2, \mathbb{F}_2)$. The stabilizer also contains a copy of $\mathrm{GL}(2, \mathbb{F}_2)$ stabilizing V. To identify the rest of the stabilizer we choose a symplectic basis $x_0, x_1, x_2, y_0, y_1, y_2$ such that
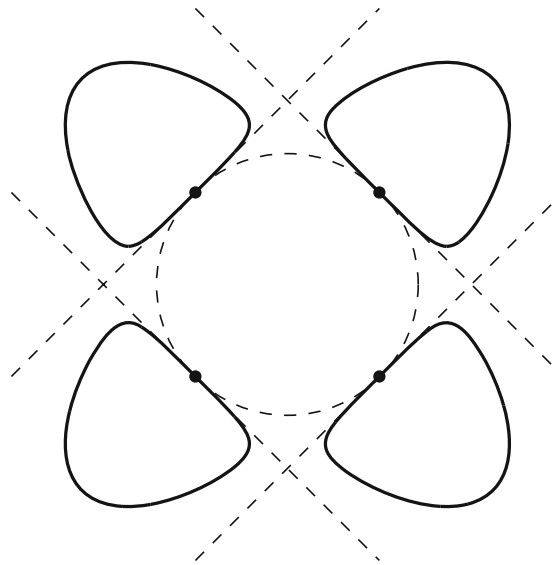
Fig. 1 A syzygetic tetrad on a plane quartic (the figure is reproduced from [7])

V is spanned by x_0 and x_1 and $V^\perp$ is spanned by x_0, x_1, x_2 and y_2; this is possible by Witt's lemma. The elements of $\mathrm{Sp}(6, \mathbb{F}_2)$ of the form

$$A_{i,j}\colon x_j \mapsto x_j + x_i, \quad y_i \mapsto y_i + y_j,$$
$$B_{i,j}\colon y_i \mapsto y_i + x_j, \quad y_j \mapsto y_j + x_i$$

generate a non-abelian special 2-group G such that its center $Z(G)$ is an abelian group isomorphic to $\mathbb{F}_2^3$ and such that $G/Z(G)$ is isomorphic to $\mathbb{F}_2^4$. Thus, G is an extension $G = \mathbb{F}_2^3.\mathbb{F}_2^4$. These parts can be shown to constitute the full stabilizer; more precisely, they fit together in a semidirect product $(\mathrm{Sp}(2, \mathbb{F}_2) \times \mathrm{GL}(2, \mathbb{F}_2)) \times \mathbb{F}_2^3.\mathbb{F}_2^4$.

Remark 3.7 Of course, $\mathrm{Sp}(2, \mathbb{F}_2) \cong \mathrm{GL}(2, \mathbb{F}_2) \cong S_3$ so one could in principle say that the stabilizer is $(S_3 \times S_3) \times \mathbb{F}_2^3.\mathbb{F}_2^4$. This is the approach of [15]. However, we found the above approach to be more transparent.

We denote the moduli space of plane quartics marked with a syzygetic tetrad by $\mathcal{Q}_{\mathrm{syz}}$.

3.7 Azygetic triads of Steiner complexes

A pair $\{\Sigma(u), \Sigma(v)\}$ of Steiner complexes is called *azygetic* if $\langle u, v \rangle = 1$ and it is called *syzygetic* if $\langle u, v \rangle = 0$.

Definition 3.8 A triple $\{\Sigma(u), \Sigma(v), \Sigma(w)\}$ of Steiner complexes is called *azygetic* if the vectors u, v and w form the nonzero vectors of a non-isotropic plane in $\mathrm{Jac}(C)[2]$.

Recall that $\mathrm{Jac}(C)[2] \cong \mathbb{F}_2^6$. Thus, there are $(64 - 1)(64 - 2)/|\mathrm{GL}(2, \mathbb{F}_2)| = 651$ planes in $\mathrm{Jac}(C)[2]$. We have seen that 315 of these are isotropic so there are

$651 - 315 = 336$ non-isotropic planes. In other words, there are 336 azygetic triads on a plane quartic.

