On finiteness theorems of polynomial functions

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Dedicated to Takuo Fukuda for his 80th birthday

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Abstract

Let d be a positive integer. We show a finiteness theorem for semialgebraic $\mathcal{RL}$ triviality of a Nash family of Nash functions defined on a Nash manifold, generalising Benedetti–Shiota's finiteness theorem for semialgebraic $\mathcal{RL}$ equivalence classes appearing in the space of real polynomial functions of degree not exceeding d. We also prove Fukuda's claim, Theorem 1.3, and its semialgebraic version Theorem 1.4, on the finiteness of the local $\mathcal{R}$ types appearing in the space of real polynomial functions of degree not exceeding d.

Keywords Polynomial function · Nash function · Topological equivalence · Semialgebraic equivalence

Mathematics Subject Classification 14P10 · 32S15 · 32S45 · 57R45

1 Introduction

Let d be a positive integer, and let us denote by $P(n, 1, d; \mathbb{R})$ the set of polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ of degree not exceeding d. The space $P(n, 1, d; \mathbb{R})$ is identified with the Euclidean space $\mathbb{R}^{L+1}$, regarded as the coefficient space of all polynomials in $P(n, 1, d; \mathbb{R})$. This L is given as $_{n+d}C_d - 1$.

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We say that two polynomials $f, g: \mathbb{R}^n \to \mathbb{R}$ are *topologically $\mathcal{RL}$ equivalent* (respectively *topologically $\mathcal{R}$ equivalent*), if there exist homeomorphisms $\sigma: \mathbb{R}^n \to \mathbb{R}^n$ and $\tau: \mathbb{R} \to \mathbb{R}$ such that $\tau \circ f = g \circ \sigma$ (respectively $f = g \circ \sigma$). Furthermore we say that two polynomials $f, g: \mathbb{R}^n \to \mathbb{R}$ are *semialgebraically $\mathcal{RL}$ equivalent* (respectively *semialgebraically $\mathcal{R}$ equivalent*) if we can take the σ and τ as semialgebraic homeomorphisms (respectively σ semialgebraic).

In [7], Fukuda established the following finiteness theorem for polynomials in $P(n, 1, d; \mathbb{R})$.

Theorem 1.1 (Fukuda's finiteness theorem) *The number of topological $\mathcal{RL}$ types appearing in $P(n, 1, d; \mathbb{R})$ is finite.*

On the other hand, Benedetti and Shiota strengthened Fukuda's finiteness theorem in [2] as follows.

Theorem 1.2 (Benedetti–Shiota's finiteness theorem) *The number of semialgebraic $\mathcal{RL}$ types appearing in $P(n, 1, d; \mathbb{R})$ is finite.*

Based on the above finiteness theorems and their proofs, we discuss finiteness properties for $\mathcal{RL}$ or $\mathcal{R}$ equivalence of polynomials in Sect. 4. In this introduction we mention two local results from those finiteness properties. Let us denote the polynomials without free term by $P_0(n, 1, d; \mathbb{R}) = \{f \in P(n, 1, d; \mathbb{R}) \mid f(0) = 0\}$, naturally identified with the Euclidean space $\mathbb{R}^L$. We say that two polynomials $f, g \in P(n, 1, d; \mathbb{R})$ are *locally topologically $\mathcal{R}$ equivalent at $0 \in \mathbb{R}^n$, and write $\mathcal{R}_0$*, if there exists a local homeomorphism $\sigma: (U \subset \mathbb{R}^n, 0) \to (V \subset \mathbb{R}^n, 0)$ between two open neighbourhoods of the origin, such that $f_{/U} = g_{/V} \circ \sigma$; and that two polynomials $f, g \in P(n, 1, d; \mathbb{R})$ are *locally semialgebraically $\mathcal{R}$ equivalent at $0 \in \mathbb{R}^n$, and write $\mathcal{R}_0$*, if we can take the σ as a local semialgebraic homeomorphism. Then we will prove the following new finiteness results for the local $\mathcal{R}_0$ equivalence of polynomial functions. For the reasons given in Sect. 4.4, we call the first result Fukuda's local $\mathcal{R}$ finiteness theorem.

Theorem 1.3 (Fukuda's local $\mathcal{R}$ finiteness theorem [9]) *The number of local topological $\mathcal{R}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite. In addition, each topological $\mathcal{R}_0$ equivalence class in $P_0(n, 1, d; \mathbb{R})$ is a semialgebraic subset of $P_0(n, 1, d; \mathbb{R})$.*

Theorem 1.4 (Local semialgebraic $\mathcal{R}_0$ finiteness theorem) *The number of local semialgebraic $\mathcal{R}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite.*

As a generalisation of the above finiteness results (Theorems 1.1 and 1.2) for topological or semialgebraic $\mathcal{RL}$ equivalence classes appearing in the space of real polynomial functions of a fixed degree, we shall show the following result in Sect. 5.

Theorem 1.5 *Let $\{f_t: M \to \mathbb{R} \mid t \in J\}$ be a Nash family of Nash functions defined on a Nash manifold M with a semialgebraic parameter space J. Namely, the function $F: M \times J \to \mathbb{R}$ defined by $F(x, t) := f_t(x)$ is a Nash function. Then there exists a finite subdivision of J into Nash open simplices $J = Q_1 \cup \cdots \cup Q_u$ such that*

$$\{f_t \colon M \to \mathbb{R} \mid t \in Q_i\}$$

is semialgebraically $\mathcal{RL}$ trivial over each Q_i.

For the definition of a Nash open simplex, see Sect. 3.3.

For the reader's convenience, we recall several classic notions. Firstly, in Sect. 2, that of Whitney's regularity and of stratified mappings, followed by the notion of controlled tube systems for stratifications. At the end of this section we introduce the semialgebraic and Nash properties which will be used to prove our results. Secondly, in Sect. 3, we recall Thom's 1st and 2nd Isotopy Lemmas and Shiota's semialgebraic versions of Thom's Isotopy Lemmas as tools to establish topological $\mathcal{RL}$ triviality and semialgebraic $\mathcal{RL}$ triviality, respectively.

2 Preliminaries

2.1 Whitney-regularity and stratified mapping

Let X, Y be disjoint C^1 submanifolds of $\mathbb{R}^m$, and let y_0 be a point in $Y \cap \overline{X}$. We say that the pair (X, Y) satisfies *Whitney's condition* (b) *at* y_0, if for any sequences $\{x_i\}$ in X and $\{y_i\}$ in Y which tend to y_0 such that the tangent planes $T_{x_i} X$ tend to some dim X-plane $\tau \subset \mathbb{R}^n$ and the secants $\widehat{x_i y_i}$ tend (in the projective space $\mathbb{P}^{m-1}$) to some line ℓ, then we have $\ell \subset \tau$. If (X, Y) satisfies Whitney's condition (b) at any point of Y, we say that X is *Whitney* (b)-*regular over* Y. For the properties of the Whitney regularity, see Whitney [28, 29], Mather [20] and Trotman [26, 27].

Let $A \subset \mathbb{R}^m$ admit a C^1 stratification $\mathcal{S}(A)$ satisfying the *frontier condition*, namely if $\overline{X} \cap Y \neq \varnothing$ for $X, Y \in \mathcal{S}(A)$, then we have $\overline{X} \supset Y$. We say that the stratification $\mathcal{S}(A)$ of A is a *Whitney stratification*, if for any strata $X, Y \in \mathcal{S}(A)$ with $\overline{X} \supset Y$, X is Whitney (b)-regular over Y. In the sequel we will assume the frontier condition whenever we consider stratifications of subsets in Euclidean spaces.

Let X, Y be disjoint C^1 submanifolds of $\mathbb{R}^m$, and let y_0 be a point in $Y \cap \overline{X}$. Let $f \colon X \cup Y \to \mathbb{R}^r$ be a mapping such that both restricted mappings $f|_X \colon X \to \mathbb{R}^r$ and $f|_Y \colon Y \to \mathbb{R}^r$ are C^1 and of constant ranks. We say that the pair (X, Y) satisfies *Thom's condition* (a_f) *at* y_0, if for any sequence $\{x_i\}$ in X tending to y_0 such that $\ker d(f|_X)_{x_i}$ tends to κ, then we have $\ker d(f|_Y)_{y_0} \subset \kappa$, where $\ker d(f|_X)_x$ denotes the kernel of the differential of $f|_X$ at x. If (X, Y) satisfies Thom's condition (a_f) at any point of Y, we say that X is *Thom* (a_f)-*regular over* Y or *Thom regular over* Y. For the properties of the Thom (a_f) regularity, see [10, 12, 20, 25].

Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^r$ and let $f \colon A \to B$ be a mapping. Assume that A and B admit C^1 stratifications $\mathcal{S}(A)$ and $\mathcal{S}(B)$ respectively. We call f a *stratified mapping*, if for any stratum $X \in \mathcal{S}(A)$, there exists a stratum $U \in \mathcal{S}(B)$ such that $f|_X \colon X \to U$ is a surjective C^1 submersion. In this paper, we assume all stratified mappings to be surjective. We call the stratified mapping $f \colon (A, \mathcal{S}(A)) \to (B, \mathcal{S}(B))$ *Thom regular*, if for any $X, Y \in \mathcal{S}(A)$ with $\overline{X} \supset Y$, X is Thom (a_f)-regular over Y.

2.2 Controlled tube system

Let M be a C^2 manifold, and let N be a C^2 submanifold of M. A *tube* at N in M is a triple $T = (|T|, \pi, \rho)$, where $|T|$ is an open neighbourhood of N in M, $\pi : |T| \to N$ is a submersive C^2 retraction, and ρ is a nonnegative C^2 function on $|T|$ such that $\rho^{-1}(0) = N$ and each point x of N is the unique critical point (nondegenerate) of the restriction of ρ to $\pi^{-1}(x)$. In the case when N is an open submanifold of M, we let $T = (|T|, \pi, \rho)$ where $|T| = N$, $\pi = \mathrm{id}_N$ and $\rho = 0$ for convenience. Here id_N denotes the identity mapping $\mathrm{id}_N : N \to N$.

