Higher-dimensional Calabi–Yau varieties with dense sets of rational points

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Received: 27 December 2020 / Revised: 4 August 2021 / Accepted: 11 September 2021 /
Published online: 22 November 2021
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Abstract

We construct higher-dimensional Calabi–Yau varieties defined over a given number field with Zariski dense sets of rational points. We give two elementary constructions in arbitrary dimensions as well as another construction in dimension three which involves certain Calabi–Yau threefolds containing an Enriques surface. The constructions also show that potential density holds for (sufficiently) general members of the families.

Keywords Rational points · Calabi–Yau varieties · Elliptic fibrations

Mathematics Subject Classification 14G05 · 14J32 · 14J27

1 Introduction

For a smooth projective variety X defined over a number field, one can ask whether the set of rational points is Zariski dense in X . It is expected that the set of rational points reflects the positivity of the canonical bundle ω_X (cf. the Bombieri–Lang conjecture) and it is tempting to study the intermediate case $\omega_X = \mathcal{O}_X$. In this case, X belongs to the class of special varieties introduced by Campana [7] which is conjecturally the same as that of varieties where rational points are potentially dense, that is, Zariski dense after passing to some finite field extension. An interesting subcase is given by abelian varieties, for which potential density is well-known.

Another important and yet challenging subcase is given by Calabi–Yau varieties in a *strict sense*, i.e., smooth projective varieties X with $\omega_X = \mathcal{O}_X$ and $H^0(X, \Omega_X^i) = 0$ for all $0 < i < \dim X$, which are simply connected over $\mathbb{C}$. For elliptic K3 surfaces and K3 surfaces with infinite automorphism groups, potential density holds due to Bogomolov–Tschinkel [4]. Moreover, there are several works on K3 surfaces over the

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rational numbers with Zariski dense sets of rational points; for instance, quartic K3 surfaces have long been studied [10,16,22,23].

Very little is known in higher dimensions. It is stated by Tschinkel [24, after Problem 3.5] that it would be worthwhile to find non-trivial examples of Calabi–Yau threefolds over number fields with Zariski dense sets of rational points. It is only recent that the first such examples were actually obtained: Bogomolov–Halle–Pazuki–Tanimoto [1] studied Calabi–Yau threefolds with abelian surface fibrations and showed potential density for threefolds (including simply connected ones) constructed by Gross–Popescu [11]. However, it is not immediately clear whether their method can be used to determine the minimal field extensions over which rational points become Zariski dense.

In this short note, we construct higher-dimensional Calabi–Yau varieties in a strict sense defined over a given number field with Zariski dense sets of rational points. We give two elementary constructions in arbitrary dimensions (Sects. 2, 3) as well as another construction in dimension three which involves certain Calabi–Yau threefolds containing an Enriques surface (Sect. 5). The constructions also show that potential density holds for (sufficiently) general members of the families.

The third construction is a by-product of the author’s attempt to analyze in detail the recent construction due to Ottem–Suzuki [17] of a pencil of Enriques surfaces with non-algebraic integral Hodge classes. For the third construction, our example contains no abelian surface and is a unique minimal model in its birational equivalence class, thus a theorem of Bogomolov–Halle–Pazuki–Tanimoto [1, Theorem 1.2] cannot be applied. For all the constructions, elliptic fibration structures are crucial.

We work over a number field unless otherwise specified.

2 Construction I

The first idea is to construct elliptic Calabi–Yau varieties in a strict sense (i.e., $H^0(X, \Omega_X^i) = 0$ for all $0 < i < \dim X$ and simply connected over $\mathbb{C}$) whose base spaces are rational and which admits infinitely many sections. For that purpose, we introduce variants of Schoen’s Calabi–Yau fiber products of rational elliptic surfaces [19].

We let $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a smooth hypersurface of bi-degree $(1, 3)$. Then the first projection defines an elliptic fibration $f: S \rightarrow \mathbb{P}^1$. Moreover, via the second projection, S is the blow-up of $\mathbb{P}^2$ along the nine points given by the intersection of two cubic curves, hence rational. For $n \geq 3$, we let $Y \subset \mathbb{P}^1 \times \mathbb{P}^{n-1}$ be a smooth hypersurface of bi-degree $(1, n)$. The first projection restricts to a fibration into Calabi–Yau hypersurfaces $g: Y \rightarrow \mathbb{P}^1$ and Y is again rational via the second projection. We assume that over any point in $\mathbb{P}^1$ either f or g is smooth, which is satisfied if Y is general with respect to S . We define $X = S \times_{\mathbb{P}^1} Y$ and let $\pi: X \rightarrow \mathbb{P}^1$ be the natural projection.

