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RESEARCH ARTICLE

Normalizers of maximal tori and real forms of Lie groups

Anton A. Gerasimov[1,2] · Dmitrii R. Lebedev[1,3] · Sergey V. Oblezin[4]

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Abstract

Given a complex connected reductive Lie group G with a maximal torus $H \subset G$, Tits defined an extension W_G^{T} of the corresponding Weyl group W_G. The extended group is supplied with an embedding into the normalizer $N_G(H)$ such that W_G^{T} together with H generate $N_G(H)$. In this paper we propose an interpretation of the Tits classical construction in terms of the maximal split real form $G(\mathbb{R}) \subset G$, which leads to a simple topological description of W_G^{T}. We also consider a variation of the Tits construction associated with compact real form U of G. In this case we define an extension W_G^U of the Weyl group W_G, naturally embedded into the group extension $\widetilde{U} := U \rtimes \Gamma$ of the compact real form U by the Galois group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Generators of W_G^U are squared to identity as in the Weyl group W_G. However, the non-trivial action of Γ by outer automorphisms requires W_G^U to be a non-trivial extension of W_G. This gives a specific presentation of the maximal torus normalizer of the group extension

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✉ Sergey V. Oblezin
oblezin@gmail.com

Anton A. Gerasimov
anton.a.gerasimov@gmail.com

Dmitrii R. Lebedev
lebedev.dm@gmail.com

1 Laboratory for Quantum Field Theory and Information, Institute for Information Transmission Problems, RAS, Moscow, Russia 127994

2 Interdisciplinary Scientific Center J.-V. Poncelet (CNRS UMI 2615), Independent University of Moscow, Moscow, Russia

3 Moscow Center for Continuous Mathematical Education, Bol. Vlasyevsky per. 11, Moscow, Russia 119002

4 School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK

$\widetilde{U}$. Finally, we describe explicitly the adjoint action of W_G^{T} and W_G^U on the Lie algebra of G.

Keywords Weyl group · Tits extension · Real form of complex semisimple Lie group

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1 Introduction

In the standard approach to classification of complex semisimple Lie groups the problem is reduced to an equivalent problem of classification of root data. In other words, the root data (roots/co-roots and lattices of characters/co-characters of a maximal torus $H \subset G$) define the corresponding semisimple Lie group up to an isomorphism. Curtis, Wiederhold and Williams [5] demonstrated that for classification of compact connected semisimple Lie groups G it is enough to classify the normalizers $N_G(H)$ of maximal tori $H \subset G$. The normalizer provides information about the action of the Weyl group $W_G := N_G(H)/H$ on H, but this is not enough for classification, since one also needs the precise structure of the group extension of W_G by H. Thus, for the classification problem one might replace the original object, the semisimple Lie group G, by the group extension of the finite group W_G by the abelian Lie group H. One perspective to grasp this equivalence is to look at $N_G(H)$ as a kind of degeneration of G [5]. An apparently related but more conceptual approach is based on attempts to look at $N_G(H)$ as the Lie group G defined over some non-standard base field (akin to the mysterious field $\mathbb{F}_1$ "with one element" introduced by Tits [11], probably with regard to this subject). In this way an equivalence of the two classification problems for compact semisimple Lie groups and normalizers looks like a manifestation of a general principle (due to Claude Chevalley [4]), saying that a classification of semisimple algebraic groups should not essentially depend on the nature of the base algebraically closed field.

The reasoning above demands a more detailed study of the group extension structure on $N_G(H)$. The key fact is that this extension does not split in general [1, 5, 6, 9, 10]. To get a universal description of $N_G(H)$ one should look for a set-theoretic section of the projection $N_G(H) \to W_G$ generating an extension of W_G. Such construction was proposed by Demazure [6] and Tits [9], [10]. It may be naturally formulated in terms of the Tits extension W_G^{T} of Weyl group W_G by the subgroup of order 2 elements in H. This construction allows an explicit presentation of $N_G(H)$ by generators and relations.

Although the Tits construction is known for a long time, its forthright explanation involves scheme-theoretic arguments. Precisely, for a Chevalley group scheme $\mathscr{G}$ over $\mathbb{Z}$ the Tits extension is the group of $\mathbb{Z}$-points of the normalizer of the $\mathbb{Z}$-split torus of $\mathscr{G}$. In this paper we use set-theoretic arguments to explain the Tits construction in the case of complex reductive Lie groups (for recent discussion on Tits groups see e.g. [1, 7, 8]). After reminding general results on normalizers of maximal tori in Sect. 2 we revisit the Tits construction in Sect. 3. We stress that the Tits group construction is defined for the split real form $G(\mathbb{R}) \subset G$ of a complex semisimple group G. This enables us to present in Proposition 3.5 a simple purely topological description of the Tits extension of the Weyl group W_G (our considerations appear to be pretty close to the final section of [2]).

As a variation of the Tits construction we consider its analog for the maximal compact group $U \subset G$. Precisely, we define an extension W_G^U of W_G embedded into a semi-direct product $\widetilde{U} = U \rtimes \Gamma$ of the maximal compact subgroup $U \subset G$ and the Galois group $\Gamma = \mathrm{Gal}\,(\mathbb{C}/\mathbb{R})$ acting in the standard (non-twisted) way on U. In this case the natural generators of the corresponding extension W_G^U of the Weyl group are squared to identity (in contrast with the case of the Tits group), while the non-trivial extension of W_G arises from the action of Γ via on W_G^U by outer automorphisms. The main result of this paper (Theorem 4.5 in Sect. 4) describes the structure of maximal tori normalizers in the Galois group extension $\widetilde{U}$ of the compact connected semisimple Lie group U. Note that our construction is dealing with the extended group $\widetilde{U}$ and thus, it differs from the constructions given in [1, 7]. In Sect. 5 we calculate explicitly the adjoint action of the Tits group and of its unitary analog on the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,(G)$. This action, in contrast with the adjoint action on $\mathfrak{h} \subset \mathfrak{g}$, depends on the lift of W_G into G. Finally in Sect. 6 we provide details of the proof of Theorem 4.5.