Let $A \subset M$, and let $\mathcal{S}(A) = \{X_i\}$ be a C^2 stratification of A (the strata are C^2 submanifolds). A *tube system* for the stratification $\{X_i\}$ consists of one tube $T_i = (|T_i|, \pi_i, \rho_i)$ at each X_i. We say that a tube system $\{T_i\}$ is *controlled* if for any i and j with $X_i \subset \overline{X_j}$, the following commutative relations hold:

$$\pi_i \circ \pi_j(x) = \pi_i(x), \quad \rho_i \circ \pi_j(x) = \rho_i(x) \quad \text{for } x \in |T_i| \cap |T_j|.$$

We call $\{T_i\}$ *weakly controlled* if the first relation holds.

Let M and P be C^2 manifolds, let $f : A \to B$ be a mapping between $A \subset M$ and $B \subset P$. Let $\mathcal{S}(A) = \{W_k\}$ and $\mathcal{S}(B) = \{V'_j\}$ be C^2 stratifications, and let $\{T_k = (|T_k|, \pi_k, \rho_k)\}$ and $\{T'_j = (|T_j|', \pi'_j, \rho'_j)\}$ be tube systems for $\{W_k\}$ and $\{V'_j\}$, respectively. We call $\{T_k\}$ *controlled over* $\{T'_j\}$ if $\{T_k\}$ satisfies the following relations:

(1) For any k and ℓ with $W_k \subset \overline{W_\ell}$, $\pi_k \circ \pi_\ell(x) = \pi_k(x)$ on $|T_k| \cap |T_\ell|$.
(2) For any k and ℓ with $W_k \subset \overline{W_\ell}$ and $f(W_k) \cup f(W_\ell) \subset V'_j$ for some j, $\rho_k \circ \pi_\ell(x) = \rho_k(x)$ on $|T_k| \cap |T_\ell|$.
(3) For any k and j with $f(W_k) \subset V'_j$, $f|_{W_k}$ of class C^2 and $f(|T_k|) \subset |T'_j|$ and $f \circ \pi_k = \pi'_j \circ f$ on $A \cap |T_k|$.

2.3 Semialgebraic and Nash properties

In this subsection we recall some important semialgebraic and Nash properties which can be useful tools when we apply Thom's Isotopy Lemmas to a family of polynomial functions or more generally to a family of Nash functions. We first recall the notions of a Nash manifold and of a Nash mapping.

Let $s = 1, 2, \ldots, \infty, \omega$. A C^s submanifold of $\mathbb{R}^m$ is called a C^s *Nash manifold*, if it is semialgebraic in $\mathbb{R}^m$. In this paper, a submanifold always means a regular submanifold. By Malgrange [19], a C^∞ Nash manifold is a C^ω manifold. We simply call a C^∞ Nash manifold a *Nash manifold*.

Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be C^s Nash manifolds. A C^r mapping $f : M \to N$, $r \leqslant s$, is called a C^r *Nash mapping*, if the graph of f is semialgebraic in $\mathbb{R}^m \times \mathbb{R}^n$. It is known also by [19] that a C^∞ Nash mapping is a C^ω mapping. We simply call a C^∞ Nash mapping a *Nash mapping*.

Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be semialgebraic sets. A continuous mapping $f : A \to B$ is called a *semialgebraic mapping*, if the graph of f is semialgebraic in $\mathbb{R}^m \times \mathbb{R}^n$. Note that we demand the continuity for a semialgebraic mapping in this paper.

Theorem 2.1 (Tarski–Seidenberg Theorem [21]) *Let A be a semialgebraic set in $\mathbb{R}^k$, and let $f : \mathbb{R}^k \to \mathbb{R}^s$ be a semialgebraic mapping. Then $f(A)$ is semialgebraic in $\mathbb{R}^s$.*

Throughout this paper a *Nash open simplex* means a Nash manifold which is Nash diffeomorphic to an open simplex in some Euclidean space.

Theorem 2.2 (Łojasiewicz's Semialgebraic Triangulation Theorem [17, 18]) *Given a finite system of bounded semialgebraic sets X_α in $\mathbb{R}^q$, there exist a simplicial decomposition $\mathbb{R}^q = \bigcup_a C_a$ and a semialgebraic automorphism τ of $\mathbb{R}^q$ such that*

(1) *each X_α is a finite union of some of the $\tau(C_a)$,*
(2) *$\tau(C_a)$ is a Nash manifold in $\mathbb{R}^q$ and τ induces a Nash diffeomorphism $C_a \to \tau(C_a)$ for every a, in other words, $\tau(C_a)$ is a Nash open simplex for every a.*

Remark 2.3 (1) Concerning the proof of Semialgebraic Triangulation Theorem, see also Hironaka [11] or Coste [4].

(2) There is a Nash embedding of $\mathbb{R}^q$ into $\mathbb{R}^{q+1}$ via $S^q \subset \mathbb{R}^{q+1}$. Let us remark that a semialgebraic subset of $\mathbb{R}^q$ can be regarded as a bounded semialgebraic subset of $\mathbb{R}^{q+1}$.

The existence of a finite Whitney stratification of Nash class for a semialgebraic set was established by Łojasiewicz [17, 18].

We next recall Fukuda's Lemma in [7], concerning the existence of compatible Whitney stratifications related to a Nash stratified mapping.

Lemma 2.4 (Fukuda's Lemma) *Let M and N be Nash manifolds, and let $f : M \to N$ be a Nash mapping. Given semialgebraic subsets $A_1, \ldots, A_a$ of M and semialgebraic subsets $B_1, \ldots, B_b$ of N, there exist finite Whitney stratifications of Nash class $\mathcal{S}(M)$ of M compatible with $A_1, \ldots, A_a$ and $\mathcal{S}(N)$ of N compatible with $B_1, \ldots, B_b$ such that $f : (M, \mathcal{S}(M)) \to (N, \mathcal{S}(N))$ is a Nash stratified mapping.*

3 Thom's Isotopy Lemmas

Thom's Isotopy Lemmas have been used widely in singularity theory, e.g. to establish the topological stability theory, to show the topological triviality of a family of sets or mappings, to solve the finite C^0 determinacy problem of map-germs, and so on. Therefore they are not always given in the same form. In this section we mention Thom's Isotopy Lemmas and their semialgebraic versions in a suitable form for our purposes.

3.1 Thom's 1st and 2nd isotopy Lemmas

Thom's 1st and 2nd Isotopy Lemmas were presented by René Thom [25] as very useful tools in singularity theory as mentioned above. For the detailed proofs, see Mather [20] or Gibson, Wirthmüller, du Plessis and Looijenga [10]. We also mention Fukuda [7] and du Plessis and Wall [6] as further references.

Now let us recall Thom's Isotopy Lemmas. Let M, N and I be C^∞ manifolds, let $f : M \to N$ and $q : N \to I$ be C^∞ mappings, and let $A \subset M$ and $B \subset N$ be closed subsets such that $f(A) \subset B$.

Theorem 3.1 (Thom's 1st Isotopy Lemma) *Let $q|_B : B \to I$ be proper. Suppose that B admits a locally finite C^∞ Whitney stratification $\mathcal{S}(B)$ such that $q : (B, \mathcal{S}(B)) \to (I, \{I\})$ is a stratified mapping. Then the stratified set $(B, \mathcal{S}(B))$ is locally C^0 trivial over I.*

Theorem 3.2 (Thom's 2nd Isotopy Lemma) *Let I be connected, and $f|_A : A \to N$ and $q|_B : B \to I$ be proper. Suppose that A and B admit locally finite C^∞ Whitney stratifications $\mathcal{S}(A)$ and $\mathcal{S}(B)$ respectively such that $f : (A, \mathcal{S}(A)) \to (B, \mathcal{S}(B))$ is a Thom regular stratified mapping and $q : (B, \mathcal{S}(B)) \to (I, \{I\})$ is a stratified mapping. Then the stratified mapping f is topologically trivial over I, namely, there are homeomorphisms $H : (q \circ f)^{-1}(P_0) \times I \to A$ and $h : q^{-1}(P_0) \times I \to B$, for some $P_0 \in I$, preserving the natural stratifications such that $h^{-1} \circ f \circ H = f|_{(q \circ f)^{-1}(P_0)} \times \mathrm{id}_I$ and $q \circ h : q^{-1}(P_0) \times I \to I$ is the canonical projection, where id_I is the identity map on I.*

3.2 Semialgebraic versions of Thom's isotopy Lemmas

Shiota established in [23] semialgebraic versions of Thom's Isotopy Lemmas. We recall them in this subsection.

Let $A \subset \mathbb{R}^a$, $B \subset \mathbb{R}^b$ be semialgebraic subsets, let I be a Nash open simplex, and let $f : A \to B$ and $q : B \to I$ be proper Nash mappings.

Theorem 3.3 (Shiota's semialgebraic version of Thom's 1st Isotopy Lemma) *Suppose that B admits a finite Whitney stratification of Nash class $\mathcal{S}(B)$ such that $q : (B, \mathcal{S}(B)) \to (I, \{I\})$ is a stratified mapping. Then the stratified set $(B, \mathcal{S}(B))$ is semialgebraically trivial over I.*

Theorem 3.4 (Shiota's semialgebraic version of Thom's 2nd Isotopy Lemma) *Suppose that A and B admit finite Whitney stratifications of Nash class $\mathcal{S}(A)$ and $\mathcal{S}(B)$ respectively such that $f : (A, \mathcal{S}(A)) \to (B, \mathcal{S}(B))$ is a Thom regular stratified mapping and $q : (B, \mathcal{S}(B)) \to (I, \{I\})$ is a stratified mapping. Then the stratified mapping f is semialgebraically trivial over I, namely, there are semialgebraic homeomorphisms $H : (q \circ f)^{-1}(P_0) \times I \to A$ and $h : q^{-1}(P_0) \times I \to B$, for some $P_0 \in I$, preserving the natural stratifications such that $h^{-1} \circ f \circ H = f|_{(q \circ f)^{-1}(P_0)} \times \mathrm{id}_I$ and $q \circ h : q^{-1}(P_0) \times I \to I$ is the canonical projection.*

3.3 Semialgebraic triviality after finite subdivision

Let $A \subset \mathbb{R}^a$, $B \subset \mathbb{R}^b$ be semialgebraic sets, let Q be a Nash open simplex and let $f : A \to B$ and $q : B \to Q$ be proper Nash mappings. Let us assume that A and B admit finite Whitney stratifications of Nash class $\mathcal{S}(A)$ and $\mathcal{S}(B)$, respectively. In addition, we assume that $f : (A, \mathcal{S}(A)) \to (B, \mathcal{S}(B))$ and $q : (B, \mathcal{S}(B)) \to (Q, \{Q\})$ are stratified mappings. Under this setting, we prepare some notations.

