Stable vector bundles on the families of curves

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Abstract

We offer a new approach to proving the Chen–Donaldson–Sun theorem which we demonstrate with a series of examples. We discuss the existence of a construction of a special metric on stable vector bundles over the surfaces formed by a family of curves and its relation to the one-dimensional cycles in the moduli space of stable bundles on curves.

Keywords Vector bundle · Stability · Families of curves

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1 Introduction

Stability of vector bundles on different geometric spaces has been an object of study for a long time deriving from work in algebraic geometry. The concept of stability arose naturally from the question of whether a space which could parameterize all bundles of a given rank exists, moreover, a subsequent question was if there is a special class of vector bundles which could be naturally parametrized in the same way. Significant progress in the direction has been made by Mumford [19,20], Maruyama [16–18],

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Narasimhan–Seshadri [21], Yau [25], Donaldson, Uhlenbeck–Yau [12, 13, 24], Hitchin [14], Corlette [10], Simpson [23] and others. The Donaldson–Uhlenbeck–Yau theorem relates Hermitian–Yang–Mills connections to an algebraic property of the holomorphic vector bundle of being stable.

There are several results concerning the existence of special metrics on stable bundles over complex projective manifolds. One of the first improvements in this question was made by Ruadhaí Dervan and Julius Ross, see [11] for more details. One of the recent ones is the Chen–Donaldson–Sun theorem [7–9] which claims the existence of a flat unitary metric on a stable bundle with Chern classes c_1, c_2 equal zero. It was obtained by considering a special Futaki-type functional on the space of Hermitian metrics and proving the existence of a metric with a minimal value of the functional. The resulting metric happens to be flat in case of $c_2 \geq 0$ which implied the result, see [7–9] for more details.

Recall that all Kähler metrics in the class $[\omega]$ can be parametrized by smooth functions in the space of Kähler potentials $\mathcal{H}_\omega = \{\varphi \in C^\infty(M) : \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0\}$. Then, for $\varphi \in \mathcal{H}_\omega$, the scalar curvature is $R_\varphi = -g_\varphi^{i\bar{j}} \partial_i \partial_{\bar{j}} \log(\det(g_{i\bar{j}} + \varphi_{i\bar{j}}))$ and the constant scalar curvature equation reads as $R_\varphi = R'$, where R' is a constant depending only on the manifold, class and complex structure $(M, [\omega], J)$.

Conjecture 1.1 (Yau–Tian–Donaldson Conjecture) (X, L) admits a CSMK metric if and only if it is K -stable.

The YTD conjecture is proven for Fano and tropical varieties. In this article, we show how one-dimensional cycles in the moduli space of vector bundles on curves are associated to bundles on families of curves which are stable on the restriction to every curve of the family. We present a connection between existence of a special metric on a class of surfaces formed by the families of curves and its' natural relation to one-dimensional cycles in the moduli space of stable bundles on curves.

Suppose that B is a family of curves, $B = \{C_\lambda\}_{\lambda \in \Lambda}$, where Λ is a parameter set. Assume that the vector bundle E on B has a stable restriction on any curve C_λ . If, in addition, the first Chern class of the restriction E_λ of E on any curve C_λ is trivial, then by [21] there is a flat Hermitian metric $h_{i,j}$ on E_λ which is unique modulo multiplication by a constant. Thus, it defines the unique flat Hermitian connection on E_λ and the unique modulo conjugation irreducible unitary representation $\rho_\lambda : \pi_1(C_\lambda) \rightarrow U(n)$.

The choice of the flat Hermitian metric $h_{i,j}$ on E_λ for any $\lambda \in B$ defines a Hermitian metric h on E over B . The metric h is not unique, but depends on a real-valued invertible function of Λ corresponding to the choice of the actual Hermitian metric on E_λ . Hence we obtain a family of such metrics on E and corresponding Hermitian connections Θ_h . The latter defines Chern forms representing all Chern classes of E . We are particularly interested in the local properties of the $(2, 2)$ -form on B representing the second Chern class $c_2(E)$ obtained from Θ_h when B is a two-dimensional complex surface. We can express the second Chern class via the curvature of the cycle $\mathcal{S}$ in the moduli space of vector bundles, which corresponds to a particular vector bundle E on the base B :

$$c_2(E) = \frac{\mathrm{tr}(\Theta(\mathcal{S})^2) - \mathrm{tr}^2(\Theta(\mathcal{S}))}{8\pi^2}.$$

The following result can be obtained by a direct calculation (see Sects. 6 and 7).