Lemma 2.1 *The fiber product X is a Calabi–Yau n -fold in a strict sense.*

Remark 2.2 If $S_{\mathbb{C}}$ is very general, it is classical that the elliptic fibration $f: S_{\mathbb{C}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ admits infinitely many sections, which implies that the same holds for the natural projection $X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$. This provides examples of Calabi–Yau varieties in a strict sense

defined over $\mathbb{C}$ containing infinitely many rational divisors in all dimensions ≥ 3 . We will construct below such an example over a given number field.

Proof of Lemma 2.1 It is immediate to see that X is smooth. The fiber product X is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^{n-1}$ of a hypersurface of tri-degree $(1, 3, 0)$ and that of tri-degree $(1, 0, n)$. We have $\omega_X = \mathcal{O}_X$ by the adjunction formula and an easy computation shows that $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < n$.

For simple connectedness over $\mathbb{C}$, Schoen proved this result for $n = 3$ (see [18, Lemma 1.1] for the strategy). In fact, his method works for $n \geq 3$ and the argument goes as follows. Let $U \subset \mathbb{P}_{\mathbb{C}}^1$ be the open subset over which $\pi: X_{\mathbb{C}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is smooth and let $V = \pi^{-1}(U)$. The natural map $\pi|_V: V \rightarrow U$ is topologically locally trivial and we let F be a fiber. We note that $\pi: X_{\mathbb{C}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ admits a section since f and g do, so does $\pi|_V: V \rightarrow U$. Then we have a commutative diagram

$$\begin{array}{ccccc} \pi_1(F) & \longrightarrow & \pi_1(V) & \overset{\curvearrowright}{\longrightarrow} & \pi_1(U) \\ & & \downarrow & & \downarrow \\ & & \pi_1(X_{\mathbb{C}}) & \overset{\curvearrowright}{\longrightarrow} & \pi_1(\mathbb{P}_{\mathbb{C}}^1) \end{array}$$

where the upper row is exact by the homotopy long exact sequence. Chasing the diagram and using the fact that $\pi_1(\mathbb{P}_{\mathbb{C}}^1)$ is trivial, we are reduced to showing that $\pi_1(F)$ has the trivial image in $\pi_1(X_{\mathbb{C}})$. Writing $F = F_1 \times F_2$, where F_1 is a fiber of f and F_2 is a fiber of g , we see that

$$\pi_1(F) = \pi_1(F_1) \times \pi_1(F_2).$$

Now it is enough to verify that the images of $\pi_1(F_1)$ and $\pi_1(F_2)$ in $\pi_1(X_{\mathbb{C}})$ are trivial. This is immediate since $\pi_1(F_1) \rightarrow \pi_1(X_{\mathbb{C}})$ (resp. $\pi_1(F_2) \rightarrow \pi_1(X_{\mathbb{C}})$) factors through the fundamental group of a section of $X_{\mathbb{C}} \rightarrow S_{\mathbb{C}}$ (resp. $X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$) and since $S_{\mathbb{C}}$ (resp. $Y_{\mathbb{C}}$) is simply connected (it is rational). $\square$

We give a construction of X defined over $\mathbb{Q}$ such that $X(\mathbb{Q})$ is Zariski dense in X . We start from constructing S defined over $\mathbb{Q}$ such that the elliptic fibration $f: S \rightarrow \mathbb{P}^1$ admits infinitely many sections over $\mathbb{Q}$, or equivalently, the generic fiber $E/\mathbb{Q}(t)$ admits a $\mathbb{Q}(t)$ -rational point and the Mordel–Weil group $E(\mathbb{Q}(t))$ has a positive rank. The construction is as follows.

