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RESEARCH ARTICLE

On the differential transcendentality of the Morita *p*-adic gamma function

Elżbieta Adamus[1]

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Abstract
In p-adic analysis one can find an analog of the classical gamma function defined on the ring of p-adic integers. In 1975, Morita defined the p-adic gamma function Γ_p by a suitable modification of the function $n \mapsto n!$. In this note we prove that for any given prime number p the Morita p-adic gamma function Γ_p is differentially transcendental over $\mathbb{C}_p(X)$. The main result is an analog of the classical Hölder's theorem, which states that Euler's gamma function Γ does not satisfy any algebraic differential equation whose coefficients are rational functions.

Keywords p-Adic gamma function · Analog of gamma function · Differentially transcendental function · Hölder's theorem

Mathematics Subject Classification 11S80 · 12H05

1 Introduction

The problem of classifying functions in terms of closed-form expressions has a long history. Depending on the context we consider some set of basic functions and a list of admissible operations. Closed-form expressions use only a finite number of those admissible operations. Similarly we can ask if a given equation or a system of equations have a closed-form solution. We distinguish *algebraic* functions as solutions of polynomial equations. A function which is not algebraic is called *transcendental*.

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✉ Elżbieta Adamus
esowa@agh.edu.pl

1 Faculty of Applied Mathematics, AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Kraków, Poland

In a similar way we can classify functions using differential field extensions. If A is a differential ring we denote by $A\{Y\}$ the ring of *differential polynomials in the indeterminate* Y. It is exactly the polynomial ring $A[Y_i]$ in the indeterminates Y_i, $i \in \mathbb{N} \cup \{0\}$, together with the unique derivation determined by $Y_i' := Y_{i+1}$ and $Y := Y_0$. We distinguish *differentially algebraic* functions as solutions of algebraic differential equations. The corresponding complement is the class of *differentially transcendental* functions. Of course all differentially transcendental functions are transcendental. An example of a transcendental but not differentially transcendental function is e^x, which is a solution of the differential equation $y'(x) - y(x) = 0$.

Another examples of closed-form expressions are *elementary* functions. The notion was introduced by Joseph Liouville in the 1830s. This class of functions is closed under arithmetic operations, composition and differentiation, but not under integration. There are many well-known examples of elementary functions with non-elementary antiderivative, e.g. e^{-x^2}, $\frac{\sin x}{x}$ or $\sin x^2$. In general, the integral of a function expressible in closed-form may not be so. Liouville in fact considered a wider class of functions, namely *Liouvillian* functions. All algebraic functions are elementary and the class of Liouvillian functions contains all elementary functions and their integrals. All Liouvillian functions are differentially algebraic, but not conversely. This leads us to the characterization of linear differential equations *solvable by quadratures* or *solvable in finite terms* by means of their differential Galois group. With a given homogeneous linear differential equation we associate a differential field extension, namely the Picard–Vessiot extension. This extension is Liouvillian if and only if the identity component of its differential Galois group is solvable. In particular algebraic functions correspond to finite differential Galois groups. For more detailed information see for example [1, Chapter 6]. Solutions of the Bessel's differential equation $x^2 y''(x) + xy'(x) + (x^2 - a^2)\, y(x) = 0$, $a \in \mathbb{C}$, may be not Liouvillian. The computation of the Galois group of the Bessel equation can be found in [7, Appendix]. And finally there exist functions such as Euler's gamma function $\Gamma(z)$, which are not solutions of any algebraic differential equation.

The Galois theory also has its topological version. This theory was developed by Khovanskii (see for example [6]). It uses the *monodromy group* instead of the Galois group. In case of algebraic functions the monodromy group is isomorphic to the Galois group of the associated extension of the field of rational functions. For a detailed information see [6, Section 4.4.3]. The monodromy group is defined for many functions for which the Galois group is not. This approach to studying functions representable by quadratures involves topological obstructions related to branching. One can consider even wider class of functions than those representable by quadratures, namely meromorphic functions. In [6, Chapter 5], Khovanskii pointed out that the composition of functions is not an algebraic operation and that in differential algebra this operation is replaced with a differential equation describing it. However, this does not work for functions like $\Gamma(z)$. Topological Galois theory is dealing with such functions. However, this method uses branching, so it cannot be used to prove that a particular single-valued meromorphic function is not representable by quadratures.