Theorem 1.2 *The second Chern class is represented locally by the formula*

$$c_2(E) = \sum_{i,j} -\frac{\det(D_{ij}) \det(\overline{D}_{ij})}{\det(h_{ij})} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2.$$

Corollary 1.3 *The form representing the second Chern class $c_2(E)$ on B is nonnegative, since $\frac{\det(D_{ij}) \det(\overline{D}_{ij})}{\det(h_{ij})} \geq 0$.*

Corollary 1.4 *Assume that $(-\det \partial^2 h_{i,j})$ is nonzero somewhere on B , then $c_2(E)$ is semipositive. In particular, if the base Λ is a compact curve and the representation ρ_λ varies modulo conjugation on Λ , then $c_2(E)$ has a representative which is a semipositive $(2, 2)$ -form.*

This fact yields a possibility for a constructive approach to the proof of the Chen–Donaldson–Sun theorem for a projective surface X . The theorem claims that $c_2(E) > 0$ for a stable vector bundle with $c_1(E) = 0$ unless it is induced from a unitary representation of $\pi_1(B)$. Our explicit local calculation implies this result for a compact surface $f: B \rightarrow C$ fibered with all smooth fibres $f^{-1}(c)$, $c \in C$, over a smooth curve and a bundle E which has stable restrictions on the fibers $f^{-1}(c)$.

Remark 1.5 We should note that Theorem 1.2 holds not only for stable bundles $E \rightarrow B$ on our class of surfaces (see [6] for more details). In particular, if the bundle E is not stable, but restricts to a stable one to a curve $C \subset B$, this result holds as well.

We also discuss other possible applications and generalizations of the above approach.

We should note that several results in a similar direction have recently appeared on arXiv [1,2,5]. We should also mention the work of O’Grady [22] in a similar direction.

2 Preliminary facts and background

2.1 Bogomolov’s restriction theorem, cones of polarizations and effective divisors

Let us recall a number of facts following the exposition presented in the book [15]. In the case of smooth projective surfaces we have a nondegenerate hyperbolic scalar product on the group $\text{Pic}_R V$ and duality between the cone of polarizations and the cone of effective divisors.

Let Num denote the free $\mathbf{Z}$ -module $\text{Pic}(X)$ factorized by the numerical equivalence. The intersection product defines an integral quadratic form on Num .

The Hodge Index Theorem says, that, over $\mathbf{R}$, the positive definite part is one-dimensional. In other words, $\text{Num}_{\mathbf{R}}$ carries the Minkowski metric. For any class $u \in \text{Num}_{\mathbf{R}}$ let $|u| = |u^2|^{1/2}$.

Denote as A a subcone consisting of all ample divisors. A polarization of X is a ray $\mathbf{R}_{>0}.H$, where $H \in A$. Let $\mathcal{H}$ denote the set of rays in K_+ . This set can be identified with the hyperbolic manifold $\{H \in K_+ : |H| = 1\}$.

The hyperbolic metric β is defined as follows: for points $[H], [H'] \in \mathcal{H}$ let

$$\beta([H], [H']) = \operatorname{arccosh} \frac{H \cdot H'}{|H| \cdot |H'|}.$$

Denote the open cone $K^+ = \{D \in \operatorname{Num}_R : D^2 > 0, D \cdot H > 0 \text{ for all ample divisors } H\}$. Note that the second condition is added only to pick one of the two connected components of the set of all D with $D^2 > 0$. This cone contains the cone of ample divisors and in turn is contained in the cone of effective divisors. K^+ satisfies the following property: $D \in K^+$ iff $D \cdot L > 0$ for all $L \in \overline{K^+} \setminus \{0\}$.

