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RESEARCH ARTICLE

# K-stable Fano threefolds of rank 2 and degree 30

**Ivan Cheltsov**[1] · **Jihun Park**[2,3]

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## Abstract
We find all K-stable smooth Fano threefolds in the family No. 2.22.

**Keywords** Fano threefolds · K-stability · Kähler–Einstein metric

**Mathematics Subject Classification** 14J45 · 14J30 · 32Q20

Let $X$ be a smooth Fano threefold. Then $X$ belongs to one of the 105 families, which are labeled as No. 1.1, No. 1.2, . . ., No. 9.1, No. 10.1. See [2], for the description of these families. If $X$ is a general member of the family No. $\mathcal{N}$, then [2, Main Theorem] gives

$$X \text{ is K-polystable} \iff \mathcal{N} \notin \left\{ \begin{array}{l} 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, \\ 3.14, 3.16, 3.18, 3.21, 3.22, 3.23, \\ 3.24, 3.26, 3.28, 3.29, 3.30, 3.31, \\ 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, \\ 5.2 \end{array} \right\}.$$

The goal of this note is to find all K-polystable smooth Fano threefolds in the family No. 2.22. This family contains both K-polystable and non-K-polystable smooth

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✉ Ivan Cheltsov
I.Cheltsov@ed.ac.uk

Jihun Park
wlog@postech.ac.kr

1 School of Mathematics, University of Edinburgh, Edinburgh, Scotland

2 Center for Geometry and Physics, Institute for Basic Science, Pohang, Korea

3 Department of Mathematics, POSTECH, Pohang, Korea

Fano threefolds, and a conjectural characterization of all K-polystable members has been given in [2, Section 7.4]. We will confirm this conjecture—this will complete the description of all K-polystable smooth Fano threefolds of Picard rank 2 and degree 30 started in [2].

Starting from now, we suppose that $X$ is a smooth Fano threefold in the family No. 2.22. Then $X$ can be described both as the blow-up of $\mathbb{P}^3$ along a smooth twisted quartic curve, and the blow-up of $V_5$, the unique smooth threefold No. 1.15, along an irreducible conic. More precisely, there are a smooth twisted quartic curve $C_4 \subset \mathbb{P}^3$, a smooth conic $C \subset V_5$, and a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\phi} \\ \mathbb{P}^3 & \overset{\psi}{\dashrightarrow} & V_5, \end{array}$$

where $\pi$ is the blow-up of $\mathbb{P}^3$ along $C_4$, $\phi$ is the blow-up of $V_5$ along $C$, and $\psi$ is given by the linear system of cubic surfaces containing $C_4$. Here, $V_5$ is embedded in $\mathbb{P}^6$ as described in [2, Section 5.10]. All smooth Fano threefolds in the family No. 2.22 can be obtained in this way.

The curve $C_4$ is contained in a unique smooth quadric surface $Q \subset \mathbb{P}^3$, and $\phi$ contracts the proper transform of this surface. Note that

$$\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathbb{P}^3, C_4) \cong \mathrm{Aut}(Q, C_4).$$

Choosing appropriate coordinates on $\mathbb{P}^3$, we may assume that $Q$ is given by $x_0x_3 = x_1x_2$, where $[x_0 : x_1 : x_2 : x_3]$ are coordinates on $\mathbb{P}^3$. Fix the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$([u : v], [x : y]) \mapsto [xu : xv : yu : yv],$$

where $([u : v], [x : y])$ are coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$. Swapping $[u : v]$ and $[x : y]$ if necessary, we may assume that $C_4$ is a divisor of degree $(1, 3)$ in $Q$, so that $C_4$ is given in $Q$ by

$$uf_3(x, y) = vg_3(x, y)$$

for some non-zero cubic homogeneous polynomials $f_3(x, y)$ and $g_3(x, y)$.

Let $\sigma : C_4 \to \mathbb{P}^1$ be the map given by the projection $([u : v], [x : y]) \mapsto [u : v]$. Then $\sigma$ is a triple cover, which is ramified over at least two points. After an appropriate change of coordinates $[u : v]$, we may assume that $\sigma$ is ramified over $[1 : 0]$ and $[0 : 1]$. Then both $f_3$ and $g_3$ have multiple zeros in $\mathbb{P}^1$. Changing coordinates $[x : y]$, we may assume that these zeros are $[0 : 1]$ and $[1 : 0]$, respectively. Keeping in mind that the curve $C_4$ is smooth, we see that $C_4$ is given by

$$u(x^3 + ax^2y) = v(y^3 + by^2x)$$

for some complex numbers $a$ and $b$, after a suitable scaling of the coordinates. If $a = b = 0$, then the curve $C_4$ is given by $ux^3 = vy^3$, which gives $\mathrm{Aut}(X) \cong \mathrm{Aut}(Q, C_4) \cong \mathbb{G}_m \rtimes \boldsymbol{\mu}_2$. In this case, the threefold $X$ is known to be K-polystable [2, Section 4.4].

***Example*** Suppose that $ab = 0$, but $a \neq 0$ or $b \neq 0$. We can scale the coordinates further and swap them if necessary, and assume that the curve $C_4$ is given by

$$ux^3 = v(y^3 + y^2x).$$

In this case, the threefold $X$ is not K-polystable [2, Section 7.4].

A conjecture in [2, Section 7.4] says that the non-K-polystable Fano threefold described in this example is the unique non-K-polystable smooth Fano threefold in the family No. 2.22. Let us show that this is indeed the case. To do this, we may assume that $a \neq 0$ and $b \neq 0$. Then, scaling the coordinates, we may assume that $C_4$ is given by

$$u(x^3 + \lambda x^2 y) = v(y^3 + \lambda y^2 x) \qquad (\bigstar)$$

for some non-zero complex number $\lambda$. Since the curve $C_4$ is smooth, we must have $\lambda \neq \pm 1$. Moreover, if $\lambda = \pm 3$, then we can change the coordinates on $Q$ in such a way that $C_4$ would be given by $ux^3 = v(y^3 + y^2x)$, so that $X$ is not K-polystable in this case.

We know from [2] that $X$ is K-stable if $C_4$ is given by ($\bigstar$) with $\lambda$ general. In particular, we know from [2, Section 4.4] that the threefold $X$ is K-stable when $\lambda = \pm\sqrt{3}$. Our main result is the following theorem.