Notation 3.5 For $t \in Q$, let A_t and B_t denote $(q \circ f)^{-1}(t) \cap A$ and $q^{-1}(t) \cap B$, respectively, and let

$$\mathcal{S}(A)_t := \{(q \circ f)^{-1}(t) \cap X \mid X \in \mathcal{S}(A)\} \text{ and } \mathcal{S}(B)_t := \{q^{-1}(t) \cap Y \mid Y \in \mathcal{S}(B)\}.$$

Note that for any $t \in Q$, $\mathcal{S}(A)_t$ and $\mathcal{S}(B)_t$ are stratifications of A_t and B_t, respectively, and $f_t := f|_{A_t} : (A_t, \mathcal{S}(A)_t) \to (B_t, \mathcal{S}(B)_t)$ is a stratified mapping. Then, using the same argument as in the proof of [13, Proposition 4.5] together with Theorem 3.4, we can show the following lemma.

Lemma 3.6 *Suppose that, for any $t \in Q$, the stratified mapping $f_t := f|_{A_t} : (A_t, \mathcal{S}(A)_t) \to (B_t, \mathcal{S}(B)_t)$ is (a_{f_t})-regular. Then, subdividing Q into finitely many Nash open simplices if necessary, the stratified mapping $f : (A, \mathcal{S}(A)) \to (B, \mathcal{S}(B))$ is semialgebraically trivial over Q (in the sense of Theorem 3.4).*

Sketch of the proof Similarly to the proof of the original Thom's 2nd Isotopy Lemma, the cotrolled tube systems also take an important roll in the proof of the semialgebraic version. In fact, we can weaken the assumptions of the Whitney's regularity and the Thom's regularity in the semialgebraic version of Thom's 2nd Isotopy Lemma as follows:

There exist a cotrolled C^2-Nash tube system $\{T_j^B = (|T_j^B|, \pi_j^B, \rho_j^B)\}$ for $\{U_j \in \mathcal{S}(B)\}$ and a C^2-Nash tube system $\{T_i^A = (|T_i^A|, \pi_i^A, \rho_i^A)\}$ for $\{X_i \in \mathcal{S}(A)\}$ controlled over $\{T_j^B\}$ such that

(1) for each j, $q \circ \pi_j^B = q$ on $B \cap |T_j^B|$;

(2) for any j and j' with $U_j \subset \overline{U_{j'}}$,

　(∗) $(\pi_j^B, \rho_j^B)|_{U_{j'} \cap |T_j^B|} : U_{j'} \cap |T_j^B| \to U_j \times \mathbb{R}$ is a C^2 submersion;

(3) for any i and i' with $f(X_i) \subset U_j$, $f(X_{i'}) \subset U_{j'}$ and $X_i \subset \overline{X_{i'}}$, if $U_j = U_{j'}$, then

　(∗) $(\pi_i^A, \rho_i^A)|_{X_{i'} \cap |T_i^A|} : X_{i'} \cap |T_i^A| \to X_i \times \mathbb{R}$ is a C^2 submersion,

and if $U_j \ne U_{j'}$, then for any $x \in X_{i'} \cap |T_i^A|$,

　(∗∗) $d(\pi_i^A|_{X_{i'}})_x : \ker d(f|_{X_{i'}})_x \to \ker d(f|_{X_i})_{\pi_i^A(x)}$ is surjective.

If $U_{j'}$ (respectively $X_{i'}$) is Whitney (b)-regular over U_j (respectively X_i), then condition (∗) in (2) (respectively (3)) is satisfied for any tube system by shrinking the tube if necessary. Note that our $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are Whitney stratications. On the other hand, if $X_{i'}$ is Thom (a_f)-regular over X_i, then condition (∗∗) in (3) is satisfied for any controlled tube systems by shrinking the tubes if necessary. We are assuming here a weaker condition than Thom (a_f)-regularity, namely we are assuming (a_{f_t})-regularity for any $t \in Q$.

Let $X, Y \in \mathcal{S}(A)$ with $X \subset \overline{Y}$, and let $U, V \in \mathcal{S}(B)$ with $U \subset \overline{V}$ such that $f(X) \subset U$ and $f(Y) \subset V$. Assume that X and U admit C^2-Nash tube systems $T_X = (|T_X|, \pi_X, \rho_X)$ and $T_U = (|T_U|, \pi_U, \rho_U)$ such that $f \circ \pi_X = \pi_U \circ f$ in $Y \cap |T_X|$ and $q \circ \pi_U = q$ in $B \cap |T_U|$. Let Ξ be the set of $y \in Y \cap |T_X|$ such that $q \circ f(y) = t$

and $d(\pi_X|_{Y_t})_y\colon \ker d(f_t|_{Y_t})_y \to \ker d(f_t|_{X_t})_{\pi_X(y)}$ is not surjective. Then, under the assumption of (a_{f_t})-regularity for any $t \in Q$, we can show that $q \circ f(\overline{\Xi} \cap X)$ is a semialgebraic subset of Q of dimension less than $\dim Q$, where $\overline{\Xi}$ denotes the closure of Ξ in A. Repeating the same argument after taking a finite subdivision of the parameter space into Nash open simplices (using Theorem 2.2), we can show the following assertion.

Assertion 3.7 *Subdividing Q into finitely many Nash open simplices and shrinking the tube T_X if necessary, for any $y \in Y \cap |T_X|$,*

(∗∗) $d(\pi_X|_Y)_y\colon \ker d(f|_Y)_y \to \ker d(f|_X)_{\pi_X(y)}$ *is surjective.*

Let us assume that a cotrolled C^2-Nash tube system $\{T_j^B = (|T_j^B|, \pi_j^B, \rho_j^B)\}$ for $\{U_j \in \mathcal{S}(B)\}$ satisfying (1) and a C^2-Nash tube system $\{T_i^A = (|T_i^A|, \pi_i^A, \rho_i^A)\}$ for $\{X_i \in \mathcal{S}(A)\}$ controlled over $\{T_j^B\}$ are given. Then, using Assertion 3.7 and Theorem 2.2, we can show that the latter condition in (3) including (∗∗) is satisfied after taking a finite subdivision of Q into Nash open simplices and shrinking the tubes if necessary.

On the other hand, the existence theorem of a controlled C^r-Nash tube system $(r < \infty)$ is shown in [23, Lemma II.6.10] (see also [23, Lemmas I.1.3 and I.1.3′]). Modifying the proof, we can show the existence of a controlled C^2-Nash tube system $\{T_j^B\}$ for $\{U_j \in \mathcal{S}(B)\}$ satisfying (1) and a C^2-Nash tube system $\{T_i^A\}$ controlled over $\{T_j^B\}$, under our setting. Therefore, taking a finite subdivision of Q into Nash open simplices if necessary, the hypothesis of the semialgebraic version of Thom's 2nd Isotopy Lemma with the weakened assumptions is satisfied. Then the statement of the lemma follows from this modified Isotopy Lemma. ☐

4 Finiteness for some equivalences of polynomial functions

4.1 Finiteness for $\mathcal{R}_0\mathcal{L}_0$-equivalence of polynomial functions

In this subsection we consider some restricted versions of Theorems 1.1 and 1.2. Let us recall $P_0(n, 1, d; \mathbb{R}) := \{f \in P(n, 1, d; \mathbb{R}) \mid f(0) = 0\}$. We say that two polynomials $f, g \in P(n, 1, d; \mathbb{R})$ are *topologically $\mathcal{R}_0\mathcal{L}_0$ equivalent* (respectively *semialgebraically $\mathcal{R}_0\mathcal{L}_0$ equivalent*), if there exist homeomorphisms (respectively semialgebraic homeomorphisms) $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\sigma(0) = 0$ and $\tau\colon \mathbb{R} \to \mathbb{R}$ with $\tau(0) = 0$ such that $\tau \circ f = g \circ \sigma$ on $\mathbb{R}^n$.

If we apply the same argument as in the proof of Theorem 1.1 we can show that the number of topological $\mathcal{R}\mathcal{L}$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite. Then, by taking the stratifications $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbb{R}^n \times P_0(n, 1d; \mathbb{R})$ and $\mathbb{R} \times P_0(n, 1, d; \mathbb{R})$ so that $\mathcal{S}_1$ and $\mathcal{S}_2$ are compatible with $\{0\} \times P_0(n, 1, d; \mathbb{R})$ where $0 \in \mathbb{R}^n$ and $\{0\} \times P_0(n, 1, d; \mathbb{R})$ where $0 \in \mathbb{R}$, respectively, we conclude that σ preserves $0 \in \mathbb{R}^n$ and τ preserves $0 \in \mathbb{R}$. Actually when restricted to $P_0(n, 1, d; \mathbb{R})$, the fact that τ preserves $0 \in \mathbb{R}$ is a consequence of the fact that σ preserves $0 \in \mathbb{R}^n$. Thus we have shown the following stronger result.

Theorem 4.1 *The number of topological $\mathcal{R}_0\mathcal{L}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite.*

Using a similar argument to the proof of the above theorem with the semialgebraic versions of Thom's Isotopy Lemmas, we can show the following theorem in the semialgebraic case.