For any pair of sheaves G, G' with nonzero rank let

$$\xi_{G', G} := \frac{c_1(G')}{\operatorname{rk}(G')} - \frac{c_1(G)}{\operatorname{rk}(G)} \in \operatorname{Num}_R.$$

This invariant in the case of a destabilizing sheaf plays the role of a “destabilizing” point in the corresponding cone.

Theorem 2.1 *Let X be a smooth projective variety of dimension n and H an ample divisor on X . If F is a semistable torsion free sheaf, then*

$$\Delta(F) \cdot H^{n-1} \geq 0.$$

Theorem 2.2 (Bogomolov’s effective restriction ’78 [3,4]) *Let W be a locally free sheaf of rank $r \geq 2$ on an algebraic surface with $c_1(W) = 0$. Assume that W is μ -stable with respect to an ample class $H \in K^+ \cap \operatorname{Num}$ and $C \subset X$ be a smooth curve with $[C] = nH$. Let $2n \geq \frac{R}{r} \Delta(F) + 1$. Then $F|_C$ is a stable sheaf.*

A consequence of the theorem is that depending on c_1 and rank, the restriction of a vector bundle E to curves (in particular to the ones with a relatively small number of singularities) is stable.

3 Moduli section

Let us introduce a new object arising from every vector bundle on particular families. Suppose we have B , a family of curves, and $\mathcal{E}$, a vector bundle on it. We can say that $\mathcal{E}|_{C_\lambda}$ restricted to a point of the family, i.e. a curve C_λ , is a bundle on a Riemann surface with the corresponding unitary connection ∇ . The information on stability of $\mathcal{E}$ is contained in its second Chern class $c_2(\mathcal{E})$.

Definition 3.1 Every vector bundle $\mathcal{E}$ on B , which is stable on the restriction to any curve of the family B , is called a *smooth section of the family of moduli spaces* M_{C_λ} , $\lambda \in \Lambda$,

$$\mathcal{E} \in \Gamma(M_{C_\lambda})_{\lambda \in \Lambda}.$$

Motivation for the definition. The vector bundle $\mathcal{E}$ on the family of curves B on the restriction to the point $\mathcal{E}|_{C_{\lambda_0}}$, i.e. a particular curve C_{λ_0} , is a vector bundle, arising from the representation of the fundamental group of $\pi_1(C_{\lambda_0})$ (by the Narasimhan–Seshadri theorem). Therefore we can simply consider the following fibration on B : the fiber on every curve C_λ corresponds to its moduli space M_{C_λ} . Consequently, to “fix a vector bundle on the family B ” simply means that on each fiber, i.e. on the corresponding moduli space, we have to choose one element. It identically repeats the definition.

To supplement our definition we should note that stable bundles come from:

- Stable bundles on normalizations of the singular curves: $\mathcal{E} \rightarrow \widehat{C}$, where $\nu: \widehat{C} \rightarrow C$ is a normalization. One can get these bundles from representations of a fundamental group of the normalization $\pi_1(\widehat{C})$ and thus the destabilizing sheaf is not induced on the initial curve C .
- Bundles which are unstable on the normalization $\widehat{C}$, but are stable on the singular curve C (parabolic semistable vector bundles). They appear in the following way: we can take an unstable bundle $\widehat{\mathcal{E}}$ (i.e. with destabilizing subsheaf $\mathcal{F}$) on the normalization $\widehat{C}$. When we arrive to a base C it transforms to a point $\mathcal{E}^{\text{sing}}$. In such a way we obtain an identification of the parameters of the destabilizing sheaf $\mathcal{F}$ with the structure of the parabolic bundle on C . It means that on $\widehat{\mathcal{E}} \rightarrow \widehat{X}$, where $\widehat{\mathcal{E}}$ is an unstable bundle at the point on the normalization $\widehat{X}$ ($\widehat{X}$ denotes X blown up in a node of a singular curve C , $\nu: \widehat{C} \rightarrow C$). Then it has a normal destabilizing subsheaf $\mathcal{F} \subset \widehat{\mathcal{E}}$ such that $c_1(\mathcal{F}) - \left(\frac{\text{rk } \mathcal{F}}{k}\right)c_1(\mathcal{E}) \in K^+$.