**Theorem** *Suppose that $C_4$ is given in ($\bigstar$) with $\lambda \notin \{0, \pm 1, \pm 3\}$. Then $X$ is K-stable.*

Let us prove this theorem. We suppose that $C_4$ is given by ($\bigstar$) with $\lambda \notin \{0, \pm 1, \pm 3\}$. Then the triple cover $\sigma : C_4 \to \mathbb{P}^1$ is ramified in four distinct points $P_1, P_2, P_3, P_4$, which implies that $\mathrm{Aut}(Q, C_4)$ is a finite group, since

$$\mathrm{Aut}(Q, C_4) \subset \mathrm{Aut}(C_4, P_1 + P_2 + P_3 + P_4).$$

Without loss of generality, we may assume that

$$P_1 = ([1:0], [0:1]) = [0:1:0:0],$$
$$P_2 = ([0:1], [1:0]) = [0:0:1:0],$$

where we use both the coordinates on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^3$ simultaneously.

Observe that the group $\mathrm{Aut}(Q, C_4)$ contains an involution $\tau$ that is given by

$$([u:v], [x:y]) \mapsto ([v:u], [y:x]).$$

Let us identify $\mathrm{Aut}(\mathbb{P}^3, C_4) = \mathrm{Aut}(Q, C_4)$ using the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ fixed above. Then $\tau$ is given by $[x_0 : x_1 : x_2 : x_3] \mapsto [x_3 : x_2 : x_1 : x_0]$. Note that $\tau$ swaps $P_1$ and $P_2$, and the $\tau$-fixed points in $C_4$ are $([1:1], [1:1])$ and $([1:-1], [1:-1])$, which are not ramification points of the triple cover $\sigma$. This shows that $\tau$ swaps the points $P_3$ and $P_4$. In fact, the group $\mathrm{Aut}(Q, C_4)$ is larger than its subgroup $\langle\tau\rangle \cong \boldsymbol{\mu}_2$. Indeed, one can change coordinates $([u : v], [x : y])$ on $Q$ so that

$$P_1 = ([1:0], [0:1]),$$
$$P_4 = ([0:1], [1:0]),$$

and the curve $C_4$ is given by

$$u(x^3 + \lambda' x^2 y) = v(y^3 + \lambda' y^2 x)$$

for some complex number $\lambda' \notin \{0, \pm 1, \pm 3\}$. This gives an involution $\iota \in \mathrm{Aut}(Q, C_4)$ such that $\iota(P_1) = P_4$ and $\iota(P_2) = P_3$. Let $G$ be the subgroup $\langle \tau, \iota \rangle \subset \mathrm{Aut}(Q, C_4) = \mathrm{Aut}(\mathbb{P}^3, C_4)$. Then $G \cong \boldsymbol{\mu}_2^2$. Note that the group $\mathrm{Aut}(\mathbb{P}^3, C_4)$ can be larger for some $\lambda \in \mathbb{C} \setminus \{0, \pm 1, \pm 3\}$. For instance, if $\lambda = \pm\sqrt{3}$, then $\mathrm{Aut}(\mathbb{P}^3, C_4) \cong \mathfrak{A}_4$, c.f. [2, Example 4.4.6].

The $G$-action on $C_4$ is faithful, so that the curve $C_4$ does not contain $G$-fixed points. Hence, the quadric $Q$ does not contain $G$-fixed points, since otherwise $Q$ would contain a $G$-invariant curve of degree $(1, 0)$, which would intersect $C_4$ by a $G$-fixed point. So, in particular, we see that $\mathbb{P}^3$ contains finitely many $G$-fixed points. Since the $G$-action on $\mathbb{P}^3$ is given by 4-dimensional linear representation of the group $G$, we conclude this representation splits as a sum of four distinct one-dimensional representations, which implies that the space $\mathbb{P}^3$ contains exactly four $G$-fixed points. Denote these points by $O_1, O_2, O_3, O_4$. These four points are not co-planar. For every $1 \leqslant i < j \leqslant 4$, let $L_{ij}$ be the line in $\mathbb{P}^3$ that passes through $O_i$ and $O_j$. Then the lines $L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}$ are $G$-invariant, and they are the only $G$-invariant lines in $\mathbb{P}^3$. For each $1 \leqslant i \leqslant 4$, let $\Pi_i$ be the plane in $\mathbb{P}^3$ determined by the three points $\{O_1, O_2, O_3, O_4\} \setminus \{O_i\}$. Then the four planes $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are the only $G$-invariant planes in $\mathbb{P}^3$.

***Remark*** Each plane $\Pi_i$ intersects $C_4$ at four distinct points. Indeed, if $|\Pi_i \cap C_4| < 4$, then $\Pi_i \cap C_4$ is a $G$-orbit of length 2, and $\Pi_i$ is tangent to $C_4$ at both the points of this orbit. Therefore, without loss of generality, we may assume that the intersection $\Pi_i \cap C_4$ is just the fixed locus of the involution $\tau$. Then $\Pi_i \cap C_4 = ([1:1], [1:1]) \cup ([1:-1], [1:-1])$, so that $\Pi_i|_Q$ is a smooth conic that is given by

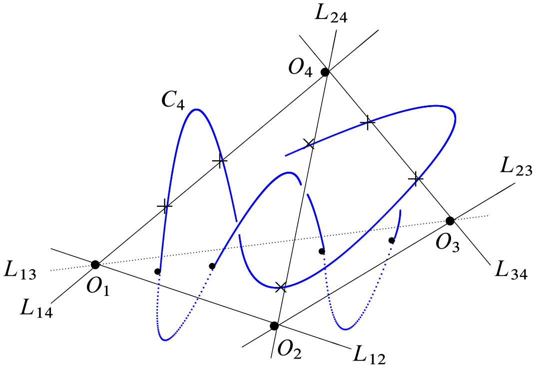
$$a(vx - uy) = b(ux - vy)$$

for some $[a : b] \in \mathbb{P}^1$. But the conic $\Pi_i|_Q$ cannot tangent $C_4$ at the points $([1:1], [1:1])$ and $([1:-1], [1:-1])$, so that $|\Pi_i \cap C_4| = 4$.

The curve $C_4$ contains exactly three $G$-orbits of length 2, and these $G$-orbits are just the fixed loci of the involutions $\tau, \iota, \tau \circ \iota$ described earlier. Let $L$, $L'$ and $L''$ be the three lines in $\mathbb{P}^3$ such that $L \cap C_4$, $L' \cap C_4$ and $L'' \cap C_4$ are the fixed loci of the involutions $\tau$, $\iota$ and $\tau \circ \iota$, respectively. Then $L$, $L'$ and $L''$ are $G$-invariant lines, so that they are three lines among $L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}$. In fact, it easily follows from Remark that the lines $L$, $L'$, $L''$ meet at one point. Therefore, we may assume that $L \cap L' \cap L'' = O_4$ and $L = L_{14}$, $L' = L_{24}$, $L'' = L_{34}$. Then

$$\Pi_1 \cap C_4 = (L' \cap C_4) \cup (L'' \cap C_4),$$
$$\Pi_2 \cap C_4 = (L \cap C_4) \cup (L'' \cap C_4),$$
$$\Pi_3 \cap C_4 = (L \cap C_4) \cup (L' \cap C_4).$$

On the other hand, the intersection $\Pi_4 \cap C_4$ is a $G$-orbit of length 4.