Remark 3.9 A triple of three mutually syzygetic Steiner complexes is called a *syzygetic triad of Steiner complexes*. It can be shown that if $\{\Sigma(u), \Sigma(v), \Sigma(w)\}$ is a syzygetic triad of Steiner complexes, then

$$\Sigma(u) \cup \Sigma(v) \cup \Sigma(w) = \Theta^-.$$

3.7.1 Stabilizer subgroup

As mentioned above, there is a symmetric group S_6 permuting pairs of elements of each Steiner complex (recall that a Steiner complex consists of six pairs of odd theta characteristics). There is also a symmetric group S_3 permuting the three Steiner complexes. This explains why the stabilizer subgroup in $\mathrm{Sp}(6, \mathbb{F}_2)$ of an azygetic tetrad of Steiner complexes can be identified with the product $S_3 \times S_6$. It is a maximal subgroup of $\mathrm{Sp}(6, \mathbb{F}_2)$ of cardinality 4320 and index 336.

  We denote the moduli space of plane quartics marked with an azygetic triad by $\mathcal{Q}_{\text{azy}}$.

3.8 Enneads and the Study quadric

The maximal subgroups of $\mathrm{Sp}(6, \mathbb{F}_2)$ of cardinality 1512 and index 960 were somewhat mysterious for some time but have now been studied extensively by Dye [20], Edge [24, 25], Frame [27] and Study [45] (to mention a few).

3.8.1 Constructions

Cayley and Hesse denoted the 28 bitangents of a plane quartic C by indexing them with pairs of objects from a set of eight objects (Sect. 3.2 expands on this perspective). Study observed that one may take the eight elements to be eight variables $x_1, \ldots, x_8$ and the 28 pairs to be the 28 monomials $x_i x_j$, $1 \leqslant i < j \leqslant 8$. Then, much of the geometry of the 28 bitangents can be explored via the *Study quadric*

$$S = \sum_{1 \leqslant i < j \leqslant 8} x_i x_j, \quad x_i \in \mathbb{F}_2. \tag{3.1}$$

For instance, S defines a variety $V(S)$ in $\mathbb{P}^7(\mathbb{F}_2)$ with 135 points corresponding to the 135 Göpel subsets. The lines in $\mathbb{P}^7(\mathbb{F}_2)$ fall into different classes depending on the number of points they have in common with $V(S)$; given a point P outside $V(S)$ there are exactly 28 lines through P which do not meet $V(S)$, 63 lines through P which meet $V(S)$ once and 36 lines through P which intersect $V(S)$ in two $\mathbb{F}_2$-points (corresponding to the 28 bitangents, the 63 Steiner complexes and the 36 Cayley octads of C).

  The form of S given in equation (3.1) depends on the chosen coordinates for $\mathbb{P}^7(\mathbb{F}_2)$ but there are many choices of coordinates for $\mathbb{P}^7(\mathbb{F}_2)$ which preserve the form of S.

To investigate the matter further, let P_i denote the point whose ith coordinate is 1 and whose other coordinates are 0 and let P_9 denote the point whose coordinates are all 1. The points $P_1, \ldots, P_9$ then all lie on $V(S)$ and the points $P_1, \ldots, P_8$ naturally correspond to the above choice of coordinates. Furthermore, any choice of eight points among $P_1, \ldots, P_9$ corresponds to another choice of coordinates which leaves S invariant.

Recall that the function $\Phi \colon \mathbb{P}^7(\mathbb{F}_2) \times \mathbb{P}^7(\mathbb{F}_2) \to \mathbb{F}_2$ given by

$$\Phi(x, y) = \sum_{\substack{i,j=1 \\ i \neq j}}^{8} x_i y_j$$

is called the *polar form* with respect to S and that two points P and Q in $\mathbb{P}^7(\mathbb{F}_2)$ such that $\Phi(P, Q) = 0$ are called *conjugate* with respect to S. We see that no two of the points $P_1, \ldots, P_9$ are conjugate with respect to S. Moreover, the chord joining any pair of the points $P_1, \ldots, P_9$ is not contained in $V(S)$. To see this, note that the chord L joining P_i and P_j contains three points and the point different from P_i and P_j has precisely two nonzero coordinates—$V(S)$ does not contain any points with precisely two nonzero coordinates.

It turns out that these two properties characterize 9-tuples coming from choices of coordinates which gives S the form of equation (3.1).

Definition 3.10 A set of nine points on $V(S)$ such that

- no two points are conjugate with respect to S, and
- no chord between two points is contained in $V(S)$

is called an *ennead*.

