RESEARCH ARTICLE

Geodesic flows of c-projectively equivalent metrics are quantum integrable

Jan Schumm[1] ⓘ

Abstract

Given two c-projectively equivalent metrics on a Kähler manifold, we show that canonically constructed Poisson-commuting integrals of motion of the geodesic flow, linear and quadratic in momenta, also commute as quantum operators. The methods employed here also provide a proof of a similar statement in the case of projective equivalence. We also investigate the addition of potentials, i.e. the generalization to natural Hamiltonian systems. We show that commuting operators lead to separation of variables for Schrödinger's equation.

1 C-projective geometry, integrals and quantization rules

Definition 1.1 A *Kähler manifold* (of arbitrary signature) is a manifold $\mathcal{M}^{2n}$ of real dimension $2n$ endowed with the following objects:

- a (pseudo-)Riemannian metric g and its associated Levi-Civita connection ∇,
- a complex structure J, i.e. an endomorphism on the space of vector fields with $J^2 = -\mathrm{Id}$,
- g and J must be compatible in the sense that $g(JX, Y) = -g(X, JY)$ and $\nabla J = 0$,
- we denote by Ω the two-form $\Omega(X, Y) = g(JX, Y)$.

✉ Jan Schumm
  jsmaths@gmx.net

1   Friedrich-Schiller-Universität Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany

Definition 1.2 A regular curve $\gamma : I \to \mathcal{M}$ is called *J-planar* if there exist functions $A, B : I \to \mathbb{R}$ such that

$$\nabla_{\dot\gamma}\,\dot\gamma = A(t)\,\dot\gamma + B(t)J(\dot\gamma) \tag{1}$$

is fulfilled on I. Here $\dot\gamma$ denotes the tangent vector to γ.

This is a natural generalization of geodesics on (pseudo-)Riemannian manifolds that in arbitrary parametrization are solutions of the equation $\nabla_{\dot\gamma}\,\dot\gamma = A(t)\,\dot\gamma$. Similarly, the property of a curve to be J-planar survives under reparametrization.

Definition 1.3 Let $g, \tilde{g}$ be two Kähler metrics (of arbitrary signature) on $(\mathcal{M}, J)$. They are called *c-projectively equivalent* if and only if every J-planar curve of g is also a J-planar curve of $\tilde{g}$. (If every J-planar curve of g is also a $\tilde{J}$-planar curve of $\tilde{g}$ and vice versa then the complex structures $J, \tilde{J}$ coincide up to a sign, so we did not restrict ourselves in defining c-projective equivalence for the case where both metrics are Kähler w.r.t. the same complex structure.)

This is the Kähler analogue of *projective equivalence* on (pseudo-)Riemannian manifolds and it was proposed by Otsuki and Tashiro [13]. For a thorough introduction to c-projective geometry see [4]. For completeness, we recall the definition of projective equivalence, since Theorem 2.3 is the projective analogue of Theorem 2.2. Since their proofs run in parallel, all statements about the projective setting will be phrased as remarks placed after their c-projective counterparts.

Definition 1.4 Let $g, \tilde{g}$ be two (pseudo-)Riemannian metrics on a manifold $\mathcal{M}$. They are called *projectively equivalent* if and only if every unparametrized geodesic of g is also an unparametrized geodesic of $\tilde{g}$.

1.1 The tensor *A*

Theorem 1.5 (see [6], and also [4, Section 5]) *Two (pseudo-)Riemannian metrics $g, \tilde{g}$ that are Kähler on a manifold $(\mathcal{M}, J)$ are c-projectively equivalent if and only if the tensor*

$$A^i_j \overset{\text{def}}{=} \left| \frac{\det \tilde{g}}{\det g} \right|^{\frac{1}{2(n+1)}} \tilde{g}^{il} g_{lj}, \quad where \;\; \tilde{g}^{il} \tilde{g}_{lm} = \delta^i_m,$$

satisfies the equation

$$\nabla_k A_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \bar{\lambda}_i \Omega_{jk} + \bar{\lambda}_j \Omega_{ik}, \tag{2}$$

where

$$\lambda \overset{\text{def}}{=} \frac{1}{4}\operatorname{tr} A, \quad \lambda_i \overset{\text{def}}{=} \nabla_i \lambda \;\; and \;\; \bar{\lambda}_i = J^j_i \lambda_j.$$

Here and throughout the rest of the paper we use the Einstein sum convention. An index preceded by a comma is meant to indicate a covariant derivative. Raising and lowering indices is always by means of g: $\lambda^i = g^{ij}\lambda_j$, where $g^{is}g_{sj} = \delta^i_j$. A covariant (*c*-)*projectively equivalent metric* $\tilde{g}$, as stated above will be the sole exception: $\tilde{g}^{is}\tilde{g}_{sj} = \delta^i_j$.

Remark 1.6 (*projective case* [2]) Two (pseudo-)Riemannian metrics g, $\tilde{g}$ on a manifold $\mathcal{M}$ are projectively equivalent if and only if the tensor

$$A^i_j \overset{\text{def}}{=} \left|\frac{\det \tilde{g}}{\det g}\right|^{\frac{1}{n+1}} \tilde{g}^{il}g_{lj}, \quad \text{where} \;\; \tilde{g}^{il}\tilde{g}_{lm} = \delta^i_m, \tag{3}$$

satisfies the equation

$$\nabla_k A_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} \;\; \text{with} \;\; \lambda \overset{\text{def}}{=} \frac{1}{2}\operatorname{tr} A, \;\; \lambda_i \overset{\text{def}}{=} \nabla_i \lambda. \tag{4}$$

Definition 1.7 We shall call Hermitian (g-self-adjoint and J-commuting) solutions of (2) *c-projectively compatible* or simply *c-compatible* with (g, J). Likewise, symmetric solutions of (4) shall be called *projectively compatible* or simply *compatible* with g.

1.2 Conserved quantities of the geodesic flow

Throughout this paper we shall canonically identify symmetric contravariant tensors with polynomials on $T^*\mathcal{M}$ via the isomorphisms $^\flat$ and $^\#$:

$$^\flat: T^{a_1\ldots a_l} \mapsto T^{a_1\ldots a_l} p_{a_1} \ldots p_{a_l},$$
$$^\#: T^{a_1\ldots a_l} p_{a_1} \ldots p_{a_l} \mapsto T^{(a_1\ldots a_l)}.$$

By the parentheses we mean symmetrisation with the appropriate combinatorial factor: $T^{(a_1\ldots a_l)} = 1/l! \sum_{(b_1\ldots b_l)=\pi(a_1\ldots a_l)} T^{b_1\ldots b_l}$, where π means any permutation.

Let (g, J, A) be c-compatible on $\mathcal{M}$ and consider the one-parameter family

$$\overset{t}{K}{}^{ij} \overset{\text{def}}{=} \sqrt{\det(t\operatorname{Id} - A)}\, (t\operatorname{Id} - A)^{-1}{}^i_l\, g^{lj}. \tag{5}$$

Throughout this paper the root is to be taken in such a way that we simply halve the powers of the eigenvalues in $\det(t\operatorname{Id} - A)$. This is well defined because all eigenvalues of A are of even multiplicity. In particular, $\sqrt{\det(t\operatorname{Id} - A)}$ can be negative and it is smooth also near points where $\det(t\operatorname{Id} - A) = 0$. With the tensors $\overset{t}{K}$ we associate the functions $\overset{t}{I}: T^*\mathcal{M} \to \mathbb{R}$

$$\overset{t}{I} \overset{\text{def}}{=} \overset{t}{K}{}^{ij} p_i p_j. \tag{6}$$

Theorem 1.8 (Topalov [14], see also [4, Section 5]) *Let (g, J, A) be c-compatible. Then for any pair of real numbers (v, w) the quantities $\overset{v}{I}$ and $\overset{w}{I}$ Poisson-commute, i.e. $\{\overset{v}{I}, \overset{w}{I}\} = 0$.*

Remark 1.9 The quantities $\overset{t}{K}$ are well-defined for all values of t: if we denote by $2n$ the dimension of the manifold, then $\overset{t}{K}$ is a polynomial of degree $n - 1$. This is a consequence of $J^2 = -\mathrm{Id}$, the antisymmetry of J with respect to g, the commutativity of A with J and the construction of $\overset{t}{K}$ from A. In particular the coefficient of t^{n-1} of $\overset{t}{K}{}^{ij}$ is g^{ij}, which can be easily confirmed when looking at (5) in a frame such that A is in Jordan normal form.

Remark 1.10 (projective case) Let (g, A) be compatible. We shall define

$$\overset{t}{K}{}^{ij} \overset{\text{def}}{=} \det(t\mathrm{Id} - A)\,(t\mathrm{Id} - A)^{-1}{}^{i}_{l}\,g^{lj}$$

for the projective case. Then for any pair $s, t \in \mathbb{R}$ the quantities $\overset{s}{I} \overset{\text{def}}{=} \overset{s}{K}{}^{ij} p_i p_j$ and $\overset{t}{I} \overset{\text{def}}{=} \overset{t}{K}{}^{ij} p_i p_j$ are commuting integrals of the geodesic flow for g [2, 12]. The projective and the c-projective cases differ merely by the power of the determinant.

For a c-compatible structure (g, J, A) there also exists a one-parameter family of commuting integrals of the geodesic flow that are linear in momenta given by

$$\overset{t}{L} = \overset{t}{V}{}^{j} p_j, \quad \overset{t}{V}{}^{j} = J^{j}_{k} g^{ki} \nabla_i \sqrt{\det(t\mathrm{Id} - A)}. \tag{7}$$

The commutation relations

$$\{\overset{t}{L}, \overset{s}{I}\} = 0 \tag{8}$$

also hold. The number of functionally independent integrals within the family $\overset{t}{I}$ is equal to the degree of the minimal polynomial of A. The number of functionally independent integrals within $\overset{t}{L}$ is equal to the number of nonconstant eigenvalues of A. Furthermore the integrals $\overset{t}{I}$ are functionally independent from $\overset{t}{L}$, see [4, Section 5]. Thus the canonically constructed integrals are sufficient in number for Liouville-integrability of the geodesic flow if all eigenvalues of A are non-constant (then they are automatically pairwise different and each eigenvalue is of multiplicity two [4, Lemma 5.16]).

1.3 Quantization rules and commutators of operators

We adopt the quantization rules introduced by Carter [5] and Duval and Valent [7, Section 3]. The formulae they give construct differential operators that are independent

of coordinate choice and symmetric with respect to the scalar product (the bar indicates complex conjugation)

$$\langle \phi, \psi \rangle = \int_{\mathcal{M}} \bar{\varphi} \psi$$

on the completion of the space $\mathcal{H}(\mathcal{M})$ of compactly supported half-densities on $\mathcal{M}$. For more details and a reasoning we refer to [5, 7] and the references therein. It is sufficient for our purposes to recall the quantization formulae they give: for a homogeneous polynomial $P_m \colon T^*\mathcal{M} \to \mathbb{R}$ of degree m we construct its symmetric contravariant tensor via $^\#$ and compose with the covariant derivative:

$$
\begin{aligned}
P_0 &\mapsto \widehat{P_0} \stackrel{\text{def}}{=} P_0^\# \, \mathrm{Id}, \\
P_1 &\mapsto \widehat{P_1} \stackrel{\text{def}}{=} \frac{i}{2} \left((P_1^\#)^j \circ \nabla_j + \nabla_j \circ (P_1^\#)^j \right), \\
P_2 &\mapsto \widehat{P_2} \stackrel{\text{def}}{=} -\nabla_j \circ (P_2^\#)^{jk} \circ \nabla_k, \\
P_3 &\mapsto \widehat{P_3} \stackrel{\text{def}}{=} -\frac{i}{2} \left(\nabla_j \circ (P_3^\#)^{jkl} \circ \nabla_k \circ \nabla_l + \nabla_j \circ \nabla_k \circ (P_3^\#)^{jkl} \circ \nabla_l \right).
\end{aligned}
\tag{9}
$$

For polynomials that are not homogeneous the quantization shall be done by quantizing the homogeneous parts and adding the results. So far we have been considering polynomials of degree two on the cotangent bundle of degree at most two and covariant tensors of valence at most $(2, 0)$. These correspond to differential operators of degree at most 2. But the commutator of two such second order operators generally is an operator of order three. Later on we can facilitate the expression for the commutator of the quantum operators of two polynomials of degree two by using the quantum operator of the Poisson bracket of the two polynomials of degree two.

2 Results

The main result of this paper is a quantum version of Theorem 1.8: using the quantization rules (9) from [5] and [7] we construct differential operators from symmetric covariant tensors and show that these differential operators commute.

Let (g, J, A) be c-compatible and $\overset{t}{K}, \overset{t}{I}$ denote the associated Killing tensors and integrals of the geodesic flow. By (9) their associated quantum operators are

$$\widehat{\overset{t}{I}}(f) \stackrel{\text{def}}{=} -\nabla_j \circ \overset{t}{K}{}^{jk} \circ \nabla_k f.$$

(The different letters I and K must not confuse the reader, for $\overset{t}{I}{}^\# = \overset{t}{K}$.)

Remark 2.1 Recalling Remark 1.9, $\widehat{\overset{t}{I}}$ is a polynomial of degree $n - 1$ in t and its lead coefficient is $-\Delta_g$, the negative of the Laplace–Beltrami operator of g.

Theorem 2.2 *Let* (g, J, A) *be* c-compatible. *Then for any pair* (v, w) *the operators* $\overset{v}{\hat{I}}$ *and* $\overset{w}{\hat{I}}$ *commute, i.e.* $[\overset{v}{\hat{I}}, \overset{w}{\hat{I}}] = 0$.

This is a new result for both the Riemannian as well as the pseudo-Riemmanian case.

Theorem 2.3 *Let* (g, A) *be* projectively *compatible. Then for any pair* (v, w) *the operators* $\overset{v}{\hat{I}}$ *and* $\overset{w}{\hat{I}}$ *commute, i.e.* $[\overset{v}{\hat{I}}, \overset{w}{\hat{I}}] = 0$.

Remark 2.4 Theorem 2.3 was already proven by Matveev [10, 11]. The proof that we give however uses only C^3-smoothness whereas the original proof used C^8. The proof that will be given here runs in parallel with the proof of Theorem 2.2. A series of remarks to the proof of Theorem 2.2 will thus provide the proof of Theorem 2.3, giving the intermediate steps for the projective case and pointing out the analogues and differences.

We then improve the result of Theorem 2.2 by adding potential terms to these second order differential operators, finding commuting quantum observables for certain natural Hamiltonian systems.