Since $C_4$ is $G$-invariant, the action of the group $G$ lifts to the threefold $X$, so that we also identify $G$ with a subgroup of the group $\mathrm{Aut}(X)$. Let $E$ be the $\pi$-exceptional surface, let $\widetilde{Q}$ be the proper transform of the quadric $Q$ on the threefold $X$, let $H_1$, $H_2$, $H_3$ and $H_4$ be the proper transforms on $X$ of the $G$-invariant planes $\Pi_1$, $\Pi_2$, $\Pi_3$ and $\Pi_4$, respectively, and let $H$ be the proper transform on $X$ of a general hyperplane in $\mathbb{P}^3$. Then

$$-K_X \sim 2\widetilde{Q} + E \sim \widetilde{Q} + 2H_1 \sim \widetilde{Q} + 2H_2 \sim \widetilde{Q} + 2H_3 \sim \widetilde{Q} + 2H_4 \sim 4H - E,$$

and the surfaces $E$, $\widetilde{Q}$, $H_1$, $H_2$, $H_3$, $H_4$ are $G$-invariant. Observe that $\widetilde{Q} \cong Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $H_1$, $H_2$, $H_3$, $H_4$ are smooth del Pezzo surfaces of degree 5.

**Claim** *Let $S$ be a possibly reducible $G$-invariant surface in $X$ such that $-K_X \sim_{\mathbb{Q}} \mu S + \Delta$, where $\Delta$ is an effective $\mathbb{Q}$-divisor, and $\mu$ is a positive rational number such that $\mu > 4/3$. Then $S$ is one of the surfaces $\widetilde{Q}$, $H_1$, $H_2$, $H_3$, $H_4$.*

***Proof*** This follows from the fact that the cone $\mathrm{Eff}(X)$ is generated by $E$ and $\widetilde{Q}$. □

Suppose $X$ is not K-stable. Since $\mathrm{Aut}(X)$ is finite, the threefold $X$ is not K-polystable. Then, by [3, Corollary 4.14], there is a $G$-invariant prime divisor $F$ over $X$ with $\beta(F) \leqslant 0$, see [2, Section 1.2] for the precise definition of $\beta(F)$. Let us seek for a contradiction.

Let $Z$ be the center of $F$ on $X$. Then $Z$ is not a surface by [2, Theorem 3.7.1], so that $Z$ is either a $G$-invariant irreducible curve or a $G$-fixed point. In the latter case, the point $\pi(Z)$ must be one of the $G$-fixed points $O_1$, $O_2$, $O_3$, $O_4$, so that the point $Z$ is not contained in $\widetilde{Q} \cup E$. Let us use the Abban–Zhuang theory [1] to show that $Z$ does not lie on $\widetilde{Q} \cup E$ in the former case.

**Lemma** *The center $Z$ cannot be contained in $\widetilde{Q} \cup E$.*

***Proof*** We suppose that $Z \subset \widetilde{Q} \cup E$. Then $Z$ is an irreducible $G$-invariant curve, because neither $\widetilde{Q}$ nor $E$ contains $G$-fixed points. Let us use notations introduced in [2, Section 1.7]. Namely, we fix $u \in \mathbb{R}_{\geqslant 0}$. Then

$$- K_X - u\widetilde{Q} \sim_{\mathbb{R}} (4-2u)H + (u-1)E \sim_{\mathbb{R}} (1-u)\widetilde{Q} + 2H,$$

so that $- K_X - u\widetilde{Q}$ is nef for $0 \leqslant u \leqslant 1$, and not pseudo-effective for $u > 2$. Thus, we have

$$P(- K_X - u\widetilde{Q}) = \begin{cases} - K_X - u\widetilde{Q} & \text{if } 0 \leqslant u \leqslant 1, \\ (4-2u)H & \text{if } 1 \leqslant u \leqslant 2, \end{cases}$$

and

$$N(- K_X - u\widetilde{Q}) = \begin{cases} 0 & \text{if } 0 \leqslant u \leqslant 1, \\ (u-1)E & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

If $Z \subset \widetilde{Q}$, then [2, Corollary 1.7.26] gives

$$1 \geqslant \frac{A_X(F)}{S_X(F)} \geqslant \min\left\{\frac{1}{S_X(\widetilde{Q})}, \frac{1}{S(W^{\widetilde{Q}}_{\bullet,\bullet}; Z)}\right\},$$

where

$$S_X(\widetilde{Q}) = \frac{1}{(-K_X)^3} \int_0^2 \mathrm{vol}(-K_X - u\widetilde{Q})\, du = \frac{1}{(-K_X)^3} \int_0^2 \big(P(-K_X - u\widetilde{Q})\big)^3 du$$

and

$$S\big(W^{\widetilde{Q}}_{\bullet,\bullet}; Z\big) = \frac{3}{(-K_X)^3}\Bigg\{ \int_0^2 \big(P(-K_X - u\widetilde{Q})^2 \cdot \widetilde{Q}\big) \cdot \mathrm{ord}_Z\big(N(-K_X - u\widetilde{Q})|_{\widetilde{Q}}\big)\, du$$
$$+ \int_0^2 \int_0^\infty \mathrm{vol}\big(P(-K_X - u\widetilde{Q})|_{\widetilde{Q}} - vZ\big)\, dv du\Bigg\}.$$

Therefore, we conclude that $S(W^{\widetilde{Q}}_{\bullet,\bullet}; Z) \geqslant 1$, because $S_X(\widetilde{Q}) < 1$, see [2, Theorem 3.7.1]. Similarly, if $Z \subset E$, then we get $S(W^{E}_{\bullet,\bullet}; Z) \geqslant 1$.