Theorem 4.2 *The number of semialgebraic $\mathcal{R}_0\mathcal{L}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite.*

4.2 Finiteness for $\mathcal{R}\mathcal{L}$-equivalence of polynomial function germs

We say that two polynomials $f, g \colon \mathbb{R}^n \to \mathbb{R}$ are *locally topologically $\mathcal{R}\mathcal{L}$ equivalent* at $0 \in \mathbb{R}^n$ and write $\mathcal{R}\mathcal{L}_0$, if there exist local homeomorphisms $\sigma \colon (U \subset \mathbb{R}^n, 0) \to (V \subset \mathbb{R}^n, 0)$, U, V two open neighbourhoods of $0 \in \mathbb{R}^n$, and $\tau \colon (J \subset \mathbb{R}, f(0)) \to (K \subset \mathbb{R}, g(0))$, J, K two open intervals, such that for their appropriate restrictions we have $\tau \circ f = g \circ \sigma$. Similarly, two polynomials $f, g \colon \mathbb{R}^n \to \mathbb{R}$ are *locally semialgebraically $\mathcal{R}\mathcal{L}$ equivalent* at $0 \in \mathbb{R}^n$ if we can take the σ and τ as local semialgebraic homeomorphisms, and write $\mathcal{R}\mathcal{L}_0$.

In the case of $f, g \in P_0(n, 1, d; \mathbb{R})$ the above equivalence relation $\mathcal{R}\mathcal{L}_0$, is in fact the local version of the previous $\mathcal{R}_0\mathcal{L}_0$ equivalence relation, i.e. $\mathcal{R}\mathcal{L}_0$ coincides with $\mathcal{R}_0\mathcal{L}_0$ in this case. In this paper local will mean only local at $0 \in \mathbb{R}^n$ and at the corresponding images, and the meaning of local will be as described above, i.e. equivalence of germs of polynomial functions at $0 \in \mathbb{R}^n$.

In the original version of [7], Fukuda proved the following finiteness theorem for local topological $\mathcal{R}\mathcal{L}$ equivalence classes appearing in the $P_0(n, 1, d; \mathbb{R})$. This theorem is an immediate consequence of Theorem 4.1.

Theorem 4.3 *The number of local topological $\mathcal{R}_0\mathcal{L}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite.*

As a corollary of Theorem 4.3, we have the following.

Corollary 4.4 (Fukuda's local $\mathcal{R}\mathcal{L}_0$ finiteness theorem) *The number of local topological $\mathcal{R}\mathcal{L}_0$ types appearing in $P(n, 1, d; \mathbb{R})$ is finite.*

Example 4.5 Let $I = [0, 1]$ be a closed interval, let $f_t \colon (\mathbb{R}, 0) \to \mathbb{R}$, $t \in I$, be the polynomial defined by $f_t(x) := t^2 - x^2$, and let

$$\mathcal{F} := \{f_t \in P(1, 1, 2; \mathbb{R}) \mid t \in I\}.$$

Then there is a unique local topological $\mathcal{R}\mathcal{L}_0$ type appearing in $\mathcal{F}$, but the local topological $\mathcal{R}_0$ types appearing in $\mathcal{F}$ have the cardinal number of the continuum since the $\mathcal{R}_0$ equivalence preserves the image. This example shows, in particular, that Fukuda's finiteness result, Theorem 1.1, cannot be generalised to $\mathcal{R}_0$.

Using a similar argument to the proof of Theorem 4.3 with the semialgebraic versions of Thom's Isotopy Lemmas, we can show the following finiteness theorem.

Theorem 4.6 *The number of local semialgebraic $\mathcal{R}_0\mathcal{L}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite.*

We have a similar corollary also in the semialgebraic case.

Corollary 4.7 (Local semialgebraic $\mathcal{RL}_0$ finiteness theorem) *The number of local semialgebraic $\mathcal{RL}_0$ types appearing in $P(n, 1, d; \mathbb{R})$ is finite.*

4.3 General setting for finiteness of polynomial functions

Let $(X, \sim_X)$ and $(Y, \sim_Y)$ be two sets endowed with some equivalence relations, and let $\pi \colon X \to Y$ be a surjective mapping. If $a \sim_X b$ implies $\pi(a) \sim_Y \pi(b)$ (or respectively $\pi(a) \sim_Y \pi(b)$ implies $a \sim_X b$) then whenever the quotient space of X is finite, the quotient space of Y is finite (or respectively, whenever the quotient space of Y is finite, the quotient space of X is finite). In particular, in the case when $f \sim_X g$ if and only if $\tilde{f} := \pi(f) \sim_Y \tilde{g} := \pi(g)$, the quotient space of Y is finite if and only if the quotient space of X is finite. We are going to apply these observations for the relations and situations mentioned below.

For $a \in \mathbb{R}^n$, let μ_a denote the translation in the direction $\vec{a}$ on $\mathbb{R}^n$, and for $c \in \mathbb{R}$, let τ_c denote the translation in the direction $\vec{c}$ on $\mathbb{R}$.

[1] Local equivalence case

Local topological equivalence of polynomials

Case 1. Let $X = P \times \mathbb{R}^n = P(n, 1, d; \mathbb{R}) \times \mathbb{R}^n$, $Y = P_0 = \{f \in P \mid f(0) = 0\}$, and $\pi(f, a) = \tilde{f} = f \circ \mu_a - f(a) = \tau_{-f(a)} \circ f \circ \mu_a$, so that $\tilde{f}(0) = 0$. Consider the case where $\sim_X = \mathcal{R}\vec{\mathcal{L}}$ is the local right-left equivalence with translation (defined below) $\tau \circ f \circ \phi = g$, where ϕ is a local homeomorphism and τ is a translation on $\mathbb{R}$, and $\sim_Y = \mathcal{R}_0$ is the local topological right equivalence.

Given $(f, a), (g, b) \in P(n, 1, d; \mathbb{R}) \times \mathbb{R}^n$, we say that they are locally topologically $\mathcal{R}\vec{\mathcal{L}}$ equivalent if there exists a local homeomorphism $\phi \colon (\mathbb{R}^n, b) \to (\mathbb{R}^n, a)$ such that $\tau_{g(b)-f(a)} \circ f \circ \phi = g$. Therefore we have

$$(\tau_{-f(a)} \circ f \circ \mu_a) \circ (\mu_{-a} \circ \phi \circ \mu_b) = \tau_{-g(b)} \circ g \circ \mu_b.$$

Since $\mu_{-a} \circ \phi \circ \mu_b$ is a local homeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\tilde{f}$ and $\tilde{g}$ are locally topologically $\mathcal{R}_0$ equivalent.

Conversely, assume that $\tilde{f}$ and $\tilde{g}$ are locally topologically $\mathcal{R}_0$ equivalent. Then there exists a local homeomorphism $\phi \colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $(\tau_{-f(a)} \circ f \circ \mu_a) \circ \phi = \tau_{-g(b)} \circ g \circ \mu_b$. Therefore we have $\tau_{g(b)-f(a)} \circ f \circ (\mu_a \circ \phi \circ \mu_{-b}) = g$. Since $\mu_a \circ \phi \circ \mu_{-b}$ is a local homeomorphism $(\mathbb{R}^n, b) \to (\mathbb{R}^n, a)$, i.e. (f, a) and (g, b) are $\mathcal{R}\vec{\mathcal{L}}$ equivalent in our sense.

We have shown that $(f, a) \sim_X (g, b)$ if and only if $\pi(f, a) = \tilde{f} \sim_Y \tilde{g} = \pi(g, b)$.

Case 2. Let $X = P$, $Y = P_0 = P_0(n, 1, d; \mathbb{R})$ and $\pi(f) = \tilde{f} = f - f(0) = \tau_{-f(0)} \circ f$, i.e. the case when both $a = b = 0$. Consider the case when $\sim_X = \mathcal{R}\vec{\mathcal{L}}$ is the local right-left equivalence with translation as above, and $\sim_Y = \mathcal{R}_0$ is the local topological right equivalence.

Using a similar argument to Case 1, we can easily show in this case that $f \sim_X g$ if and only if $\tilde{f} \sim_Y \tilde{g}$.

Case 3. Let $X = P$, $Y = P_0 = P_0(n, 1, d; \mathbb{R})$ and $\pi(f) = \tilde{f} = f - f(0) = \tau_{-f(0)} \circ f$. Both equivalences are the local topological right-left equivalence $\mathcal{RL}$.

Using a similar argument to the global topological equivalence case below, we can show also in this local case that $f \sim_X g$ if and only if $\tilde{f} \sim_Y \tilde{g}$.

By the above observations and discussions, we have the following.

Note 4.8 (i) Consider Case 1: By Theorem 1.3, the number of local topological $\mathcal{R}_0$ orbits in P_0 is finite. Therefore the number of local topological $\mathcal{R}\tilde{\mathcal{L}}$ orbits in $P \times \mathbb{R}^n$ is finite.

(ii) Consider Case 2: By Theorem 1.3, the number of local topological $\mathcal{R}_0$ orbits in P_0 is finite. Therefore the number of local topological $\mathcal{R}\tilde{\mathcal{L}}$ orbits in P is finite.

(iii) Consider Case 3: By Theorem 4.3, the number of local topological $\mathcal{RL}$ orbits in P_0 is finite. Therefore the number of local topological $\mathcal{RL}$ orbits at $0 \in \mathbb{R}^n$ in P is finite. This is also a consequence of the previous fact (ii).

Local semialgebraic equivalence of polynomials In the semialgebraic equivalence case, we can show a similar note to Note 4.8, using Theorems 1.4 and 4.6.

[2] Global equivalence case

Topological equivalence of polynomial functions
Let $X = P = P(n, 1, d; \mathbb{R})$, $Y = P_0 = \{f \in P \mid f(0) = 0\}$ and $\pi(f) = \tilde{f} = f - f(0) = \tau_{-f(0)} \circ f$. Both equivalences are the topological right-left equivalences $\mathcal{RL}$.

Let $f, g \in P(n, 1, d; \mathbb{R})$. Assume that f and g are topologically $\mathcal{RL}$ equivalent. Then there exist homeomorphisms $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ and $\psi \colon \mathbb{R} \to \mathbb{R}$ such that $\psi \circ f \circ \phi = g$. Therefore we have

$$(\tau_{-g(0)} \circ \psi \circ \tau_{f(0)}) \circ (\tau_{-f(0)} \circ f) \circ \phi = \tau_{-g(0)} \circ g.$$

It follows that $\tilde{f}$ and $\tilde{g}$ are topologically $\mathcal{RL}$ equivalent.