Theorem 2.5 *Let* (g, J, A) *be c-compatible. Let*

$$\overset{t}{\hat{I}} \overset{\text{def}}{=} -\nabla_j \circ \overset{t}{K}{}^{jk} \circ \nabla_k, \quad \overset{t}{K}{}^{ij} \overset{\text{def}}{=} \sqrt{\det(t\mathrm{Id} - A)}\,(t\mathrm{Id} - A)^{-1}{}^i_l\, g^{lj}$$

be as in Theorem 2.2. *Let*

$$\overset{t}{\hat{L}} = \frac{i}{2}\left(\nabla_j \circ \overset{t}{V}{}^j + \overset{t}{V}{}^j \circ \nabla_j\right), \quad \overset{t}{V}{}^j = J^j_k g^{ki} \nabla_i \sqrt{\det(t\mathrm{Id} - A)} \tag{10}$$

be the differential operators associated with the canonical Killing vector fields of g. *Let* $\overset{\mathrm{nc}}{E} = \{\varrho_1, \dots, \varrho_r\}$ *be the set of non-constant eigenvalues of* A. *Let* $\overset{\mathrm{c}}{E} = \{\varrho_{r+1}, \dots, \varrho_{r+R}\}$ *be the set of constant eigenvalues and* $E = \overset{\mathrm{nc}}{E} \cup \overset{\mathrm{c}}{E}$. *Denote by* $m(\varrho_i)$ *the algebraic multiplicity of* ϱ_i. *Let the family of potentials* $\overset{t}{U}$, *parametrized by* t, *be given by*

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2 - 1} f_i \tag{11}$$

with $d f_l \circ A = \varrho_l d f_l$ *for all* $l = 1, \dots, r + R$ *and with* $d f_l$ *proportional to* $d\varrho_l$ *for all* l *for which* ϱ_l *is non-constant. Let associated operators* $\overset{t}{\hat{U}}$ *act on functions by mere multiplication, i.e. for any point* p *we have* $(\overset{t}{\hat{U}}(f))(p) = \overset{t}{U}(p) f(p)$. *Then the operators*

$$\overset{t}{\hat{Q}} \overset{\text{def}}{=} \overset{t}{\hat{I}} + \overset{t}{\hat{U}}, \quad \overset{t}{\hat{L}} \tag{12}$$

commute within the one-parameter-families as well as crosswise, i.e. for all values of $t, s \in \mathbb{R}$:

$$[\overset{t}{\widehat{Q}}, \overset{s}{\widehat{Q}}] = [\overset{t}{\widehat{Q}}, \overset{s}{\widehat{L}}] = [\overset{t}{\widehat{L}}, \overset{s}{\widehat{L}}] = 0. \tag{13}$$

Remark 2.6 Recalling Remark 1.9, $\overset{t}{\widehat{Q}}$ is a polynomial of degree $n - 1$ in t, whose lead coefficient is the Schrödinger operator $-\Delta_g + \widehat{U}$, where U is simply a name introduced for the lead coefficient of $\overset{t}{U}$.

Remark 2.7 Since $\overset{t}{V}{}^j = J_k^j g^{ki} \nabla_i \sqrt{\det(t\mathrm{Id} - A)}$ is a Killing vector field for any choice of the real parameter t [4, Section 5], we have $\overset{t}{\widehat{L}} = \frac{i}{2}(\nabla_j \circ \overset{t}{V}{}^j + \overset{t}{V}{}^j \circ \nabla_j) = i\overset{t}{V}{}^j \circ \nabla_j$.

Remark 2.8 1. We do not discuss whether the $\overset{t}{U}$ are smooth at all points of the manifold. Smoothness is guaranteed at points that have a neighbourhood in which the number of different eigenvalues is constant (see Definition 2.11 of regular points below), provided that the f_i are smooth.

2. Formula (11) generally allows $\overset{t}{U}$ to be complex-valued. The conditions under which $\overset{t}{U}$ is real for any choice of $t \in \mathbb{R}$ are the following: for any real eigenvalue ϱ_i of A the corresponding function f_i must be real-valued. For all pairs $(\varrho_i, \varrho_j = \bar{\varrho}_i)$ of complex-conjugate eigenvalues of A the corresponding functions f_i and f_j must be complex conjugate to each other: $f_i = \bar{f}_j$.

3. The potentials that are admissible to be added to the quantum operators are the same that may be added to the Poisson commuting integrals. In the proof we show that the quantization imposes no stronger conditions on the potential than classical integrability and then use the the conditions imposed by the Poisson-brackets to find the allowed potentials.

Theorem 2.9 *Let (g, J, A) be c-compatible and A semi-simple. Let $\overset{t}{\widehat{I}}, \overset{t}{\widehat{L}}$ be as in Theorem 2.5. Then, for the operators*

$$\overset{t}{\widehat{Q}} \overset{\mathrm{def}}{=} \overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \quad \overset{t}{\widehat{L}} \tag{14}$$

the commutation relations $[\overset{t}{\widehat{Q}}, \overset{s}{\widehat{Q}}] = [\overset{t}{\widehat{Q}}, \overset{s}{\widehat{L}}] = 0$ are satisfied if and only if the potentials are of the form (11) with the sole exception that a function of t alone may be added to $\overset{t}{\widehat{U}}$.

A result similar to Theorems 2.5 and 2.9 for projective geometry has been published in [2, Theorem 8]. Lemma 3.15 explains how this theorem is similar in nature to the ones at hand.

Corollary 2.10 *Let* (g, J, A), $\overset{t}{\widehat{I}}$, $\overset{t}{\widehat{L}}$, $\overset{t}{\widehat{U}}$ *be as in Theorem* 2.5. *Let* $\widehat{I}_{(l)}$, $\widehat{L}_{(l)}$, $\widehat{U}_{(l)}$ *be the coefficients of* t^l *in* $\overset{t}{\widehat{I}}$, $\overset{t}{\widehat{L}}$, $\overset{t}{\widehat{U}}$ *respectively. Then the commutation relations*

$$\big[\overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \widehat{I}_{(l)} + \widehat{U}_{(l)}\big] = \big[\overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \widehat{L}_{(l)}\big] = \big[\overset{t}{\widehat{L}}, \widehat{I}_{(m)} + \widehat{U}_{(m)}\big]$$
$$= \big[\overset{t}{\widehat{L}}, \widehat{L}_{(l)}\big] = \big[\overset{t}{\widehat{L}}, \widehat{I}_{(m)} + \widehat{U}_{(m)}\big] = 0, \tag{15}$$

$$\big[\widehat{I}_{(l)} + \widehat{U}_{(l)}, \widehat{I}_{(m)} + \widehat{U}_{(m)}\big] = \big[\widehat{L}_{(l)}, \widehat{L}_{(m)}\big] = \big[\widehat{I}_{(l)} + \widehat{U}_{(l)}, \widehat{L}_{(m)}\big] = 0 \tag{16}$$

hold true for any value of t *and any values* $l, m \in \{1, \dots, n-1\}$.

Equations (16) *are equivalent to*

$$\big[\overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}\big] = \big[\overset{t}{\widehat{L}}, \overset{s}{\widehat{L}}\big] = \big[\overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \overset{s}{\widehat{L}}\big] = 0. \tag{17}$$

Equations (15), (16), (17) *remain true if a function of* t *alone is added to* $\overset{t}{\widehat{U}}$ *and constants* $c_{(l)}$ *are added to* $\widehat{U}_{(l)}$. *If all eigenvalues of* A *are non-constant and* $\overset{t}{\widehat{I}}$, $\overset{t}{\widehat{V}}$ *are as in Theorem* 2.5 *then no other than the described* $\widehat{U}_{(l)}$ *can be found such that the commutation relations above hold.*

Lastly, we shall show how the search for common eigenfunctions of the operators can be reduced to differential equations in lower dimension in appropriate coordinates around regular points. In particular, if all eigenvalues of A are non-constant we can reduce it to ordinary differential equations only. Moreover, the case where all eigenvalues of A are non-constant provides an example of *reduced separability* of Schrödinger's equation as described in [1].

Definition 2.11 Let (g, J, A) be c-compatible. A point $x \in \mathcal{M}$ is called *regular with respect to* A if in a neighbourhood of x the number of different eigenvalues of A is constant and for each eigenvalue ϱ either $d\varrho \neq 0$ or ϱ is constant in a neighbourhood of x. The set of regular points shall be denoted by $\mathcal{M}^0$.

Definition 2.12 Let (g, J, A) be c-compatible on $\mathcal{M}$. A *local normal coordinate system for* $\mathcal{M}$ is a coordinate system where (g, J, A) assume the form of Example 2.13. Existence of such coordinates in the neighbourhood of regular points is guaranteed by [3, Theorem 1.6].

Example 2.13 (*General example for c-compatible structures* (g, J, ω, A) [3, Example 5]) Let $2n \geqslant 4$ and consider an open subset W of $\mathbb{R}^{2n}$ of the form $W = U \times V \times S_1 \times \cdots \times S_L \times S_{L+1} \times \cdots \times S_{L+Q}$ for open subsets $V, U \subseteq \mathbb{R}^r$, $S_\gamma \subseteq \mathbb{R}^{4m_{c_\gamma}}$ for $\gamma = 1, \dots, L$ and $S_\gamma \subseteq \mathbb{R}^{2m_{c_\gamma}}$ for $\gamma = L+1, \dots, L+Q$. Let $t_1, \dots, t_l$ be the coordinates on V and let the coordinates on U be separated into l complex coordinates $z_1, \dots, z_l$ and q real coordinates $x_{l+1}, \dots, x_{l+q}$ and introduce the tuple $(\chi_1, \dots, \chi_r) = (z_1, \bar{z}_1, \dots, z_l, \bar{z}_l, x_{l+1}, \dots, x_{l+q})$. Suppose the following data is given on these open subsets:

- Kähler structures $(g_\gamma, J_\gamma, \omega_\gamma)$ on S_γ for $\gamma = 1, \ldots, L + Q$.
- For each $\gamma = 1, \ldots, L + Q$, a parallel Hermitian endomorphism $A_\gamma : TS_\gamma \to TS_\gamma$ for (g_γ, J_γ). For $\gamma = 1, \ldots, L$, A has a pair of complex conjugate eigenvalues $c_\gamma, \bar{c}_\gamma$ of equal algebraic multiplicity $m(c_\gamma) = m(\bar{c}_\gamma)$. For $\gamma = L+1, \ldots, L+Q$, A has a single real eigenvalue c_γ of algebraic multiplicity $m(c_\gamma)$.
- Holomorphic functions $\sigma_j(z_j)$ for $1 \leqslant j \leqslant l$ and smooth functions $\sigma_j(x_j)$ for $l+1 \leqslant j \leqslant r$.

Moreover, we choose one-forms $\alpha_1, \ldots, \alpha_r$ on $S = S_1 \times \cdots \times S_{L+Q}$ that satisfy

$$
\mathrm{d}\alpha_i = (-1)^i \sum_{\gamma=1}^{L+Q} \omega_\gamma(A_\gamma^{r-i}\cdot, \cdot) \tag{18}
$$

where A_γ^{r-i} denotes the $(r - i)$th power of A_γ.

To facilitate the expressions for the c-compatible structure that will be constructed, the following expressions shall be introduced: the tuple $E = (\varrho_1, \ldots, \varrho_n) = (\sigma_1, \bar{\sigma}_1, \ldots, \sigma_l, \bar{\sigma}_l, \sigma_{l+1}, \ldots, \sigma_{l+q}, c_1, \bar{c}_1, \ldots, c_L, \bar{c}_L, c_{L+1}, \ldots, c_{L+Q})$ contains the designated eigenvalues for A. Their algebraic multiplicities shall be denoted by $(m(\varrho_l), l = 1, \ldots, r + R) = (2, \ldots, 2, m(c_1), m(\bar{c}_1), \ldots)$. The non-constant eigenvalues shall be collected in order in $\overset{\text{nc}}{E} = (\varrho_1, \ldots, \varrho_r) = (\sigma_1, \bar{\sigma}_1, \ldots, \sigma_l, \ldots, \bar{\sigma}_l, \sigma_{l+1}, \ldots, \sigma_{l+q})$ and the collection of constant eigenvalues shall be referenced as $\overset{\text{c}}{E} = E \setminus \overset{\text{nc}}{E}$. The quantity Δ_i for $i = 1, \ldots, r$ is given by $\Delta_i = \prod_{\varrho \in \overset{\text{nc}}{E}\setminus\{\varrho_i\}} (\varrho_i - \varrho)$. The function μ_i denotes the elementary symmetric polynomial of degree i in the variables $\overset{\text{nc}}{E}$ and $\mu_i(\widehat{\varrho}_s)$ denotes the elementary symmetric polynomial of degree i in the variables $\overset{\text{nc}}{E} \setminus \{\varrho_s\}$. We shall further define the one-forms $\vartheta_1, \ldots, \vartheta_r$ on W via $\vartheta_i = \mathrm{d}t_i + \alpha_i$.

Suppose that at every point of W the elements of $\overset{\text{nc}}{E}$ are mutually different and different from the constants $c_1, \bar{c}_1, \ldots, c_{L+Q}$ and their differentials are non-zero. Then (g, ω, J) given by the formulae

$$
g = \sum_{i=1}^{r} \varepsilon_i \Delta_i \mathrm{d}\chi_i^2 + \sum_{i=0}^{r} (-1)^i \mu_i \sum_{\gamma=1}^{L+Q} g_\gamma(A_\gamma^{r-i}\cdot, \cdot)
$$

$$
+ \sum_{i,j=1}^{r} \left[\sum_{s=1}^{r} \frac{\mu_{i-1}(\widehat{\varrho}_s)\mu_{j-1}(\widehat{\varrho}_s)}{\varepsilon_s \Delta_s} \left(\frac{\partial \varrho_s}{\partial \chi_s}\right)^2 \right] \vartheta_i \vartheta_j, \tag{19}
$$

$$
\omega = \sum_{i=1}^{r} \mathrm{d}\mu_i \wedge \vartheta_i + \sum_{i=0}^{r} (-1)^i \mu_i \sum_{\gamma=1}^{L+Q} \omega_\gamma(A_\gamma^{r-i}\cdot, \cdot),
$$

$$d\chi_i \circ J = -\frac{1}{\varepsilon_i \Delta_i} \frac{\partial \varrho_i}{\partial \chi_i} \sum_{j=1}^{r} \mu_{j-1}(\widehat{\varrho_i}) \vartheta_i,$$

$$\vartheta_i \circ J = (-1)^{i-1} \sum_{j=1}^{r} \varepsilon_j \varrho_j^{r-i} \left(\frac{\partial \varrho_j}{\partial \chi_j}\right)^{-1} d\chi_j$$

(20)

is Kähler, where $(\varepsilon_1, \ldots, \varepsilon_{2l}, \varepsilon_{2l+1}, \ldots, \varepsilon_r) = (-1/4, \ldots, -1/4, \pm 1, \ldots, \pm 1)$ determine the signature of g.

With local coordinates $\overset{\gamma}{y}$ on S_γ we write $\alpha_i = \sum_{\gamma,q} \overset{\gamma}{\alpha}_{iq} d\overset{\gamma}{y}_q$ and $A_\gamma = \sum_{p,q} (A_\gamma)_p^q d\overset{\gamma}{y}_p \otimes \partial_{\overset{\gamma}{y}_q}$. Then the endomorphism A given by

$$A = \sum_{s=1}^{r} \varrho_s \, d\chi_s \otimes \partial_{\chi_s} + \sum_{i,j=1}^{r} (\mu_i \delta_{1j} - \delta_{i(j-1)}) \, \vartheta_i \otimes \partial_{t_j}$$
$$+ \sum_{\gamma=1}^{L+Q} \sum_{p,q} (A_\gamma)_p^q d\overset{\gamma}{y}_p \otimes \left(\partial_{\overset{\gamma}{y}_q} - \sum_{i=1}^{r} \overset{\gamma}{\alpha}_{iq} \partial_{t_i}\right)$$

(21)

is c-compatible with (g, J, ω).

Remark 2.14 (ω *as an exterior derivative*) The formula for ω in (19) can at least locally be expressed in a very concise manner: Since A_γ is parallel and ω_γ is closed for all $\gamma = 1, \ldots, L+Q$, we can at least locally find a one-form α_0 on S such that

$$d\alpha_0 = \sum_{\gamma=1}^{L+Q} \omega_\gamma(A_\gamma^r \cdot, \cdot).$$

Then if we define the one-form ϑ_0 on W simply as $\vartheta_0 = \alpha_0$, it is easy to check that

$$\omega = d \sum_{i=0}^{r} \mu_i \vartheta_i.$$

Theorem 2.15 ([3, Theorem 1.6 / Example 5]) *Suppose* (g, J, A) *are c-compatible on* $\mathcal{M}$ *of real dimension* $2n$. *Assume that in a small neighbourhood* $W \subseteq \mathcal{M}^0$ *of a regular point, A has*

- $r = 2l + q$ *non-constant eigenvalues on* W *which separate into l pairs of complex-conjugate eigenvalues* $\varrho_1, \bar{\varrho}_1, \ldots, \varrho_l, \bar{\varrho}_l \colon W \to \mathbb{C}$ *and q real eigenvalues* $\varrho_{l+1}, \ldots, \varrho_{l+q} \colon W \to \mathbb{R}$,
- $R = 2L + Q$ *constant eigenvalues which separate into L pairs of complex conjugated eigenvalues* $c_1, \bar{c}_1, \ldots, c_L, \bar{c}_L$ *and Q real eigenvalues* $c_{L+1}, \ldots, c_{L+Q}$,

then the Kähler structure (g, J, ω) *and A are given on W by the formulas of Example* 2.13.