A satisfactory Galois theory for linear differential equations was developed by Ellis Kolchin under the assumption that the subfield of constants of a considered differential field is algebraically closed. This theory is known as the Picard–Vesssiot theory. This theory can be also developed in case of formally real differential fields with a real closed field of constants and over formally p-adic differential fields with a p-adically closed field of constants (see [4]). As a result it is possible to characterise formally real Liouvillian extensions of real differential fields with a real closed field of constants (see [2]). These results have been generalized for partial differential real and p-adic fields in [3].

In this note we prove that for any given prime number p the Morita p-adic gamma function Γ_p is differentially transcendental over $\mathbb{C}_p(X)$.

2 Preliminaries

If $K \subset L$ is a differential field extension, then an element $f \in L$ is called *differentially algebraic* over K if there is a non-zero differential polynomial $P \in K\{Y\}$ such that $P(f) = 0$. If f is not differentially algebraic over K then we say that f is *differentially transcendental* over K.

One can consider the field of rational functions $\mathbb{C}(z)$ of a single complex variable z as a differential field equipped with the standard derivation $\frac{d}{dz}$. Functions which are differentially transcendental over $\mathbb{C}(z)$ are also called *transcendentally transcendental* or *hypertranscendental*. A well-known example of a differentially transcendental function is *Euler's gamma function* $\Gamma(z)$. If we consider the Euler integral

$$\int_0^\infty t^{z-1} e^{-t}\, dt,$$

it converges absolutely on $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. For any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$ we define

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Using the equality $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ one can extend the definition on the set $\mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, where $\mathbb{Z}_{\leqslant 0} = \{k \in \mathbb{Z} : k \leqslant 0\}$. Observe that $\Gamma(z+n)$ is holomorphic for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > -n$. We use the identity theorem for holomorphic functions to obtain the meromorphic function $\Gamma\colon \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0} \to \mathbb{R}_+$. Hölder's theorem states that Γ does not satisfy any algebraic differential equation whose coefficients are rational functions. For a proof see the paper by Totik [10].

Theorem 2.1 (Hölder, 1887) *The gamma function* Γ *is differentially transcendental over* $\mathbb{C}(z)$.

3 The Morita *p*-adic gamma function

In p-adic analysis one can find an analog of the classical gamma function defined on the ring of p-adic integers. In 1975 in [8], Morita defined the p-adic gamma function Γ_p by a suitable modification of the function $n \mapsto n!$.

We denote by $\mathbb{Z}_p$ and $\mathbb{Q}_p$ the ring of p-adic integers and the field of p-adic numbers, respectively. By $\mathbb{Z}_p^\times$ we denote the group of invertible elements in $\mathbb{Z}_p$ and by $\mathbb{C}_p$ we denote the completion of the algebraic closure of the field $\mathbb{Q}_p$. The factorial function $n \mapsto n!$ cannot be extended by continuity to a function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$, such that $f(n) = nf(n-1)$ for all integers greater than 0. When p is odd, one needs to consider the *restricted factorial* $n!*$ given by

$$n!* = \prod_{\substack{1\leqslant j\leqslant n\\ p\nmid j}} j$$

and use the generalization of the Wilson congruence, namely

$$\prod_{\substack{a\leqslant j<a+p^\nu\\ p\nmid j}} j \equiv -1 \mod p^\nu,$$

where a and $\nu \geqslant 1$ are integers. As a consequence we obtain that

$$f(n) := (-1)^n \prod_{\substack{1\leqslant j<n\\ p\nmid j}} j, \quad n \geqslant 2,$$

satisfies $f(a) = f(a + mp^\nu) \bmod p^\nu$ for $m \in \mathbb{Z}, m \geqslant 1$.