A blow-down map $\widehat{\nu}$ induces a morphism

$$\widehat{\nu}: \mathcal{F} \rightarrow \mathcal{E}_p,$$

where $\mathcal{E}_p$ is a flag variety of type determined by a fixed quasiparabolic structure, i.e. a flag $\mathcal{E}_p = F_1\mathcal{E}_p \supset F_2\mathcal{E}_p \supset \cdots \supset F_r\mathcal{E}_p$ and weights $\alpha_1, \dots, \alpha_r$ attached to $F_1\mathcal{E}_p, \dots, F_r\mathcal{E}_p$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1$, $k_1 = \dim F_1\mathcal{E}_p - \dim F_2\mathcal{E}_p$, $k_r = \dim F_r\mathcal{E}_p$ the multiplicities of $\alpha_1, \dots, \alpha_r$.

These are parabolic bundles. Thus we get a strata, which is glued to our families of moduli spaces of vector bundles. It is known that the set of all parabolic semistable bundles is an open subset of a suitable Hilbert scheme, which has the usual properties, i.e. non-singular, irreducible and of a given dimension. This open set could be mapped to a product of Grassmannians and flag varieties.

Remark 3.2 If $C_\lambda = C$ is a constant curve, then all the moduli spaces are the same $M_{C_\lambda} = M_C = M$, thus every vector bundle on B is decoded by a subset $\mathcal{S} \subset M$ in the moduli space M :

$$\mathcal{S} = \{m_\lambda \in M : \lambda \in \Lambda\}.$$

Remark 3.3 The moduli space M_B of all the vector bundles on B consists of all the sections $\mathcal{S}$ of the family of the corresponding moduli spaces M_{C_λ} .

Remark 3.4 If $C_\lambda = C_1$ and $\Lambda = C_2$, we have a vector bundle on $C_1 \times C_2$. Then every subset $\mathcal{C}_1 \subset M_{C_2}$ or $\mathcal{C}_2 \subset M_{C_1}$ will form a cycle in the corresponding moduli space of vector bundles.

This way we have a correspondence between families of stable vector bundles and the sections of families of moduli spaces of vector bundles over a given base.

4 Ehresmann-induced connection for a smooth family of curves

Let $\mathcal{E} \rightarrow B$ be a vector bundle such that $\mathcal{E}|_{f^{-1}(\lambda)}$ for $\lambda \in C$. Coordinates on $\mathcal{E} \rightarrow B$ could be described as a pair of representations (τ_λ, ρ) , where τ_λ corresponds to a horizontal bundle and ρ corresponds to a vertical bundle. The fiber is equal to $\mathcal{E}|_{(x,y)} = E_\rho \cap E_{\tau_\lambda} = E|_{(\rho, \tau_\lambda)}$.

As is well known, along E_ρ, E_{τ_λ} there exist Ehresmann connections which can be viewed as horizontal subbundles H_ρ, H_{τ_λ} :

$$TE_\rho = H_\rho \oplus V_\rho, \quad TE_{\tau_\lambda} = H_{\tau_\lambda} \oplus V_{\tau_\lambda}.$$

Denote $\mathcal{C}_1 := \{C \subset (\mathcal{M}_{C_\lambda})_{\lambda \in C}\}$ and $\mathcal{C}_2 := \{(C_\lambda)_{\lambda \in C} \subset \mathcal{M}_C\}$. Thus we have two families of Ehresmann connections $\{H_\rho : \rho \in \mathcal{C}_1\}$ and $\{H_{\tau_\lambda} : \tau_\lambda \in \mathcal{C}_2\}$.

We are seeking for the connection H on the bundle $\mathcal{E}$ such that $T\mathcal{E} = H \oplus V$, where V is a tangent space to a fiber. Note that $V = T(E_\rho \cap E_{\tau_\lambda}) = (H_\rho \oplus V_\rho) \cap (H_{\tau_\lambda} \oplus V_{\tau_\lambda})$.