A local description of Kähler structures (g, J, ω) with Riemannian signature that admit a Hermitian solution of 4 goes back to [1], using the language of Hamiltonian 2-forms.

Theorem 2.16 *Let (g, J, A) be c-compatible on $\mathfrak{M}$. Let A be semi-simple and let all constant eigenvalues be real. Let (g, J, ω, A) be given by the formulas of Example 2.13 and adopt the naming conventions of Example 2.13.*

Let ψ be a simultaneous eigenfunction of

$$\overset{t}{Q} \overset{\text{def}}{=} -\nabla_j \circ \overset{t}{K}^{jk} \circ \nabla_k + \overset{t}{U}, \quad \overset{t}{K}^{ij} \overset{\text{def}}{=} \sqrt{\det(t\mathrm{Id} - A)}\,(t\mathrm{Id} - A)^{-1\,i}_{l}\,g^{lj}$$

and

$$\overset{t}{L} = \frac{i}{2}\left(\nabla_j \circ \overset{t}{V}^j + \overset{t}{V}^j \circ \nabla_j\right), \quad \overset{t}{V}^j = J^j_k\,g^{ki}\nabla_i\sqrt{\det(t\mathrm{Id} - A)}$$

for all t, where

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}}\,(t - \varrho_i)^{m(\varrho_i)/2 - 1}\,f_i \tag{22}$$

with $\mathrm{d}f_l \circ A = \varrho_l \mathrm{d}f_l$ for all $l = 1, \ldots, r + R$ and with $\mathrm{d}f_l$ proportional to $\mathrm{d}\varrho_l$ for all l for which ϱ_l is non-constant. Then there exist constants $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{r+R-1}, \omega_1, \ldots, \omega_r$, such that ψ satisfies the following ODE:

$$\frac{-1}{\varepsilon_k \varrho'_k}\,\partial_{\chi_k}\varrho'_k \prod_{\varrho_\gamma \in \overset{c}{E}} (\varrho_k - \varrho_\gamma)^{m(\varrho_\gamma)/2}\,\partial_{\chi_k}\psi$$

$$+ \sum_{i,j=1}^{r} \frac{\varepsilon_k(-\varrho_k)^{2r-i-j}}{(\varrho'_k)^2} \prod_{\varrho_\gamma \in \overset{c}{E}} (\varrho_k - \varrho_\gamma)^{m(\varrho_\gamma)/2}\,\omega_i \omega_j + f_r \psi = \sum_{i=0}^{n-1} \lambda_i \varrho_k^i, \tag{23}$$

$$i\partial_{t_q}\psi = \omega_q \psi$$

for $q = 1, \ldots, r$, where the λ_i are given by

$$\sum_{i=0}^{n-1} \lambda_i s^i = \prod_{\varrho_\gamma \in \overset{c}{E}} (s - \varrho_\gamma)^{m(\varrho_\gamma)/2 - 1} \sum_{j=0}^{r+R-1} \tilde{\lambda}_j$$

and ψ also fulfills the partial differential equations:

$$\sum_{j=0}^{r+R-1} \tilde{\lambda}_j \varrho_\gamma^j = -\prod_{\substack{c \\ \varrho_c \in E \setminus \{\varrho_\gamma\}}} (\varrho_\gamma - \varrho_c) \left[\frac{1}{|\det g_\gamma|^{1/2}} \, \partial_{\substack{\gamma \\ y_i}} g_\gamma^{ij} |\det g_\gamma|^{1/2} \, \partial_{\substack{\gamma \\ y_j}} \psi \right.$$

$$-i \sum_{q=1}^{r} \frac{1}{|\det g_\gamma|^{1/2}} \, \partial_{\substack{\gamma \\ y_i}} g_\gamma^{ij} |\det g_\gamma|^{1/2} \, \overset{\gamma}{\alpha}_{qj} \omega_q \psi$$

$$-i \sum_{q=1}^{r} g_\gamma^{ij} \overset{\gamma}{\alpha}_{qi} \omega_q \partial_{\substack{\gamma \\ y_j}} \psi$$

$$\left. - \sum_{p,q=1}^{r} g_\gamma^{ij} \overset{\gamma}{\alpha}_{qi} \overset{\gamma}{\alpha}_{pj} \omega_q \omega_p \psi \right]$$

$$+ \frac{1}{\prod_{\substack{c \\ \varrho_c \in E \setminus \{\varrho_\gamma\}}} (\varrho_\gamma - \varrho_c)^{m(\varrho_c)/2 - 1}} \, f_\gamma \psi \qquad (24)$$

for $\gamma = r+1, \ldots, r+R$.

The converse is also true: If a function ψ satisfies equations (23) *and* (24) *for some constants $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{r+R-1}, \omega_1, \ldots, \omega_r$, then it is an eigenfunction of $\overset{t}{\widehat{Q}}$ and $\overset{t}{\widehat{L}}$.*

A particular application of Theorem 2.16 is in the treatment of the Laplace–Beltrami operator in the case where the number of integrals is maximal: if $\mathcal{M}$ is compact and Riemannian its eigenfunctions provide means to construct a countable basis in $L^2(\mathcal{M})$, see e.g. [8]. Because the integrals are self-adjoint and commute pairwise as well as with the Laplace–Beltrami operator they leave each others eigenspaces invariant and thus there exists a countable basis of $L^2(\mathcal{M})$ that consists of simultaneous eigenfunctions of the commuting operators. In our case, since Δ is within the span of our one-parameter family of operators we can find such basis by solving the simultaneous eigenvalue problems corresponding to the integrals only. In the case where all eigenvalues of A are non-constant this reduces to ODE.

3 Proof of the results

3.1 Basic facts

We shall provide some formulae that will be used throughout the proof of Theorem 2.2. We will be mainly working with (2). Unless it is stated otherwise we are working in the *c-projective* setting with only some remarks providing the analogue formulae for the *projective setting*.

We start by computing the covariant derivative of $\det(A)$. By using Jacobi's formula $d \det M = \mathrm{tr}(\mathrm{Ad}(M) \, dM)$, we get

$$\nabla_k \det(A) = \det(A) A^{-1}{}^q_r \nabla_k A^r_q$$
$$= \det(A) A^{-1}{}^q_r \left[\lambda^r g_{qk} + \lambda_q \delta^r_k + g^{rs} \bar{\lambda}_s \Omega_{qk} + \bar{\lambda}_q g^{rs} \Omega_{sk} \right] \quad (25)$$
$$= 4 \det(A) A^{-1}{}^s_k \lambda_s.$$

Equation (2) was used to expand the covariant derivative of A and then the properties of J interacting with g were exploited to obtain this.

Lemma 3.1 ([4, Section 5]) *A and thus $t\mathrm{Id} - A$ and $(t\mathrm{Id} - A)^{-1}$ are self-adjoint with respect to $\nabla^2 \lambda$: $A^{-1}{}^j_l \lambda_{j,k} = A^{-1}{}^j_k \lambda_{j,l}$. The last claim is of course true only if $(t\mathrm{Id} - A)^{-1}$ exists.*

That is, if we consider the second derivative of λ in the way that it maps two vector fields ξ, η to a scalar function on $\mathcal{M}$ then we have $\nabla^2 \lambda(A\xi, \eta) = \nabla^2 \lambda(\xi, A\eta)$ for all (ξ, η).

Proof of Lemma 3.1 Without loss of generality we may assume that A is invertible. Otherwise we may choose ε such that $\varepsilon \mathrm{Id} - A$ is invertible. Then we can apply the same procedure that we will apply to $\ln \det A$ to $\ln \det(\varepsilon \mathrm{Id} - A)$ instead. We can therefrom show that $(\varepsilon \mathrm{Id} - A)^{-1}$ is self-adjoint with respect to $\nabla^2 \lambda$. And thus by linear algebra $\varepsilon \mathrm{Id} - A$ and consequentially A are also $\nabla^2 \lambda$-self-adjoint.

Now to the main argument: We compute the second covariant derivative of $\ln \det A$ using (25), (2), the general identity $\mathrm{d}A^{-1} = A^{-1} \cdot (\mathrm{d}A) \cdot A^{-1}$ as well as the antisymmetry of J with respect to g:

$$\frac{1}{4} \nabla_k \nabla_l \ln \det A = \lambda_j A^{-1}{}^j_p g^{ps} \lambda_s A^{-1}{}^q_l g_{qk} + \lambda_j A^{-1}{}^j_k A^{-1}{}^q_l \lambda_q$$
$$+ \lambda_j A^{-1}{}^j_p g^{ps} \bar{\lambda}_s A^{-1}{}^q_l \Omega_{qk} \quad (26)$$
$$- \lambda_j A^{-1}{}^j_p J^p_k \bar{\lambda}_q A^{-1}{}^q_l + A^{-1}{}^j_l \lambda_{j,k}.$$

The left-hand side is symmetric with respect to (k, l). The first, second and fourth terms on the right-hand side are symmetric as well. The third term vanishes. Consequently the last term must be symmetric as well. Thus A^{-1} is self-adjoint with respect to $\nabla^2 \lambda$. By means of linear algebra, the self-adjointness with respect to $\lambda_{j,k}$ is also true for $(t\mathrm{Id} - A)$ or $(t\mathrm{Id} - A)^{-1}$. The latter of course is only true, provided that t is not chosen to be within the spectrum of A. This concludes the proof of Lemma 3.1. $\qquad \square$

Remark 3.2 (*projective setting*) The formula $A^j_l \lambda_{j,k} = A^j_k \lambda_{j,l}$ is true for the projective setting as well. Performing the same computations as in Lemma 3.1 gives the intermediate results $\nabla_k \det(t\mathrm{Id} - A) = -2 \det(t\mathrm{Id} - A)(t\mathrm{Id} - A)^{-1}{}^s_k \lambda_s$ and $\frac{1}{2} \nabla_k \nabla_l \ln \det A = \lambda_j A^{-1}{}^j_p g^{ps} \lambda_s A^{-1}{}^q_l g_{qk} + \lambda_j A^{-1}{}^j_k A^{-1}{}^q_l \lambda_q + A^{-1}{}^j_l \lambda_{j,k}$ to which the same logic is applied as in the c-projective case.

Lemma 3.3 *Let S be an endomorphism on the space of vector fields on $\mathcal{M}$ with the following properties: $J \circ S = S \circ J$, $A \circ S = S \circ A$, $\nabla^2 \lambda(S\cdot, \cdot) = \nabla^2 \lambda(\cdot, S\cdot)$ and*

$g(S\cdot, \cdot) = g(\cdot, S\cdot)$. *Then the formula*

$$R^r_{ijk}A_{rl}S^{ij} - R^r_{ijl}A_{rk}S^{ij} = 0 \tag{27}$$

is valid.

The sign of the Riemann tensor is chosen so that $R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{ks}\Gamma^s_{lj} - \Gamma^i_{ls}\Gamma^s_{kj}$. It is obviously not essential for the formula but the proof would need a few changes in the signs.

Proof of Lemma 3.3 In the proof we reuse and extend the ideas used in the proof of equation (12) in [9]. We inspect the second derivative of A:

$$
\begin{aligned}
R^r_{jkl}A_{ir} + R^r_{ikl}A_{rj} &= (\nabla_l\nabla_k - \nabla_k\nabla_l)A_{ij} \\
&= \lambda_{i,l}g_{jk} - \lambda_{i,k}g_{jl} + \lambda_{j,l}g_{ik} - \lambda_{j,k}g_{il} \\
&\quad + \bar\lambda_{i,l}\Omega_{jk} - \bar\lambda_{i,k}\Omega_{jl} + \bar\lambda_{j,l}\Omega_{ik} - \bar\lambda_{j,k}\Omega_{il}.
\end{aligned}
\tag{28}
$$

The first equality is the Ricci identity and is true for any $(0, 2)$-tensor. The second equality comes from differentiating (2). We continue by adding the equation with itself three times after performing cyclic permutations of (j, k, l). The three terms rising from the first term on the left-hand side of (28) vanish due to the Bianchi identity. On the right-hand side only terms involving $\bar\lambda$ remain:

$$
\begin{aligned}
R^r_{ikl}A_{rj} &+ R^r_{ijk}A_{rl} + R^r_{ilj}A_{rk} \\
&= \left(+ \bar\lambda_{i,l}\Omega_{jk} + \bar\lambda_{i,k}\Omega_{lj} + \bar\lambda_{i,j}\Omega_{kl} \right. \\
&\qquad \left. - \bar\lambda_{i,k}\Omega_{jl} - \bar\lambda_{i,l}\Omega_{kj} - \bar\lambda_{i,j}\Omega_{lk} \right) + (i \leftrightarrow j)
\end{aligned}
\tag{29}
$$

$+ (i \leftrightarrow j)$ means that the preceding term is to be added again, but with indices i and j interchanged. We now multiply this equation with S^{ij}. Since S is g-self-adjoint and commutes with J, we have $\bar\lambda_{i,j}S^{ij} = 0$. Using this and using again that S commutes with J, the right-hand side simplifies as follows:

$$R^r_{ikl}A_{rj}S^{ij} + R^r_{ijk}A_{rl}S^{ij} + R^r_{ilj}A_{rk}S^{ij} = 4\left(S^j_k\lambda_{j,l} - S^j_l\lambda_{j,k}\right).$$

$S \circ A$ is g-self-adjoint because S and A are g-self-adjoint and commute. As a consequence, we have that $R^r_{ikl}A_{rj}S^{ij}$ vanishes on the left-hand side. Further utilizing the symmetry of the curvature tensor as well as the self-adjointness of S with respect to $\nabla^2\lambda$ we reach the desired result:

$$R^r_{ijk}A_{rl}S^{ij} - R^r_{ijl}A_{rk}S^{ij} = 0. \qquad \square$$

Remark 3.4 (*projective case*) In the projective case Lemma 3.3 also holds true. The procedure involves the same steps as in the proof of Lemma 3.3. Removing the terms involving Ω from (28) gives the intermediate step for the projective case. Equation

(29) trivially simplifies to $R^r_{ikl}A_{rj} + R^r_{ijk}A_{rl} + R^r_{ilj}A_{rk} = 0$. After multiplication with S^{ij} the first term vanishes with the same argument as in the c-projective case and the result is obtained. Since the proof of Corollary 3.5 involves only linear algebra, the arguments are exactly the same in the projective and the c-projective case.

Corollary 3.5 *Linear algebra applied to* (27) *provides the formula*

$$(v\mathrm{Id} - A)^{-1lr} R^k_{jir} S^{ij} - (v\mathrm{Id} - A)^{-1kr} R^l_{jir} S^{ij} = 0. \tag{30}$$

Proof Consider the endomorphism $R^r_{ijl}S^{ij}$ on the space of vector fields. Raising and lowering indices in (27) shows that it commutes with A. Consequently it also commutes with $(v\mathrm{Id} - A)$ and thus with $(v\mathrm{Id} - A)^{-1}$. Standard index manipulations using the symmetries of the curvature tensor imply the result.   □

3.2 Proof of Theorem 2.2

The family $\overset{t}{\widehat{I}}$ is a polynomial in t and therefore continuous. It is therefore sufficient to show that the commutator vanishes for all v and w that are not in the spectrum of A. Otherwise we can consider two sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ where none of the elements of the sequence are in the spectrum of A and that converge to v and w. Then for each of the pairs (v_n, w_m) from the sequences the commutator $[\overset{v_n}{\widehat{I}}, \overset{w_m}{\widehat{I}}]$ will vanish and consequently it will vanish in the limit $(m, n) \to \infty$.