The *Morita p-adic gamma function* is the continuous function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ extending

$$f(n) := (-1)^n \prod_{\substack{1\leqslant j<n\\ p\nmid j}} j = (-1)^n (n-1)!*,$$

for $n \geqslant 2$. One can observe that the values of Γ_p belong to $\mathbb{Z}_p^\times$. Recall that $\mathbb{Z}_p = p\mathbb{Z}_p \cup \mathbb{Z}_p^\times$, where $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p : \|x\|_p = 1\}$ and $p\mathbb{Z}_p = \{x \in \mathbb{Z}_p : \|x\|_p < 1\}$ is the only maximal ideal of the ring $\mathbb{Z}_p$. What is more

$$\Gamma_p(x+1) = h_p(x)\cdot\Gamma_p(x), \quad \text{where} \quad h_p(x) = \begin{cases} -x; & x \in \mathbb{Z}_p^\times, \\ -1; & x \in p\mathbb{Z}_p. \end{cases} \tag{1}$$

It can be proved that one can also define Γ_2 and that the formula given above holds also for $p = 2$. For more detailed information see [9, Section 7.1.7].

4 Main result

Let us note that Γ_p may be extended to a nonempty ball of $\mathbb{C}_p$, so it is a locally analytic function on $\mathbb{C}_p$. It has a power series expansion that converges for $\|x\|_p < p^{-(2p-1)/(p(p-1))}$. Details can be found in [5]. Consider the field $\mathbb{C}_p(X)$ of rational functions with coefficients in $\mathbb{C}_p$. We claim the following.

Theorem 4.1 *For a given prime number p, the Morita p-adic gamma function Γ_p is differentially transcendental over $\mathbb{C}_p(X)$.*

In the following proof we consider the *antilexicographical ordering* of monomials in $\mathbb{C}_p(X)[Y_0, Y_1, \dots, Y_n]$. If we consider $\alpha, \beta \in \mathbb{Z}^{n+1}$, $\alpha \geqslant 0$, $\beta \geqslant 0$, then $\alpha > \beta$ if in the vector difference $\alpha - \beta$ the leftmost nonzero entry is negative. For $Y = (Y_0, Y_1, \dots, Y_n)$ we write $Y^\alpha > Y^\beta$ if $\alpha > \beta$. Thus we have $Y_0 < Y_1 < \cdots < Y_n$.

Proof Suppose on the contrary, that Γ_p is not differentially transcendental over $\mathbb{C}_p(X)$. If Γ_p is differentially algebraic over $\mathbb{C}_p(X)$, then so is its restriction $g\colon p\mathbb{Z}_p \to \mathbb{Z}_p^\times$, for all $x \in p\mathbb{Z}_p$, $g(x) = \Gamma_p(x)$. Hence there exists a polynomial $P \in \mathbb{C}_p[X][Y_0, Y_1, \dots, Y_n]$ such that $P(X, Y_0, Y_1, \dots, Y_n) \not\equiv 0$ and

$$P(x, g(x), g'(x), \dots, g^{(n)}(x)) = 0 \quad \text{for all } x \in p\,\mathbb{Z}_p. \tag{2}$$

Without loss of generality one can assume that P contains a monomial having a non-zero power of one of the indeterminates $Y_0, Y_1, \dots, Y_n$. This is due to the fact that P cannot define a function which is zero on $g(p\mathbb{Z}_p)$. Let $q(X)\,Y_0^{a_0}Y_1^{a_1}\dots Y_n^{a_n}$, where $q \in \mathbb{C}_p[X]$, be the leading term of P, i.e. the one with the biggest $(a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$ with respect to the antilexicographical ordering. Let us denote it by $\mathrm{LT}(P)$. Assume that from all such polynomials we choose P which is minimal, in the sense that it has a minimal leading term $\mathrm{LT}(P)$ and moreover $\deg q$ is minimal and the leading coefficient of q is 1.

For every $x \in p\mathbb{Z}_p$ we have $1 = \|1\|_p \neq \|x\|_p < 1$. Hence $\|x + 1\|_p = \max\{\|x\|_p, 1\} = 1$ and that is equivalent to $x + 1 \in \mathbb{Z}_p^\times$. By (1) for every $x \in p\mathbb{Z}_p$ we obtain

$$\begin{aligned}
\Gamma_p(x+p) &= -(x+p-1)\,\Gamma_p(x+p-1)\\
&= (-1)^2(x+p-1)(x+p-2)\,\Gamma_p(x+p-2)\\
&= (-1)^{p-1}(x+p-1)(x+p-2)\cdots(x+1)\,\Gamma_p(x+1)\\
&= (-1)^p(x+p-1)(x+p-2)\cdots(x+1)\,\Gamma_p(x).
\end{aligned}$$