Fix an isomorphism $\widetilde{Q} \cong Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $E|_{\widetilde{Q}}$ is a divisor in $\widetilde{Q}$ of degree $(1, 3)$. For $(a, b) \in \mathbb{R}^2$, let $\mathcal{O}_{\widetilde{Q}}(a, b)$ be the class of a divisor of degree $(a, b)$ in $\mathrm{Pic}(\widetilde{Q}) \otimes \mathbb{R}$. Then

$$P(-K_X - u\widetilde{Q})|_{\widetilde{Q}} \sim_{\mathbb{R}} \begin{cases} \mathcal{O}_{\widetilde{Q}}(3-u, u+1) & \text{if } 0 \leqslant u \leqslant 1, \\ \mathcal{O}_{\widetilde{Q}}(4-2u, 4-2u) & \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

Therefore, if $Z = E \cap \widetilde{Q}$, then

$$\begin{aligned} S\big(W^{\widetilde{Q}}_{\bullet,\bullet}; Z\big) &= \frac{1}{10}\Bigg\{ \int_1^2 2(4-2u)^2(u-1)\,du \\ &\quad + \int_0^1 \int_0^\infty \mathrm{vol}\big(\mathcal{O}_{\widetilde{Q}}(3-u-v, u+1-3v)\big)\,dv du \\ &\quad + \int_1^2 \int_0^\infty \mathrm{vol}\big(\mathcal{O}_{\widetilde{Q}}(4-2u-v, 4-2u-3v)\big)\,dv du \Bigg\} \\ &= \frac{2}{30} + \frac{1}{10}\Bigg\{ \int_0^1 \int_0^{\frac{u+1}{3}} 2(u+1-3v)(3-u-v)\,dv du \\ &\quad + \int_1^2 \int_0^{\frac{4-2u}{3}} 2(4-2u-3v)(4-2u-v)\,dv du \Bigg\} \\ &= \frac{161}{540}. \end{aligned}$$

To estimate $S(W^{\widetilde{Q}}_{\bullet,\bullet}; Z)$ in the case when $Z \subset \widetilde{Q}$ and $Z \neq E \cap \widetilde{Q}$, observe that $|Z - \Delta| \neq \varnothing$, where $\Delta$ is the diagonal curve in $\widetilde{Q}$. Indeed, this follows from the fact that $\widetilde{Q}$ contains neither $G$-invariant curves of degree (0, 1) nor $G$-invariant curves of degree (1, 0), which in turns easily follows from the fact that the curve $C_4 \cong \mathbb{P}^1$ does not have $G$-fixed points. Thus, if $Z \subset \widetilde{Q}$ and $Z \neq E \cap \widetilde{Q}$, then

$$
\begin{aligned}
S\big(W^{\widetilde{Q}}_{\bullet,\bullet}; Z\big) &\leqslant \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}\big(P(-K_X-u\widetilde{Q})|_{\widetilde{Q}}-v\Delta\big)\,dvdu \\
&= \frac{1}{10}\Bigg\{\int_0^1\int_0^\infty \mathrm{vol}\big(\mathcal{O}_{\widetilde{Q}}(3-u-v,u+1-v)\big)\,dvdu \\
&\qquad + \int_1^2\int_0^\infty \mathrm{vol}\big(\mathcal{O}_{\widetilde{Q}}(4-2u-v,4-2u-v)\big)\,dvdu\Bigg\} \\
&= \frac{1}{10}\Bigg\{\int_0^1\int_0^{u+1} 2(u+1-v)(3-u-v)\,dvdu \\
&\qquad + \int_1^2\int_0^{4-2u} 2(4-2u-v)^2 dvdu\Bigg\} \\
&= \frac{17}{30}.
\end{aligned}
$$

Therefore, $Z\not\subset\widetilde{Q}$, and hence $Z\subset E$ and $Z\neq\widetilde{Q}\cap E$.

One has $E\cong\mathbb{F}_n$ for some integer $n\geqslant 0$. It follows from the argument as in the proof of [2, Lemma 4.4.16] that $n$ is either 0 or 2. Indeed, let $\mathbf{s}$ be the section of the projection $E\to C_4$ such that $\mathbf{s}^2=-n$, and let $\mathbf{l}$ be its fiber. Then $-E|_E\sim\mathbf{s}+k\mathbf{l}$ for some integer $k$. But

$$
-n+2k=E^3=-c_1(\mathcal{N}_{C_4/\mathbb{P}^3})=-14,
$$

so that $k=(n-14)/2$. Then

$$
\widetilde{Q}|_E\sim(2H-E)|_E\sim\mathbf{s}+(k+8)\mathbf{l}=\mathbf{s}+\frac{n+2}{2}\mathbf{l},
$$

which implies that $\widetilde{Q}|_E\not\sim\mathbf{s}$. Moreover, we know that $\widetilde{Q}|_E$ is a smooth irreducible curve, since the quadric surface $Q$ is smooth. Thus, since $\widetilde{Q}|_E\neq\mathbf{s}$, we have

$$
0\leqslant\widetilde{Q}|_E\cdot\mathbf{s}=\left(\mathbf{s}+\frac{n+2}{2}\mathbf{l}\right)\cdot\mathbf{s}=-n+\frac{n+2}{2}=\frac{2-n}{2}
$$

so that $n=0$ or $n=2$. Now, let us show that $S(W^E_{\bullet,\bullet};Z)<1$ in both cases.

For $u\geqslant 0$,

$$
-K_X-uE\sim 2\widetilde{Q}+(1-u)E,
$$

so that $-K_X - uE$ is pseudo-effective if and only if $u \leqslant 1$, and it is nef if and only if $u \leqslant 1/3$. Furthermore, if $1/3 \leqslant u \leqslant 1$, then

$$P(-K_X - uE) = (2-2u)(3H - E)$$

and $N(-K_X - uE) = (3u-1)\widetilde{Q}$. Thus, if $n = 0$, we have

$$P(-K_X - uE)|_E = \begin{cases} (1+u)\mathbf{s} + (9-7u)\mathbf{l} & \text{if } 0 \leqslant u \leqslant 1/3, \\ (2-2u)\mathbf{s} + (10-10u)\mathbf{l} & \text{if } 1/3 \leqslant u \leqslant 1. \end{cases}$$

Similarly, if $n = 2$, then

$$P(-K_X - uE)|_E = \begin{cases} (1+u)\mathbf{s} + (10-6u)\mathbf{l} & \text{if } 0 \leqslant u \leqslant 1/3, \\ (2-2u)\mathbf{s} + (12-12u)\mathbf{l} & \text{if } 1/3 \leqslant u \leqslant 1. \end{cases}$$

Recall that $Z \neq \widetilde{Q} \cap E$. Moreover, we have $Z \not\sim \mathbf{l}$, since $\pi(Z)$ is not one of the $G$-fixed points $O_1, O_2, O_3, O_4$. Thus, using [2, Corollary 1.7.26], we get

$$S\big(W^E_{\bullet,\bullet}; Z\big) = \frac{1}{10}\int_0^1\int_0^\infty \mathrm{vol}(P(u)|_E - vZ)\,dvdu \leqslant \frac{1}{10}\int_0^1\int_0^\infty \mathrm{vol}(P(u)|_E - v\mathbf{s})\,dvdu,$$

because the divisor $|Z - \mathbf{s}| \neq \varnothing$.