Conversely, assume that $\tilde{f}$ and $\tilde{g}$ are topologically $\mathcal{RL}$ equivalent. Then there exist homeomorphisms $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ and $\psi \colon \mathbb{R} \to \mathbb{R}$ such that $\psi \circ (\tau_{-f(0)} \circ f) \circ \phi = \tau_{-g(0)} \circ g$. Therefore we have $(\tau_{g(0)} \circ \psi \circ \tau_{-f(0)}) \circ f \circ \phi = g$. Hence f and g are topologically $\mathcal{RL}$ equivalent.

It follows that $f \sim_X g$ if and only if $\tilde{f} \sim_Y \tilde{g}$.

We have the following note in the global topological equivalence case.

Note 4.9 By Theorem 1.1, the number of topological $\mathcal{RL}$ orbits in P is finite. Therefore the numbers of topological $\mathcal{RL}$ orbits in P and P_0 are simultaneously finite.

Semialgebraic equivalence of polynomial functions
In this case we can show a similar note to Note 4.9, using Theorem 1.2.

The above considerations show that one can concentrate on proving finiteness results for $P_0 = P_0(n, 1, d; \mathbb{R})$, for instance all local cases are a consequence of

the local topological right equivalence on P_0. Examples 4.5 and 4.14 show that we do not have local or global finiteness for the $\mathcal{R}$ equivalence in $P(n, 1, d; \mathbb{R})$.

4.4 Finiteness for $\mathcal{R}$-equivalence of polynomial function germs

The finiteness results may hold for a stronger equivalence than topological $\mathcal{R}\mathcal{L}$ equivalence. In fact, after proving Theorem 4.3, Fukuda in his lectures [9] pointed out that the following local finiteness theorem holds.

Theorem 1.3 (Fukuda's local $\mathcal{R}_0$ finiteness theorem) *The number of local topological $\mathcal{R}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite. In addition, each local topological $\mathcal{R}_0$ equivalence class in $P_0(n, 1, d; \mathbb{R})$ is a semialgebraic subset of $P_0(n, 1, d; \mathbb{R})$.*

Apart from mentioning the result in [9], it seems that Fukuda has not written up its statement and his idea of proof (we could not find the result in the published literature). Therefore we decided to give a proof for Fukuda's local $\mathcal{R}$ finiteness result in this paper. The latter semialgebraicness statement follows from the proof of the first one.

Before we start the proof, we prepare for some notion. We say that two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are *locally topologically $\mathcal{R}$-$\mathcal{J}\mathcal{L}$ equivalent*, if there exist local homeomorphisms $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\tau : (\mathrm{Im}(f), 0) \to (\mathrm{Im}(g), 0)$ such that $\tau \circ f = g \circ \sigma$, where $\mathrm{Im}(f)$ denotes the image of f.

Remark 4.10 Suppose that $f, g \in P_0(n, 1, d; \mathbb{R})$ are locally topologically $\mathcal{R}$-$\mathcal{J}\mathcal{L}$ equivalent with local homeomorphisms $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\tau : (\mathrm{Im}(f), 0) \to (\mathrm{Im}(g), 0)$. If τ is the identity mapping on $\mathrm{Im}(f)$ around $0 \in \mathbb{R}$, then f and g are locally topologically $\mathcal{R}_0$ equivalent.

Proof of Theorem 1.3 Using the Curve Selection Lemma, we can easily see the following.

Property 4.11 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a C^ω function germ. Suppose that $0 \in \mathbb{R}^n$ is a singular point of f. Then the singular points set $S(f)$ of f is contained in $f^{-1}(0)$ as germs at $0 \in \mathbb{R}^n$.*

As mentioned before, $P_0(n, 1, d; \mathbb{R})$ is naturally identified with the Euclidean space $\mathbb{R}^L$. Let us express the correspondence by $\mathbb{R}^L \ni r \leftrightarrow f_r \in P_0(n, 1, d; \mathbb{R})$. Using Theorem 2.1, we can show that $\{(f_r(S(f_r)), r) \in \mathbb{R} \times \mathbb{R}^L\}$ is a semialgebraic subset of $\mathbb{R} \times \mathbb{R}^L$. Therefore we have the following observation from Theorem 2.2.

Observation 4.12 Let $Q \subset \mathbb{R}^L$ be a Nash manifold of dimension ℓ. Then

$$\overline{\{(f_r(S(f_r)), r) \in \mathbb{R} \times Q \mid f_r \in P_0(n, 1, d; \mathbb{R})\} \backslash \{0\} \times Q} \cap \{0\} \times Q$$

is a semialgebraic subset of $\{0\} \times Q$ of dimension less than ℓ, where the closure is taken in $\mathbb{R} \times Q$.

Let $F : (\mathbb{R}^n \times \mathbb{R}^L, \{0\} \times \mathbb{R}^L) \to (\mathbb{R}, \{0\})$ be a polynomial function defined by $F(x, r) := f_r(x)$. Then we define the map

$$\Gamma : (\mathbb{R}^n \times \mathbb{R}^L, \{0\} \times \mathbb{R}^L) \to (\mathbb{R} \times \mathbb{R}^L, \{0\} \times \mathbb{R}^L)$$

by $\Gamma := (F, \mathrm{id}_{\mathbb{R}^L})$, namely $\Gamma(x, r) = (f_r(x), r)$. Let $q \colon \mathbb{R} \times \mathbb{R}^L \to \mathbb{R}^L$ be the canonical projection. For $Q \subset \mathbb{R}^L$, we set $F_Q := F|_{\mathbb{R}^n \times Q}$.

Let us consider the situation when Thom's 2nd Isotopy Lemma (Theorem 3.2) is applicable to the above map-germs Γ and q after stratifying $\mathbb{R}^n \times \mathbb{R}^L$ near $\{0\} \times \mathbb{R}^L$ and $\mathbb{R} \times \mathbb{R}^L$ near $\{0\} \times \mathbb{R}^L$ and taking a finite subdivision of $\mathbb{R}^L$ into connected Nash manifolds (in order to see Theorem 4.3, that is the finiteness for local topological $\mathcal{RL}$ types appearing in $P_0(n, 1, d; \mathbb{R})$). Note that for $f \in P_0(n, 1, d; \mathbb{R})$, $\mathrm{Im}(f) = \{0\}$ as germs at $0 \in \mathbb{R}$ if and only if f is a 0-mapping. Set $\{0\text{-map}\} \subset \mathbb{R}^L$ as Q_0. Therefore we consider the situation when Γ and q are stratified in the following way. Namely, there exists a finite partition of $\mathbb{R}^L \setminus Q_0$ into connected Nash manifolds $\mathbb{R}^L \setminus Q_0 = \bigcup_{i=1}^u Q_i$ such that for each $i = 1, \ldots, u$, there are closed semialgebraic neighbourhoods A_i of $\{0\} \times Q_i$ in $\mathbb{R}^n \times Q_i$ and C_i of $\{0\} \times Q_i$ in $\mathbb{R} \times Q_i$ with the following properties: Set $B_i := C_i \cap \mathrm{Im}(\Gamma|_{A_i})$.

(1) (Case I) If $f \in Q_i$ takes both positive and negative values as a function germ at $0 \in \mathbb{R}^n$, then we have $C_i \subset \mathrm{Im}(\Gamma|_{A_i})$.

   (Case II) If $f \in Q_i$ does not take negative values as a function germ at $0 \in \mathbb{R}^n$, then we have $C_i \cap \{y \in \mathbb{R} \mid y \geqslant 0\} \times Q_i \subset \mathrm{Im}(\Gamma|_{A_i})$.

  (Case III) If $f \in Q_i$ does not take positive values as a function germ at $0 \in \mathbb{R}^n$, then we have $C_i \cap \{y \in \mathbb{R} \mid y \leqslant 0\} \times Q_i \subset \mathrm{Im}(\Gamma|_{A_i})$.

(2) $\Gamma|_{A_i} \colon A_i \to B_i$ and $q|_{B_i} \colon B_i \to Q_i$ are proper.

(3) A_i and B_i admit finite Whitney stratifications of Nash class $\mathcal{S}(A_i)$ and $\mathcal{S}(B_i)$, respectively, such that $\Gamma \colon (A_i, \mathcal{S}(A_i)) \to (B_i, \mathcal{S}(B_i))$ is a Thom regular stratified mapping and $q \colon (B_i, \mathcal{S}(B_i)) \to (Q_i, \{Q_i\})$ is a stratified mapping.

By Theorem 2.2 and Lemma 2.4, taking a finite subdivision of Q_i and substratifying $\mathcal{S}(A_i)$ and $\mathcal{S}(B_i)$ if necessary, we may assume that

(4) The stratification $\mathcal{S}(A_i)$ is compatible with $\{0\} \times Q_i$, $\{(S(f_r), r) \mid r \in Q_i\}$ and $F_{Q_i}^{-1}(0)$, the stratification $\mathcal{S}(B_i)$ is compatible with $\{0\} \times Q_i$ and $\{f_r(S(f_r)), r) \mid r \in Q_i\}$, and Q_i is a Nash open simplex.

By Observation 4.12 and Theorem 2.4, after taking a finite subdivision of Q_i into Nash open simplices if necessary, we can assume that

$$\overline{\{(f_r(S(f_r)), r) \in \mathbb{R} \times Q_i \mid f_r \in P_0(n, 1, d; \mathbb{R})\} \setminus \{0\} \times Q_i} \cap \{0\} \times Q_i = \varnothing,$$

$1 \leqslant i \leqslant u$. Taking a further subdivision of Q_i into finite Nash open simplices if necessary, we may assume that if $U \cap \{0\} \times Q_i \neq \varnothing$ for $U \in \mathcal{S}(B_i)$, then $U = \{0\} \times Q_i$, and that

$$\left\{ (f_r(S(f_r)), r) \in \mathbb{R} \times Q_i \mid f_r \in P_0(n, 1, d; \mathbb{R}) \right\} = U$$

as set-germs at $\{0\} \times Q_i$, if it is not empty. In addition, if $\overline{V} \cap \{0\} \times Q_i \neq \varnothing$ for $V \in \mathcal{S}(B_i)$, then $\overline{V} \supset \{0\} \times Q_i = U$. Let us denote by V_1 and V_2 such two adjacent strata of $\mathcal{S}(B_i)$ to U in Case I, and by V such only one adjacent stratum of $\mathcal{S}(B_i)$ to U in Cases II and III. We set $V_0 := U$.