2 Normalizers of maximal tori and Weyl groups

We start with recalling standard facts on normalizers of maximal tori and the associated Weyl groups. Let G be a complex connected semisimple Lie group, $H \subset G$ be a maximal torus and $N_G(H)$ be its normalizer in G. Then there is the following exact sequence:

$$1 \longrightarrow H \longrightarrow N_G(H) \xrightarrow{p} W_G \longrightarrow 1, \tag{2.1}$$

where p is the projection on the finite group $W_G := N_G(H)/H$, the Weyl group of G. The Weyl group W_G does not actually depend on the choice of $H \subset G$, and thus appears to be an invariant of G. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and let I be the set of vertices of the Dynkin diagram associated to G, where $|I| = \mathrm{Lie}(\mathfrak{g})$. Let $(\Delta, \Delta^\vee)$ be the root-coroot system corresponding to G, $\{\alpha_i, i \in I\}$ be a set of positive simple roots, and $\{\alpha_i^\vee, i \in I\}$ be the corresponding set of positive simple co-roots. Let $\|a_{ij}\|$, $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$, be the Cartan matrix of $(\Delta, \Delta^\vee)$. The Weyl group W_G has a simple presentation in terms of generators and relations. Precisely, W_G is generated by simple root reflections $\{s_i, i \in I\}$ subjected to

$$s_i^2 = 1, \tag{2.2}$$
$$\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}, \quad i \neq j, \quad i, j \in I, \tag{2.3}$$

where $m_{ij} = 2, 3, 4, 6$ for $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively. Equivalently these relations may be written in the Coxeter form:

$$s_i^2 = 1, \quad (s_i s_j)^{m_{ij}} = 1, \quad i \neq j, \qquad i, j \in I.$$

The exact sequence (2.1) defines the canonical action of W_G on H. Let $h_i \in \mathfrak{h} = \mathrm{Lie}\,(H)$ be the generators corresponding to the simple co-roots $\alpha_i^\vee$, then the W_G-action on $\mathfrak{h} \subset \mathfrak{g}$ and on its dual is as follows:

$$\begin{aligned} s_i(h_j) &= h_j - \langle \alpha_i, \alpha_j^\vee \rangle\, h_i = h_j - a_{ji} h_i, \\ s_i(\alpha_j) &= \alpha_j - \langle \alpha_j, \alpha_i^\vee \rangle\, \alpha_i = \alpha_j - a_{ij} \alpha_i. \end{aligned} \tag{2.4}$$

The exact sequence (2.1) does not split in general, i.e. $N_G(H)$ is not necessarily isomorphic to a semi-direct product of W_G and H. A delicate situation in this regard is described by the following result due to [1, 5].

Theorem 2.1 *Assume G is a simple complex Lie group and let $Z(G)$ be the center of G. Then the exact sequence (2.1) splits in the following and only the following cases:*

- *Type A_ℓ, $\ell \geqslant 1$, $\ell \neq 3$, such that $|Z(G)|$ is odd;*
- *Type B_ℓ, $\ell > 1$, for the adjoint form;*
- *Type D_ℓ, $\ell > 2$, for all forms except* Spin (2ℓ)*;*
- *Type G_2.*

Thus to have an explicit description of the normalizer $N_G(H)$ one should look for appropriate set-theoretic section of the projection map p in (2.1) generating some extension of W_G. In the following section we provide the construction of the resulting extension of the Weyl group by a finite group. Let us note that for a normal finite subgroup $Z \subset G$ one has: if (2.1) splits for G then it splits for G/Z as well. In the following, for simplicity, we consider only the case of simply-connected complex groups.

3 The Tits extension of Weyl group

To describe the extension (2.1) in terms of generators and relations Tits proposed the following extension W_G^{T} of the Weyl group W_G by a discrete group [9, 10] (closely related results were obtained by Demazure [6]).

Definition 3.1 Let $A = \|a_{ij}\|$ be the Cartan matrix corresponding to a semisimple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and let $m_{ij} = 2, 3, 4, 6$ for $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively. The Tits group W_G^{T} is an extension of the Weyl group W_G by an abelian group $\mathbb{Z}_2^{|I|}$ generated by $\{\tau_i, \theta_i, i \in I\}$ subjected to the following relations:

$$(\tau_i)^2 = \theta_i, \quad \theta_i\theta_j = \theta_j\theta_i, \quad \theta_i^2 = 1, \tag{3.1}$$

$$\tau_i\theta_j = \theta_i^{-a_{ji}}\theta_j\tau_i, \tag{3.2}$$

$$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}} = \underbrace{\tau_j\tau_i\tau_j\cdots}_{m_{ij}}, \quad i \neq j, \qquad i, j \in I, \tag{3.3}$$

where the abelian subgroup is generated by $\{\theta_i, i \in I\}$.

Let $\{h_i, e_i, f_i : i \in I\}$ be the Chevalley–Serre generators of the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,(G)$, satisfying the standard relations

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_j, \tag{3.4}$$

$$\mathrm{ad}_{e_i}^{1-a_{ij}}(e_j) = 0, \quad \mathrm{ad}_{f_i}^{1-a_{ij}}(f_j) = 0, \tag{3.5}$$

where $A = \|a_{ij}\|$ is the Cartan matrix, i.e. $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.

According to [2] (see also [9]) there exists a subset $\{\zeta_i, i \in I\} \subset H$ of canonical elements of order two satisfying the following relations:

$$s_i(\zeta_j) = \zeta_j \zeta_i^{-a_{ji}},$$

where s_i, $i \in I$, are the generators of the Weyl group W_G (2.2), (2.3).

Theorem 3.2 (Demazure–Tits) *Let W_G^{T} be the Tits group associated with the complex semisimple simply connected Lie group G, then the map*

$$\tau_i \mapsto \dot{s}_i := e^{f_i} e^{-e_i} e^{f_i}, \quad \theta_i \mapsto \zeta_i, \qquad i \in I, \tag{3.6}$$

defines an embedding of the Tits group W_G^{T} into $N_G(H)$ such that $p(W_G^{\mathrm{T}}) = W_G$ for the projection p in (2.1). In particular, the normalizer group $N_G(H)$ is generated by H and by the image of the Tits group, so that the following relations hold:

$$\mathrm{Ad}_{\dot{s}_i}(h) = s_i(h), \quad \textit{for all} \;\; h \in \mathfrak{h} = \mathrm{Lie}(H), \quad i \in I.$$

Example 3.3 In the standard faithful two-dimensional representation $\phi\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{End}\,(\mathbb{C}^2)$ given by (6.3) we have

$$\phi(\dot{s}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \phi(\dot{s})^2 = \phi(\zeta) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The appearance of the Tits extension W_G^{T} via a specific choice of a set-theoretic section looks a bit ad hoc. As it is mentioned in Introduction one may use a scheme-theoretic argument to support this particular choice of extension of W_G. In the following we propose a set-theoretic argument based on consideration of the split real form $G(\mathbb{R})$ of G to elucidate the construction of W_G^{T}. For the split real form $G(\mathbb{R}) \subset G$ there is an analog of (2.1):

$$1 \longrightarrow H(\mathbb{R}) \longrightarrow N_{G(\mathbb{R})}(H(\mathbb{R})) \xrightarrow{p} W_G \longrightarrow 1, \tag{3.7}$$

with the real split maximal torus given by the intersection

$$H(\mathbb{R}) = H \cap G(\mathbb{R}),$$

of the complex maximal torus with the split real subgroup. Note that the set-theoretic section of (3.7) defining W_G^{T} provides a set-theoretic section of (2.1) thus embedding W_G^{T} into $N_{G(\mathbb{R})}(H(\mathbb{R}))$. The group $H(\mathbb{R})$ allows the product decomposition

$$H(\mathbb{R}) = MA, \quad M := H(\mathbb{R}) \cap K,$$

where $K \subset G(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$, M is isomorphic to the group $\mathbb{Z}_2^{|I|}$ and $A \subset H(\mathbb{R})$ is the connected exponential group $A = \exp(\mathfrak{h}(\mathbb{R}))$ without torsion. Therefore, $H(\mathbb{R})$ is not connected and consists of $2^{|I|}$ components, and the group M may be identified with the discrete group of connected components of $H(\mathbb{R})$:

$$M = \pi_0(H(\mathbb{R})).$$

Considering the groups of connected components of the topological groups entering (3.7) we obtain the induced exact sequence

$$1 \longrightarrow \pi_0(H(\mathbb{R})) \longrightarrow \pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) \longrightarrow W_G \longrightarrow 1, \tag{3.8}$$

so that $|\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R})))| = |M| \cdot |W_G| = |W_G^{\mathrm{T}}|$. This provides a canonical extension of W_G by the abelian group M of order $2^{|I|}$.