Consequently, if $n = 0$, then

$$\begin{aligned} &S\big(W^E_{\bullet,\bullet}; Z\big) \\ &\quad\leqslant \frac{1}{10}\Bigg\{\int_0^{\frac{1}{3}}\int_0^\infty \mathrm{vol}\big((1+u)\mathbf{s} + (9-7u)\mathbf{l} - v\mathbf{s}\big)\,dvdu \\ &\qquad\qquad + \int_{\frac{1}{3}}^1\int_0^\infty \mathrm{vol}\big((2-2u)\mathbf{s} + (10-10u)\mathbf{l} - v\mathbf{s}\big)\,dvdu\Bigg\} \\ &\quad= \frac{1}{10}\Bigg\{\int_0^{\frac{1}{3}}\int_0^{1+u} 2(1+u-v)(9-7u)\,dvdu \\ &\qquad\qquad + \int_{\frac{1}{3}}^1\int_0^{2-2u} 2(2-2u-v)(10-10u)\,dvdu\Bigg\} \\ &\quad= \frac{1783}{3240}. \end{aligned}$$

Similarly, if $n=2$, then

$$
\begin{aligned}
S\big(W^E_{\bullet,\bullet};Z\big)
&\leqslant \frac{1}{10}\Bigg\{\int\limits_0^{\frac{1}{3}}\int\limits_0^{\infty}\mathrm{vol}\big((1+u)\mathbf{s}+(10-6u)\mathbf{l}-v\mathbf{s}\big)\,dvdu\\
&\qquad+\int\limits_{\frac{1}{3}}^{1}\int\limits_0^{\infty}\mathrm{vol}\big((2-2u)\mathbf{s}+(12-12u)\mathbf{l}-v\mathbf{s}\big)\,dvdu\Bigg\}\\
&=\frac{1}{10}\Bigg\{\int\limits_0^{\frac{1}{3}}\int\limits_0^{1+u}2(1+u-v)(9+v-7u)\,dvdu\\
&\qquad+\int\limits_{\frac{1}{3}}^{1}\int\limits_0^{2-2u}2(2-2u-v)(10+v-10u)\,dvdu\Bigg\}\\
&=\frac{157}{270}.
\end{aligned}
$$

In both cases, we have $S(W^E_{\bullet,\bullet};Z)<1$, which is a contradiction. □

Now, we prove our main technical result using the Abban–Zhuang theory, see also [2, Section 1.7].

**Proposition** *The center $Z$ is not contained in $H_1\cup H_2\cup H_3\cup H_4$.*

***Proof*** We first suppose that $Z\subset H_1\cup H_2\cup H_3$. Without loss of generality, we may assume that $Z\subset H_1$. Then $\pi(Z)\subset\Pi_1$. Therefore, we see that one of the following two subcases are possible:

- either $\pi(Z)$ is one of the $G$-fixed points $O_2$, $O_3$, $O_4$, or
- $Z$ is a $G$-invariant irreducible curve in $H_1$.

We will deal with these subcases separately. In both subcases, we let $S=H_1$ for simplicity. Recall that $S$ is a smooth del Pezzo surface of degree 5, the surface $S$ is $G$-invariant, and the action of the group $G$ on the surface $S$ is faithful. Note also that $Z\not\subset\widetilde{Q}$ by Lemma.

Let us use notations introduced in [2, Section 1.7]. Take $u\in\mathbb{R}_{\geqslant 0}$. Then

$$
-K_X-uS\sim_{\mathbb{R}}(4-u)H-E\sim_{\mathbb{R}}\widetilde{Q}+(2-u)H\sim_{\mathbb{R}}(u-1)\widetilde{Q}+(2-u)(3H-E).
$$

Let $P(u)=P(-K_X-uS)$ and $N(u)=N(-K_X-uS)$. Then

$$
P(u)=\begin{cases}-K_X-uS & \text{if } 0\leqslant u\leqslant 1,\\ (2-u)(3H-E) & \text{if } 1\leqslant u\leqslant 2,\end{cases}
$$

and

$$N(u)=\begin{cases}0 & \text{if } 0\leqslant u\leqslant 1,\\ (u-1)\widetilde{Q} & \text{if } 1\leqslant u\leqslant 2.\end{cases}$$

Note that $S_X(S)<1$, see [2, Theorem 3.7.1]. In fact, one can compute $S_X(S)=17/30$.

Let $\varphi\colon S\to\Pi_1$ be the birational morphism induced by $\pi$. Then $\varphi$ is a $G$-equivariant blow-up of the four intersection points $\Pi_1\cap C_4$. Let $\ell$ be the proper transform on $S$ of a general line in $\Pi_1$, and let $e_1,e_2,e_3,e_4$ be $\varphi$-exceptional curves, and let $\ell_{ij}$ be the proper transform on the surface $S$ of the line in $\Pi_1$ that passes through $\varphi(e_i)$ and $\varphi(e_j)$, where $1\leqslant i<j\leqslant 4$. Then the cone $\overline{\mathrm{NE}}(S)$ is generated by the curves $e_1$, $e_2$, $e_3$, $e_4$, $\ell_{12}$, $\ell_{13}$, $\ell_{14}$, $\ell_{23}$, $\ell_{24}$, $\ell_{34}$. Recall also that

$$\Pi_1\cap C_4=(L_{24}\cap C_4)\cup(L_{34}\cap C_4).$$

Therefore, we may assume that $L_{24}\cap C_4=\varphi(e_1)\cup\varphi(e_2)$ and $L_{34}\cap C_4=\varphi(e_3)\cup\varphi(e_4)$, so that we have $\varphi(\ell_{12})=L_{24}$ and $\varphi(\ell_{34})=L_{34}$.

Observe that, the group $\mathrm{Pic}^G(S)$ is generated by the divisor classes $\ell$, $e_1+e_2$, $e_3+e_4$, because both $L_{24}\cap C_4$ and $L_{34}\cap C_4$ are $G$-orbits of length 2. Therefore, if $Z$ is a curve, then $\varphi(Z)$ is a curve of degree $d\geqslant 1$, so that

$$Z\sim d\ell-m_{12}(e_1+e_2)-m_{34}(e_3+e_4)$$

for some non-negative integers $m_{12}$ and $m_{34}$, which gives

$$\begin{aligned}Z&\sim(d-2m_{12})\ell+m_{12}(2\ell-e_1-e_2-e_3-e_4)+(m_{12}-m_{34})(e_3+e_4)\\&\sim(d-2m_{12})(\ell_{12}+e_1+e_2)+m_{12}(\ell_{12}+\ell_{34})+(m_{12}-m_{34})(e_3+e_4)\end{aligned}$$

and

$$\begin{aligned}Z&\sim(d-2m_{34})\ell+m_{34}(2\ell-e_1-e_2-e_3-e_4)+(m_{34}-m_{12})(e_1+e_2)\\&\sim(d-2m_{34})(\ell_{34}+e_3+e_4)+m_{34}(\ell_{12}+\ell_{34})+(m_{34}-m_{12})(e_1+e_2).\end{aligned}$$

Moreover, if $Z\neq\ell_{12}$ and $Z\neq\ell_{34}$, then $d-2m_{12}=Z\cdot\ell_{12}\geqslant 0$ and $d-2m_{34}=Z\cdot\ell_{34}\geqslant 0$. Hence, if $Z$ is a curve, then $|Z-\ell_{12}|\neq\varnothing$ or $|Z-\ell_{34}|\neq\varnothing$.