Consequently,

$$g(x+p) = f(x)g(x) \quad \text{for all } \ x \in p\,\mathbb{Z}_p, \tag{3}$$

where $f(x) = (-1)^p (x+p-1)(x+p-2)\cdots(x+1)$. We compute

$$\begin{aligned} g'(x+p) &= f'(x)g(x) + f(x)g'(x), \\ g''(x+p) &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x), \\ &\dots \\ g^{(n)}(x+p) &= \sum_{k=0}^{n} f^{(n-k)}(x)g^{(k)}(x). \end{aligned}$$

By (2) and (3) for every $x \in p\mathbb{Z}_p$ we have

$$P\left(x+p, f(x)g(x), f'(x)g(x) + f(x)g'(x), \dots, \sum_{k=0}^{n} f^{(n-k)}(x)g^{(k)}(x)\right) = 0.$$

We define

$$\begin{aligned} &Q(X, Y_0, Y_1, \dots, Y_n) \\ &\qquad := P\left(x+p, f(x)Y_0, f'(x)Y_0 + f(x)Y_1, \dots, \sum_{k=0}^{n} f^{(n-k)}(x)Y_k\right). \end{aligned}$$

Consequently, $Q(x, g(x), g'(x), \dots, g^{(n)}(x)) = 0$ for every $x \in p\mathbb{Z}_p$. Since P is minimal, P must divide Q. Applying the Euclidean algorithm to $\mathrm{LT}(P)$ and $\mathrm{LT}(Q) = q(X+p) \cdot f(X)^{a_0+a_1+\dots+a_n} Y_0^{a_0} Y_1^{a_1} \dots Y_n^{a_n}$ we conclude that there exists $R(X) \in \mathbb{C}_p[X]$ such that

$$Q(X, Y_0, Y_1, \dots, Y_n) = R(X)P(X, Y_0, Y_1, \dots, Y_n). \tag{4}$$

More precisely, $R(X) = \frac{q(X+p)}{q(X)} \cdot f(X)^{a_0+a_1+\dots+a_n}$. Since $\deg f \geqslant 1$ and $a_0 + a_1 + \dots + a_n \neq 0$, then $\deg R \geqslant 1$. $\mathbb{C}_p$ is algebraically closed, so the nonzero polynomial R must have a root $x_0 \in \mathbb{C}_p$. Substituting x_0 into (4) we obtain

$$P\left(x_0+p, f(x_0)Y_0, f'(x_0)Y_0 + f(x_0)Y_1, \dots, \sum_{k=0}^{n} f^{(n-k)}(x_0)Y_k\right) = 0.$$

If $f(x_0) \neq 0$, then $X - x_0 - p$ divides P, a contradiction with the minimality of P. Therefore $f(x_0) = 0$. A change of variables then yields

$$P(x_0 + p, 0, Z_1, \dots, Z_n) = 0,$$

for some $x_0 \in \{1-p, 2-p, \ldots, -1\}$. Substituting $X = x_0 + p$ and $Y_0 = 0$ into (4) gives us

$$P\left(x_0 + 2p, 0, f(x_0+p)Y_1, \ldots, \sum_{k=1}^{n} f^{(n-k)}(x_0+p)Y_k\right) \\ = R(x_0+p)P(x_0+p, 0, Y_1, \ldots, Y_n) = 0.$$

After performing a suitable change of variables we obtain

$$P(x_0 + 2p, 0, Z_1, \ldots, Z_n) = 0.$$

By induction

$$P(x_0 + mp, 0, Z_1, \ldots, Z_n) = 0 \quad \text{for all} \quad m \in \mathbb{Z}, \quad m \geqslant 1.$$

Hence $P(X, 0, Z_1, \ldots, Z_n) = 0$ and Y_0 divides P. We obtain a contradiction with the minimality of P. Thus g is differentially transcendental over $\mathbb{C}_p(X)$. As a result Γ_p is differentially transcendental over $\mathbb{C}_p(X)$. $\square$

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