Explicitly, the group of connected components may be identified with the quotients by the connected normal subgroup A

$$\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) \simeq N_{G(\mathbb{R})}(H(\mathbb{R}))/A,$$

and we have the exact sequence

$$1 \longrightarrow A \longrightarrow N_{G(\mathbb{R})}(H(\mathbb{R})) \longrightarrow \pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) \longrightarrow 1. \tag{3.9}$$

Lemma 3.4 *The exact sequence (3.9) splits and thus $\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R})))$ allows an embedding into $N_{G(\mathbb{R})}(H(\mathbb{R}))$.*

Proof The extension (3.9) is an instance of extensions of $\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R})))$ by A. Such extensions are classified by the group $H^2(\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))), A)$. The triviality of this group follows from the fact that $\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R})))$ is a finite group and A is an instance of abelian torsion-free group. By the standard cohomology argument (see e.g. [3, Chapter VI]), the second cohomology of any finite group with coefficients in a torsion-free abelian group is trivial. Thus the extension (3.9) is necessarily trivial and therefore the required embedding exists. □

Up to now we have constructed a canonical extension of W_G given by (3.8). It is easy to see that this extension is isomorphic to the Tits group.

Proposition 3.5 *The following isomorphism holds:*

$$\pi_0(N_{G(\mathbb{R})}(H(\mathbb{R}))) \simeq W_G^{\mathrm{T}}. \tag{3.10}$$

Proof Recall that the images $\dot{s}_i$, ζ_i, $i \in I$, of the Tits generators belong to the maximally split real subgroup $G(\mathbb{R}) \subset G$. Thus we have the embedding W_G^{T} into $N_{G(\mathbb{R})}(H(\mathbb{R}))$ providing section of the exact sequence

$$1 \longrightarrow A \longrightarrow N_{G(\mathbb{R})}(H(\mathbb{R})) \longrightarrow W_G^{\mathrm{T}} \longrightarrow 1.$$

Note that splitting of this exact sequence may be independently verified by using an argument similar to the one used in our proof of Lemma 3.4. Therefore the group $N_{G(\mathbb{R})}(H(\mathbb{R}))$ allows a representation as semidirect product $N_{G(\mathbb{R})}(H(\mathbb{R})) = A \rtimes W_G^{\mathrm{T}}$. By considering the connected components we deduce the assertion (3.10).□

Example 3.6 For maximal split form $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$ we have

$$H(\mathbb{R}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \, \lambda \in \mathbb{R}^* \right\}, \quad H(\mathbb{R}) = MA,$$
$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \, \lambda \in \mathbb{R}_+^* \right\}, \quad M = \{\pm \mathrm{Id}\}.$$

Elements $g \in N_{\mathrm{SL}_2(\mathbb{R})}(H(\mathbb{R}))$ are defined by the condition that for each $\lambda \in \mathbb{R}^*$ there exists $\tilde{\lambda} \in \mathbb{R}^*$ such that

$$g \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{\lambda} & 0 \\ 0 & \tilde{\lambda}^{-1} \end{pmatrix} g, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

One might check that the normalizer group $N_{\mathrm{SL}_2(\mathbb{R})}(H(\mathbb{R}))$ is a union of two components

$$N_{\mathrm{SL}_2}(H(\mathbb{R})) = N_1 \sqcup N_s,$$

where N_1 is a set of diagonal elements with $c = b = 0$, $ad = 1 \neq 0$, and N_s is the set of anti-diagonal elements with $a = d = 0$, $cb = -1$. Each of the co-sets N_1, N_s splits further into two connected components

$$N_1 = N_1^+ \sqcup N_1^-, \quad N_s = N_s^+ \sqcup N_s^-,$$

depending on the sign of the entries c, d in the last row of g.

The group $\pi_0(N_{\mathrm{SL}_2(\mathbb{R})}(H(\mathbb{R})))$ is isomorphic to the quotient of $N_{\mathrm{SL}_2(\mathbb{R})}(H(\mathbb{R}))$ by $A \simeq \mathbb{R}_+^*$ and consists of four elements corresponding to the connected components $N_1^{\pm}$, $N_s^{\pm}$ of the group $N_{\mathrm{SL}_2}(H(\mathbb{R}))$, allowing the following parameterization:

$$N_1^+ = A, \quad N_s^+ = \dot{s}A, \quad N_1^- = \theta A, \quad N_s^- = \theta \dot{s} A,$$

where

$$\dot{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta = (\dot{s})^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta \dot{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Clearly, the group $\pi_0(N_{\mathrm{SL}_2(\mathbb{R})}(H(\mathbb{R})))$ is generated by representatives of the connected components $N^{\pm}_{1,s}$, so indeed it is isomorphic to the order four cyclic group generated by $\dot{s}$, in concordance with (3.10).

4 Weyl group and Galois extension of the compact real form

As we have demonstrated in the previous section the Tits group extension W_G^{T} appears quite naturally if we consider the split real subgroup $G(\mathbb{R}) \subset G$. This motivates to look for analogs of the Tits construction associated with other real forms of G.

Let $\{h_i, e_i, f_i; i \in I\}$ be the Chevalley–Serre generators of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, satisfying (3.4), (3.5). We fix the split real structure by assuming the generators $\{h_i, -e_i, -f_i-; i \in I\}$ to be real. Let $\top\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan anti-involution, associated with the split real structure:

$$\top\colon e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad h_i \mapsto h_i, \quad i \in I. \tag{4.1}$$

Let $U \subset G$ be the connected compact real form of the Lie group G:

$$U = \{g \in G : g^{\dagger} g = 1\},$$

where $g \mapsto g^{\dagger}$ is the composition of the Cartan anti-involution (4.1) with the complex conjugation associated with the split real structure. The Galois group $\Gamma := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}_2$ of the extension $\mathbb{R} \subset \mathbb{C}$ is generated by γ, $\gamma^2 = 1$. The group Γ acts both on G and on U by the complex conjugation, so let us introduce the following semidirect products:

$$\widetilde{U} := U \rtimes \Gamma \subset \widetilde{G} := G \rtimes \Gamma. \tag{4.2}$$

Since the generators $\{e_i, f_i, h_i; i \in I\}$ are real, they are fixed by $\gamma \in \Gamma$. Let us note that the Γ-fixed subgroup of U is a maximal compact subgroup $K \subset G(\mathbb{R})$ of the split real form $G(\mathbb{R})$.