On the other hand, if $Z$ is a curve, then [2, Corollary 1.7.26] gives

$$1\geqslant\frac{A_X(F)}{S_X(F)}\geqslant\min\left\{\frac{1}{S_X(S)},\frac{1}{S(W^S_{\bullet,\bullet};Z)}\right\}=\min\left\{\frac{30}{17},\frac{1}{S(W^S_{\bullet,\bullet};Z)}\right\},$$

where

$$S\big(W^S_{\bullet,\bullet};Z\big)=\frac{3}{(-K_X)^3}\int\limits_0^2\int\limits_0^\infty\mathrm{vol}(P(u)|_S-vZ)\,dvdu,$$

because $Z \not\subset \widetilde{Q}$. Moreover, if $S(W^S_{\bullet,\bullet}; Z) = 1$, then [2, Corollary 1.7.26] gives

$$1 \geqslant \frac{A_X(E)}{S_X(E)} = \frac{1}{S_X(S)} = \frac{30}{17},$$

which is absurd. Thus, if $Z$ is a curve, then $S(W^S_{\bullet,\bullet}; Z) > 1$, which gives

$$\begin{aligned}1 < S\big(W^S_{\bullet,\bullet}; Z\big) &= \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - vZ)\,dvdu \\ &\leqslant \max\left\{\frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu, \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{34})\,dvdu\right\},\end{aligned}$$

because $|Z - \ell_{12}| \neq \varnothing$ or $|Z - \ell_{34}| \neq \varnothing$. Note also that

$$\begin{aligned}S\big(W^S_{\bullet,\bullet}; \ell_{12}\big) &= \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu \\ &= \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{34})\,dvdu.\end{aligned}$$

Hence, if $Z$ is a curve, then the second statement in [2, Corollary 1.7.26] gives

$$1 < S\big(W^S_{\bullet,\bullet}; Z\big) \leqslant S\big(W^S_{\bullet,\bullet}; \ell_{12}\big) = \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu.$$

Let us compute $S(W^S_{\bullet,\bullet}; \ell_{12})$. For $0 \leqslant u \leqslant 1$ and $v \geqslant 0$, we have

$$P(u)_S - v\ell_{12} = (-K_X - uS)|_S - v\ell_{12} \sim_{\mathbb{R}} (4 - u - v)\ell - (1 - v)(e_1 + e_2) - e_3 - e_4.$$

Therefore, if $0 \leqslant v \leqslant 1$, then this divisor is nef, and its volume is $u^2 + 2uv - v^2 - 8u - 4v + 12$. Similarly, if $1 \leqslant v \leqslant 2 - u$, then its Zariski decomposition is

$$P(u)|_S - v\ell_{12} \sim_{\mathbb{R}} \underbrace{(4 - u - v)\ell - e_3 - e_4}_{\text{positive part}} + \underbrace{(v - 1)(e_1 + e_2)}_{\text{negative part}},$$

so that its volume is $u^2 + 2uv + v^2 - 8u - 8v + 14$. Likewise, if $2 - u \leqslant v \leqslant 3 - u$, then the Zariski decomposition of the divisor $P(u)|_S - v\ell_{12}$ is

$$P(u)\big|_S - v\ell_{12} \sim_{\mathbb{R}} \underbrace{(3 - u - v)(2\ell - e_3 - e_4)}_{\text{positive part}} + \underbrace{(v - 1)(e_1 + e_2) + (v - 2 + u)\ell_{34}}_{\text{negative part}},$$

so that its volume is $2(3-u-v)^2$. If $v>3-u$, then $P(u)|_S - v\ell_{12}$ is not pseudo-effective, so that the volume of this divisor is zero. Thus, we have

$$
\begin{aligned}
&\frac{1}{10}\int_0^1\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu\\
&\qquad= \frac{1}{10}\int_0^1\int_0^{3-u} \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu\\
&\qquad= \frac{1}{10}\bigg\{\int_0^1\int_0^1 \big(u^2+2uv-v^2-8u-4v+12\big)\,dvdu\\
&\qquad\qquad+\int_0^1\int_1^{2-u} \big(u^2+2uv+v^2-8u-8v+14\big)\,dvdu\\
&\qquad\qquad+\int_0^1\int_{2-u}^{3-u} 2(3-u-v)^2 dvdu\bigg\}\\
&\qquad= \frac{107}{120}.
\end{aligned}
$$

Similarly, if $1 \leqslant u \leqslant 2$, then

$$
P(u)|_S - v\ell_{12} \sim_{\mathbb{R}} (6-3u-v)\ell + (v+u-2)(e_1+e_2) + (u-2)(e_3+e_4).
$$

If $0 \leqslant v \leqslant 2-u$, this divisor is nef, and its volume is $5u^2+2uv-v^2-20u-4v+20$. Likewise, if $2-u \leqslant v \leqslant 4-2u$, then its Zariski decomposition is

$$
P(u)|_S - v\ell_{12} \sim_{\mathbb{R}} \underbrace{(4-2u-v)(2\ell-e_3-e_4)}_{\text{positive part}} + \underbrace{(v-2+u)(e_1+e_2+\ell_{34})}_{\text{negative part}},
$$

and its volume is $2(4-2u-v)^2$. If $v>4-2u$, this divisor is not pseudo-effective, so that

$$
\begin{aligned}
\frac{1}{10}\int_1^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu
&= \frac{1}{10}\int_1^2\int_0^{4-2u} \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu\\
&= \frac{1}{10}\Bigg\{\int_1^2\int_0^{2-u} \big(5u^2+2uv-v^2-20u-4v+20\big)\,dvdu\\
&\qquad + \int_1^2\int_{2-u}^{4-2u} 2(4-2u-v)^2\,dvdu\Bigg\}\\
&= \frac{13}{120}.
\end{aligned}
$$

Therefore, we see that

$$
\begin{aligned}
S\big(W^S_{\bullet,\bullet};\ell_{12}\big) &= \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu\\
&= \frac{1}{10}\int_0^1\int_0^\infty \mathrm{vol}(P(u)|_S - v\ell_{12})\,dvdu\\
&\qquad + \frac{1}{10}\int_1^2\int_0^\infty \mathrm{vol}(P(u)\big|_S - v\ell_{12})\,dvdu\\
&= \frac{107}{120}+\frac{13}{120} = 1,
\end{aligned}
$$

which implies, in particular, that $Z$ is not a curve.