Equations (3.12), (3.13) and (3.14) in [7] give us the general formula for the commutator of two operators formed from arbitrary homogeneous polynomials P_2, Q_2 of degree two on $T^*\mathcal{M}$:

$$[\widehat{P_2}, \widehat{Q_2}] = i\{\widehat{P_2, Q_2}\} + \frac{2}{3}\left(\nabla_j B^{jk}_{P_2,Q_2}\right)\nabla_k. \tag{31}$$

Here $\{P_2, Q_2\}$ is the Poisson bracket of the two homogeneous polynomials P_2 and Q_2 (of degree two) on $T^*\mathcal{M}$. $\{P_2, Q_2\}$ is a polynomial of degree three in momenta. We recall that the "hat" over $\{P_2, Q_2\}$ is explained in (9): This polynomial is mapped to a differential operator according to

$$\widehat{\cdot}: P_3 \mapsto \widehat{P_3} \overset{\text{def}}{=} -\frac{i}{2}\left(\nabla_j \circ P_3^{jkl} \circ \nabla_k \circ \nabla_l + \nabla_j \circ \nabla_k \circ P_3^{jkl} \circ \nabla_l\right)$$

where for a given polynomial P_3 the quantities P_3^{jkl} are chosen so that they are symmetric and $P_3 = P_3^{jkl} p_j p_k p_l$.

The tensor $B^{kl}_{P_2,Q_2}$ is given by the formula

$$\begin{aligned} B^{jk}_{P_2,Q_2} = P^{l[j}\nabla_l\nabla_m Q^{k]m} + P^{l[j}R^{k]}_{mnl}Q^{mn} - (P \leftrightarrow Q) \\ - \nabla_l P^{m[j}\nabla_m Q^{k]l} - P^{l[j}R_{lm}Q^{k]m}. \end{aligned} \tag{32}$$

The brackets around the indices mean taking the antisymmetric part. In the third term ∇_l acts only on P. The subtraction of $(P \leftrightarrow Q)$ is meant to act upon the two leftmost terms. For the two rightmost terms the antisymmetrization w.r.t. $(j \leftrightarrow k)$ is the same as if one were to antisymmetrize these terms w.r.t. $(P \leftrightarrow Q)$. For formula (32) the sign of $R^i_{\ jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{ks}\Gamma^s_{lj} - \Gamma^i_{ls}\Gamma^s_{kj}$ is important. But the reader may forget about it at once because it is not needed for our further investigations, as will be seen in the upcoming Lemmas 3.8 and 3.10.

We plug the operators $\overset{v}{\widehat{I}}$ and $\overset{w}{\widehat{I}}$ into formula (31). Using Theorem 1.8 by Topalov we get

$$[\overset{v}{\widehat{I}}, \overset{w}{\widehat{I}}] = \frac{2}{3}\left(\nabla_j \underset{I,I}{B^{jk}_{v\ w}}\right)\nabla_k. \tag{33}$$

Remark 3.6 $i\{\widehat{P_2, Q_2}\}$ in formula (31) is a differential operator of order three while the other term on the right-hand side is a differential operator of order one. Therefore it is a necessary condition for the quantities $\overset{v}{I}$ and $\overset{w}{I}$ to Poisson-commute in order for their associated differential operators to commute. This is of course a well-known fact.

Remark 3.7 (*projective case*) The fact that $\overset{t}{K}$ is a family of Killing tensors polynomial of degree $n-1$ in t and that their corresponding quadratic polynomials on $T^*\mathcal{M}$ Poisson commute pairwise can be found in [2]. Employing this instead of Theorem 1.8 brings proof in the projective case to the point where only equation (34) needs to be verified.

It remains to prove that

$$\nabla_j \underset{I,I}{B^{jk}_{v\ w}} = \nabla_j\left(\left(\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v \leftrightarrow w)\right) - \nabla_l\overset{v}{K}{}^{m[j}\nabla_m\overset{w}{K}{}^{k]l}\right.$$
$$\left. + \left(\overset{v}{K}{}^{l[j}R^{k]}_{\ mnl}\overset{w}{K}{}^{mn} - (v \leftrightarrow w)\right) - \overset{v}{K}{}^{l[j}R_{lm}\overset{w}{K}{}^{k]m}\right) = 0. \tag{34}$$

The proof will be split into three steps: first we show that $\overset{v}{K}{}^{l[j}R^{k]}_{\ mnl}\overset{w}{K}{}^{mn} - (v \leftrightarrow w) = 0$. In the second step we show that $\overset{v}{K}{}^{l[j}R_{lm}\overset{w}{K}{}^{k]m} = 0$. This will be done in Lemmas 3.8 and 3.10. These will reduce (34) to

$$\nabla_j\left(\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v \leftrightarrow w) - \nabla_l\overset{v}{K}{}^{m[j}\nabla_m\overset{w}{K}{}^{k]l}\right) = 0 \tag{35}$$

which we will show in the last step.

Lemma 3.8 $\overset{v}{K}{}^{l[j}R^{k]}_{\ mnl}\overset{w}{K}{}^{mn} - (v \leftrightarrow w) = 0.$

Proof of Lemma 3.8 It follows from Lemma 3.1 that in Corollary 3.5 we may take $S = \sqrt{\det(w\text{Id} - A)}(w\text{Id} - A)^{-1}$, i.e. $S^{ij} = \overset{w}{K}{}^{ij}$. Plugging this into equation (30)

and multiplying with $\sqrt{\det(v\mathrm{Id} - A)}$ gives

$$2\overset{v}{K}{}^{l[j} R^{k]}_{mnl} \overset{w}{K}{}^{mn} = 0. \tag{36}$$

Interchanging v and w, subtracting the result from this and dividing by 2 proves Lemma 3.8. □

Remark 3.9 (*projective case*) Lemma 3.8 is true in the projective and c-projective cases. The proof for the projective case only requires to take $S = \det(w\mathrm{Id} - A)(w\mathrm{Id} - A)^{-1}$ and multiplying with $\det(v\mathrm{Id} - A)$ instead of $\sqrt{\det(v\mathrm{Id} - A)}$.

Lemma 3.10 $\overset{v}{K}{}^{l[j} R_{lm} \overset{w}{K}{}^{k]m} = 0$

Proof of Lemma 3.10 If we let $S = \mathrm{Id}$ in Corollary 3.5, then in formula (30) the multiplication with $S^{ij} = g^{ij}$ means contraction of the Riemann tensor to the negative of the Ricci tensor. Raising and lowering indices yields that $(v\mathrm{Id} - A)^{-1}$ commutes with the Ricci tensor when both are considered as endomorphisms on the space of vector fields:

$$(v\mathrm{Id} - A)^{-1r}_{\ \ l} R^k_r - (v\mathrm{Id} - A)^{-1k}_{\ \ r} R^r_l = 0. \tag{37}$$

Of course, $(w\mathrm{Id} - A)^{-1}$ commutes with $(v\mathrm{Id} - A)^{-1}$ and the Ricci tensor as well, so by multiplying (37) with $(w\mathrm{Id} - A)^{-1l}_{\ \ j}$ and using the commutativity gives

$$(w\mathrm{Id} - A)^{-1k}_{\ \ r} R^r_l (v\mathrm{Id} - A)^{-1l}_{\ \ j} - (v\mathrm{Id} - A)^{-1k}_{\ \ r} R^r_l (w\mathrm{Id} - A)^{-1l}_{\ \ j} = 0. \tag{38}$$

After multiplication of (38) with $\sqrt{\det(v\mathrm{Id} - A)\det(w\mathrm{Id} - A)}$ and raising and lowering indices, Lemma 3.10 is proven. □

Remark 3.11 (*projective case*) Replacing the multiplication of $\sqrt{\det(v\mathrm{Id} - A)}$ $\sqrt{\det(w\mathrm{Id} - A)}$ with $\det(v\mathrm{Id} - A)\det(w\mathrm{Id} - A)$ is the only difference between the proof of Lemma 3.10 in the projective and c-projective cases.

Having established Lemmas 3.8 and 3.10 we now compute the terms $\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m}$ $- (v \leftrightarrow w)$ and $\nabla_m\overset{v}{K}{}^{l[j}\nabla_l\overset{w}{K}{}^{k]m}$ separately and then show that (35) is fulfilled to prove Theorem 2.2.

We introduce a shorthand notation:

$$\Lambda = \mathrm{grad}\,\lambda = (g^{ij}\lambda_j), \quad \bar{\Lambda} = J\,\mathrm{grad}\,\lambda, \quad \overset{t}{M} = (t\mathrm{Id} - A)^{-1}$$

for any t outside the spectrum of A.

We get

$$\nabla_k \det(t\mathrm{Id} - A) = -4\det(t\mathrm{Id} - A)\overset{t}{M}{}^s_k\lambda_s$$

in the same way as we have obtained formula (25). Using this, the general matrix identity $dC^{-1} = C^{-1} \cdot dC \cdot C^{-1}$ and (2), the covariant derivative of the Killing tensor $\overset{t}{K}$ evaluates to

$$
\begin{aligned}
\nabla_k \overset{t}{K}{}^{jl} = \sqrt{\det(t\mathrm{Id} - A)} \big[&- 2\overset{t}{M}{}^s_k \lambda_s \, \overset{t}{M}{}^j_r g^{rl} + \overset{t}{M}{}^j_p \lambda^p \overset{t}{M}{}^l_k \\
&+ \overset{t}{M}{}^j_k \overset{t}{M}{}^l_q \lambda^q - \overset{t}{M}{}^j_p g^{ps} \bar\lambda_s \overset{t}{M}{}^l_q J^q_k - \overset{t}{M}{}^l_r g^{rq} \bar\lambda_q \overset{t}{M}{}^j_p J^p_k \big].
\end{aligned}
\tag{39}
$$

Contracting the indices k and l we get

$$
\nabla_k \overset{t}{K}{}^{jk} = \sqrt{\det(t\mathrm{Id} - A)} \big[- 2\overset{t}{M}{}^j_l \overset{t}{M}{}^l_s \lambda^s + \overset{t}{M}{}^j_l \lambda^l \mathrm{tr}(\overset{t}{M}) \big].
\tag{40}
$$

For the derivative of this expression we receive

$$
\begin{aligned}
\nabla_l \nabla_m \overset{t}{K}{}^{km} = \sqrt{\det(t\mathrm{Id} - A)} \big[&2\overset{t}{M}{}^r_l \lambda_r \lambda_s g^{ts} \overset{t}{M}{}^m_t \overset{t}{M}{}^k_m + 2\overset{t}{M}{}^q_l \overset{t}{M}{}^r_q \lambda_r \lambda_s g^{ts} \overset{t}{M}{}^k_t \\
&+ 2\overset{t}{M}{}^r_l \bar\lambda_r \bar\lambda_s g^{ts} \overset{t}{M}{}^m_t \overset{t}{M}{}^k_m + 2\overset{t}{M}{}^q_l \overset{t}{M}{}^r_q \bar\lambda_r \bar\lambda_s g^{ts} \overset{t}{M}{}^k_t \\
&- \overset{t}{M}{}^r_l \lambda_r \lambda_s g^{st} \overset{t}{M}{}^k_l \mathrm{tr}(\overset{t}{M}) - \overset{t}{M}{}^r_l \bar\lambda_r \bar\lambda_s g^{st} \overset{t}{M}{}^k_l \mathrm{tr}(\overset{t}{M}) \\
&- 2\overset{t}{M}{}^k_m g(\overset{t}{M}\Lambda, \overset{t}{M}\Lambda) - 2\overset{t}{M}{}^k_r \overset{t}{M}{}^r_l g(\overset{t}{M}\Lambda, \Lambda) \\
&+ \overset{t}{M}{}^k_l g(\overset{t}{M}\Lambda, \Lambda) \mathrm{tr}(\overset{t}{M}) - 2\lambda_{s,l} g^{ps} \overset{t}{M}{}^p_t \overset{t}{M}{}^k_p \\
&+ \lambda_{s,l} g^{ps} \overset{t}{M}{}^k_p \mathrm{tr}(\overset{t}{M}) \big].
\end{aligned}
\tag{41}
$$

We shall denote by $(\overset{w}{M}\Lambda)^k$ the k-th component of $\overset{w}{M}(\Lambda)$. Now multiplying the previous equation with $\overset{v}{K}{}^{lj}$ and antisymmetrizing with respect to $(j \leftrightarrow k)$ and $(v \leftrightarrow w)$ gives

$$
\begin{aligned}
\overset{v}{K}{}^{l[j} \nabla_l \nabla_m \overset{w}{K}{}^{k]m} - (v \leftrightarrow w) = &\frac{1}{2} \sqrt{\det(v\mathrm{Id} - A)} \sqrt{\det(w\mathrm{Id} - A)} \\
&\cdot \big[\big[\big[2(\overset{v}{M}\overset{w}{M}\Lambda)^j (\overset{w}{M}{}^2\Lambda)^k + 2(\overset{v}{M}\overset{w}{M}{}^2\Lambda)^j (\overset{w}{M}\Lambda)^k \\
&- (\overset{v}{M}\overset{w}{M}\Lambda)^j (\overset{w}{M}\Lambda)^k \mathrm{tr}(\overset{w}{M}) \big] \\
&+ (\Lambda \leftrightarrow \bar\Lambda) \big] - (j \leftrightarrow k) \big] - (v \leftrightarrow w).
\end{aligned}
\tag{42}
$$

Here $(\Lambda \leftrightarrow \bar\Lambda)$ indicates that the previous bracket shall be added with Λ replaced by $\bar\Lambda$, $(j \leftrightarrow k)$ indicates antisymmetrization with respect to j and k, likewise for $(v \leftrightarrow w)$. In (42) the terms from (41) involving second derivatives of λ have cancelled out as a consequence of Lemma 3.1. When forming the right-hand side expression of (42) the terms of the second to the last row of (41) cancel each other out after the antisymmetrization $(j \leftrightarrow k)$ due to $(v\mathrm{Id} - A)$ and $(w\mathrm{Id} - A)$ commuting and being self-adjoint to g.