Definition 4.1 Let $\|a_{ij}\|$ be the Cartan matrix corresponding to a semisimple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Let W_G^U be a group generated by $\{\sigma_i, \bar{\sigma}_i; i \in I\}$ subjected to

$$\sigma_i^2 = \bar{\sigma}_i^2 = 1, \quad \sigma_i \bar{\sigma}_i = \bar{\sigma}_i \sigma_i, \tag{4.3}$$

$$\sigma_j \sigma_i \sigma_j = \bar{\sigma}_j \Pi^{1-a_{ji}}(\bar{\sigma}_i)\, \bar{\sigma}_j, \quad i \neq j, \quad i, j \in I, \tag{4.4}$$

$$\underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{m_{ij}} = \underbrace{\bar{\sigma}_j \bar{\sigma}_i \bar{\sigma}_j \cdots}_{m_{ij}}, \quad i \neq j, \quad i, j \in I, \tag{4.5}$$

where in (4.5), $m_{ij} = 2, 3, 4, 6$ for $a_{ij} a_{ji} = 0, 1, 2, 3$, respectively. Here Π is the involutive map of the generators given by $\Pi(\sigma_i) = \bar{\sigma}_i$, $\Pi(\bar{\sigma}_i) = \sigma_i$, $i \in I$.

Lemma 4.2 *For all $i, j \in I$ the following holds:*

$$(\sigma_j\bar{\sigma}_j)(\sigma_i\bar{\sigma}_i) = (\sigma_i\bar{\sigma}_i)(\sigma_j\bar{\sigma}_j). \tag{4.6}$$

Proof For both a_{ji} and a_{ij} odd, (4.4) reads

$$(\sigma_j\bar{\sigma}_j)\,\sigma_i\,(\sigma_j\bar{\sigma}_j)^{-1} = \bar{\sigma}_i, \quad (\sigma_j\bar{\sigma}_j)\,\bar{\sigma}_i\,(\sigma_j\bar{\sigma}_j)^{-1} = \sigma_i,$$

which yields $(\sigma_j\bar{\sigma}_j)(\sigma_i\bar{\sigma}_i) = (\sigma_i\bar{\sigma}_i)(\sigma_j\bar{\sigma}_j)$.

For a_{ji} even, we have two cases: $a_{ji} = a_{ij} = 0$ and $a_{ji} = -2$, $a_{ij} = -1$. In the former case (4.5) gives $\sigma_i\sigma_j = \bar{\sigma}_j\bar{\sigma}_i$ and $\sigma_j\sigma_i = \bar{\sigma}_i\bar{\sigma}_j$, which implies

$$(\sigma_j\bar{\sigma}_j)(\sigma_i\bar{\sigma}_i) = \sigma_j\,(\bar{\sigma}_j\bar{\sigma}_i)\,\sigma_i = (\sigma_j\sigma_i)(\sigma_j\sigma_i) = \bar{\sigma}_i\,(\bar{\sigma}_j\bar{\sigma}_i)\,\bar{\sigma}_j = (\bar{\sigma}_i\sigma_i)(\sigma_j\bar{\sigma}_j).$$

In the case $a_{ji} = -2$, $a_{ij} = -1$ (4.4) gives $(\sigma_j\bar{\sigma}_j)\,\sigma_i = \sigma_i(\sigma_j\bar{\sigma}_j)$ and $(\sigma_i\bar{\sigma}_i)\,\sigma_j = \bar{\sigma}_j(\sigma_i\bar{\sigma}_i)$, so that the latter equality entails $\bar{\sigma}_j(\sigma_i\bar{\sigma}_i) = (\sigma_i\bar{\sigma}_i)\,\sigma_j$. Thus we have

$$(\sigma_i\bar{\sigma}_i)(\sigma_j\bar{\sigma}_j) = \bar{\sigma}_j(\sigma_i\bar{\sigma}_i)\,\bar{\sigma}_j = (\sigma_j\bar{\sigma}_j)(\sigma_i\bar{\sigma}_i).$$

This completes our proof. □

Lemma 4.3 *The map Π acting on the generators by*

$$\Pi(\sigma_i) = \bar{\sigma}_i, \quad \Pi(\bar{\sigma}_i) = \sigma_i,$$

extends to an involutive automorphism of the group W_G^U.

Proof Clearly, the relations (4.3), (4.5) are invariant under the action of Π. The relation (4.4) transforms under the Π-action into

$$\bar{\sigma}_j\bar{\sigma}_i\bar{\sigma}_j = \sigma_j\,\Pi^{1-a_{ji}}(\sigma_i)\,\sigma_j, \quad i \neq j, \quad i, j \in I,$$

which may be equivalently written as follows:

$$(\sigma_j\bar{\sigma}_j)\,\bar{\sigma}_i\,(\sigma_j\bar{\sigma}_j)^{-1} = \Pi^{-a_{ji}}(\bar{\sigma}_i), \quad i \neq j, \quad i, j \in I. \tag{4.7}$$

For a_{ji} odd (4.7) reads

$$(\sigma_j\bar{\sigma}_j)\,\bar{\sigma}_i\,(\sigma_j\bar{\sigma}_j)^{-1} = \sigma_i,$$

which follows from (4.4) in the form $(\sigma_j\bar{\sigma}_j)\,\sigma_i\,(\sigma_j\bar{\sigma}_j)^{-1} = \bar{\sigma}_i$.

For a_{ji} even we have to prove (4.7), which reads

$$(\sigma_j\bar{\sigma}_j)\,\bar{\sigma}_i\,(\sigma_j\bar{\sigma}_j)^{-1} = \bar{\sigma}_i.$$

This follows from (4.4), $(\sigma_j\bar{\sigma}_j)\,\sigma_i\,(\sigma_j\bar{\sigma}_j)^{-1} = \sigma_i$, by multiplying both sides by $\sigma_i\bar{\sigma}_i$ and using (4.6). □

Proposition 4.4 *The group W_G^U is given by the following extension:*

$$1 \longrightarrow \mathbb{Z}_2^{|I|} \longrightarrow W_G^U \longrightarrow W_G \longrightarrow 1, \tag{4.8}$$

of the Weyl group W_G by the abelian group $\mathbb{Z}_2^{|I|} \subset W_G^U$.