Hence, we see that $\pi(Z)$ is one of the points $O_2, O_3, O_4$. Without loss of generality, we may assume that either $\pi(Z) = O_2$ or $\pi(Z) = O_4$, so that $Z \in \ell_{12}$ in both subcases. Now, using [2, Theorem 1.7.30], we see that

$$
\begin{aligned}
1 \geqslant \frac{A_X(F)}{S_X(F)} &\geqslant \min\left\{\frac{1}{S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet};Z)}, \frac{1}{S(W^S_{\bullet,\bullet};\ell_{12})}, \frac{1}{S_X(S)}\right\}\\
&= \min\left\{\frac{1}{S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet};Z)}, 1\right\},
\end{aligned}
$$

where $S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet}; Z)$ is defined in [2, Section 1.7]. In fact, [2, Theorem 1.7.30] implies the strict inequality $S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet}; Z) < 1$, because $S_X(S) < 1$. Let us compute $S(W^{S,\ell_{12}}_{\bullet,\bullet,\bullet}; Z)$.

For $0 \leqslant u \leqslant 2$ and $v \geqslant 0$, let $P(u, v)$ be the positive part of the Zariski decomposition of the divisor $P(u)|_S - v\ell_{12}$, and let $N(u, v)$ be its negative part.

If $0 \leqslant u \leqslant 1$, then

$$P(u,v) = \begin{cases} (4-u-v)\ell - (1-v)(e_1+e_2) - e_3 - e_4 & \text{if } 0 \leqslant v \leqslant 1, \\ (4-u-v)\ell - e_3 - e_4 & \text{if } 1 \leqslant v \leqslant 2-u, \\ (3-u-v)(2\ell - e_3 - e_4) & \text{if } 2-u \leqslant v \leqslant 3-u, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 1, \\ (v-1)(e_1+e_2) & \text{if } 1 \leqslant v \leqslant 2-u, \\ (v-1)(e_1+e_2) + (v-2+u)\ell_{34} & \text{if } 2-u \leqslant v \leqslant 3-u. \end{cases}$$

Similarly, if $1 \leqslant u \leqslant 2$, then

$$P(u,v) = \begin{cases} (6-3u-v)\ell + (v+u-2)(e_1+e_2) + (u-2)(e_3+e_4) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 \leqslant v \leqslant 2-u, \\ (4-2u-v)(2\ell - e_3 - e_4) \quad \text{if } 2-u \leqslant v \leqslant 4-2u, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2-u, \\ (v-2+u)(e_1+e_2+\ell_{34}) & \text{if } 2-u \leqslant v \leqslant 4-2u. \end{cases}$$

Recall from [2, Theorem 1.7.30] that

$$S\big(W^{S,\ell_{12}}_{\bullet,\bullet}; Z\big) = F_Z\big(W^{S,\ell_{12}}_{\bullet,\bullet}\big) + \frac{3}{(-K_X)^3} \int_0^2 \int_0^\infty (P(u,v)\cdot\ell_{12})^2 dv du$$

for

$$F_Z\big(W^{S,\ell_{12}}_{\bullet,\bullet}\big) = \frac{6}{(-K_X)^3} \int_0^2 \int_0^\infty (P(u,v)\cdot\ell_{12}) \operatorname{ord}_Z\big(N'_S(u)|_{\ell_{12}} + N(u,v)|_{\ell_{12}}\big) dv du,$$

where $N'_S(u)$ is the part of the divisor $N(u)|_S$ whose support does not contain $\ell_{12}$, so that $N'_S(u) = N(u)|_S$ in our case, which implies that $\operatorname{ord}_Z(N'_S(u)|_{\ell_{12}}) = 0$ for $0 \leqslant u \leqslant 2$, because $Z \notin \widetilde{Q}$. Thus, if $\pi(Z) = O_2$, then $Z \notin \ell_{34} \cup e_1 \cup e_2$, which gives $F_Z(W^{S,\ell_{12}}_{\bullet,\bullet}) = 0$. On the other hand, if $\pi(Z) = O_4$, then $Z = \ell_{12} \cap \ell_{34}$ and $Z \notin e_1 \cup e_2$, so that

$$
\begin{aligned}
F_Z\big(W^{S,\ell_{12}}_{\bullet,\bullet}\big) &= \frac{1}{5}\int_0^2\int_0^\infty (P(u,v)\cdot\ell_{12})\,\mathrm{ord}_Z\big(N(u,v)|_{\ell_{12}}\big)\,dvdu \\
&= \frac{1}{5}\Bigg\{\int_0^1\int_{2-u}^{3-u}(6-2u-2v+6)(v-2+u)\,dvdu+ \\
&\qquad +\int_1^2\int_{2-u}^{4-2u}(8-4u-2v+8)(v-2+u)\,dvdu\Bigg\} \\
&= \frac{1}{12}.
\end{aligned}
$$

Therefore, we see that

$$
\begin{aligned}
S\big(W^{S,\ell_{12}}_{\bullet,\bullet};Z\big) &\leqslant \frac{1}{12}+\frac{1}{10}\int_0^2\int_0^\infty (P(u,v)\cdot\ell_{12})^2\,dvdu \\
&= \frac{1}{12}+\frac{1}{10}\Bigg\{\int_0^1\int_0^1(2-u+v)^2\,dvdu+\int_0^1\int_1^{2-u}(4-u-v)^2\,dvdu \\
&\qquad +\int_0^1\int_{2-u}^{3-u}(6-2u-2v)^2\,dvdu \\
&\qquad +\int_1^2\int_0^{2-u}(2-u+v)^2\,dvdu \\
&\qquad +\int_1^2\int_{2-u}^{4-2u}(8-4u-2v)^2\,dvdu\Bigg\} \\
&= 1.
\end{aligned}
$$

However, as we already mentioned, one has $S(W^{S,\ell_{12}}_{\bullet,\bullet};Z) < 1$ by [2, Theorem 1.7.30]. The obtained contradiction concludes that $Z \subset H_4$.

Since $Z \not\subset H_1 \cup H_2 \cup H_3$, the center $Z$ must be a $G$-invariant curve on $H_4$. Moreover, $\pi(Z)$ cannot be one of the lines determined by the points $O_1$, $O_2$, $O_3$ on $\Pi_4$. This implies that $\pi(Z)$ is a curve of degree $d \geqslant 2$ on $\Pi_4$.