The tangent to $T\mathcal{E}$ is a tangent bundle of a moduli cycle $\mathcal{C}_1$. Locally $T\mathcal{E} = TE_\rho \oplus TE_{\tau_\lambda}$, therefore we can define a possible connection at least set-theoretically:

$$H = (H_\rho \oplus V_\rho) \oplus (H_{\tau_\lambda} \oplus V_{\tau_\lambda}) / \{(H_\rho \oplus V_\rho) \cap (H_{\tau_\lambda} \oplus V_{\tau_\lambda})\}.$$

Definition 4.1 We call a connection $\nabla_{\mathcal{E}}$ induced from flat connections from both directions the *Ehresmann-induced connection* (EIC, for short).

Theorem 4.2 Denote by $\Theta_{\mathcal{E}}$ a curvature of connection $\nabla_{\mathcal{E}}$. The obstruction to the Jordan property — failure for the tangent fields V, W to the connection to generate a Lie algebra — is exactly the curvature $\Theta_{\mathcal{E}}$.

Proof Our goal is to find an expression for the curvature tensor Θ of the connection on the vector bundle $\mathcal{E}$. Therefore, we would be able to express the second Chern class $c_2(\mathcal{E})$ explicitly, depending on a moduli cycle $\mathcal{S}$:

$$c_2(\mathcal{E}) = \frac{\text{tr}(\Theta(\mathcal{S})^2) - \text{tr}^2(\Theta(\mathcal{S}))}{8\pi^2}.$$

Notice that there exist only two “directions” on the base space B : a direction along a curve C_λ and a direction along a curve C , which parameterizes the family. When we lift vector fields from the base B to a vector bundle, we get the vector fields V, W along the “cycle/section” $\mathcal{C}_1$ and along the curve C . V and W induce diffeomorphisms φ_V, ψ_W on a vector bundle $\mathcal{E}$, which act on particular sections $s \in \Gamma(\mathcal{E})$. If we differentiate the result twice $\partial^2 \varphi_V \psi_W^{-1} \varphi_V^{-1} \psi_W(s)$, we will get $\Theta(s)$ a value of curvature on s . If we take a corresponding basis of sections of $\mathcal{E}$ and lift the basis of vector fields V, W from the base to our cycle C in the moduli space of vector bundles and look how do the diffeomorphisms act on the basis of sections, we would be able to count $\Theta_{\mathcal{E}}$ explicitly.

As is well known in this case, the obstruction to the Jordan property — failure for the tangent fields V, W to the connection to generate a Lie algebra — is the curvature $\Theta_{\mathcal{E}}$. $\square$

Note that we can use Gauss–Manin connection which acts as $\nabla_{\text{GM}}: TC \rightarrow V$ (by the reason that ∇_{GM} is flat), where V , as above, is a tangent space to a fiber.

Recall that by non-Abelian Hodge theory, the tangent space at the point (C, U) which is equal to $M_{\text{dR}}(C_u)$ is canonically identified with $H_{\text{dR}}^1(C_u, \text{End } E)$.

Therefore a Gauss–Manin connection computes the derivative of our section S . Notice that we have not only the Gauss–Manin connection, but also the other one induced from the connections from the both directions whose properties we will study in the following sections.

5 Calculation of the second Chern class

Recall that the Dolbeaut representative of the Atiyah class is given by $[\bar{\partial}, \Omega]$, where Ω is a curvature tensor on the vector bundle E . In coordinates,

$$\Omega_j^i = \sum R_{j\alpha\bar{\beta}}^i dz^\alpha \wedge d\bar{z}^\beta,$$

therefore, for $C = \Omega \wedge \Omega = \sum_k (\Omega_k^i \wedge \Omega_j^k)$. As every Ω_{ij} has a form $\Omega_j^i = \sum R_{j\alpha\bar{\beta}}^i dz^\alpha \wedge d\bar{z}^\beta$, one has

$$\begin{aligned} C_{ij} &= \sum \Omega_k^i \wedge \Omega_j^k = \sum R_{k\alpha\bar{\beta}}^i dz^\alpha \wedge d\bar{z}^\beta \wedge \sum R_{j\alpha\bar{\beta}}^k dz^\alpha \wedge d\bar{z}^\beta \\ &= \sum_k \sum_{\alpha, \beta, \gamma, \theta} \sum_{\sigma \in S_4} (-1)^\sigma R_{k\sigma(\alpha\bar{\beta})}^i R_{j\sigma(\gamma\bar{\theta})}^k dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta. \end{aligned}$$