In order to see Theorem 1.3 holds, it suffices to show that for $r_1, r_2 \in Q_i$, $i = 1, \ldots, u$, f_{r_1} and f_{r_2} are topologically $\mathcal{R}$ equivalent. Since Q_i is a Nash open simplex, there is a Nash diffeomorphism $\tau \colon Q_i \to \Delta_i$ between Q_i and an open simplex Δ_i in some Euclidean space. Let $s_1 = \tau(r_1)$ and $s_2 = \tau(r_2)$, and let $h \colon I \to \Delta_i$ be the homotopy defined by $h(t) = (1-t)s_1 + t s_2$ for $t \in I = (-\eta, 1+\eta)$. Here η is a sufficiently small positive number such that if we set $J := \tau^{-1} \circ h(I)$, $\overline{J} \subset Q_i$ where the closure is taken in $\mathbb{R}^L$. Then J is a Nash curve without end points. Therefore $\mathbb{R}^n \times J$ and $\mathbb{R} \times J$ are Nash manifolds of dimension $n+1$ and 2, respectively.

We consider the restriction of $q \circ \Gamma$ over $J \subset \mathbb{R}^L$:

$$\left(A_i \cap \mathbb{R}^n \times J, \{0\} \times J\right) \xrightarrow{\tilde{\Gamma}} \left(B_i \cap \mathbb{R} \times J, \{0\} \times J\right) \xrightarrow{\tilde{q}} J.$$

We define the vector field v on J by $v := D(\tau^{-1} \circ h)\left(\frac{\partial}{\partial t}\right)$. Note that v is tangent to $J \subset \mathbb{R}^L$.

When we show the local topological triviality of $\{f_\gamma\}_{\gamma \in J}$ using Thom's 2nd Isotopy Lemma, we lift the vector field v on J to a neighbourhood D_i of $\{0\} \times J$ in $B_i \cap \mathbb{R} \times J$ by $d\tilde{q}$ and then to a neighbourhood E_i of $\{0\} \times J$ in $A_i \cap \mathbb{R}^n \times J$ by $d(\tilde{q} \circ \tilde{\Gamma})$ so that the first lifted vector field is controlled by a controlled tube system $\{T'_j = (|T'_j|, \pi'_j, \rho'_j)\}$ for $\{V_j \in \mathcal{S}(B_i \cap \mathbb{R} \times J)\}$ and the second lifted vector field is controlled by a controlled tube system $\{T_k = (|T_k|, \pi_k, \rho_k)\}$ for $\{W_k \in \mathcal{S}(A_i \cap \mathbb{R}^n \times J)\}$ controlled over $\{T'_j\}$.

As the neighbourhood D_i of $\{0\} \times J$ in $B_i \cap \mathbb{R} \times J$, we take $[-\epsilon_0, \epsilon_0] \times J$, $[0, \epsilon_0] \times J$ and $[-\epsilon_0, 0] \times J$ ($\epsilon_0 > 0$) in Cases I, II and III, respectively. Here we take $\epsilon_0 > 0$ sufficiently small so that $[-\epsilon_0, \epsilon_0] \times J \subset V_1 \cup V_0 \cup V_2$, $[0, \epsilon_0] \times J \subset V \cup V_0$ and $[-\epsilon_0, 0] \times J \subset V \cup V_0$ in Cases I, II and III, respectively. Let $\widehat{V}_0 := \{0\} \times J$, $\widehat{V}_1 := (0, \epsilon_0) \times J$, $\widehat{V}_2 := (-\epsilon_0, 0) \times J$, $\widehat{V}_3 := \{\epsilon_0\} \times J$ and $\widehat{V}_4 := \{-\epsilon_0\} \times J$. We consider canonical tubular neighborhood systems (in $D_i \subset \mathbb{R} \times J$) for $\{\widehat{V}_j\}$ as follows: $\widehat{T}'_0 = (|\widehat{T}'_0|, \widehat{\pi}'_0, \widehat{\rho}'_0)$ where $|\widehat{T}'_0| := \{(y,s) \mid \widehat{\rho}'_0(y,s) \leqslant \epsilon_1^2\}$, $0 < \epsilon_1 \leqslant \epsilon_0$, $\widehat{\pi}'_0(y,s) = (0,s)$ and $\widehat{\rho}'_0(y,s) = y^2$; $\widehat{T}'_3 = (|\widehat{T}'_3|, \widehat{\pi}'_3, \widehat{\rho}'_3)$ where $|\widehat{T}'_3| := \{(y,s) \mid \widehat{\rho}'_3(y,s) < \epsilon_3^2\}$, $0 < \epsilon_3 \leqslant \epsilon_0$, $\widehat{\pi}'_3(y,s) = (\epsilon_0, s)$ and $\widehat{\rho}'_3(y,s) = (y - \epsilon_0)^2$; $\widehat{T}'_4 = (|\widehat{T}'_4|, \widehat{\pi}'_4, \widehat{\rho}'_4)$ where $|\widehat{T}'_4| := \{(y,s) \mid \widehat{\rho}'_4(y,s) < \epsilon_4^2\}$, $0 < \epsilon_4 \leqslant \epsilon_0$, $\widehat{\pi}'_4(y,s) = (-\epsilon_0, s)$ and $\widehat{\rho}'_4(y,s) = (y + \epsilon_0)^2$; $\widehat{T}'_1 = (|\widehat{T}'_1|, \widehat{\pi}'_1, \widehat{\rho}'_1)$ where $|\widehat{T}'_1| = \widehat{V}_1$, $\widehat{\pi}'_1 = \mathrm{id}_{\widehat{V}_1}$ and $\widehat{\rho}'_1 = 0$; $\widehat{T}'_2 = (|\widehat{T}'_2|, \widehat{\pi}'_2, \widehat{\rho}'_2)$ where $|\widehat{T}'_2| = \widehat{V}_2$, $\widehat{\pi}'_2 = \mathrm{id}_{\widehat{V}_2}$ and $\widehat{\rho}'_2 = 0$. Then we define the first lifted vector field w of v on D_i by $w(y,s) := (0, v(s))$. This is a vector field controlled by the controlled tube system of Nash class $\{\widehat{T}'_j \mid j = 0, 1, 2, 3, 4\}$, $\{\widehat{T}'_j \mid j = 0, 1, 3\}$ and $\{\widehat{T}'_j \mid j = 0, 2, 4\}$ for $\{\widehat{V}_j \in \mathcal{S}([-\epsilon_0, \epsilon_0] \times J)\}$, $\{\widehat{V}_j \in \mathcal{S}([0, \epsilon_0] \times J)\}$ and $\{\widehat{V}_j \in \mathcal{S}([-\epsilon_0, 0] \times J)\}$ in Cases I, II and III, respectively. Actually, $d\widehat{\rho}'_j(w) = 0$, $j = 0, 1, 2, 3, 4$. In particular, w is tangent to $\{(y,s) \in D_i \mid y = \text{constant}\}$ in Cases I, II and III.

Let $W_0 := \{0\} \times J \subset \mathbb{R}^n \times J$, and let $W_1, \ldots, W_d \in \mathcal{S}(A_i)$ be all the adjacent strata to W_0. As the neighbourhood E_i of $\{0\} \times J$ in $A_i \cap \mathbb{R}^n \times J$, we take $\tilde{\Gamma}^{-1}(D_i) \cap A_i$. We may assume that $E_i \subset W_0 \cup W_1 \cup \cdots \cup W_d$. Let $\mathring{D}_i$ be $(-\epsilon_0, \epsilon_0) \times J$, $[0, \epsilon_0) \times J$ and $(-\epsilon_0, 0] \times J$ in Cases I, II and III, respectively, and let $\mathring{E}_i := \tilde{\Gamma}^{-1}(\mathring{D}_i) \cap A_i$. We further let $\check{D}_i := \{\pm\epsilon_0\} \times J$, $\{\epsilon_0\} \times J$ and $\{-\epsilon_0\} \times J$ in Cases I, II and III, respectively, and let $\check{E}_i := \tilde{\Gamma}^{-1}(\check{D}_i) \cap A_i$. Set $\mathcal{S}(\mathring{E}_i) := \{W \cap \mathring{E}_i \mid W \in \mathcal{S}(A_i)\}$ and $\mathcal{S}(\check{E}_i) := \{\text{connected components of } \check{E}_i\}$. Define $\mathcal{S}(E_i) := \mathcal{S}(\mathring{E}_i) \cup \mathcal{S}(\check{E}_i)$.

Now we lift the vector field w on E_i so that the lifted vector field β is controlled by a controlled tube system of Nash class $\{\widehat{T}_k = (|\widehat{T}_k|, \widehat{\pi}_k, \widehat{\rho}_k)\}$ for $\{\widehat{W}_k \in \mathcal{S}(E_i)\}$ controlled over $\{\widehat{T}'_j\}$. The flows of β and w yield the topological triviality of $\{f_\gamma\}_{\gamma \in J}$. By construction, the flow of w yields the triviality keeping "$y = $ constant". Since $\{0\} \times J \in \mathcal{S}(E_i)$, β is tangent to $\{0\} \times J$. Therefore f_{r_1} and f_{r_2} are locally topologically $\mathcal{R}\text{-}\mathcal{IL}_0$ equivalent with local homeomorphisms $\sigma \colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\tau \colon (\mathrm{Im}(f), 0) \to (\mathrm{Im}(g), 0)$ such that τ is the identity mapping on $\mathrm{Im}(f)$. It follows from Remark 4.10 that f_{r_1} and f_{r_2} are locally topologically $\mathcal{R}_0$ equivalent. $\qquad\square$

Remark 4.13 Let us regard $P_0(n, 1, d; \mathbb{R})$ as a family of polynomials, indexed by $r \in \mathbb{R}^L$. From the proof of Theorem 4.3 using Thom's 2nd Isotopy Lemma, we can see that a finiteness result for the local $\mathcal{RL}$ triviality of the family $P_0(n, 1, d; \mathbb{R})$ holds. On the other hand, we cannot see from the proof of the above theorem whether a finiteness result for the local $\mathcal{R}$ triviality of the family $P_0(n, 1, d; \mathbb{R})$ holds or not.