Proof Introduce the elements $\eta_i := \sigma_i \bar{\sigma}_i$, $i \in I$. They have order two, and they pairwise commute by (4.6):

$$\eta_i^2 = 1, \quad \eta_i \eta_j = \eta_j \eta_i, \quad i, j \in I,$$

and are invariant under involution Π. Consider a Π-stable subgroup $H_\eta \subset W_G^U$ generated by $\{\eta_i, i \in I\}$. The relation (4.4) may be equivalently written in the following form:

$$\sigma_i \eta_j = \eta_j \eta_i^{-a_{ji}} \sigma_i, \quad \bar{\sigma}_i \eta_j = \eta_j \eta_i^{-a_{ji}} \bar{\sigma}_i. \tag{4.9}$$

Indeed, from (4.4) we have $\sigma_i \eta_j = \eta_j \Pi^{1-a_{ji}}(\bar{\sigma}_i)$, which for a_{ji} even reads $\sigma_i \eta_j = \eta_j \sigma_i$ and for a_{ji} odd (4.4) is equivalent to $\sigma_i \eta_j = \eta_j \bar{\sigma}_i = \eta_j \eta_i \sigma_i$. This implies the first equation in (4.9). The second relation in (4.9) is obtained by applying the automorphism Π to the first one. The identities (4.9) yield that the subgroup $H_\eta \subset W_G^U$ generated by η_i, $i \in I$, is normal. A proof of the fact that $|H_\eta| = 2^{|I|}$ is given in Lemma 6.4 below.

Next, we consider the quotient group $W_G^U/\mathbb{Z}_2^{|I|}$. It is generated by $s_i := \pi(\sigma_i), i \in I$, and satisfying the standard relations (2.2), (2.3) of the group W_G. Indeed, $\pi(\sigma_i) = \pi(\bar{\sigma}_i)$ implies that the relations (4.3) are mapped to the relations (2.2), relations (4.4) equivalent to (4.9) become identities, and the braid relations (4.5) of W_G^U are mapped to the braid relations (2.3) of W_G. Thus, we have a surjective homomorphism π

$$\pi : W_G^U \to W_G, \quad \eta_i \mapsto 1, \quad i \in I.$$

This gives the exact sequence (4.8). □

The following analog of Theorem 3.2 holds.

Theorem 4.5 *Let $U \subset G$ be a maximal compact subgroup of the complex semisimple simply connected Lie group G. Let (Δ, W_G) be the root system of $\mathfrak{g} = \mathrm{Lie}(G)$ with the Cartan matrix $\|a_{ij}\|$. Let γ be the generator of the Galois group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of the field extension $\mathbb{R} \subset \mathbb{C}$ and let $\imath \in \mathbb{C}$ be the imaginary unit. Then the following map:*

$$\sigma_i \mapsto \varsigma_i := e^{\frac{\imath\pi}{2}(e_i + f_i)} \gamma, \quad \bar{\sigma}_i \mapsto \bar{\varsigma}_i := e^{-\frac{\imath\pi}{2}(e_i + f_i)} \gamma, \qquad i \in I, \tag{4.10}$$

defines an injective homomorphism $W_G^U \to \widetilde{U}$, with $\widetilde{U}$ given by (4.2). The elements ς_i, $i \in I$, and γ together with the maximal torus H generate the group $N_G(H) \rtimes \Gamma$.

We give a proof of Theorem 4.5 in Sect. 6 below.

Let us stress a clear analogy between the constructions of W_G^{T} and W_G^U. On the one hand, in the Tits setting the finite group is embedded into the maximal compact subgroup $K \subset G(\mathbb{R})$ of the maximally split real form $G(\mathbb{R}) \subset G$. On the other hand, the extension W_G^U constructed above is embedded into Galois group extension $\widetilde{U}$ of maximal compact subgroup $U \subset G$ (note that the action of complex conjugation on U may be equivalently represented by the action of the Cartan anti-involution (4.1)). In this way $\widetilde{U}$ plays the role analogous to $K \subset G(\mathbb{R})$ in the Tits construction, while W_G^U looks like a "complex" analog of the finite group W_G^{T}, where the relations $\tau_i^2 = \theta_i$ are replaced by $\sigma_i \bar{\sigma}_i = \eta_i$. Let us note that for each $i \in I$ we have the cyclic subgroup $\langle \dot{s}_i : (\dot{s}_i)^4 = 1 \rangle = \mathbb{Z}_4 \subset W_G^{\mathrm{T}}$ in the Tits construction. Meanwhile in the case of W_G^U for each $i \in I$ the group $\langle \sigma_i, \bar{\sigma}_i : \sigma_i^2 = \bar{\sigma}_i^2 = (\sigma_i \bar{\sigma}_i)^2 = 1 \rangle = (\mathbb{Z}_2)^2 \subset W_G^U$ appears. Note that these two instances exhaust possible extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. It is natural to expect that with other real forms of the complex group G one can associate appropriate extensions of the Weyl group W_G. These extensions presumably would be combinations of both constructions considered above.

5 Adjoint action of the extended Weyl groups

While the action of W_G on the maximal commutative subalgebra $\mathfrak{h} = \mathrm{Lie}(H)$ is defined canonically (2.4) and does not depend on a lift of W_G into $N_G(H)$ its action on the whole Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ does depend on the lift. Above we have considered two extensions of the Weyl group W_G together with their homomorphisms into the corresponding Lie group. Here we describe their induced adjoint actions on $\mathfrak{g}$.

Proposition 5.1 *The adjoint action of the Tits group W_G^{T} on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ via homomorphism (3.6) is given by*

$$\mathrm{Ad}_{\dot{s}_i}(e_i) = -f_i, \quad \mathrm{Ad}_{\dot{s}_i}(f_i) = -e_i, \tag{5.1}$$

$$\mathrm{Ad}_{\dot{s}_i}(e_j) = e_j, \qquad \mathrm{Ad}_{\dot{s}_i}(f_j) = f_j, \quad a_{ij} = 0, \tag{5.2}$$

$$\begin{aligned} \mathrm{Ad}_{\dot{s}_i}(e_j) &= \frac{(-1)^{|a_{ij}|}}{|a_{ij}|!} \big[\underbrace{e_i, [\ldots [e_i,}_{|a_{ij}|}\ e_j] \ldots]\big], \\ \mathrm{Ad}_{\dot{s}_i}(f_j) &= \frac{1}{|a_{ij}|!} \big[\underbrace{f_i, [\ldots [f_i,}_{|a_{ij}|}\ f_j] \ldots]\big], \quad i \neq j. \end{aligned} \tag{5.3}$$

Proof Relations (5.1) are actually relations for $\mathfrak{sl}_2$ Lie subalgebras generated by $\{e_i, h_i, f_i, i \in I\}$ and may easily be checked using for example the standard faithful representation (6.3). Relations (5.2) trivially follow from the Lie algebra relations (3.4). Thus we need to prove (5.3). Let us introduce the following notation: $\dot{s}_i(a) := \mathrm{Ad}_{\dot{s}_i}(a)$. Then for the conjugated generators we have

$$\begin{aligned} [h_k, \dot{s}_i(e_j)] &= \dot{s}_i([h_{s_i(k)}, e_j]) = \langle s_i(\alpha_k^\vee), \alpha_j \rangle\, \dot{s}_i(e_j) = (a_{kj} - a_{ki} a_{ij})\, \dot{s}_i(e_j), \\ [h_k, \dot{s}_i(f_j)] &= \dot{s}_i([h_{s_i(k)}, f_j]) = -\langle s_i(\alpha_k^\vee), \alpha_j \rangle\, \dot{s}_i(f_j) = -(a_{kj} - a_{ki} a_{ij})\, \dot{s}_i(f_j). \end{aligned}$$