Remark 3.12 (*projective case*) To get the formula for $\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v \leftrightarrow w)$ in the projective case we perform the same steps, using (4) instead of the c-projective formula (2). The next three equations give projective analogues of the formulae (39), (41) and (42):

$$\nabla_k \overset{t}{K}{}^{jl} = \det(t\mathrm{Id} - A)\big[-2\overset{t}{M}{}^s_k\lambda_s \, \overset{t}{M}{}^j_r g^{rl} + \overset{t}{M}{}^j_p\lambda^p \overset{t}{M}{}^l_k + \overset{t}{M}{}^j_k \, \overset{t}{M}{}^l_q\lambda^q\big],$$

(43)

$$\begin{aligned}
\nabla_l\nabla_m \overset{t}{K}{}^{km} = \det(t\mathrm{Id} - A)\big[& \overset{t}{M}{}^r_l\lambda_r\lambda_s g^{ts}\overset{t}{M}{}^m_t\overset{t}{M}{}^k_m + \overset{t}{M}{}^q_l\overset{t}{M}{}^r_q\lambda_r\lambda_s g^{ts}\overset{t}{M}{}^k_t \\
& - \overset{t}{M}{}^r_l\lambda_r\lambda_s g^{st}\overset{t}{M}{}^k_t\mathrm{tr}(\overset{t}{M}) \\
& - \overset{t}{M}{}^k_m \, g(\overset{t}{M}\Lambda,\,\overset{t}{M}\Lambda) - \overset{t}{M}{}^k_r\overset{t}{M}{}^r_l \, g(\overset{t}{M}\Lambda,\,\Lambda) \\
& + \overset{t}{M}{}^k_l \, g(\overset{t}{M}\Lambda,\,\Lambda)\,\mathrm{tr}(\overset{t}{M}) \\
& - \lambda_{s,l}g^{ps}\overset{t}{M}{}^p_t\overset{t}{M}{}^k_p + \lambda_{s,l}g^{ps}\overset{t}{M}{}^k_p\mathrm{tr}(\overset{t}{M})\big],
\end{aligned}$$

$$\begin{aligned}
\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v\leftrightarrow w) = & \frac{1}{2}\det(v\mathrm{Id} - A)\det(w\mathrm{Id} - A) \\
& \cdot\big[[(\overset{v}{M}\overset{w}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k + (\overset{v}{M}\overset{w}{M}{}^2\Lambda)^j(\overset{w}{M}\Lambda)^k \\
& - (\overset{v}{M}\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{w}{M})] \\
& - (j\leftrightarrow k)\big] - (v\leftrightarrow w).
\end{aligned}$$

(44)

general matrix identity

$$(v\mathrm{Id} - A)^{-1} - (w\mathrm{Id} - A)^{-1} = (w - v)\cdot(v\mathrm{Id} - A)^{-1}\cdot(w\mathrm{Id} - A)^{-1}$$

which reads $\overset{v}{M} - \overset{w}{M} = (w - v)\overset{v}{M}\overset{w}{M}$ in our shorthand, as well as the trace applied to this matrix identity we expand (42):

$$\begin{aligned}
\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v\leftrightarrow w) = & \frac{1}{2}\sqrt{\det(v\mathrm{Id} - A)}\sqrt{(w\mathrm{Id} - A)}\cdot\big[[[(w - v)^{-1} \\
& [2(\overset{v}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k - 2(\overset{w}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k \\
& + 2(\overset{v}{M}\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - 2(\overset{w}{M}{}^2\Lambda)^j(\overset{w}{M}\Lambda)^k \\
& - (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{w}{M}) + (\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{w}{M})]] \\
& - (v\leftrightarrow w)\big] - (j\leftrightarrow k)\big] + (\Lambda\leftrightarrow\bar{\Lambda}).
\end{aligned}$$

We strike out terms that cancel after antisymmetrization w.r.t. $(j \leftrightarrow k)$:

$$
\begin{aligned}
= \frac{1}{2}&\sqrt{\det(v\mathrm{Id} - A)}\sqrt{\det(w\mathrm{Id} - A)} \cdot \big[\big[(w - v)^{-1} \\
&\cdot \big[2(\overset{v}{M}\Lambda)^j(\overset{w}{M^2}\Lambda)^k + 2(\overset{v}{M}\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k \\
&- (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{w}{M})\big] \\
&- (v \leftrightarrow w)\big] - (j \leftrightarrow k)\big] + (\Lambda \leftrightarrow \bar{\Lambda}).
\end{aligned}
$$

We expand $(v \leftrightarrow w)$ and $(j \leftrightarrow k)$. The sign of the antisymmetrization is caught in the prefactor $(w - v)^{-1}$:

$$
\begin{aligned}
= \frac{1}{2}&\sqrt{\det(v\mathrm{Id} - A)}\sqrt{\det(w\mathrm{Id} - A)}(w - v)^{-1} \\
\cdot\big[&2(\overset{v}{M}\Lambda)^j(\overset{w}{M^2}\Lambda)^k + 2(\overset{w}{M}\Lambda)^j(\overset{v}{M^2}\Lambda)^k \\
+&2(\overset{v}{M}\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k + 2(\overset{w}{M}\overset{v}{M}\Lambda)^j(\overset{v}{M}\Lambda)^k \\
-&2(\overset{v}{M}\Lambda)^k(\overset{w}{M^2}\Lambda)^j - 2(\overset{w}{M}\Lambda)^k(\overset{v}{M^2}\Lambda)^j \\
-&2(\overset{v}{M}\overset{w}{M}\Lambda)^k(\overset{w}{M}\Lambda)^j - 2(\overset{w}{M}\overset{v}{M}\Lambda)^k(\overset{v}{M}\Lambda)^j \\
-&(\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{w}{M}) - (\overset{w}{M}\Lambda)^j(\overset{v}{M}\Lambda)^k\mathrm{tr}(\overset{v}{M}) \\
+&(\overset{v}{M}\Lambda)^k(\overset{w}{M}\Lambda)^j\mathrm{tr}(\overset{w}{M}) + (\overset{w}{M}\Lambda)^k(\overset{v}{M}\Lambda)^j\mathrm{tr}(\overset{v}{M})\big] \\
+&(\Lambda \leftrightarrow \bar{\Lambda}).
\end{aligned}
$$

We can now apply $\overset{v}{M} - \overset{w}{M} = (w - v)\overset{v}{M}\overset{w}{M}$ and $\mathrm{tr}(\overset{v}{M}) - \mathrm{tr}(\overset{w}{M}) = (w - v)\mathrm{tr}(\overset{v}{M}\overset{w}{M})$ in the opposite direction as before, pairing terms (1,8), (2,7), (3,6), (4,5), (9,12), (10,11) in the bracket:

$$
\begin{aligned}
= \frac{1}{2}&\sqrt{\det(v\mathrm{Id} - A)}\sqrt{\det(w\mathrm{Id} - A)} \\
\cdot\big[\big[&- 2(\overset{v}{M^2}\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - 2(\overset{v}{M}\Lambda)^j(\overset{v}{M}\overset{w}{M^2}\Lambda)^k \\
+&(\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{v}{M}\cdot\overset{w}{M})\big] \\
-&(j \leftrightarrow k)\big] + (\Lambda \leftrightarrow \bar{\Lambda}).
\end{aligned}
\tag{45}
$$

Remark 3.13 (*projective case*) Performing the same steps on (44) gives

$$
\begin{aligned}
\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v \leftrightarrow w) = \frac{1}{2}&\det(v\mathrm{Id} - A)\det(w\mathrm{Id} - A) \\
\cdot\big[\big[&- (\overset{v}{M^2}\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - (\overset{v}{M}\Lambda)^j(\overset{v}{M}\overset{w}{M^2}\Lambda)^k \\
+&(\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{v}{M}\cdot\overset{w}{M})\big] - (j \leftrightarrow k)\big]
\end{aligned}
\tag{46}
$$

for the projective scenario.

We have now worked $\overset{v}{K}{}^{l[j}\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v \leftrightarrow w)$ into a suitable form. From (39) we now compute $\nabla_m\overset{v}{K}{}^{l[j}\nabla_l\overset{w}{K}{}^{k]m}$:

$$
\begin{aligned}
\nabla_m\overset{v}{K}{}^{l[j}\nabla_l\overset{w}{K}{}^{k]m} = {} & \frac{1}{2}\sqrt{\det(v\mathrm{Id}-A)}\sqrt{\det(w\mathrm{Id}-A)} \\
& \cdot\Big[\big[-2(\overset{v}{M}{}^2\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - 2(\overset{v}{M}\Lambda)^j(\overset{v}{M}\overset{w}{M}{}^2\Lambda)^k \\
& + (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\mathrm{tr}(\overset{v}{M}\cdot\overset{w}{M})\big] \\
& - (j\leftrightarrow k)\big] - (\Lambda\leftrightarrow\bar\Lambda).
\end{aligned}
\tag{47}
$$

In this equation $(j \leftrightarrow k)$ yields the same result as $(v \leftrightarrow w)$.

Remark 3.14 (*projective case*) By means of (43) the projective analogue of (47) evaluates to

$$
\begin{aligned}
\nabla_m\overset{v}{K}{}^{l[j}\nabla_l\overset{w}{K}{}^{k]m} = {} & \frac{1}{2}\det(v\mathrm{Id}-A)\det(w\mathrm{Id}-A) \\
& \cdot\Big[\big[-2(\overset{v}{M}{}^2\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - 2(\overset{v}{M}\Lambda)^j(\overset{v}{M}\overset{w}{M}{}^2\Lambda)^k \\
& + (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\,\mathrm{tr}(\overset{v}{M}\cdot\overset{w}{M})\big] - (j\leftrightarrow k)\big].
\end{aligned}
\tag{48}
$$

We see that the right-hand side expression is equal to the right-hand side expression of (46). Thus if we plug (46) and (48) into (35) then both terms cancel each other and (35) is satisfied without even having to carry out the differentiation, concluding the proof of Theorem 2.3. The fact that in the projective case $B_{\underset{1,1}{v}\,w}^{jk}$ vanishes, whereas in the c-projective case $\nabla_j B_{\underset{1,1}{v}\,w}^{jk}$ vanishes but $B_{\underset{1,1}{v}\,w}^{jk}$ does not is the most significant difference between the projective and the c-projective cases.

We now compare (45) and (47): they are the same except that the first is symmetric with respect to $(\Lambda \leftrightarrow \bar\Lambda)$ while the latter is antisymmetric. Subtracting both consequently yields

$$
\begin{aligned}
\overset{v}{K}{}^{l[j}&\nabla_l\nabla_m\overset{w}{K}{}^{k]m} - (v\leftrightarrow w) - \nabla_m\overset{v}{K}{}^{l[j}\nabla_l\overset{w}{K}{}^{k]m} \\
& = \sqrt{\det(v\mathrm{Id}-A)}\sqrt{\det(w\mathrm{Id}-A)} \\
& \quad \cdot\Big[-2(\overset{v}{M}{}^2\overset{w}{M}\bar\Lambda)^j(\overset{w}{M}\bar\Lambda)^k - 2(\overset{v}{M}\bar\Lambda)^j(\overset{v}{M}\overset{w}{M}{}^2\bar\Lambda)^k \\
& \quad + (\overset{v}{M}\bar\Lambda)^j(\overset{w}{M}\bar\Lambda)^k\mathrm{tr}(\overset{v}{M}\cdot\overset{w}{M})\Big] - (v\leftrightarrow w).
\end{aligned}
\tag{49}
$$

It remains to show that (35) is fulfilled, that is to apply ∇_j to this expression and show that it vanishes. In the computation we use

- the compatibility condition (2),
- Jacobi's formula for the derivative of the determinant,
- $d(A^{-1}) = -A^{-1} \cdot d(A) \cdot A^{-1}$

to expand the left-hand side expression of (35). We then immediately strike out terms that vanish individually due to the self-adjointness of A with respect to g and the antisymmetry of J with respect to g:

$$
\begin{aligned}
\nabla_j \big(\overset{v}{K}{}^{l[j} \nabla_l \nabla_m &\overset{w}{K}{}^{k]m} \ -(v \leftrightarrow w) - \nabla_m \overset{v}{K}{}^{l[j} \nabla_l \overset{w}{K}{}^{k]m} \big) \\
&= -2\sqrt{\det(v\mathrm{Id} - A)}\sqrt{\det(w\mathrm{Id} - A)} \\
&\quad \cdot \big[2g(\overset{v}{M}\overset{w}{M}\bar\Lambda, \overset{v}{M}\overset{w}{M}\bar\Lambda)(\overset{w}{M}\Lambda - \overset{v}{M}\Lambda)^k \\
&\qquad -2g(\overset{v}{M}\overset{w}{M}\bar\Lambda, \overset{w}{M}\bar\Lambda - \overset{v}{M}\bar\Lambda)(\overset{v}{M}\overset{w}{M}\Lambda)^k \\
&\qquad +2g(\bar\Lambda, \overset{v}{M}\overset{w}{M}\bar\Lambda)](\overset{v}{M}\overset{w}{M}(\overset{w}{M}\Lambda - \overset{v}{M}\Lambda))^k \\
&\quad -2g(\bar\Lambda, \overset{w}{M}\bar\Lambda - \overset{v}{M}\bar\Lambda)(\overset{v}{M}{}^2\overset{w}{M}{}^2\Lambda)^k \\
&\quad +\bar\lambda_s g^{ls} \overset{v}{M}{}^m_l \overset{w}{M}{}^t_m \overset{v}{M}{}^j_t \bar\lambda_{p,j} g^{pq} \overset{w}{M}{}^k_q \\
&\quad -\bar\lambda_s g^{ls} \overset{w}{M}{}^m_l \overset{v}{M}{}^t_m \overset{w}{M}{}^j_t \bar\lambda_{p,j} g^{pq} \overset{v}{M}{}^k_q \\
&\quad +\bar\lambda_s g^{ps} \overset{v}{M}{}^j_p \bar\lambda_{t,j} g^{tr} \overset{v}{M}{}^l_r \overset{v}{M}{}^m_l \overset{w}{M}{}^k_m \\
&\quad -\bar\lambda_s g^{ps} \overset{w}{M}{}^j_p \bar\lambda_{t,j} g^{tr} \overset{v}{M}{}^l_r \overset{w}{M}{}^m_l \overset{v}{M}{}^k_m \\
&\quad +\mathrm{tr}(\overset{v}{M}\overset{w}{M}) \cdot \big[g(\bar\Lambda, \overset{v}{M}\overset{w}{M}\bar\Lambda)(\overset{w}{M}\Lambda - \overset{v}{M}\Lambda)^k \\
&\qquad -g(\bar\Lambda, \overset{w}{M}\bar\Lambda - \overset{v}{M}\bar\Lambda)(\overset{v}{M}\overset{w}{M}\Lambda)^k \\
&\qquad +\bar\lambda_s g^{ps} \overset{v}{M}{}^j_p \bar\lambda_{t,j} g^{tr} \overset{w}{M}{}^k_r - \bar\lambda_s g^{ps} \overset{w}{M}{}^j_p \bar\lambda_{t,j} g^{tr} \overset{v}{M}{}^k_r \big]. \quad (50)
\end{aligned}
$$

As a consequence of Lemma 3.1 the terms involving second derivatives of λ cancel each other out in this expression. The other terms cancel each other out after applying $\overset{v}{M} - \overset{w}{M} = (w - v)\overset{v}{M}\overset{w}{M}$. Thus Theorem 2.2 is proven.

3.3 Addition of potential: Proof of Theorems 2.5 and 2.9

3.3.1 Four equivalent problems

Let (g, J, A) be *c-compatible*.

Lemma 3.15 *Let* $K = g^{ij} p_i p_j$,

$$
\overset{t}{I} \overset{\mathrm{def}}{=} \overset{t}{K}{}^{jk} p_j p_k, \quad \overset{t}{K}{}^{ij} \overset{\mathrm{def}}{=} \sqrt{\det(t\mathrm{Id} - A)}\, (t\mathrm{Id} - A)^{-1}{}^i_l g^{lj}
$$

and

$$\overset{t}{L} = \overset{t}{V}^j p_j, \quad \overset{t}{V}^j = J_k^j g^{ki} \nabla_i \sqrt{\det(t\mathrm{Id} - A)}$$

as well as the corresponding differential operators according to the quantization rules stated earlier.

Then the following four problems are equivalent: describe all functions $U, \overset{t}{U}$, such that

(1) $\{\overset{s}{I} + \overset{s}{U}, K + U\} = 0$ *and* $\{\overset{s}{I} + \overset{s}{U}, \overset{t}{L}\} = 0$ *for all* $t, s \in \mathbb{R}$,

(2) $[\overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}, \widehat{K} + \widehat{U}] = 0$ *and* $[\overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}, \overset{t}{\widehat{L}}] = 0$ *for all* $t, s \in \mathbb{R}$,

(3) $\{\overset{s}{I} + \overset{s}{U}, \overset{t}{I} + \overset{t}{U}\} = 0$ *and* $\{\overset{s}{I} + \overset{s}{U}, \overset{t}{L}\} = 0$ *for all* $t, s \in \mathbb{R}$,

(4) $[\overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}, \overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}] = 0$ *and* $[\overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}, \overset{t}{\widehat{L}}] = 0$ *for all* $t, s \in \mathbb{R}$.

Proof of Lemma 3.15 To do so, we show that

i. $\{\overset{t}{I} + \overset{t}{U}, K + U\} = 0$ for all $t \in \mathbb{R} \Leftrightarrow [\overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \widehat{K} + \widehat{U}] = 0$ for all $t \in \mathbb{R} \Leftrightarrow \overset{t}{K}_j^i \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^j}$ for all $t \in \mathbb{R}$,

ii. $\{\overset{t}{I} + \overset{t}{U}, K + U\} = 0$ for all $t \in \mathbb{R} \Leftrightarrow \{\overset{t}{I} + \overset{t}{U}, \overset{s}{I} + \overset{s}{U}\} = 0$ for all $s, t \in \mathbb{R}$,

iii. $\{\overset{t}{I} + \overset{t}{U}, \overset{s}{I} + \overset{s}{U}\} = 0$ for all $s, t \in \mathbb{R} \Leftrightarrow [\overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}] = 0$ for all $s, t \in \mathbb{R}$
$\Leftrightarrow \overset{t}{K}_j^i \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}_j^i \frac{\partial \overset{t}{U}}{\partial x^i}$ for all $s, t \in \mathbb{R}$,

iv. $\{\overset{t}{I} + \overset{t}{U}, \overset{s}{L}\} = 0$ for all $t, s \in \mathbb{R} \Leftrightarrow [\overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}, \overset{s}{\widehat{L}}] = 0$ for all $t, s \in \mathbb{R} \Leftrightarrow \mathrm{d}\overset{t}{U}(\overset{s}{V}) = 0$ for all $s, t \in \mathbb{R}$.