In the proof of Theorem 1.3 we could assume that

$$\overline{\{(f_r(S(f_r)), r) \in \mathbb{R} \times Q_i \mid f_r \in P_0(n, 1, d; \mathbb{R})\} \backslash \{0\} \times Q_i} \cap \{0\} \times Q_i = \varnothing.$$

Therefore there exists a positive constant $\epsilon_0 > 0$ such that in Cases I, II and III, $[-\epsilon_0, \epsilon_0] \times J$, $[0, \epsilon_0] \times J$ and $[-\epsilon_0, 0] \times J$, respectively, is contained in

$$B_i \cap \mathbb{R} \times J \backslash \overline{\{(f_s(S(f_s)), s) \in \mathbb{R} \times J \mid f_s \in P_0(n, 1, d; \mathbb{R})\} \backslash \{0\} \times J},$$

where the closure is taken in $\mathbb{R} \times J$. Remember that $\overline{J} \subset Q_i$, where the closure is taken in $\mathbb{R}^L$. In general, even if we take a positive constant $\epsilon_0 > 0$ arbitrarily small, in Cases I, II and III, $[-\epsilon_0, \epsilon_0] \times Q_i$, $[0, \epsilon_0] \times Q_i$ and $[-\epsilon_0, 0] \times Q_i$, respectively, may not be contained in

$$B_i \backslash \overline{\{(f_r(S(f_r)), r) \in \mathbb{R} \times Q_i \mid f_r \in P_0(n, 1, d; \mathbb{R})\} \backslash \{0\} \times Q_i}.$$

In addition, taking a finite subdivision of Q_i into Nash open simplices, we cannot avoid this situation. Therefore we cannot always see a local $\mathcal{R}$ triviality over Q_i, using a similar argument to the case of local $\mathcal{R}$ equivalence.

Example 4.14 Let $I = [0, 1]$ be a closed interval, let $g_t \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $t \in I$, be a polynomial defined by $g_t(x) := t^2 - (x - t)^2$, and let

$$\mathcal{G} := \{g_t \in P_0(1, 1, 2; \mathbb{R}) \mid t \in I\}.$$

Then the number of local topological $\mathcal{R}$ types appearing in $\mathcal{G}$ is 2, but the global topological $\mathcal{R}$ types appearing in $\mathcal{G}$ as global polynomial functions have the cardinal number of the continuum. Note this family is $\mathcal{R}$ equivalent in $P(n, 1, 2; \mathbb{R})$ with the family in Example 4.5.

By [24, Fact 2.1] ([23, Theorem II.7.1]), we have the following fact in the case of semialgebraic equivalence.

Theorem 4.15 *Let $X \subset Y \subset \mathbb{R}^n$ be semialgebraic sets, and let $f, g: Y \to \mathbb{R}$ be semialgebraic functions with $f^{-1}(0) = g^{-1}(0) = X$. If the germs of f and g at X are semialgebraically $\mathcal{RL}$ equivalent, then the germs of f and g are semialgebraically $\mathcal{R}$ equivalent or the germs of f and $-g$ are so. Here we can choose the semialgebraic homeomorphisms of equivalence to be the identity mapping on X.*

As a corollary of this theorem, we have the following result.

Corollary 4.16 *Let $f, g \in P_0(n, 1, d; \mathbb{R})$. If f and g are locally semialgebraically $\mathcal{R}_0 \mathcal{L}_0$ equivalent, then f and g are locally semialgebraically $\mathcal{R}_0$ equivalent or f and $-g$ are locally semialgebraically $\mathcal{R}_0$ equivalent.*

Proof Suppose that $f, g \in P_0(n, 1, d; \mathbb{R})$ are locally semialgebraically $\mathcal{R}_0 \mathcal{L}_0$ equivalent. Therefore there exist semialgebraic homeomorphisms $\sigma: U \to V$ with $\sigma(0) = 0$ and $\tau: P \to Q$ with $\tau(0) = 0$ such that $\tau \circ f \circ \sigma = g$, where U and V (respectively P and Q) are open semialgebraic neighbourhoods of 0 in $\mathbb{R}^n$ (respectively 0 in $\mathbb{R}$). Set $\tilde{f} := f \circ \sigma$ and $\tilde{g} := g|_U$. Then $\tilde{f}, \tilde{g}: U \to \mathbb{R}$ are semialgebraic functions, and $\tilde{f}^{-1}(0) = \tilde{g}^{-1}(0)$ denoted by X after this. In addition, $\tilde{f}$ and $\tilde{g}$ are locally semialgebraically $\mathcal{R}_0 \mathcal{L}_0$ equivalent. Here the source homeomorphism of the local semialgebraic $\mathcal{R}_0 \mathcal{L}_0$ equivalence is the identity mapping on U. Therefore $\tilde{f}$ and $\tilde{g}$ are locally semialgebraically $\mathcal{R}_0 \mathcal{L}_0$ equivalent as function germs at $X \subset U$. By Theorem 4.15, $\tilde{f}$ and $\tilde{g}$ are semialgebraically $\mathcal{R}_0$ equivalent or $\tilde{f}$ and $-\tilde{g}$ are semialgebraically $\mathcal{R}_0$ equivalent as function germs at X. Since we can choose the semialgebraic homeomorphisms of equivalence to be the identity mapping on X, it follows that f and g are semialgebraically $\mathcal{R}_0$ equivalent or f and $-g$ are semialgebraically $\mathcal{R}_0$ equivalent as function germs at $0 \in \mathbb{R}^n$. □

Combining this corollary with Theorem 4.6, we have the following finiteness theorem in the semialgebraic case.

Theorem 1.4 *The number of local semialgebraic $\mathcal{R}_0$ types appearing in $P_0(n, 1, d; \mathbb{R})$ is finite.*

We may call the above result *Shiota's local semialgebraic $\mathcal{R}_0$ finiteness theorem.*

At the end of this subsection, we make an interesting remark on $\mathcal{R}_0$ equivalence of polynomial function germs, which was shown by Shiota.

Remark 4.17 Shiota proved that there exist two homogeneous polynomials $f, g:$ $(\mathbb{R}^7, 0) \to (\mathbb{R}, 0)$ of the same degree with isolated singularities which are locally topologically $\mathcal{R}_0$ equivalent but not locally semialgebraically $\mathcal{R}_0$ equivalent. (See [23, Example II.7.9] or [24, Fact 2.3]).

5 A generalisation

In this section we treat a generalisation of the Fukuda–Benedetti–Shiota theorem. We can read Fukuda's finiteness theorem (respectively Benedetti–Shiota's one) from its proof as a finiteness result on topological $\mathcal{RL}$ triviality (respectively semialgebraic $\mathcal{RL}$ one) for a polynomial family of polynomial functions as follows:

Theorem 5.1 *Given a polynomial family of polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$, there exists a finite subdivision of the parameter space into Nash open simplices such that over each Nash open simplex the family of polynomial functions is topologically $\mathcal{RL}$ trivial* (*respectively semialgebraically $\mathcal{RL}$ trivial*).

We define the notions of *topological $\mathcal{RL}$ equivalence* and *semialgebraic $\mathcal{RL}$ equivalence* for Nash functions defined on a Nash manifold M in a similar way to the above polynomial case with $M = \mathbb{R}^n$. Then we can generalise Theorem 5.1 to the following form.

Theorem 1.5 *Let $\{f_t \colon M \to \mathbb{R} \mid t \in J\}$ be a Nash family of Nash functions defined on a Nash manifold M with a semialgebraic parameter space J. Namely, the function $F \colon M \times J \to \mathbb{R}$ defind by $F(x, t) := f_t(x)$ is a Nash function. Then there exists a finite subdivision of J into Nash open simplices $J = Q_1 \cup \cdots \cup Q_u$ such that*

$$\{f_t \colon M \to \mathbb{R} \mid t \in Q_i\}$$

is semialgebraically $\mathcal{RL}$ trivial over each Q_i.

Remark 5.2 In the case where $M = \mathbb{R}^n$, Bochnak, Coste and Roy showed in [3] a finiteness theorem for semialgebraic triviality of a semialgebraic family of semialgebraic functions defined on $\mathbb{R}^n$. This is a generalisation of the Fukuda–Benedetti–Shiota theorem from the polynomial function case to the semialgebraic function case. On the other hand, Theorem 1.5 is a generalisation of it to a finiteness result for semilagebraic triviality of a Nash family of Nash functions defined on a general Nash manifold.

Proof of Theorem 1.5 The Fukuda–Benedetti–Shiota theorem is an algebraic result. When we show the global triviality using the 2nd Thom's Isotopy Lemma (Theorem 3.2) or its semialgebraic version (Theorem 3.4), we can consider the projectivisation in order to get the properness of mappings. On the other hand, Theorem 1.5 is a generalisation of the Fukuda–Benedetti–Shiota theorem to the Nash case. We first reduce the Nash case to the algebraic one in the proof of Theorem 1.5. This reduction process follows from similar arguments to those discussed in [14, 15].

Let us consider the composition of the mapping $(F, \mathrm{id}_J) \colon M \times J \to \mathbb{R} \times J$ defined by $(F, \mathrm{id}_J)(x, t) := (f_t(x), t)$ and the canonical projection $q \colon \mathbb{R} \times J \to J$. Here we recall a useful result by Shiota.