These relations fix the r.h.s. of (5.3) up to coefficients. Let us calculate the coefficients by taking into account only the terms of the right weights. We have

$$\dot{s}_i(e_j) = e^{f_i} e^{-e_i} e^{f_i}\, e_j\, e^{-f_i} e^{e_i} e^{-f_i} = \frac{(-1)^{|a_{ij}|}}{|a_{ij}|!}\left(\mathrm{Ad}^{|a_{ij}|}_{e^{f_i} e_i e^{-f_i}}(e_j)\right) + \cdots ,$$

where we have used the Serre relations (3.5) and denote by $\cdots$ the terms of the "wrong" weight. Taking into account

$$\mathrm{Ad}_{e^{f_i}}(e_i) = e_i + \cdots ,$$

we obtain the first relation in (5.3). The second relation is obtained quite similarly using the following equality (for a proof see Lemma 6.1):

$$e^{f_i} e^{-e_i} e^{f_i} = e^{-e_i} e^{f_i} e^{-e_i}.$$

In this case we have

$$\dot{s}_i(f_j) = e^{-e_i} e^{f_i} e^{-e_i}\, f_j\, e^{e_i} e^{-f_i} e^{e_i} = \frac{1}{|a_{ij}|!}\left(\mathrm{Ad}^{|a_{ij}|}_{e^{-e_i} f_i e^{e_i}}(f_j)\right) + \cdots .$$

Taking into account

$$\mathrm{Ad}_{e^{-e_i}}(f_i) = f_i + \cdots ,$$

we obtain the second relation in (5.3). □

Let us stress that there is a simple way to get rid of sign factors in (5.1) and (5.3). Define a new set of generators $\tilde{e}_i = -e_i$, $\tilde{f}_i = f_i$. Then we have

$$\begin{aligned}
&\mathrm{Ad}_{\dot{s}_i}(\tilde{e}_i) = \tilde{f}_i, \quad \mathrm{Ad}_{\dot{s}_i}(\tilde{f}_i) = \tilde{e}_i,\\
&\mathrm{Ad}_{\dot{s}_i}(\tilde{e}_j) = \tilde{e}_j, \quad \mathrm{Ad}_{\dot{s}_i}(\tilde{f}_j) = \tilde{f}_j, \quad a_{ij} = 0,\\
&\mathrm{Ad}_{\dot{s}_i}(\tilde{e}_j) = \frac{1}{|a_{ij}|!}\big[\underbrace{\tilde{e}_i, [\ldots[\tilde{e}_i,}_{|a_{ij}|}\, \tilde{e}_j]\ldots]\big],\\
&\mathrm{Ad}_{\dot{s}_i}(\tilde{f}_j) = \frac{1}{|a_{ij}|!}\big[\underbrace{\tilde{f}_i, [\ldots[\tilde{f}_i,}_{|a_{ij}|}\, \tilde{f}_j]\ldots]\big], \quad i \neq j.
\end{aligned}$$

Now we describe the action on $\mathfrak{g}$ of the Weyl group extension W_G^U introduced in Sect. 4. It is convenient to express it in terms of purely imaginary generators $\imath e_i, \imath f_i$, $i \in I$.

Proposition 5.2 *The elements of the group W_G^U act on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ via homomorphism (4.10) as follows:*

$$\mathrm{Ad}_{\varsigma_i}(\imath e_i) = -\imath f_i, \quad \mathrm{Ad}_{\varsigma_i}(\imath f_i) = -\imath e_i, \tag{5.4}$$

and

$$\mathrm{Ad}_{\varsigma_i}(\imath e_j) = -\imath e_j, \quad \mathrm{Ad}_{\varsigma_i}(\imath f_j) = -\imath f_j, \quad a_{ij} = 0, \tag{5.5}$$

$$\mathrm{Ad}_{\varsigma_i}(\imath e_j) = -\frac{1}{|a_{ij}|!}\,\underbrace{[\imath e_i, [\ldots[\imath e_i}_{|a_{ij}|},\, \imath e_j]\ldots]],$$

$$\mathrm{Ad}_{\varsigma_i}(\imath f_j) = -\frac{1}{|a_{ij}|!}\,\underbrace{[\imath f_i, [\ldots[\imath f_i}_{|a_{ij}|},\, \imath f_j]\ldots]], \quad i \neq j. \tag{5.6}$$

Proof Taking into account (3.4) we have

$$e^{\imath\pi t\,\mathrm{ad}_{h_i}}(e_j) = e_j e^{\imath\pi t a_{ij}}, \quad e^{\imath\pi t\,\mathrm{ad}_{h_i}}(f_j) = f_j e^{-\imath\pi t a_{ij}}.$$

Next, we use the following representation for ς_i (see Lemma 6.1 for details):

$$\varsigma_i = \dot{s}_i e^{\frac{\imath\pi}{2}h_i}\gamma, \quad i \in I,$$

and Proposition 5.1 to obtain (5.6) and (5.4). ◻

6 Proof of Theorem 4.5

We start with establishing an explicit relation between the generators ς_i, $\bar{\varsigma}_i$ (4.10) and the Tits generators $\dot{s}_i$.

Lemma 6.1 *For each $i \in I$ the following identities hold:*

$$\dot{s}_i := e^{f_i}e^{-e_i}e^{f_i} = e^{-e_i}e^{f_i}e^{-e_i} = e^{\frac{\imath\pi}{4}h_i}e^{\frac{\imath\pi}{2}(e_i+f_i)}e^{-\frac{\imath\pi}{4}h_i}, \quad \dot{s}_i^2 = e^{\imath\pi h_i}. \tag{6.1}$$

Thus the generators $\{\varsigma_i, \bar{\varsigma}_i, i \in I\}$ defined by (4.10) may be represented as follows:

$$\begin{aligned} \varsigma_i &= e^{-\frac{\imath\pi}{2}h_i}\dot{s}_i\gamma = e^{-\frac{\imath\pi}{4}h_i}\dot{s}_i\, e^{\frac{\imath\pi}{4}h_i}\gamma, \\ \bar{\varsigma}_i &= e^{\frac{\imath\pi}{2}h_i}\dot{s}_i\gamma = e^{\frac{\imath\pi}{4}h_i}\dot{s}_i e^{-\frac{\imath\pi}{4}h_i}\gamma, \\ \varsigma_i\bar{\varsigma}_i &= \dot{s}_i^2 = e^{\imath\pi h_i}. \end{aligned} \tag{6.2}$$

Proof The identities (6.1) follow from the corresponding relations in $\mathrm{SL}_2 \subset G$, using the standard faithful two-dimensional representation $\phi\colon \mathrm{SL}_2 \to \mathrm{End}(\mathbb{C}^2)$,

$$\phi(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \phi(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \phi(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3}$$