It is implied that all equations are to hold for all choices of its parameters, we shall not specify it each and every time again.

To iii: We use the linearity of the commutator:

$$[\overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}, \overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}] = [\overset{s}{\widehat{I}}, \overset{t}{\widehat{I}}] + [\overset{s}{\widehat{I}}, \overset{t}{\widehat{U}}] + [\overset{s}{\widehat{U}}, \overset{t}{\widehat{I}}] + [\overset{s}{\widehat{U}}, \overset{t}{\widehat{U}}]. \tag{51}$$

The term $[\overset{s}{\widehat{I}}, \overset{t}{\widehat{I}}]$ vanishes due to Theorem 2.2 and $[\overset{s}{\widehat{U}}, \overset{t}{\widehat{U}}]$ vanishes trivially since the operators corresponding to the potentials act merely by multiplication. In [7] or by direct computation we have $[\overset{s}{\widehat{I}}, \overset{t}{\widehat{U}}] = \widehat{\{\overset{s}{I}, \overset{t}{U}\}}$. Consequently $[\overset{s}{\widehat{I}} + \overset{s}{\widehat{U}}, \overset{t}{\widehat{I}} + \overset{t}{\widehat{U}}] = \widehat{\{\overset{s}{I}, \overset{t}{U}\}} + \widehat{\{\overset{s}{U}, \overset{t}{I}\}}$. Since quantization is a linear map and only the zero polynomial is mapped to a vanishing differential operator, we have that $[\overset{s}{\widehat{Q}}, \overset{t}{\widehat{Q}}] = 0$ if and only if $\{\overset{s}{I}, \overset{t}{U}\} + \{\overset{s}{U}, \overset{t}{I}\} = 0$. This in turn is true if and only if $\{\overset{s}{I}, \overset{t}{U}\}^{\#} + \{\overset{s}{U}, \overset{t}{I}\}^{\#} = 0$. Expressing this in terms of $\overset{t}{K}, \overset{t}{U}, \overset{s}{K}, \overset{s}{U}$ and lowering an index and rearranging terms yields $\overset{t}{K}_j^i \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}_j^i \frac{\partial \overset{t}{U}}{\partial x^i}$. Likewise using the fact that $\{\overset{s}{I}, \overset{t}{I}\} = 0$, we have $\{\overset{t}{I} + \overset{t}{U}, \overset{s}{I} + \overset{s}{U}\} = \{\overset{s}{I}, \overset{t}{U}\} + \{\overset{s}{U}, \overset{t}{I}\}$.

Statement i can be seen analogously to iii, since K lies in the span of $\overset{t}{K}$.

To ii: It suffices to show the equivalence of the rightmost equations of items i and iii: Fix an arbitrary value for t in $\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i}$ and choose pairwise different values $(s_1, \ldots, s_n)$ for s. Add the resulting equations, weighting the ith equation with $(-1)^{n-1}\mu_{n-1}(\widehat{s_i})/\prod_{i \neq j}(s_i - s_j)$. Here $\mu_{n-1}(\widehat{s_i})$ is the elementary symmetric polynomial of degree $n - 1$ in the variables $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. On the right-hand side this gives the coefficient of s^{n-1} of $\overset{s}{K}$ which is the identity operator (when considered as a (1,1)-tensor) acting on the differential of $\overset{t}{U}$. On the left hand side we identify the sum $\sum_{i=1}^{n}(-1)^{n-1}\mu_{n-1}(\widehat{s_i})/\left(\prod_{i \neq j}(s_i - s_j)\right)\frac{\partial \overset{s_i}{U}}{\partial x^i}$ with the differential of U and thus arrive at $\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^j}$. For the other direction, consider two arbitrary values s and t and the equations

$$\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^j}, \quad \overset{s}{K}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{s}{U}}{\partial x^j}. \tag{52}$$

We muliply the first equation with $\overset{s}{K}{}^j_k$, and use the commutativity of $\overset{t}{K}$ with $\overset{s}{K}$ (again considered as mapping one-forms to one-forms):

$$\overset{t}{K}{}^j_k \overset{s}{K}{}^i_j \frac{\partial U}{\partial x^i} = \overset{s}{K}{}^j_k \frac{\partial \overset{t}{U}}{\partial x^j}. \tag{53}$$

Now we can use the second equation of (52) to replace $\overset{s}{K}{}^i_j \frac{\partial U}{\partial x^i}$ with $\frac{\partial \overset{s}{U}}{\partial x^j}$ arriving back at $\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i}$, as we desired.

To iv: Whenever one applies the quantization rules (9) to a linear polynomial $\overset{s}{L}$ and a polynomial $\overset{t}{I}$ of second degree in the momentum variables on $T^*\mathcal{M}$ and takes the commutator of the operators, then combining equations (3.8) and (3.9) from [7] gives us the formula

$$[\widehat{\overset{t}{I}}, \widehat{\overset{s}{L}}] = i\widehat{\{\overset{t}{I}, \overset{s}{L}\}} - \frac{i}{2}\nabla_j\left(\overset{t}{K}{}^{jk}\nabla_k(\nabla_l \overset{s}{V}{}^l)\right). \tag{54}$$

It can be obtained via explicit calculation. The second term on the right-hand side of (54) acts on functions by mere multiplication. It vanishes in our case because $\overset{t}{V}$ is a Killing vector field and thus divergence free. To show that the first term on the right-hand side of (54) vanishes we use that $\widehat{\{\overset{t}{I}, \overset{s}{L}\}} = -\nabla_j \circ (\{\overset{t}{I}, \overset{s}{V}\}^\#)^{jk} \circ \nabla_k$. So, we must show that $\{\overset{t}{I}, \overset{s}{V}\}^\#$ vanishes in order for $\{\overset{t}{I}, \overset{s}{L}\}$ to vanish. Inspection of the components of $\{\overset{t}{I}, \overset{s}{L}\}^\#$ reveals that they are simply the components of the Lie derivative of $\overset{t}{K}$ with respect to $\overset{t}{V}$:

$$(\{\overset{t}{I}, \overset{s}{V}\}^{\#})^{jk} = \overset{s}{V}{}^i \partial_i \overset{t}{K}{}^{jk} - \partial_i (\overset{s}{V}{}^j) \overset{t}{K}{}^{ik} - \partial_i (\overset{s}{V}{}^k) \overset{t}{K}{}^{ji} = (\mathcal{L}_{\overset{s}{V}} \overset{t}{K})^{jk}.$$

Applying the Leibniz rule gives

$$\begin{aligned}
(\mathcal{L}_{\overset{s}{V}} \overset{t}{K})^{jk} = {}& \mathcal{L}_{\overset{s}{V}} \left(\sqrt{\det(t\mathrm{Id} - A)}\right) (t\mathrm{Id} - A)^{-1}{}^i_l g^{lj} \\
& + \sqrt{\det(t\mathrm{Id} - A)} \, (\mathcal{L}_{\overset{s}{V}} (t\mathrm{Id} - A)^{-1})^i_l g^{lj} \\
& + \sqrt{\det(t\mathrm{Id} - A)} \, (t\mathrm{Id} - A)^{-1}{}^i_l (\mathcal{L}_{\overset{s}{V}} g)^{lj}.
\end{aligned}$$

From this we see that, since the flow of $\overset{s}{V}$ preserves A (and thus $\det A$ and functions thereof) [3, Lemma 2.2] and since $\overset{s}{V}$ also is a Killing vector field, the term $\widehat{\{\overset{t}{I}, \overset{s}{V}\}}$ in (54) vanishes. Using this, a direct calculation immediately reveals that both $\{\overset{t}{I} + \overset{t}{U}, \overset{s}{L}\} = 0$ and $[\widehat{\overset{t}{I} + \overset{t}{U}}, \overset{s}{L}] = 0$ reduce to the same expression, namely $d\overset{t}{U}(\overset{s}{V}) = 0$ for all $s, t \in \mathbb{R}$. $\qquad\square$

Lemma 3.16 *Let (g, J, A) be c-compatible and $\overset{t}{K}$ be defined as in (5). Consider a simply connected domain where the number of different eigenvalues of A is constant. Let A be semi-simple. Let $\overset{nc}{E} = \{\varrho_1, \ldots, \varrho_r\}$ be the set of non-constant eigenvalues of A. Let $\overset{c}{E} = \{\varrho_{r+1}, \ldots, \varrho_{r+R}\}$ be the set of constant eigenvalues and $E = \overset{nc}{E} \cup \overset{c}{E}$. Denote by $m(\varrho_i)$ the algebraic multiplicity of ϱ_i. Let U be a function such that $\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i}$ is exact for all values of t and let $\overset{t}{U}$ be such that*

$$\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^j} \tag{55}$$

is satisfied for all values of t. Then up to addition of a function of the single variable t the family of functions $\overset{t}{U}(t, x)$ may be written as

$$\overset{t}{U} = \prod_{\varrho_l \in \overset{c}{E}} (t - \varrho_l)^{m(\varrho_l)/2 - 1} \overset{t}{\tilde{U}} \tag{56}$$

where $\overset{t}{\tilde{U}}$ is a polynomial of degree $r - 1$ in t. Equally $\overset{t}{U}$ can be written as

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2 - 1} f_i \tag{57}$$

where f_i are functions on $\mathcal{M}$. The functions f_i may however not be chosen arbitrarily.

Proof of Lemma 3.16 Because we assumed that A is semi-simple, we can factorize $\overset{t}{K}$ into $\overset{t}{K} = \prod_{l=1}^{r+R} (t - \varrho_l)^{m_l/2-1} \overset{t}{\widetilde{K}}$, with $\overset{t}{\widetilde{K}}$ being a polynomial of degree $r + R - 1$.

$$\prod_{\substack{\varrho_l \in E \\ c}} (t - \varrho_l)^{m(\varrho_l)/2-1} \overset{t}{\widetilde{K}}{}_j^i \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^i} \quad \text{for all } t \in \mathbb{R}. \tag{58}$$

We used that for the non-constant eigenvalues $\varrho_1, \ldots, \varrho_r$ the multiplicities are 2 [3, Lemma 2.2]. Thus in the product on the left-hand side all factors corresponding to non-constant eigenvalues are equal to 1. We observe that upon addition of a function of the single variable t to $\overset{t}{U}$ the equation above is still satisfied. This allows us to choose an arbitrary point x_0 and an arbitrary function $U_0(t)$ and assume that $\overset{t}{U}(x_0, t) = U_0(t)$. Since the left-hand side of (58) is a polynomial in t and is exact for all t, each of the coefficients must be exact. This allows us to integrate the terms of (58) individually:

$$\overset{t}{U}(t, x) = U_0(t) + \int_{x_0}^x \overset{t}{K}(dU) = U_0(t) + \prod_{\substack{\varrho_l \in E \\ c}} (t - \varrho_l)^{m(\varrho_l)/2-1} \sum_{i=0}^r t^i \int_{x_0}^x \widetilde{K}_{(i)}(dU). \tag{59}$$

The last step of this calculation makes use of the fact that if ϱ_l is of multiplicity $m_l \geqslant 4$ then ϱ_l is constant [3, Lemma 2.2]. The integral is meant to be taken along any path connecting x_0 and x and the $\widetilde{K}_{(i)}$ is the coefficient of t^i in $\overset{t}{\widetilde{K}}$. Again $\overset{t}{\widetilde{K}}$ and $\widetilde{K}_{(i)}$ are considered as $(1, 1)$ tensors mapping one-forms to one-forms. So for any value of t the value of $\overset{t}{U}$ at x is uniquely defined by its value at x_0 and the function U. Formula (59) proves the claim that $\overset{t}{U}$ can be written in the form (56) where on the right-hand side $U_0(t)$ takes the role of the possible addition of a function of t alone. Evidently, we have $\overset{t}{\widetilde{U}}(x) = \sum_{i=0}^{r+R} t^i \int_{x_0}^x \widetilde{K}_{(i)}(dU)$. Since $\overset{t}{\widetilde{U}}$ is a polynomial of degree $r + R - 1$ it is uniquely defined by its values at the $r + R$ different eigenvalues of A. Via the Lagrange interpolation formula we have

$$\overset{t}{\widetilde{U}} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)}{(\varrho_i - \varrho_l)} \tilde{f}_i \tag{60}$$

for some functions $\tilde{f}_i$. Introducing $f_i = \prod_{\varrho_l \in E \setminus \{\varrho_i\}} (\varrho_i - \varrho_l)^{m(\varrho_l)/2-1} \tilde{f}_i$ the potential U can be written as

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2-1} f_i \tag{61}$$

concluding the proof of Lemma 3.16. □

Lemma 3.17 *Let (g, J, A) be c-compatible and $\overset{t}{K}$ as in (5). Let A be semi-simple. Let $\overset{nc}{E} = \{\varrho_1, \ldots, \varrho_r\}$ be the set of non-constant eigenvalues of A. Let $\overset{c}{E} = \{\varrho_{r+1}, \ldots, \varrho_{r+R}\}$ be the set of constant eigenvalues and $E = \overset{nc}{E} \cup \overset{c}{E}$. The multiplicity of ϱ_l is denoted by $m(\varrho_l)$. Let*

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2 - 1} f_i \tag{62}$$

and let

$$\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^j} \tag{63}$$

be satisfied for all values of t. Then for all values of i the relation $d f_i \circ A = \varrho_i d f_i$ must be satisfied. In other words: The differentials of the functions f_i are eigenvectors of A with eigenvalues ϱ_i, where A is considered as to map one-forms to one-forms.

Proof of Lemma 3.17 We consider $\overset{t}{K}$ as a $(1, 1)$ tensor field. Using our assumption that A is semi-simple we rewrite equation (63) in terms of the quantities $\overset{t}{\tilde{U}}$, $\tilde{f}_i$ and $\overset{t}{\tilde{K}}$ defined by

$$\overset{t}{\tilde{U}} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)}{(\varrho_i - \varrho_l)} \tilde{f}_i,$$

$$f_i = \prod_{\varrho_l \in E \setminus \{\varrho_i\}} (\varrho_i - \varrho_l)^{m(\varrho_l)/2 - 1} \tilde{f}_i,$$

$$\overset{t}{K} = \prod_{\varrho_l \in E \setminus \{\varrho_i\}} (t - \varrho_l)^{m(\varrho_l)/2 - 1} \overset{t}{\tilde{K}}.$$

Then we use the fact that eigenvalues of multiplicity $m_l \geqslant 4$ are constant [3, Lemma 2.2] and equation (63) transforms into

$$\overset{t}{\tilde{K}}(dU) = d\overset{t}{\tilde{U}} \tag{64}$$

by dividing out the common factors.