Denote by $(\alpha) = (\alpha, \beta, \gamma, \theta)$ a multindex consisting of four variables. Then,

$$\begin{aligned} (\Omega^4)_{ii} &= \sum_j (\Omega^2)_{ij} \wedge (\Omega^2)_{ji} \\ &= \sum_j \sum_{(\alpha)} \left[\sum_k \sum_{\sigma \in S_4} (-1)^\sigma R_{k\sigma' \cdot \sigma(\alpha\bar{\beta})}^i R_{j\sigma' \cdot \sigma(\gamma\bar{\theta})}^k \right] \\ &\quad \cdot \left[\sum_k \sum_{\sigma \in S_4} (-1)^\sigma R_{k\sigma' \cdot \sigma(\alpha'\bar{\beta}')}^j R_{i\sigma' \cdot \sigma(\gamma'\bar{\theta}')}^k \right] dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta. \end{aligned}$$

Recall that

$$R_{j\bar{k}\alpha\bar{\beta}} = \sum h_{i\bar{k}} R_{j\alpha\bar{\beta}}^i = -\partial_{\bar{\beta}} \partial_\alpha h_{j\bar{k}} + \sum h^{a\bar{b}} \partial_\alpha h_{j\bar{b}} \partial_{\bar{\beta}} h_{a\bar{k}}.$$

Recall that $(h^{\beta\gamma}) = h^{-1}$ and

$$R_{i\bar{j}\alpha}^{\beta} = -h^{\beta\gamma} \left(\partial^2 \frac{h_{\alpha\bar{\gamma}}}{\partial z^i \partial \bar{z}^j} - h^{\delta\mu} \frac{\partial h_{\alpha\mu}}{\partial z^i} \frac{\partial h_{\delta\bar{\gamma}}}{\partial \bar{z}^j} \right)$$

and the curvature

$$\Omega = \sum R_{i\bar{j}\alpha}^{\beta} dz^i \wedge d\bar{z}^j \otimes e^{*\alpha} \otimes e_{\beta}.$$

For our expression we would need

$$R_{\sigma'(k) \cdot \sigma(\alpha\bar{\beta})}^j = h^{jk} \left(-\partial_{\sigma' \cdot \sigma(\bar{\beta})} \partial_{\sigma' \cdot \sigma(\alpha)} h_{i\bar{k}} + \sum h^{a\bar{b}} \partial_{\sigma' \cdot \sigma(\alpha)} h_{i\bar{b}} \partial_{\sigma' \cdot \sigma\beta} h_{a\bar{k}} \right).$$

Then we will have

$$\begin{aligned} (\Omega^4)_{ii} = & \sum_j \sum_{\alpha} \left[\sum_k \sum_{\sigma \in S_4} (-1)^{\sigma} h^{ik} h^{kj} h^{jk} h^{ki} \right. \\ & \cdot \left(-\partial_{\sigma(\bar{\beta})} \partial_{\sigma(\alpha)} h_{i\bar{k}} + \sum h^{a\bar{b}} \partial_{\sigma(\alpha)} h_{i\bar{b}} \partial_{\sigma\beta} h_{a\bar{k}} \right) \\ & \cdot \left(-\partial_{\sigma(\bar{\theta})} \partial_{\sigma(\gamma)} h_{k\bar{j}} + \sum h^{a\bar{b}} \partial_{\sigma(\gamma)} h_{k\bar{b}} \partial_{\sigma\theta} h_{a\bar{j}} \right) \Big] \\ & \cdot \left[\sum_k \sum_{\sigma \in S_4} (-1)^{\sigma} \left(-\partial_{\sigma(\bar{\beta})} \partial_{\sigma(\alpha)} h_{j\bar{k}} + \sum h^{a\bar{b}} \partial_{\sigma(\alpha)} h_{j\bar{b}} \partial_{\sigma\beta} h_{a\bar{k}} \right) \right. \\ & \cdot \left. \left(-\partial_{\sigma(\bar{\theta})} \partial_{\sigma(\gamma)} h_{k\bar{i}} + \sum h^{a\bar{b}} \partial_{\sigma(\gamma)} h_{k\bar{b}} \partial_{\sigma\theta} h_{a\bar{i}} \right) \right] dz^{\alpha} \wedge dz^{\beta} \wedge d\bar{z}^{\gamma} \wedge d\bar{z}^{\theta}. \end{aligned}$$