Direct calculations show that

$$\phi(\dot{s}) = \phi(e^f e^{-e} e^f) = \phi(e^{-e} e^f e^{-e}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\phi(\dot{s}^2) = \phi(e^{-e} e^f e^{-e} e^{-e} e^f e^{-e}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \phi(e^{\imath\pi h}),$$

$$\phi\big(e^{-\frac{\imath\pi}{4}h} \dot{s} e^{\frac{\imath\pi}{4}h}\big) = \phi\big(e^{\frac{\imath\pi}{2}(e+f)}\big) = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}, \quad \phi\big(e^{\frac{-\imath\pi}{2}(e+f)}\big) = \begin{pmatrix} 0 & -\imath \\ -\imath & 0 \end{pmatrix}.$$

Then (6.1), (6.2) follow from the faithfulness of ϕ. □

Lemma 6.2 *The following relations hold:*

$$\varsigma_i^2 = \bar{\varsigma}_i^2 = 1, \quad i \in I. \tag{6.4}$$

Proof Direct calculation gives

$$\begin{aligned} \varsigma_i^2 &= e^{-\frac{\imath\pi}{4}h_i} \dot{s}_i e^{\frac{\imath\pi}{4}h_i} \gamma e^{-\frac{\imath\pi}{4}h_i} \dot{s}_i e^{\frac{\imath\pi}{4}h_i} \gamma \\ &= e^{-\frac{\imath\pi}{4}h_i} \dot{s}_i e^{\frac{\imath\pi}{4}h_i} e^{\frac{\imath\pi}{4}h_i} \dot{s}_i e^{-\frac{\imath\pi}{4}h_i} \\ &= e^{-\frac{\imath\pi}{4}h_i} \dot{s}_i e^{\frac{\imath\pi}{2}h_i} \dot{s}_i e^{-\frac{\imath\pi}{4}h_i} \\ &= e^{-\frac{\imath\pi}{4}h_i} e^{-\frac{\imath\pi}{2}h_i} e^{-\frac{\imath\pi}{4}h_i} (\dot{s}_i)^2 = e^{-\imath\pi h_i} \cdot e^{\imath\pi h_i} = 1. \end{aligned}$$

The identity $\bar{\varsigma}_i^2 = 1$ follows from $\bar{\varsigma}_i = \gamma \varsigma_i \gamma$, $i \in I$. □

Now let us verify that the generators $\varsigma_i, \bar{\varsigma}_i$ and $\xi_i = \varsigma_i \bar{\varsigma}_i$, $i \in I$, satisfy the remaining defining relations (4.9), (4.5) for the group W_G^U:

$$\varsigma_i \xi_j = \xi_j \xi_i^{-a_{ji}} \varsigma_i, \quad \bar{\varsigma}_i \xi_j = \xi_j \xi_i^{-a_{ji}} \bar{\varsigma}_i, \tag{6.5}$$

and

$$\underbrace{\varsigma_i \varsigma_j \varsigma_i \cdots}_{m_{ij}} = \underbrace{\bar{\varsigma}_j \bar{\varsigma}_i \bar{\varsigma}_j \cdots}_{m_{ij}}, \quad i \neq j, \quad i, j \in I, \tag{6.6}$$

where $m_{ij} = 2, 3, 4, 6$ for $a_{ij} a_{ji} = 0, 1, 2, 3$, respectively.

The first identity in (6.5) follows from (6.2) and (3.2):

$$\varsigma_i \xi_j \varsigma_i^{-1} = e^{-\frac{\pi\imath}{2}h_i} \dot{s}_i \gamma e^{\pi\imath h_j} \gamma \dot{s}_i^{-1} e^{\frac{\pi\imath}{2}h_i} = e^{-\frac{\pi\imath}{2}h_i} e^{\pi\imath(h_j - a_{ji}h_i)} e^{\frac{\pi\imath}{2}h_i} = \xi_j \xi_i^{-a_{ji}}.$$

The other identity in (6.5) follows from $\bar{\varsigma}_i = \gamma \varsigma_i \gamma$, $i \in I$.

For the relation (6.6), on the left-hand side we have

$$\underbrace{\varsigma_i \varsigma_j \varsigma_i \cdots}_{m_{ij}} = \underbrace{\big(e^{-\frac{\pi\imath}{2}h_i} \dot{s}_i e^{\frac{\pi\imath}{2}h_j} \dot{s}_j \cdots\big)}_{m_{ij}} \gamma^{m_{ij}} = \exp \frac{\pi\imath}{2} \underbrace{(-h_i + s_i h_j - s_i s_j h_i + \ldots)}_{m_{ij}} \underbrace{(\dot{s}_i \dot{s}_j \cdots)}_{m_{ij}} \gamma^{m_{ij}},$$

and on the right-hand side:

$$\underbrace{\bar{\varsigma}_j\bar{\varsigma}_i\bar{\varsigma}_j\cdots}_{m_{ij}}=\underbrace{\left(e^{\frac{\pi\imath}{2}h_j}\dot{s}_je^{-\frac{\pi\imath}{2}h_i}\dot{s}_i\cdots\right)}_{m_{ij}}\gamma^{m_{ij}}=\exp\frac{\pi\imath}{2}\underbrace{(h_j-s_jh_i+s_js_ih_j-\ldots)}_{m_{ij}}\underbrace{(\dot{s}_j\dot{s}_i\cdots)}_{m_{ij}}\gamma^{m_{ij}}.$$

Since $\underbrace{\dot{s}_i\dot{s}_j\cdots}_{m_{ij}}=\underbrace{\dot{s}_j\dot{s}_i\cdots}_{m_{ij}}$ holds due to (3.3), the identity (6.6) reduces to the following:

$$\exp\frac{\pi\imath}{2}\underbrace{\big(-h_i+s_ih_j-s_is_jh_i+\ldots\big)}_{m_{ij}}=\exp\frac{\pi\imath}{2}\underbrace{\big(h_j-s_jh_i+s_js_ih_j-\ldots\big)}_{m_{ij}}.\quad(6.7)$$

In turn the identity (6.7) may be proved by invoking the following fact.

Lemma 6.3 *For each pair $i,j\in I$, $i\neq j$, the following holds:*

$$\underbrace{(1-s_j+s_is_j-\ldots)}_{m_{ij}}h_i=0.\quad(6.8)$$

Proof Consider the order $2m_{ij}$ Coxeter subgroup of W_G generated by a pair of the simple root reflections s_i,s_j, $i\neq j$:

$$\langle s_i,s_j:(s_is_j)^{m_{ij}}=s_i^2=s_j^2=1\rangle\subset W_G.$$

This group is isomorphic to the dihedral group $D_{m_{ij}}\subset O_2(\mathbb{R})$ of symmetries of m_{ij}-gone in the real plane $V_{ij}=\mathbb{R}h_i\oplus\mathbb{R}h_j$. The dihedral group may be equivalently written in the following form:

$$D_{m_{ij}}=\big\langle t,r:t^{m_{ij}}=r^2=1,rtr^{-1}=t^{-1}\big\rangle,\quad t=s_is_j,\quad r=s_i.$$