We keep the same notations as in the beginning of the proof, i.e., put $S = H_4$ and let $\varphi\colon S \to \Pi_1$ be birational morphism induced by $\pi$. As before, $\varphi$ is a $G$-equivariant blow-up of the four intersection points $\Pi_4 \cap C_4$ which consist of a $G$- orbit of length

4. We also denote by $\ell$ the proper transform on $S$ of a general line in $\Pi_4$ and by $e_1$, $e_2$, $e_3$, $e_4$ the four $\varphi$-exceptional curves. In addition, denote by $\mathcal{C}$ the proper transform of a general conic passing through the four points $\Pi_4 \cap C_4$.

Since the group $\mathrm{Pic}^G(S)$ is generated by the divisor classes $\ell$, $e_1+e_2+e_3+e_4$, we have

$$Z \sim d\ell - m(e_1+e_2+e_3+e_4).$$

where $m$ is a non-negative integer. By taking intersection with the proper transforms of the lines on $\Pi_4$ passing through $\varphi(e_i)$, $\varphi(e_j)$, we obtain $d \geqslant 2m$. Since $d \geqslant 2$, this implies that $|Z - \mathcal{C}| \neq \varnothing$. Note that $\mathcal{C} \not\subset \widetilde{Q}$. By the same argument as before, we obtain

$$\begin{aligned}1 < S\big(W^S_{\bullet,\bullet}; Z\big) &= \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - vZ)\,dvdu\\ &\leqslant \frac{1}{10}\int_0^2\int_0^\infty \mathrm{vol}(P(u)|_S - v\mathcal{C})\,dvdu = S\big(W^S_{\bullet,\bullet}; \mathcal{C}\big),\end{aligned}$$

where $P(u)$ is the positive part of $-K_X - uS$ as before. Let us compute $S(W^S_{\bullet,\bullet}; \mathcal{C})$.

Similarly to the notations used earlier in the proof, we denote by $P(u,v)$ the positive part of the Zariski decomposition of the divisor $P(u)|_S - v\mathcal{C}$ for $0 \leqslant u \leqslant 2$ and $v \geqslant 0$, and we denote by $N(u,v)$ its negative part. If $0 \leqslant u \leqslant 1$, then

$$P(u,v) = \begin{cases}(4-u-2v)\ell - (1-v)(e_1+e_2+e_3+e_4) & \text{if } 0 \leqslant v \leqslant 1,\\ (4-u-2v)\ell & \text{if } 1 \leqslant v \leqslant \frac{4-u}{2},\end{cases}$$

and

$$N(u,v) = \begin{cases}0 & \text{if } 0 \leqslant v \leqslant 1,\\ (v-1)(e_1+e_2+e_3+e_4) & \text{if } 1 \leqslant v \leqslant \frac{4-u}{2}.\end{cases}$$

Similarly, if $1 \leqslant u \leqslant 2$, then

$$P(u,v) = \begin{cases}(6-3u-2v)\ell + (v+u-2)(e_1+e_2+e_3+e_4) & \text{if } 0 \leqslant v \leqslant 2-u,\\ (6-3u-2v)\ell & \text{if } 2-u \leqslant v \leqslant \frac{6-3u}{2},\end{cases}$$

and

$$N(u,v) = \begin{cases}0 & \text{if } 0 \leqslant v \leqslant 2-u,\\ (v+u-2)(e_1+e_2+e_3+e_4) & \text{if } 2-u \leqslant v \leqslant \frac{6-3u}{2}.\end{cases}$$

This gives

$$
\begin{aligned}
1 < S\big(W^{S}_{\bullet,\bullet};\mathcal{C}\big) &= \frac{1}{10}\Bigg\{\int\limits_0^1\int\limits_0^1 (P(u)|_S - v\mathcal{C})^2 dvdu + \int\limits_0^1\int\limits_1^{\frac{4-u}{2}} \big((4-u-2v)\ell\big)^2 dvdu \\
&\quad + \int\limits_1^2\int\limits_0^{2-u} (P(u)|_S - v\mathcal{C})^2 dvdu \\
&\quad + \int\limits_1^2\int\limits_{2-u}^{\frac{6-3u}{2}} \big((6-3u-2v)\ell\big)^2 dvdu\Bigg\} \\
&= \frac{1}{10}\Bigg\{\int\limits_0^1\int\limits_0^1 (4-u-2v)^2 - 4(1-v)\, dvdu \\
&\quad + \int\limits_0^1\int\limits_1^{\frac{4-u}{2}} (4-u-2v)^2 dvdu \\
&\quad + \int\limits_1^2\int\limits_0^{2-u} (6-3u-2v)^2 - 4(2-u-v)\, dvdu \\
&\quad + \int\limits_1^2\int\limits_{2-u}^{\frac{6-3u}{2}} (6-3u-2v)^2 dvdu\Bigg\} \\
&= \frac{23}{40},
\end{aligned}
$$

which is a contradiction. This completes the proof of the proposition. □

**Corollary** *Both $Z$ and $\pi(Z)$ are irreducible curves, and $\pi(Z)$ is not entirely contained in $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4 \cup Q$.*

Using [2, Lemma 1.4.4], we see that $\alpha_{G,Z}(X) < 3/4$. Now, using [2, Lemma 1.4.1], we see that there are a $G$-invariant effective $\mathbb{Q}$-divisor $D$ on the threefold $X$ and a positive rational number $\mu < 3/4$ such that $D \sim_{\mathbb{Q}} -K_X$ and $Z$ is contained in the locus $\mathrm{Nklt}(X, \mu D)$. Moreover, it follows from Claim that $\mathrm{Nklt}(X, \mu D)$ does not contain $G$-irreducible surfaces except maybe for $\widetilde{Q}$, $H_1$, $H_2$, $H_3$, $H_4$. Now, applying [2, Corollary A.1.13] to $(\mathbb{P}^3, \mu\pi(D))$, we see that $\pi(Z)$ must be a $G$-invariant line in $\mathbb{P}^3$. But this is impossible by Corollary, since all $G$-invariant lines in $\mathbb{P}^3$ are contained in $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$.

The obtained contradiction completes the proof of our Theorem.

## References

1. Abban, H., Zhuang, Z.: K-stability of Fano varieties via admissible flags. Forum Math. Pi **10**, Art. No. e15 (2022)
2. Araujo, C., Castravet, A.-M., Cheltsov, I., Fujita, K., Kaloghiros, A.-S., Martinez-Garcia, J., Shramov, C., Süß, H., Viswanathan, N.: The Calabi problem for Fano threefolds (2021). MPIM preprint 2021-31
3. Zhuang, Z.: Optimal destabilizing centers and equivariant K-stability. Invent. Math. **226**(1), 195–223 (2021)