Theorem 5.3 (Shiota [22]) *A Nash manifold is Nash diffeomorphic to a nonsingular, affine algebraic variety.*

By this theorem, we can replace the Nash manifold M with a nonsingular, affine algebraic variety W. On the other hand, by Theorem 2.2, J is a finite union of Nash open simplices, and a Nash open simplex is Nash diffeomorphic to some Euclidean space. Therefore, in order to show our finiteness result, we may assume from the beginning that J is an Euclidean space $\mathbb{R}^d$. Then the above composed mapping becomes the following:

$$W \times \mathbb{R}^d \xrightarrow{\;\Delta\;} \mathbb{R} \times \mathbb{R}^d \xrightarrow{\;q\;} \mathbb{R}^d.$$

Note that $\Delta \colon W \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ is a Nash family of Nash mappings between non-singular, affine algebraic varieties.

We next apply the Artin–Mazur theorem (Artin and Mazur [1], Coste, Ruiz and Shiota [5], Shiota [22]) for a Nash family of Nash mappings, modified to our situation. For an algebraic variety $X \subset \mathbb{R}^p$, we denote by $\mathrm{Reg}(X)$ the smooth part of X.

Theorem 5.4 (Artin–Mazur Theorem) *Let $W \subset \mathbb{R}^m$ be the above nonsingular algebraic variety, and let $\Delta \colon W \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ be the above Nash mapping. Then there exists a Nash mapping $H \colon W \times \mathbb{R}^d \to \mathbb{R}^b$ with the following property:*

Let $\tau \colon (W \times \mathbb{R}^d) \times (\mathbb{R} \times \mathbb{R}^d) \times \mathbb{R}^b \to W \times \mathbb{R}^d$ and $\pi \colon (W \times \mathbb{R}^d) \times (\mathbb{R} \times \mathbb{R}^d) \times \mathbb{R}^b \to \mathbb{R} \times \mathbb{R}^d$ be the canonical projections, let $G = (\Delta, H) \colon W \times \mathbb{R}^d \to (\mathbb{R} \times \mathbb{R}^d) \times \mathbb{R}^b$ be the Nash mapping defined by $G(x, t) = (\Delta(x, t), H(x, t))$, and let X be the Zariski closure of graph G. Then there is a union L of some connected components of X with $\dim L = \dim X$ and $L \subset \mathrm{Reg}(X)$ such that $\tau|_L \colon L \to W \times \mathbb{R}^d$ is a t-level preserving Nash diffeomorphism and $(\pi|_L) \circ (\tau|_L)^{-1} = \Delta$.

Therefore we can regard the Nash family of Nash mappings $\Delta \colon W \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ between nonsingular, algebraic varieties as the restriction of some canonical projection π to a union of connected components L of an algebraic variety X. In this way, we can reduce the Nash case to the algebraic case.

We next consider the projectivisation so that the mappings π and q between projectivisated spaces are proper. Let us denote by $\mathbb{R}P_{[r]}$ the product of r real projective lines $\mathbb{R}P^1 \times \cdots \times \mathbb{R}P^1$. We recall that W is an algebraic variety in $\mathbb{R}^m$. We denote by $\widehat{X}$ the projectivisation of X in $(\mathbb{R}P_{[m]} \times \mathbb{R}P_{[d]}) \times (\mathbb{R}P^1 \times \mathbb{R}P_{[d]}) \times \mathbb{R}P_{[b]}$. In this paper we regard $\mathbb{R}^r \subset \mathbb{R}P_{[r]}$ and $L \subset X \subset \widehat{X}$.

Let

$$\widehat{\pi} \colon (\mathbb{R}P_{[m]} \times \mathbb{R}P_{[d]}) \times (\mathbb{R}P^1 \times \mathbb{R}P_{[d]}) \times \mathbb{R}P_{[b]} \to \mathbb{R}P^1 \times \mathbb{R}P_{[d]}$$

and $\widehat{q} \colon \mathbb{R}P^1 \times \mathbb{R}P_{[d]} \to \mathbb{R}P_{[d]}$ be the canonical projections. We set $\widehat{\Pi} := \widehat{\pi}|_{\widehat{X}}$. Then we consider the composition of projections

$$(\mathbb{R}P_{[m]} \times \mathbb{R}P_{[d]}) \times (\mathbb{R}P^1 \times \mathbb{R}P_{[d]}) \times \mathbb{R}P_{[b]} \supset \widehat{X} \xrightarrow{\widehat{\Pi}} \mathbb{R}P^1 \times \mathbb{R}P_{[d]} \xrightarrow{\widehat{q}} \mathbb{R}P_{[d]}.$$

Note that $\widehat{X}$ is an algebraic variety and $\widehat{\Pi}$ is a family of polynomial functions defined on $\widehat{X}$ with the parameter space $\mathbb{R}P_{[d]}$. Using a similar argument to Fukuda [7, 8] on (a_{f_t})-regular stratification with Łojasiewicz's Semialgebraic Triangulation Theorem and Fukuda's Lemma for a stratified mapping, we have the following assertion.

Assertion 5.5 *There exist finite Whitney stratifications of Nash class $\mathcal{S}(\widehat{X})$ and $\mathcal{S}(\mathbb{R}P^1 \times \mathbb{R}P_{[d]})$ of $\widehat{X}$ and $\mathbb{R}P^1 \times \mathbb{R}P_{[d]}$, respectively, and a finite subdivision of $\mathbb{R}P_{[d]}$ into Nash open simplices $\mathbb{R}P_{[d]} = R_1 \cup R_2 \cup \cdots \cup R_e$ which satisfy the following:*

(1) *Let $\mathcal{S}(\mathbb{R}P_{[d]}) = \{R_1, R_2, \ldots, R_e\}$. Then the projections*

$$\widehat{\Pi} \colon (\widehat{X}, \mathcal{S}(\widehat{X})) \to (\mathbb{R}P^1 \times \mathbb{R}P_{[d]}, \mathcal{S}(\mathbb{R}P^1 \times \mathbb{R}P_{[d]})),$$
$$\widehat{q} \colon (\mathbb{R}P^1 \times \mathbb{R}P_{[d]}, \mathcal{S}(\mathbb{R}P^1 \times \mathbb{R}P_{[d]})) \to (\mathbb{R}P_{[d]}, \mathcal{S}(\mathbb{R}P_{[d]}))$$

are proper stratified mappings.

(2) *For any $t \in R_i$, $1 \leqslant i \leqslant e$, the stratified mapping*

$$\widehat{\Pi}_t \colon (\widehat{X}_t, \mathcal{S}(\widehat{X})_t) \to \big((\mathbb{R}P^1 \times \mathbb{R}P_{[d]})_t, \mathcal{S}(\mathbb{R}P^1 \times \mathbb{R}P_{[d]})_t\big)$$

is $(a_{\widehat{\Pi}_t})$-regular.

(3) *The stratification $\mathcal{S}(\widehat{X})$ is compatible with $L \subset X$ and $(q \circ \Pi)^{-1}(R_i)$, $1 \leqslant i \leqslant e$, and the stratification $\mathcal{S}(\mathbb{R}P^1 \times \mathbb{R}P_{[d]})$ is compatible with $\widehat{\Pi}(\widehat{X})$, $\widehat{\Pi}(L) = \pi(L)$, $\mathbb{R} \times \mathbb{R}^d$ and $q^{-1}(R_i) = R_i \times \mathbb{R}P_{[d]}$, $1 \leqslant i \leqslant e$.*

(4) *The stratification $\mathcal{S}(\mathbb{R}P_{[d]})$ is compatible with $\mathbb{R}^d$. (Therefore let us assume that $\mathbb{R}^d = R_1 \cup R_2 \cup \cdots \cup R_c$ for $c < e$ after this.)*

We set $\widehat{X}^{(i)} := \widehat{X} \cap (q \circ \widehat{\Pi})^{-1}(R_i)$, $\mathcal{S}(\widehat{X}^{(i)}) := \{(q \circ \widehat{\Pi})^{-1}(R_i) \cap W \mid W \in \mathcal{S}(\widehat{X})\}$, $\widehat{\Pi}^{(i)} := \widehat{\Pi}|_{\widehat{X}^{(i)}}$, $\mathcal{S}(\mathbb{R}P^1 \times R_i) := \{\widehat{q}^{-1}(R_i) \cap U \mid U \in \mathcal{S}(\mathbb{R}P^1 \times \mathbb{R}P_{[d]})\}$ and $\widehat{q}^{(i)} := \widehat{q}|_{\mathbb{R}P^1 \times R_i}$, $1 \leqslant i \leqslant c$. Let us apply Lemma 3.6 to the composition of stratified mappings

$$\big(\widehat{X}^{(i)}, \mathcal{S}(\widehat{X}^{(i)})\big) \xrightarrow{\widehat{\Pi}^{(i)}} \big(\mathbb{R}P^1 \times R_i, \mathcal{S}(\mathbb{R}P^1 \times R_i)\big) \xrightarrow{\widehat{q}^{(i)}} (R_i, \{R_i\}),$$

$1 \leqslant i \leqslant c$. Then, subdividing R_i into finitely many Nash open simplices if necessary, the stratified mapping

$$\widehat{\Pi}^{(i)} \colon \big(\widehat{X}^{(i)}, \mathcal{S}(\widehat{X}^{(i)})\big) \to \big(\mathbb{R}P^1 \times R_i, \mathcal{S}(\mathbb{R}P^1 \times R_i)\big)$$

is semialgebraically trivial over R_i, $1 \leqslant i \leqslant c$. Let $L_i := L \cap (q \circ \widehat{\Pi})^{-1}(R_i)$. By construction, $\mathcal{S}(\widehat{X}^{(i)})$ is compatible with L_i and $\mathcal{S}(\mathbb{R}P^1 \times R_i))$ is compatible with $\widehat{\Pi}(L_i) = \pi(L_i)$ and $\mathbb{R} \times R_i$, $1 \leqslant i \leqslant c$. Therefore $\pi|_{L_i} \colon L_i \to \mathbb{R} \times R_i$ is semialgebraically trivial over R_i, $1 \leqslant i \leqslant c$. It follows that there exists a finite subdivision of J into Nash open simplices $J = Q_1 \cup \cdots \cup Q_u$ such that

$$\{f_t \colon M \to \mathbb{R} \mid t \in Q_i\}$$

is semialgebraically $\mathcal{RL}$ trivial over each Q_i. This completes the proof of Theorem 1.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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