Let $C \subset \mathbb{P}^2$ be an elliptic curve defined over $\mathbb{Q}$ with a Zariski dense set of $\mathbb{Q}$ -rational points. Let $O \in C(\mathbb{Q})$ (resp. $P \in C(\mathbb{Q})$) be the origin (resp. a non-torsion point). Let $D \subset \mathbb{P}^2$ be another elliptic curve defined over $\mathbb{Q}$ which goes through both O and P and which intersects transversally with C . Let $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ be the hypersurface of bi-degree $(1, 3)$ defined over $\mathbb{Q}$ corresponding to the pencil of elliptic curves generated by C and D . A zero-section of $f: S \rightarrow \mathbb{P}^1$ is given by the (-1) -curve over O and the Mordel–Weil group $E(\mathbb{Q}(t))$ has a positive rank since the image of the specialization homomorphism $E(\mathbb{Q}(t)) \rightarrow C(\mathbb{Q})$ has a positive rank. We conclude that S has the desired property.

Let Y be smooth, defined over $\mathbb{Q}$, and general so that over any point in $\mathbb{P}^1$ either f or g is smooth. Then $X = S \times_{\mathbb{P}^1} Y$ is a Calabi–Yau n -fold in a strict sense defined over $\mathbb{Q}$. Moreover the elliptic fibration $X = S \times_{\mathbb{P}^1} Y \rightarrow Y$ admits infinitely many sections over $\mathbb{Q}$ by construction. We note that the set $Y(\mathbb{Q})$ is Zariski dense in Y since Y is rational over $\mathbb{Q}$. The following theorem is now immediate:

Theorem 2.3 *The set $X(\mathbb{Q})$ is Zariski dense in X .*

3 Construction II

We let $X \subset (\mathbb{P}^1)^{n+1}$ be a smooth hypersurface of multi-degree $(2, \dots, 2)$. The following is an immediate consequence of the Lefschetz hyperplane section theorem:

Lemma 3.1 *If $n \geq 2$, then X is a Calabi–Yau n -fold in a strict sense.*

We give a construction of X defined over $\mathbb{Q}$ such that $X(\mathbb{Q})$ is Zariski dense in X . The construction relies on the following theorem due to Bogomolov–Tschinkel. We recall that a multi-section of an elliptic fibration is *saliently ramified* if it is ramified in a point which lies in a smooth elliptic fiber.

Theorem 3.2 ([2,3]) *Let $\phi: \mathcal{E} \rightarrow B$ be an elliptic fibration over a number field K . If there exists a saliently ramified multi-section $\mathcal{M}$ such that $\mathcal{M}(K)$ is Zariski dense in $\mathcal{M}$, then $\mathcal{E}(K)$ is Zariski dense in $\mathcal{E}$.*

Proof We sketch the proof for the convenience of the reader. We may assume that $\phi: \mathcal{E} \rightarrow B$ is smooth by shrinking B , and let $\phi_{\mathcal{J}}: \mathcal{J} = \text{Pic}_{\mathcal{E}/B}^0 \rightarrow B$ be the corresponding Jacobian fibration (see [6, Chapters 8, 9] for the basic properties), which is an abelian scheme. We consider a family of line bundles of degree 0

$$d\Delta_{\mathcal{E}} - \mathcal{E} \times_B \mathcal{M} \in \text{Pic}(\mathcal{E} \times_B \mathcal{E}) / \text{Pic}(\mathcal{E}) \subset \text{Pic}_{\mathcal{E}/B}(\mathcal{E}),$$

where $d = \deg(\mathcal{M}/B)$, and let

$$\tau: \mathcal{E} \rightarrow \mathcal{J}, \quad p \mapsto dp - \mathcal{M}_{\phi(p)}$$

be the induced morphism. It is direct to see that τ is étale finite. Then $\tau(\mathcal{M})$ cannot be contained in the m -torsion part $\mathcal{J}[m]$ for any positive integer m , otherwise $\phi|_{\mathcal{M}}: \mathcal{M} \rightarrow B$ would be smooth, which contradicts the assumption. Now Merel’s theorem (or Mazur’s theorem when $K = \mathbb{Q}$) implies that there exists a non-empty Zariski open subset $U \subset B$ such that for any $b \in \phi(\mathcal{M}(K)) \cap U$ there exists a rational point $p_b \in \mathcal{M}(K) \cap \mathcal{E}_b$ such that $\tau(p_b)$ is non-torsion on the fiber $\mathcal{J}_b$. Finally, the fiberwise action of the Jacobian fibration on $\mathcal{E}$ translates rational points on $\mathcal{M}$, which concludes the proof. $\square$