The right-hand side can be rewritten: consider $\overset{t}{\tilde{U}}$ where, rather than choosing a constant value for the parameter t we fill in the lth eigenvalue of A. Then we have $\overset{\varrho_l}{\tilde{U}} = \tilde{f}_l$. Taking the differential and rearranging the terms gives

$$d\tilde{f}_l - \frac{\partial \overset{t}{\tilde{U}}}{\partial t}\bigg|_{t=\varrho_l} d\varrho_l = d\overset{\varrho_l}{\tilde{U}} - \frac{\partial \overset{t}{\tilde{U}}}{\partial t}\bigg|_{t=\varrho_l} d\varrho_l = d\overset{t}{\tilde{U}}|_{t=\varrho_l}. \tag{65}$$

We evaluate (64) at $t = \varrho_l$ and plug in (65):

$$\overset{\varrho_l}{\tilde{K}}(dU) = d\tilde{f}_l - \left.\frac{\partial \overset{t}{\tilde{U}}}{\partial t}\right|_{t=\varrho_l} d\varrho_l. \tag{66}$$

Because we assumed A to be semi-simple we can decompose dU into one-forms υ_l such that $\upsilon_l \circ A = \varrho_l \upsilon_l$. From the definition of $\overset{t}{\tilde{K}}$ we have that $\overset{t}{\tilde{K}}(\upsilon_l) = \prod_{\varrho_m \in E \setminus \{\varrho_l\}} (t - \varrho_m)\upsilon_l$, again because we assumed A to be semi-simple. Evaluating at $t = \varrho_k$ yields

$$\overset{\varrho_k}{\tilde{K}}(\upsilon_l) = \left(\prod_{\varrho_m \in E \setminus \{\varrho_l\}} (\varrho_k - \varrho_m)\right) \upsilon_l. \tag{67}$$

In particular this means that if $k \neq l$ then $\overset{\varrho_k}{\tilde{K}}(d\varrho_l)$ is zero. Plugging this into 66 we get that on the left hand side only $\overset{\varrho_l}{\tilde{K}}(dU) = \overset{\varrho_l}{\tilde{K}}(\upsilon_l)$, which we express via (67):

$$\left(\prod_{\varrho_m \in E \setminus \{\varrho_l\}} (\varrho_l - \varrho_m)\right) \upsilon_l = d\tilde{f}_l - \left.\frac{\partial \overset{t}{\tilde{U}}}{\partial t}\right|_{t=\varrho_l} d\varrho_l. \tag{68}$$

Since $d\varrho_l \circ A = \varrho_l d\varrho_l$ [3, Lemma 2.2] and $\upsilon_l \circ A = \varrho_l \upsilon_l$ we have that $d\tilde{f}_l \circ A = \varrho_l d\tilde{f}_l$. The way in which $\tilde{f}_l$ was constructed then implies $df_l \circ A = \varrho_l df_l$, concluding the proof of Lemma 3.17. □

Lemma 3.18 *Let (g, J, A) be c-compatible, $\overset{t}{K}$ as in (5) and $\overset{t}{V}$ as in (7). $\overset{nc}{E} = \{\varrho_1, \ldots, \varrho_r\}$ denotes the set of non-constant eigenvalues of A. $\overset{c}{E} = \{\varrho_{r+1}, \ldots, \varrho_{r+R}\}$ denotes the set of constant eigenvalues and $E = \overset{nc}{E} \cup \overset{c}{E}$. The multiplicity of ϱ_l is denoted by $m(\varrho_l)$. Let*

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2-1} f_i. \tag{69}$$

Let furthermore $df_l \circ A = \varrho_l df_l$ for all values of $l = 1, \ldots, r + R$. Let $d\overset{t}{U}(\overset{s}{V}) = 0$ be satisfied for all values of $s, t \in \mathbb{R}$. Then $d\overset{t}{U}(\overset{s}{V}) = 0$ is satisfied for all values of $s, t \in \mathbb{R}$ if and only if for each eigenvalue ϱ_k of A the differential df_k is proportional to $d\varrho_k$ at all points where $d\varrho_k \neq 0$.

Corollary 3.19 *If ϱ_l is a non-constant real eigenvalue and $d\varrho_l \neq 0$ in the neighbourhood of a given point then locally f_l can be expressed as a smooth function of ϱ_l. Likewise if ϱ_l is a non-constant complex eigenvalue and $d\varrho_l \neq 0$ in the neighbourhood of a given point then locally f_l can be expressed as a holomorphic function of ϱ_l.*

Proof of Lemma 3.18 The condition $d\overset{t}{U}(\overset{s}{V}) = 0$ for all $s, t \in \mathbb{R}$ is equivalent to $d\overset{t}{U}(\mathrm{span}\{J\,\mathrm{grad}\,\varrho_i, i = 1, \ldots, r\}) = 0$, because $\mathrm{span}\{\overset{t}{V}, t \in \mathbb{R}\} = \mathrm{span}\{J\,\mathrm{grad}\,\varrho_i, i = 1, \ldots, r\}$. From (69) we see that $d\overset{t}{U}$ involves (with some coefficients) only the differentials of the eigenvalues of A and the differentials of the functions f_i. Thus $d\overset{t}{U}(J\,\mathrm{grad}\,\varrho_i)$ is a linear combination of $d\varrho_j$ and df_j applied to $J\,\mathrm{grad}\,\varrho_i$. But $d\varrho_j(J\,\mathrm{grad}\,\varrho_i)$ is zero for all values of i, j: if $i = j$, then $d\varrho_j(J\,\mathrm{grad}\,\varrho_i) = 0$ due to the fact that J is antisymmetric with respect to g and if $i \neq j$ then $d\varrho_j(J\,\mathrm{grad}\,\varrho_i) = 0$, because A is g-self-adjoint and $\mathrm{grad}\,\varrho_i$ and $\mathrm{grad}\,\varrho_j$ are eigenvectors of A with different eigenvalues. Thus $d\overset{t}{U}(J\,\mathrm{grad}\,\varrho_i)$ is a linear combination of $\{df_j(J\,\mathrm{grad}\,\varrho_i), j = 1, \ldots, r + R\}$. But because we assumed that for all l, $df_l \circ A = \varrho_l df_l$ and because $A\,\mathrm{grad}\,\varrho_i = \varrho_i\,\mathrm{grad}\,\varrho_i$ [3, Lemma 2.2] and A commutes with J and is g-self-adjoint we get that $d\overset{t}{U}(J\,\mathrm{grad}\,\varrho_i)$ is some coefficient times $df_i(J\,\mathrm{grad}\,\varrho_i)$.

From (69) we see that this coefficient is

$$\prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2 - 1}.$$

But for a given value of t this can only vanish at points on $\mathcal{M}$ where t is equal to an eigenvalue of A. So at each point on the manifold we can choose a value for t such that this expression is non-zero and thus $df_i(J\,\mathrm{grad}\,\varrho_i)$ must vanish for all values of i. If we consider a value i such that ϱ_i is constant then $df_i(J\,\mathrm{grad}\,\varrho_i) = 0$ is trivially satisfied. If ϱ_i is non-constant then its multiplicity is 2 [3, Lemma 2.2] and at all points where $d\varrho_i \neq 0$ the set $\{\mathrm{grad}\,\varrho_i, J\,\mathrm{grad}\,\varrho_i\}$ is an orthogonal basis of the ϱ_i-eigenspace of A. It follows that at such points df_i may be written as a linear combination of $d\varrho_i$ and $d\varrho_i \circ J$. Plugging this decomposition into $df_i(J\,\mathrm{grad}\,\varrho_i) = 0$ we conclude that df_i is proportional to $d\varrho_i$ at all points where $d\varrho_i \neq 0$ and Lemma 3.18 is proven. $\qquad\square$

Lemma 3.20 *Let (g, J, A) be c-compatible. $\overset{nc}{E} = \{\varrho_1, \ldots, \varrho_r\}$ denotes the set of non-constant eigenvalues of A. $\overset{c}{E} = \{\varrho_{r+1}, \ldots, \varrho_{r+R}\}$ denotes the set of constant eigenvalues and $E = \overset{nc}{E} \cup \overset{c}{E}$. The multiplicity of ϱ_l is denoted by $m(\varrho_l)$. Let*

$$\overset{t}{K}{}^{ij} \overset{\text{def}}{=} \sqrt{\det(t\,\mathrm{Id} - A)}\,(t\,\mathrm{Id} - A)^{-1}{}^i_l\,g^{lj}, \quad \overset{t}{V}{}^j = J^j_k g^{ki}\nabla_i\sqrt{\det(t\,\mathrm{Id} - A)}$$

and

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2 - 1} f_i \tag{70}$$

with $df_l \circ A = \varrho_l df_l$ for all $l = 1, \ldots, r$ and df_l proportional to $d\varrho_l$ for all l for which ϱ_l is non-constant.

Then

$$\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i} \quad \text{for all } s,t \in \mathbb{R} \quad \text{and} \quad d\overset{t}{U}(\overset{s}{V}) = 0 \quad \text{for all } s,t \in \mathbb{R}. \quad (71)$$

Proof of Lemma 3.20 $d\overset{t}{U}(\overset{s}{V}) = 0$ for all $s,t \in \mathbb{R}$ is fulfilled because df_i and $d\varrho_i$ evaluate to zero when applied to $J\mathrm{grad}\,\varrho_j$ for all $i, j = 1, \ldots, r+R$.

To see $\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i}$, we compute $d\overset{t}{U}$ using the fact that non-constant eigenvalues of A are of multiplicity 2 [3, Lemma 2.2]:

$$d\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \left(\frac{t-\varrho_l}{\varrho_i - \varrho_l}\right)^{m(\varrho_l)/2} (t-\varrho_i)^{m(\varrho_i)/2-1} \left[df_i - \sum_{\varrho_p \in E\setminus\{\varrho_i\}} \frac{m(\varrho_p)/2}{\varrho_i - \varrho_p} f_i d\varrho_i\right]$$

$$- \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_k\}} \frac{t-\varrho_l}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} \sum_{\underset{nc}{\varrho_p \in E\setminus\{\varrho_i\}}} \prod_{\varrho_l \in E\setminus\{\varrho_p\}} (t-\varrho_l)^{m(\varrho_l)/2} f_i \frac{d\varrho_p}{\varrho_p - \varrho_i}. \quad (72)$$

Considering $\overset{s}{K}$ as a $(1,1)$-tensor mapping one-forms to one-forms and using that for all $i = 1, \ldots, r$: $d\varrho_i \circ A = \varrho_i d\varrho_i$ and $df_i \circ A = \varrho_i df_i$ we have

$$\overset{s}{K}(d\varrho_i) = \left(\prod_{\varrho_l \in E\setminus\{\varrho_i\}} (s-\varrho_l)^{m_l/2}\right)(s-\varrho_i)^{m_i/2-1}d\varrho_i,$$

$$\overset{s}{K}(df_i) = \left(\prod_{\varrho_l \in E\setminus\{\varrho_i\}} (s-\varrho_l)^{m_l/2}\right)(s-\varrho_i)^{m_i/2-1}df_i. \quad (73)$$

Again we consider $\overset{s}{K}$ as a $(1,1)$-tensor acting on the differential of $\overset{t}{U}$. By combining (72) and (73) and using that the non-constant eigenvalues have multiplicity 2 we get

$$\overset{s}{K}(d\overset{t}{U}) = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \left(\frac{(s-\varrho_l)(t-\varrho_l)}{\varrho_i - \varrho_l}\right)^{m_l/2} ((t-\varrho_i)(s-\varrho_i))^{m_i/2-1}$$

$$\times \left[df_i - \sum_{p\neq i} \frac{m_p/2}{\varrho_i - \varrho_p} f_i d\varrho_i\right]$$

$$- \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_k\}} \frac{1}{(\varrho_i - \varrho_l)^{m_l/2}}$$

$$\times \sum_{\underset{nc}{\varrho_p \in E\setminus\{\varrho_i\}}} \prod_{\varrho_l \in E\setminus\{\varrho_p\}} ((s-\varrho_l)(t-\varrho_l))^{m_l/2} f_i \frac{d\varrho_p}{\varrho_p - \varrho_i}. \quad (74)$$

The right-hand side is apparently symmetric when exchanging s and t and as a consequence $\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i}$ is fulfilled, concluding the proof of Lemma 3.20. $\qquad\square$

Proof of Theorem 2.5 The theorem results as a combination of the proofs of Lemmata 3.15 and 3.20.   □

Proof of Theorem 2.9 Combining the proofs of Lemmata 3.15, 3.16, 3.17, 3.18 and 3.20.   □

3.4 Proof of Theorem 2.16: common eigenfunctions

We first show that ψ is an eigenfunction of $\overset{s}{\hat{L}}$ for all values of s if and only if it is an eigenfunction of ∂_{t_i} for all values of i. A direct computation from (19), (20), (21) shows that

$$\overset{s}{\hat{L}}\psi = \prod_{\varrho_p \in \overset{c}{E}} (s - \varrho_p)^{m(\varrho_p)/2} \sum_{q=1}^{r} s^{r-q} \partial_{t_q} \psi. \tag{75}$$

Now suppose that ψ is an eigenfunction of $\overset{s}{\hat{L}}$ for all values of s and denote the eigenvalue by $\overset{s}{\hat{\omega}}$. Then $\overset{s}{\hat{\omega}}$ must be a polynomial in s of degree $r + R - 1$, because $\overset{s}{\hat{L}}$ is a polynomial in s of degree $r + R - 1$. From the equation above we see that $\overset{s}{\hat{\omega}}$ must have a zero of order $m(\varrho_p)/2$ at $s = \varrho_p$ for all constant eigenvalues ϱ_p. Thus we can write $\overset{s}{\hat{\omega}} = \prod_{\varrho_p \in \overset{c}{E}} (s - \varrho_p)^{m(\varrho_p)/2} \sum_{q=1}^{r} s^{r-q} i\omega_q$. But because polynomials are equal if and only if all their coefficients are equal we get that $\overset{s}{\hat{L}}\psi = \overset{s}{\hat{\omega}}$ for all $s \in \mathbb{R}$ if and only if $i\partial_{t_q}\psi = \omega_q \psi$ for $q = 1, \ldots, r$.

To obtain the other separated equations we work with the family of second order differential operators $\overset{s}{\hat{K}}$. Since the metric is not given in terms of the coordinate basis and the one-forms α and ϑ are not unique it poses an obstruction to using the standard formula for the Laplacian and $\overset{t}{\hat{K}}$. The workaround is quick and simple though.

Let $\{X_i, i = 1, \ldots, n\}$ be a set of n linearly independent differentiable vector fields on a manifold $\mathcal{M}^n$ and denote by $\{\beta^i, i = 1, \ldots, n\}$ its dual basis, i.e. $\beta^i(X_j) = \delta^i_j$. We shall denote by T the matrix relating the coordinate vector fields ∂_i and the vector fields X_i: $T^i_j X_i = \partial_j$. Then for an arbitrary symmetric $(2, 0)$-tensor the following formula is easily obtained via the product rule for partial derivatives:

$$\frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|} \overset{s}{K}^{ij} \partial_j = \frac{\det T}{\sqrt{|\det g|}} X_s \frac{\sqrt{|\det g|}}{\det T} T^s_i \overset{s}{K}^{ij} T^k_j X_j$$
$$- X_s(T^s_i) \overset{s}{K}^{ij} T^k_j X_k + \frac{X_s(\det T)}{\det T} T^s_i \overset{s}{K}^{ij} T^k_j X_j \tag{76}$$

where on the right-hand side the X's are to be interpreted as the directional derivative in the sense $X_s = (T^{-1})^i_s \partial_i$ and in the last two terms $X_s(\cdot)$ is meant as to only act on the expression in the parentheses.

The quantity $\frac{\det g}{\det T^2}$ is simply the determinant of the matrix with (i, j)th component $g(X_i, X_j)$ and $T_i^s \overset{s}{K}{}^{ij} T_j^k$ are the components of $\overset{s}{K}$ in the basis $\{X_i, i = 1, \ldots, n\}$. If $\{X_i, i = 1, \ldots, n\}$ are the coordinate vector fields belonging to some coordinate system then the last two terms cancel out and one arrives at the well-known fact that the left-hand side expression is independent of the choice of coordinates.