There are precisely 960 enneads (and, thus, $960 \cdot 9!$ different choices of coordinates preserving equation (3.1)).

3.8.2 Stabilizer subgroup

Dye [20] uses character theory to identify the stabilizer G of an ennead as a finite group of order 1512 containing the group $\mathrm{PSL}(2, \mathbb{F}_8)$ as a maximal simple subgroup of index 3. Up to isomorphism, there are exactly two such groups. The character table of G is given by Littlewood [36, p. 279]. The character table is not that of $\mathrm{PSL}(2, \mathbb{F}_8) \times \mathbb{F}_3$ so G must be the other possibility, namely the projective semilinear group $\mathrm{P\Gamma L}(2, \mathbb{F}_8)$—this group is also known as the Ree group $\mathrm{Ree}(3)$. The paper [25] of Edge is devoted to describing the action explicitly, we refer to his paper for details. Dye has investigated G in several works, see for instance [20–23].

We denote the moduli space of plane quartics marked with an ennead by $\mathcal{Q}_{\mathrm{enn}}$.

3.9 Octonionic structures

In the above, we have considered seven of the eight maximal subgroups of $\mathrm{Sp}(6, \mathbb{F}_2)$. The eighth maximal subgroup is isomorphic to the Chevalley group $G_2(2)$. It has size

Table 3 Various structures associated to a plane quartic curve, their stabilizers, the size of the stabilizer, the number of inequivalent structures and whether or not the stabilizer is a maximal subgroup of $\mathrm{Sp}(6, \mathbb{F}_2)$

Structure	Stabilizer	Size	#	Max.?
Bitangent (odd theta)	$W(E_6)$	51840	28	Yes
Cayley octad (even theta)	S_8	40320	36	Yes
Steiner complex	$\mathbb{F}_2^5 \rtimes S_6$ (alt. $W(D_6)$)	23040	63	Yes
Göpel subset (max. isotropic subspace)	$\mathrm{PGL}(3, \mathbb{F}_2) \times \mathbb{F}_2^6$	10752	135	Yes
Aronhold heptad	S_7	5040	288	No
Syzygetic tetrad (isotropic plane)	$(\mathrm{Sp}(2, \mathbb{F}_2) \times \mathrm{GL}(2, \mathbb{F}_2)) \times \mathbb{F}_2^3.\mathbb{F}_2^4$	4608	315	Yes
Azygetic triad	$S_3 \times S_6$	4320	336	Yes
Ennead	$\mathrm{P\Gamma L}(2, \mathbb{F}_8)$ (alt. $\mathrm{Ree}(3)$)	1512	960	Yes
Octonionic structure	$G_2(2)$	12096	120	Yes

12096 and index 120. It can be identified with the stabilizer of a point outside of the Study quadric (recall from Sect. 3.8 that the Study quadric $V(S)$ is a hypersurface in $\mathbb{P}^7(\mathbb{F}_2)$ containing 135 points so there are $255 - 135 = 120$ points outside of $V(S)$).

Apart from the aobve observation, we have not found any structure associated to a plane quartic C which is fixed by $G_2(2)$ in the literature. The group $G_2(2)$ is isomorphic to the automorphism group of the integral octonions and the above observation does not shed any light on this fact. There seem to be some hints to what $G_2(2)$ fixes in terms of bitangents and octonions in [39, Section 4], but we have not been able to come up with any meaningful structure. Thus, the only definition at hand of the anticipated "octonionic structure" is a point outside the Study quartic. It would of course be very interesting to find a more meaningful definition.

We denote the moduli space of plane quartics marked with an octonionic structure by $\mathcal{Q}_{\mathrm{OS}}$.

3.10 Summary

We summarize the results in Table 3.

4 Cohomological computations

In this section we compute the cohomology groups of the moduli spaces of plane quartic curves marked with the various structures from Sect. 3. Using Propositions 2.8 and 2.9, the results follow straightforwardly from the following result.

Theorem 4.1 ([8]) *For each i, the cohomology group $H_{\mathrm{dR}}^i(\mathcal{Q}[2])$ of the moduli space of smooth plane quartic curves with level two structure is pure of Tate type (i, i). Its structure as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$ is as given in Table 1.*

Also, recall from Sect. 2.4 that $\mathcal{Q}[2]$ is minimally pure (both in the mixed Hodge theory and étale senses).