We have two projectors in the plane $V_{ij}=\mathbb{R}h_i\oplus\mathbb{R}h_j$:

$$P_\pm=\frac{1\pm s_i}{2}:V_{ij}\to V_{ij},\quad P_\pm^2=P_\pm,\quad P_\pm P_\mp=0,$$

such that

$$P_+h_i=0,\quad P_-h_i=h_i.$$

Therefore, the identity (6.8) is equivalent to the following:

$$\underbrace{(1-s_j+s_is_j-\cdots)}_{m_{ij}}(1-s_i)\cdot h_i=\sum_{g\in D_{m_{ij}}}(-1)^g g\cdot h_i=0,\quad(6.9)$$

where $(-1)^g := \det(g)$ is the sign character of $D_{m_{ij}} \subset O_2(\mathbb{R})$. The kernel of the sign character is a normal subgroup,

$$\mathbb{Z}_{m_{ij}} = \langle t : t^{m_{ij}} = 1 \rangle \subset D_{m_{ij}},$$

which consists of the rotations by $\frac{2\pi k}{m_{ij}}$, $0 \leqslant k < m_{ij}$ of the plane V_{ij}. The non-trivial co-set in $D_{m_{ij}}/\langle t \rangle$ consists of the reflections $\{r_k = t^k r : 0 \leqslant k < m_{ij}\}$ with $r_0 = r$ being a reflection sending h_i to $-h_i$. Thus we have $(-1)^{r_k} = \det(r_k) = -1$ and

$$D_{m_{ij}} = \left\{ t^k : 0 \leqslant k < m_{ij} \right\} \sqcup \left\{ r_k = r t^k : 0 \leqslant k < m_{ij} \right\},$$

hence the identity (6.9) reads

$$\sum_{g \in D_{m_{ij}}} (-1)^g g \cdot h_i = (1 - r) \sum_{k=0}^{m_{ij}-1} t^k \cdot h_i.$$

Now in the group algebra $\mathbb{C}[D_{m_{ij}}]$ the following identity holds:

$$t^{m_{ij}} - 1 = (t - 1) \sum_{k=0}^{m_{ij}-1} t^k = 0.$$

Since t acts in the faithful representation V_{ij} without fixed vectors, we infer that $\sum_{k=0}^{m_{ij}-1} t^k \cdot h_i = 0$ and thus prove (6.9). □

Lemma 6.4 *The elements $\eta_i = \sigma_i \bar{\sigma}_i$, $i \in I$, generate a subgroup $H_\eta \subset W_G^U$ of order $|H_\eta| = 2^{|I|}$.*

Proof By (6.4), (6.5) and (6.6), the elements $\{\varsigma_i, \bar{\varsigma}_i, i \in I\} \subset \widetilde{U} = U \rtimes \Gamma$ satisfy the defining relations of the group W_G^U from Definition 4.1. Moreover, the images $\varsigma_i \bar{\varsigma}_i = e^{\imath \pi h_i} \in U$ of the elements $\eta_i = \sigma_i \bar{\sigma}_i$, $i \in I$, generate the subgroup $H_\eta \simeq H^{(2)}$ of order two points in the maximal torus $H \subset G$, so that the order of H_η should be not less then $2^{|I|}$ and thus $|H_\eta| = 2^{|I|}$ holds. □

We complete our proof of Theorem 4.5 by verifying injectivity of the homomorphism $\psi : W_G^U \to \widetilde{U}$. By Proposition (4.4), W_G^U has a structure of the group extension:

$$1 \longrightarrow \mathbb{Z}_2^{|I|} \longrightarrow W_G^U \longrightarrow W_G \longrightarrow 1. \tag{6.10}$$

Let $\mathscr{W}_G^U = \psi(W_G^U) \subset \widetilde{U}$, then the $\mathscr{W}_G^U$-action on $\mathfrak{h} = \mathrm{Lie}\,(H)$ implies the existence of the surjective homomorphism $\pi : \mathscr{W}_G^U \to W_G$. By Lemma 6.4, $\mathscr{W}_G^U$ contains a normal abelian subgroup generated by $\{\psi(\eta_i) = \varsigma_i \bar{\varsigma}_i = e^{\imath \pi h_i}, i \in I\} \subset U$, which is isomorphic to $H^{(2)} \simeq \mathbb{Z}_2^{|I|}$. Clearly, the normal abelian subgroup $H^{(2)}$ acts trivially on $\mathfrak{h}$, hence it is in the kernel of the surjective homomorphism π, which entails $|W_G^U| \leqslant$

$|\mathscr{W}_G^U|$. On the other hand, the existence of homomorphism $\psi : W_G^U \to \mathscr{W}_G^U$ implies that $|W_G^U| \geqslant |\mathscr{W}_G^U|$ and hence $|W_G^U| = |\mathscr{W}_G^U|$. Thus for $\mathscr{W}_G^U$ we have the following exact sequence:

$$1 \longrightarrow \mathbb{Z}_2^{|I|} \longrightarrow \mathscr{W}_G^U \longrightarrow W_G \longrightarrow 1.$$

Taking into account (6.10) this provides a proof of injectivity of ψ, and therefore, of Theorem 4.5.

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References

1. Adams, J., He, X.: Lifting of elements of Weyl groups. J. Algebra **485**, 142–165 (2017). arXiv:math/1608.00510 [math.RT]
2. Borel, A., Tits, J.: Groupes réductifs. Inst. Hautes Études Sci. Publ. Math. **27**, 55–150 (1965)
3. Brown, K.S.: Cohomology of Groups. Graduate Texts in Mathematics, vol. 87. Springer, New York (1982)
4. Chevalley, C.: Classification des Groupes Algébriques semi-simples. In: Cartier, P. (ed.) Collected Works, vol. 3. Springer, Berlin (2005)
5. Curtis, M., Wiederhold, A., Williams, B.: Normalizers of maximal tori. In: Hilton, P. (ed.) Localization in Group Theory and Homotopy Theory, and Related Topics. Lecture Notes in Mathematics, vol. 418, pp. 31–47. Springer, Berlin (1974)
6. Demazure, M.: Schémas en groupes réductifs. Bull. Soc. Math. France **93**, 369–413 (1965)
7. Dwyer, W.G., Wilkerson, C.W.: Normalizers of tori. Geom. Topol. **9**, 1337–1380 (2005)
8. Neumann, F.: A theorem of Tits, normalizers of maximal tori and fibrewise Bousfield–Kan completions. Publ. Res. Inst. Math. Sci. **35**(5), 711–723 (1999)
9. Tits, J.: Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples. Inst. Hautes Études Sci. Publ. Math. **31**, 21–55 (1966)
10. Tits, J.: Normalisateurs de tores. I. Groupes de Coxeter étendus. J. Algebra **4**, 96–116 (1966)
11. Tits, J.: Sur les analogues algébriques des groupes semi-simples complexes. In: Colloque d'Algèbre Supérieure. tenu à Bruxelles du 19 au 22 décembre 1956, pp. 261–289. Centre Belge de Recherches Mathématiques, Établissements Ceuterick, Louvain (1957)