6 Curvature of $E \rightarrow C$

Let $\varphi: C \rightarrow C$ be a fibration of curves over a curve. Then we can form an induced connection D bearing it from $E|_C$ and every curve $E|_{\varphi^{-1}(c)}$, $c \in C$.

Thus we can differentiate a section of a bundle E using connection D induced from the flat connections from both directions. In this case the action of a connection D along $E|_C$ will be zero, but the action of D along the other direction may not be zero. We can again consider a matrix of the second derivatives of the metric $h_{i\bar{j}}$. Let us denote the basis of a local frame field as $s_U = (s_1, \dots, s_r, s'_1, \dots, s'_l)$, where $s = (s_1, \dots, s_r)$ is a frame field along $E|_{\varphi^{-1}(c)}$ and $s' = (s'_1, \dots, s'_l)$ is a local frame field along $E|_C$. Thus

$$D(s) = \nabla_{\varphi^{-1}(c)}(s) + \nabla_C(s').$$

Therefore, when we form a corresponding matrix of an action of connection D on the local frame section, it is $D(s) = (\nabla_{\varphi^{-1}(c)} + \nabla_C)(s_1, \dots, s_r, s'_1, \dots, s'_l)$.

Let us again consider the coordinates $(z_1, \bar{z}_1, z_2, \bar{z}_2)$. Now the first part of coordinates corresponds to coordinates on the curve C and the second part to those on the curve $\varphi^{-1}(c)$. Thus, locally, we have $s_U = (s_1, \dots, s_r, s'_1, \dots, s'_r)$. Denote $h_{ij} = h(s_i, s_j)$, $h'_{ij} = h(s_i, s'_j)$, $h''_{ij} = h(s'_i, s'_j)$. Notice that $\partial^2 h_{ij}$, $\partial^2 h'_{ij}$, $\partial^2 h''_{ij}$ vanish along $(z_1, \bar{z}_1)$ and nowhere else.

Consequently, we will have

$$\partial^2(h_{ij}) = \begin{pmatrix} 0 & 0 & a_{\bar{i},i} & a_{\bar{i},j} \\ 0 & 0 & a_{\bar{j},i} & a_{\bar{j},j} \\ a_{i,i} & a_{i,j} & * & * \\ a_{i,j} & a_{j,j} & * & * \end{pmatrix}.$$

7 Second Chern class for $E \rightarrow \mathcal{C}$

To analyze a formula for Ω^4 we have to understand how the first derivatives ∂h_{ij} involved into the expression

$$\begin{aligned} (\Omega^4)_{ii} = & \sum_j \sum_\alpha \left[\sum_k \sum_{\sigma \in S_4} (-1)^\sigma \left(-\partial_{\sigma(\bar{\beta})} \partial_{\sigma(\alpha)} h_{i\bar{k}} + \sum h^{a\bar{b}} \partial_{\sigma(\alpha)} h_{i\bar{b}} \partial_{\sigma\beta} h_{a\bar{k}} \right) \right. \\ & \cdot \left(-\partial_{\sigma(\bar{\theta})} \partial_{\sigma(\gamma)} h_{k\bar{j}} + \sum h^{a\bar{b}} \partial_{\sigma(\gamma)} h_{k\bar{b}} \partial_{\sigma\theta} h_{a\bar{j}} \right) \Big] \\ & \cdot \left[\sum_k \sum_{\sigma \in S_4} (-1)^\sigma \left(-\partial_{\sigma(\bar{\beta})} \partial_{\sigma(\alpha)} h_{j\bar{k}} + \sum h^{a\bar{b}} \partial_{\sigma(\alpha)} h_{j\bar{b}} \partial_{\sigma\beta} h_{a\bar{k}} \right) \right. \\ & \cdot \left(-\partial_{\sigma(\bar{\theta})} \partial_{\sigma(\gamma)} h_{k\bar{i}} + \sum h^{a\bar{b}} \partial_{\sigma(\gamma)} h_{k\bar{b}} \partial_{\sigma\theta} h_{a\bar{i}} \right) \Big] dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta \end{aligned}$$