Recall that the generating series of the Betti numbers of a topological space X is called the *Poincaré series* of X, i.e.

$$P_X(t) = \sum_{i \geqslant 0} \dim (H^i(X)) t^i.$$

If $P_X(t)$ is a polynomial, it is called the *Poincaré polynomial* of X.

Theorem 4.2 *The Poincaré polynomials of the moduli spaces introduced in Sect. 3 are as follows:*

Moduli space	Poincaré polynomial
$\mathcal{Q}_{btg}$	$1 + t^5 + 2t^6$
$\mathcal{Q}_{CO}$	$1 + t + t^5 + 4t^6$
$\mathcal{Q}_{AH}$	$1 + t + t^3 + 4t^4 + 6t^5 + 6t^6$
$\mathcal{Q}_{SC}$	$1 + t + 2t^5 + 5t^6$
$\mathcal{Q}_{GS}$	$1 + t + 2t^5 + 11t^6$
$\mathcal{Q}_{syz}$	$1 + t + t^4 + 7t^5 + 13t^6$
$\mathcal{Q}_{azy}$	$1 + t + t^3 + 3t^4 + 8t^5 + 9t^6$
$\mathcal{Q}_{enn}$	$1 + 3t^3 + 11t^4 + 13t^5 + 11t^6$
$\mathcal{Q}_{OS}$	$1 + 2t^5 + 7t^6$

For each of the above spaces and for each i, the cohomology group H^i is a pure Hodge structure of type (i, i).

Proof We apply Proposition 2.8 (or Proposition 2.9 in the étale case) to Theorem 4.1 to see that in each case we just need to find the invariant part of Table 1 with respect to the relevant stabilizer subgroup.

Let str be one of the above structures, let $\mathcal{Q}_{str}$ be the corresponding moduli space and let $G \subset \mathrm{Sp}(2g, \mathbb{F}_2)$ be the corresponding stabilizer subgroup. The character χ_G of $\mathrm{Sp}(2g, \mathbb{F}_2)$ corresponding to G is the character of $\mathrm{Ind}_G^{\mathrm{Sp}(2g, \mathbb{F}_2)} \mathrm{Triv}$, the induced representation from the trivial representation of G to $\mathrm{Sp}(2g, \mathbb{F}_2)$. By Frobenius reciprocity (see e.g. [28, Corollary 3.20]), we obtain the G-invariants of $H^i(\mathcal{Q}[2])$ by taking the inner product (in the sense of character theory) of χ_G and the character corresponding to $H^i(\mathcal{Q}[2])$.

The characters of $H^i(\mathcal{Q}[2])$ are given in Table 1 and the characters for the stabilizer subgroups G are given in the table below. Most of these characters can be found in [15, p. 46]. The only exception is the character corresponding to $\mathcal{Q}_{AH}$ (the only stabilizer subgroup which is not maximal in $\mathrm{Sp}(2g, \mathbb{F}_2)$). However, we can compute the character corresponding to S_7 quite easily (but tediously) in a number of ways—most straightforward is perhaps to use an explicit embedding of S_7 into $\mathrm{Sp}(6, \mathbb{F}_2)$, see for instance [7, p. 60]. □

Remark 4.3 The cohomology of $\mathcal{Q}_{btg}$ is known since before, see [37, Corollary 4.5] and [46, Theorem 1.1]. See also [26] for some computations and constructions related to $\mathcal{Q}_{CO}$.

Moduli space	Character
$\mathcal{Q}_{\text{btg}}$	$\phi_{1a} + \phi_{27a}$
$\mathcal{Q}_{\text{CO}}$	$\phi_{1a} + \phi_{35b}$
$\mathcal{Q}_{\text{AH}}$	$\phi_{1a} + \phi_{27a} + \phi_{35b} + \phi_{105b} + \phi_{120a}$
$\mathcal{Q}_{\text{SC}}$	$\phi_{1a} + \phi_{27a} + \phi_{35b}$
$\mathcal{Q}_{\text{GS}}$	$\phi_{1a} + \phi_{15a} + \phi_{35b} + \phi_{84a}$
$\mathcal{Q}_{\text{syz}}$	$\phi_{1a} + \phi_{27a} + \phi_{35b} + \phi_{84a} + \phi_{168a}$
$\mathcal{Q}_{\text{azy}}$	$\phi_{1a} + \phi_{27a} + \phi_{35b} + \phi_{105b} + \phi_{168a}$
$\mathcal{Q}_{\text{enn}}$	$\phi_{1a} + \phi_{70a} + \phi_{84a} + \phi_{105b} + \phi_{280a} + \phi_{420a}$
$\mathcal{Q}_{\text{OS}}$	$\phi_{1a} + \phi_{35a} + \phi_{84a}$