In our case we choose $(X_i) = \left(\partial_{\chi_i}, \partial_{t_i}, \partial_{\overset{\gamma}{y_i}} - \sum_{p=1}^r \overset{\gamma}{\alpha}_{pi} \partial_{t_p}\right)$. The dual basis consists of the one-forms $(\mathrm{d}\chi_i, \mathrm{d}t_i + \alpha_i, \mathrm{d}\overset{\gamma}{y_i})$. Taking i as the column index and j as the row index we have the components T_j^i given by

$$\left(\begin{array}{c|c|c} \mathrm{Id} & 0 & 0 \\ \hline 0 & \mathrm{Id} & 0 \\ \hline 0 & * & \mathrm{Id} \end{array}\right),$$

where the $*$-block contains the components of α_l as the lth column and the Id-blocks are of dimensions equal to the number of χ-, t- and y-coordinates.

From this we can conclude that $X_s(T_i^s) = (T^{-1})_s^j \partial_j T_i^s = 0$ because the one-forms α do not depend on the t-variables. Furthermore, $\det T = 1$ and thus $X_s(\det T) = 0$ for all values of s.

For our specific case (76) simplifies to

$$\frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|}\, \overset{s}{K}{}^{ij} \partial_j = \frac{\det T}{\sqrt{|\det g|}} X_s \frac{\sqrt{|\det g|}}{\det T} T_i^s \overset{s}{K}{}^{ij} T_j^k X_j. \qquad (77)$$

Here $\frac{\sqrt{|\det g|}}{\det T}$ is the determinant of the matrix of g in the basis $(\mathrm{d}\chi_i, \mathrm{d}t_i, \mathrm{d}\overset{\gamma}{y_i})$ and $T_i^s \overset{s}{K}{}^{ij} T_j^k$ is the matrix of $\overset{s}{K}$ in the basis (X_i). These quantities can be obtained from the formulae (19), (20) and (21). We can then express the vector fields (X_i) in terms of the coordinate basis to get the following result (we use that $A_\gamma = \varrho_\gamma \mathrm{Id}$, because we assumed that A is semi-simple and that all constant eigenvalues are real):

$$\nabla_i \overset{s}{K}{}^{ij} \nabla_j = \frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|}\, \overset{s}{K}{}^{ij} \partial_j$$

$$= \sum_{\substack{\mathrm{nc} \\ \varrho_k \in E}} \frac{\prod_{\varrho_l \in E \setminus \{\varrho_k\}} (s - \varrho_l)^{m(\varrho_l)/2}}{\varepsilon_k \Delta_k \prod_{\varrho_\gamma \in E}^{\mathrm{c}} (\varrho_\gamma - \varrho_k)^{m(\varrho_\gamma)/2} \varrho_k'}\, \partial_{\chi_k} \varrho_k' \prod_{\varrho_\gamma \in E}^{\mathrm{c}} (\varrho_\gamma - \varrho_k)^{m(\varrho_\gamma)/2} \partial_{\chi_k}$$

$$+ \sum_{i,j=1}^r \sum_{\substack{\mathrm{nc} \\ \varrho_k \in E}} \frac{\varepsilon_k (-\varrho_k)^{2r-i-j}}{\Delta_k (\varrho_k')^2} \prod_{\varrho_l \in E \setminus \{\varrho_k\}} (s - \varrho_l)^{m(\varrho_l)/2} \partial_{t_i} \partial_{t_j}$$

$$+ \sum_{\gamma: \varrho_\gamma \in E}^{\mathrm{c}} \frac{1}{\sqrt{|\det g_\gamma|}} \partial_{\overset{\gamma}{y_i}} \sqrt{|\det g_\gamma|} \left(\prod_{\varrho_l \in E} (s - \varrho_l)^{m(\varrho_l)/2} (s\mathrm{Id} - A_\gamma)^{-1}\right.$$

$$\left.\prod_{\substack{\mathrm{nc} \\ \varrho_k \in E}} (A_\gamma - \varrho_k \mathrm{Id})^{-1} g_\gamma^{-1}\right)^{ij} \partial_{\overset{\gamma}{y_j}}$$

$$
-\sum_{\gamma:\varrho_\gamma\in \overset{c}{E}} \frac{1}{\sqrt{|\det g_\gamma|}}\,\partial_{\overset{\gamma}{y_i}}\sqrt{|\det g_\gamma|}\Bigg(\prod_{\varrho_l\in E}(s-\varrho_l)^{m(\varrho_l)/2}(s\mathrm{Id}-A_\gamma)^{-1}
$$

$$
\prod_{\varrho_k\in \overset{nc}{E}}(A_\gamma-\varrho_k\mathrm{Id})^{-1}g_\gamma^{-1}\Bigg)^{ij\;\overset{r}{\sum_{q=1}}}\overset{\gamma}{\alpha}_{qj}\partial_{t_q}
$$

$$
-\sum_{\gamma:\varrho_\gamma\in \overset{c}{E}}\Bigg(\prod_{\varrho_l\in E}(s-\varrho_l)^{m(\varrho_l)/2}(s\mathrm{Id}-A_\gamma)^{-1}\prod_{\varrho_k\in \overset{nc}{E}}(A_\gamma-\varrho_k\mathrm{Id})^{-1}g_\gamma^{-1}\Bigg)^{ij}\overset{r}{\sum_{q=1}}\overset{\gamma}{\alpha}_{qi}\partial_{t_q}\partial_{\overset{\gamma}{y_j}}
$$

$$
-\sum_{\gamma:\varrho_\gamma\in \overset{c}{E}}\Bigg(\prod_{\varrho_l\in E}(s-\varrho_l)^{m(\varrho_l)/2}(s\mathrm{Id}-A_\gamma)^{-1}\prod_{\varrho_k\in \overset{nc}{E}}(A_\gamma-\varrho_k\mathrm{Id})^{-1}g_\gamma^{-1}\Bigg)^{ij}\overset{r}{\sum_{p,q=1}}\overset{\gamma}{\alpha}_{qi}\overset{\gamma}{\alpha}_{pj}\partial_{t_q}\partial_{t_p}.
$$

$$(78)$$

If now we suppose that ψ simultaneously satisfies the family of eigenvalue equations

$$
-\nabla_i \overset{s}{K}{}^{ij}\nabla_j\psi + \overset{s}{U}\psi = \overset{s}{\lambda}\psi \tag{79}
$$

then the left-hand side is a polynomial in s and for the equations to hold $\overset{s}{\lambda}$ must be a polynomial of degree $n-1$ in s as well and we shall write $\overset{s}{\lambda}=\sum_{l=0}^{n-1}\lambda_l s^l$. Because of our assumption that all constant eigenvalues are real and that A is semi-simple, we have $A_\gamma = c_\gamma\mathrm{Id} = \varrho_{r+\gamma}\mathrm{Id}$ for $\gamma = 1,\ldots,R$. Thus for all values of γ where $m(\varrho_\gamma) > 2$ the left-hand side of $-\nabla_j \overset{s}{K}{}^{jk}\nabla_k\psi + \overset{s}{U}\psi = \overset{s}{\lambda}\psi$ has a zero of multiplicity $m(\varrho_\gamma)/2-1$ at $s=\varrho_\gamma$. We can thus write $\overset{s}{\lambda} = \prod_{\varrho_\gamma\in \overset{c}{E}}(s-\varrho_\gamma)^{m(\varrho_\gamma)/2-1}\overset{s}{\tilde\lambda}$ where $\overset{s}{\tilde\lambda}$ is a polynomial of degree $r+R-1$ in s. $\overset{s}{\tilde\lambda}$ can be written as $\overset{s}{\tilde\lambda}=\sum_{j=0}^{r+R-1}s^j\tilde\lambda_j$. Because we assumed that the eigenvalue equation (79) is satisfied for all values of s we can split (79) into n equations, one for each coefficient. We choose a non-constant eigenvalue $\varrho_k\in \overset{nc}{E}$ and multiply the equation for the coefficient of s^l with ϱ_k^l. We do this for all n equations and add them up. If we use $\nabla_i \overset{s}{K}{}^{ij}\nabla_j = \frac{1}{\sqrt{|\det g|}}\partial_i\sqrt{|\det g|}\overset{s}{K}{}^{ij}\partial_j$ in (79) then the result of these steps is equal to replacing $\nabla_i \overset{s}{K}{}^{ij}\nabla_j\psi$ via (78) and substituting s with ϱ_k. We end up with

$$
\frac{-1}{\varepsilon_k\varrho_k'}\partial_{\chi_k}\varrho_k'\prod_{\varrho_\gamma\in \overset{c}{E}}(\varrho_k-\varrho_\gamma)^{m(\varrho_\gamma)/2}\partial_{\chi_k}\psi
$$

$$
-\sum_{i,j=1}^{r}\frac{\varepsilon_k(-\varrho_k)^{2r-i-j}}{(\varrho_k')^2}\prod_{\varrho_\gamma\in \overset{c}{E}}(\varrho_k-\varrho_\gamma)^{m(\varrho_\gamma)/2}\partial_{t_i}\partial_{t_j} + f_r\psi = \sum_{i=0}^{n-1}\lambda_i\varrho_k^i. \tag{80}
$$

Because $\partial_{t_q}\psi = i\omega_q\psi$, ψ satisfies the ordinary differential equations that have been claimed.

To obtain the separated partial differential equations that we have claimed we divide $-\nabla_j \overset{s}{K}{}^{jk}\nabla_k\psi + \overset{s}{U}\psi = \overset{s}{\lambda}\psi$ by $\prod_{\varrho_\gamma\in E} {}^{c}(s-\varrho_\gamma)^{m(\varrho_\gamma)/2-1}$. Then we choose a constant eigenvalue ϱ_γ and evaluate the result at $s = \varrho_\gamma \in \overset{c}{E}$ in the same way as before:

$$
\begin{aligned}
\sum_{j=0}^{r+R-1}\tilde\lambda_j\varrho_\gamma^j = & -\prod_{\varrho_c\in\overset{c}{E}\setminus\{\varrho_\gamma\}}(\varrho_\gamma-\varrho_c)\Bigg[\frac{1}{|\det g_\gamma|^{1/2}}\,\partial_{\overset{\gamma}{y_i}} g_\gamma^{ij}|\det g_\gamma|^{1/2}\partial_{\overset{\gamma}{y_j}}\psi \\
& -\sum_{q=1}^{r}\frac{1}{|\det g_\gamma|^{1/2}}\,\partial_{\overset{\gamma}{y_i}} g_\gamma^{ij}|\det g_\gamma|^{1/2}\overset{\gamma}{\alpha}_{qj}\partial_{t_q}\psi - \sum_{q=1}^{r} g_\gamma^{ij}\overset{\gamma}{\alpha}_{qi}\partial_{t_q}\partial_{\overset{\gamma}{y_j}}\psi \\
& +\sum_{p,q=1}^{r} g_\gamma^{ij}\overset{\gamma}{\alpha}_{qi}\overset{\gamma}{\alpha}_{pj}\partial_{t_q}\partial_{t_p}\psi\Bigg] + \frac{1}{\prod_{\varrho_c\in\overset{c}{E}\setminus\{\varrho_\gamma\}}(\varrho_\gamma-\varrho_c)^{m(\varrho_c)/2-1}}\,f_\gamma\psi.
\end{aligned}
$$
$$(81)$$

Again: using $\partial_{t_i}\psi = i\omega_i\psi$ gives us the desired result.

The last thing to prove is to show that if ψ fulfills (23) and (24) for some constants $\tilde\lambda_0,\ldots,\tilde\lambda_{r+R-1},\omega_1,\ldots,\omega_r$ then it is also an eigenfunction of $\overset{s}{\widehat{K}}$ for all real values s. To do so, we use $\partial_{t_i}\psi = i\omega_i\psi$ to obtain (80) and (81) from (23) and (24). Then for all non-constant eigenvalues ϱ_k we multiply the corresponding equation of (23) by

$$
\prod_{\varrho_i\in E\setminus\{\varrho_k\}}\left(\frac{s-\varrho_i}{\varrho_k-\varrho_i}\right)^{m(\varrho_i)/2}
$$

and for each constant eigenvalue ϱ_γ we multiply the corresponding equation of (81) by

$$
\prod_{\varrho_i\in E\setminus\{\varrho_\gamma\}}\left(\frac{s-\varrho_i}{\varrho_k-\varrho_i}\right)\prod_{\varrho_i\in\overset{c}{E}}(s-\varrho_\gamma)^{m_\gamma/2-1}
$$

and add up the results. Then we use the product rule for differentiation and the lagrange interpolation formula for polynomials and arrive at $\overset{s}{\widehat{K}}\psi = \sum_{k=1}^{n-1}\lambda_k s^k\psi$. Theorem 2.16 is proven.

4 Summary

We have generalized the integrals that Topalov [14] found for the geodesic flow of c-compatible structures (g, J, A) to certain natural Hamiltonian systems. We have proven that these integrals also commute in the quantum sense. The result was then

generalized to a class of natural Hamiltonian systems: in the case where the tensor A is semi-simple we have described all potentials that may be added to the kinetic energy term such that the resulting functions on $T^*\mathcal{M}$ still Poisson-commute pairwise and their quantum operators commute pairwise as well. The potentials that are admissible in the quantum problem are the same as in the classic case. In the case that A is not semi-simple we could present some potentials that may be added to the kinetic energy such that the modified integrals still commute in both the classical and the quantum sense; it is however not clear whether there exist more and this shall be subject to further investigation. We have tackled the question of the separation of variables for common eigenfunctions of the constructed differential operators in the case where A is semi-simple and all constant eigenvalues are real. If all eigenvalues of A are non-constant we get complete reduction to ordinary differential equations. The case where A has constant non-real eigenvalues or Jordan blocks still needs investigation.

References

1. Benenti, S., Chanu, C., Rastelli, G.: Remarks on the connection between the additive separation of the Hamilton–Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and Robertson conditions. J. Math. Phys. **43**(11), 5183–5222 (2002)
2. Bolsinov, A.V., Matveev, V.S.: Geometrical interpretation of Benenti systems. J. Geom. Phys. **44**(4), 489–506 (2003)
3. Bolsinov, A.V., Matveev, V.S., Rosemann, S.: Local normal forms for c-projectively equivalent metrics and proof of the Yano-Obata conjecture in arbitrary signature. Proof of the projective Lichnerowicz conjecture for Lorentzian metrics. Ann. Sci. Éc. Norm. Supér. **54**(6), 1465–1540 (2021)
4. Calderbank, D.M.J., Eastwood, M.G., Matveev, V.S., Neusser, K.: C-Projective Geometry. Memoirs of the American Mathematical Society, vol. 267 (1299). American Mathematical Society, Providence (2020)
5. Carter, B.: Killing tensor quantum numbers and conserved currents in curved space. Phys. Rev. D **16**(12), 3395–3414 (1977)
6. Domashev, V.V., Mikeš, J.: Theory of holomorphically projective mappings of Kählerian spaces. Math. Notes **23**(2), 160–163 (1978)
7. Duval, C., Valent, G.: Quantum integrability of quadratic Killing tensors. J. Math. Phys. **46**(5), 053516 (2005)
8. Jost, J.: Riemannian Geometry and Geometric Analysis, 7th edn. Springer, Cham (2017)
9. Kiosak, V., Matveev, V.S.: Proof of the projective Lichnerowicz conjecture for pseudo-Riemannian metrics with degree of mobility greater than two. Commun. Math. Phys. **297**(2), 401–426 (2010)
10. Matveev, V.S.: Quantum integrability for the Beltrami-Laplace operators of projectively equivalent metrics of arbitrary signatures. Chebyshevskiĭ Sb. **21**(2), 275–289 (2020)

11. Matveev, V.S., Topalov, P.J.: Quantum integrability of Beltrami–Laplace operator as geodesic equivalence. Math. Z. **238**(4), 833–866 (2001)
12. Matveev, V.S., Topalov, P.J.: Trajectory equivalence and corresponding integrals. Regul. Chaotic Dyn. **3**(2), 30–45 (1998)
13. Ōtsuki, T., Tashiro, Y.: On curves in Kaehlerian spaces. Math. J. Okayama Univ. **4**(1), 57–78 (1954)
14. Topalov, P.: Geodesic compatibility and integrability of geodesic flows. J. Math. Phys. **44**(2), 913–929 (2003)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.