behave in this case. It is clear that $\partial_{z_1} h_{ij} = 0$, $\partial_{\bar{z}_1} h_{ij} = 0$, meanwhile the derivatives from the other direction may not be zero (i.e. $\partial_{\bar{z}} h_{ij}$ is not zero, $z \in C$). If we look at the expression above, it could be deduced that the first derivatives are included in the expression $\sum h^{a\bar{b}} \partial_{\bar{z}_i} h_{k\bar{b}} \partial_{\bar{z}_j} h_{a\bar{i}}$ in pairs (so that at least one of them is always zero as the connection is flat in one direction and hence the first derivative is zero) therefore the terms involving it go to zero and what is left are the only expressions for the second derivatives. If we permute the columns or rows of the matrix, while working with the expression for Ω^4 , the essential form will stay the same. This way we get the expression

$$\begin{aligned} \kappa = \sum_i (\Omega^4)_{ii} &= \sum_{i,j,\alpha,\sigma \in S_4} \partial_{\sigma(\bar{\beta})} \partial_{\sigma(\alpha)} h_{i\bar{k}} \cdot \partial_{\sigma(\bar{\beta})} \partial_{\sigma(\alpha)} h_{j\bar{k}} \\ &\quad \cdot \partial_{\sigma(\bar{\theta})} \partial_{\sigma(\gamma)} h_{k\bar{j}} \cdot \partial_{\sigma(\bar{\beta})} \partial_{\sigma(\alpha)} h_{j\bar{k}} dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta \\ &= \sum_{i,j} \det(\partial^2 h_{ij}) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2. \end{aligned}$$

If all the expressions with the first derivatives go to zero, we get

$$\partial^2(h_{ij}) = \begin{pmatrix} 0 & 0 & a_{i,i}^- & a_{i,j}^- \\ 0 & 0 & a_{j,i}^- & a_{j,j}^- \\ a_{i,i} & a_{i,j} & * & * \\ a_{i,j} & a_{j,j} & * & * \end{pmatrix}.$$

We denote by $*$ the elements of derivatives of metric h_{ij} not in the mixed directions. Denote the matrices corresponding to the mixed directions as

$$D_{ij} = \begin{pmatrix} a_{i,i} & a_{i,j} \\ a_{i,j} & a_{j,j} \end{pmatrix}$$

and $\bar{D}_{ij}$ correspondingly. In general, for $E \rightarrow \mathcal{C}$, $\det(\partial^2(h_{ij})) = \det(D_{ij}) \det(\bar{D}_{ij})$ (in the second case the additional minor does not influence the expression of determinant by the Linear Algebra rules).

Theorem 7.1 *The resulting formula for $E \rightarrow \mathcal{C}$ is*

$$\begin{aligned} c_2(E) &= \sum_i (\Omega^4)_{ii} \\ &= \sum_{i,j} \frac{\det(\partial^2 h_{ij})}{\det(h_{ij})} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \\ &= \sum_{i,j} \frac{\det(D_{ij}) \det(\bar{D}_{ij})}{\det(h_{ij})} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2. \end{aligned}$$

Thus from the last theorem the stability result follows.

Corollary 7.2 *We have for $E \rightarrow \mathcal{C}$, $c_2(E) > 0$.*

Remark 7.3 Local positivity of c_2 -form holds outside of singular points and in fact, the impact of singular point depends on the dilation (distance on symmetric space $GL(n, \mathbb{C})/R^*U(n\mathbb{C})$), two points corresponding to unitary structures from different branches of the singular curve.

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