Remark 4.4 We can also compute the cohomology of $\mathcal{Q}_{\text{btg}}$ in the following way. The cohomology of the moduli space $Q_{\text{btg}}[2]$ of plane quartics with a marked bitangent line and level two structure is given in Table 2 as a representation of $\text{Sp}(6, \mathbb{F}_2)$. The cohomology of the quotient by $\text{Sp}(6, \mathbb{F}_2)$, i.e. the cohomology of Q_{btg}, can be read off as the invariant part, i.e. the part given by the trivial representation. This can be read off in column 1 of Table 2.

Remark 4.5 We can also compute the cohomology of $\mathcal{Q}_{\text{AH}}$ in the following way. The stabilizer G of an Aronhold heptad is isomorphic to the symmetric group S_7. The cohomology of $Q[2]$ was computed as a representation of G in $\text{Sp}(6, \mathbb{F}_2)$ in [6] (see also [7] and [9]). We reproduce the result in Table 4. We use the notation s_λ for the irreducible representation of S_7 corresponding to the partition λ of 7. In particular, s_7 denotes the trivial representation of S_7. We obtain the result by reading off the invariant part, i.e. the column corresponding to s_7.

Using the Grothendieck–Lefschetz trace formula, Theorem 4.2 and Corollary 2.6 gives the following corollary (by simply making the substitution $t = -q$ in Theorem 4.2).

Corollary 4.6 *Let q be a power of an odd prime number and let $\mathbb{F}_q$ be a finite field with q elements. The number of points over $\mathbb{F}_q$ of the moduli spaces introduced in Sect. 3 are as follows:*

Moduli space	Point count
$\mathcal{Q}_{\text{btg}}$	$q^6 - q + 2$
$\mathcal{Q}_{\text{CO}}$	$q^6 - q^5 - q + 4$
$\mathcal{Q}_{\text{AH}}$	$q^6 - q^5 - q^3 + 4q^2 - 6q + 6$
$\mathcal{Q}_{\text{SC}}$	$q^6 - q^5 - 2q + 5$
$\mathcal{Q}_{\text{GS}}$	$q^6 - q^5 - 2q + 11$
$\mathcal{Q}_{\text{syz}}$	$q^6 - q^5 + q^2 - 7q + 13$
$\mathcal{Q}_{\text{azy}}$	$q^6 - q^5 - q^3 + 3q^2 - 8q + 9$
$\mathcal{Q}_{\text{enn}}$	$q^6 - 3q^3 + 11q^2 - 13q + 11$
$\mathcal{Q}_{\text{OS}}$	$q^6 - 2q + 7$

Table 4 The cohomology of $Q[2]$ as a representation of S_7 (see [6, 7, 9])

	s_7	$s_{6,1}$	$s_{5,2}$	$s_{5,1^2}$	$s_{4,3}$	$s_{4,2,1}$	$s_{4,1^3}$	$s_{3^2,1}$	$s_{3,2^2}$	$s_{3,2,1^2}$
H^0	1	.	.	.	.	.	.	.	.	.
H^1	1	1	1	.	1	.	.	.	.	.
H^2	.	3	4	4	3	5	1	3	1	1
H^3	1	8	14	18	14	30	16	16	12	18
H^4	4	20	44	47	44	99	56	56	54	83
H^5	6	33	76	76	72	178	97	104	105	169
H^6	6	23	51	54	54	127	74	76	77	126

	$s_{3,1^4}$	$s_{2^3,1}$	$s_{2^2,1^3}$	$s_{2,1^5}$	s_{1^7}
H^0	.	.	.	.	.
H^1	.	.	.	.	.
H^2	.	.	.	.	.
H^3	4	6	3	.	.
H^4	32	31	25	6	1
H^5	71	65	64	26	3
H^6	54	54	50	22	5

Data availability All data generated or analysed during this study are included in this published article (and its supplementary information files).

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