We start from constructing a smooth hypersurface $X_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(2, 2)$ defined over $\mathbb{Q}$ such that $X_1(\mathbb{Q})$ is Zariski dense in X_1 . We set

$$\mathbb{P}^1 \times \mathbb{P}^1 = \text{Proj } \mathbb{Q}[S_1, T_1] \times \text{Proj } \mathbb{Q}[S_2, T_2].$$

For instance, we can take X_1 to be the hypersurface defined by the equation

$$S_1^2 S_2 T_2 + S_1 T_1 (S_2^2 + 2S_2 T_2 + 2T_2^2) + T_1^2 T_2 (2S_2 + 2T_2) = 0.$$

Then X_1 is an elliptic curve with a non-torsion point defined over $\mathbb{Q}$, hence with a Zariski dense set of $\mathbb{Q}$ -rational points, which can be directly seen as follows. It is immediate to see that X_1 is smooth. Moreover, if we let

$$O = ([1:0], [1:0]), P = ([4:-1], [1:1]) \in X_1(\mathbb{Q}),$$

then $P - O \in \text{Pic}^0(X_1)$ is of infinite order. Indeed, setting

$$\mathbb{P}^2 = \text{Proj } \mathbb{Q}[U, V, W],$$

the rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ given by

$$U = S_1 T_2 + T_1 S_2, \quad V = S_1 T_2 - T_1 S_2, \quad W = S_1 T_2 + T_1 S_2 + 2T_1 T_2$$

transforms X_1 into its Weierstrass model

$$V^2 W = U^3 - U W^2 + W^3,$$

which is, in terms of inhomogeneous coordinates $u = U/W$, $v = V/W$,

$$v^2 = u^3 - u + 1.$$

It is now straightforward to see that O (resp. P) is sent to the point at infinity (resp. $(u, v) = (3, 5)$), where the latter defines a point of infinite order by a theorem of Lutz and Nagell [20, VIII. Corollary 7.2] as the v -value 5 is non-zero and 5^2 does not divide $4 \cdot (-1)^3 + 27 \cdot 1^2 = 23$. The claim follows.

For $n > 1$, we set

$$(\mathbb{P}^1)^{n+1} = \text{Proj } \mathbb{Q}[S_1, T_1] \times \cdots \times \text{Proj } \mathbb{Q}[S_{n+1}, T_{n+1}]$$

and inductively define $X_n \subset (\mathbb{P}^1)^{n+1}$ to be a general hypersurface of multi-degree $(2, \dots, 2)$ defined over $\mathbb{Q}$ containing X_{n-1} as the fiber of the projection $\text{pr}_{n+1}: X_n \rightarrow \mathbb{P}^1$ over $T_{n+1} = 0$.

Lemma 3.3 *The hypersurface X_n is smooth and X_{n-1} is a saliently ramified multi-section of the elliptic fibration given by the projection $\text{pr}_{1,\dots,n-1}: X_n \rightarrow (\mathbb{P}^1)^{n-1}$ onto the first $n-1$ factors.*

Proof For smoothness, it is enough to show that X_n is smooth around X_{n-1} . This is obvious because X_{n-1} is a fiber of the flat proper morphism $\text{pr}_{n+1}: X_n \rightarrow \mathbb{P}^1$ and X_{n-1} is smooth by induction. We are reduced to showing the second assertion.

Let $B \subset (\mathbb{P}^1)^{n-1}$ be the branch locus, that is, the set of critical values of the composition $X_{n-1} \hookrightarrow X_n \xrightarrow{\text{pr}_{1,\dots,n-1}} (\mathbb{P}^1)^{n-1}$. We note that this morphism is generically finite, but not finite when $n > 3$. We only need to prove that the fiber of $\text{pr}_{1,\dots,n-1}: X_n \rightarrow (\mathbb{P}^1)^{n-1}$ over a general point in B is smooth. Let $\Sigma \subset X_n$ be the set of critical points of $\text{pr}_{1,\dots,n-1}: X_n \rightarrow (\mathbb{P}^1)^{n-1}$. By generality, it is sufficient to show that $\dim \Sigma \cap X_{n-1} = n - 3$. This can be checked by a direct computation as follows. The equation of $X_n \subset (\mathbb{P}^1)^{n+1}$ can be written as

$$S_{n+1}^2 F + S_{n+1} T_{n+1} G + T_{n+1}^2 H = 0$$

for some multi-homogeneous polynomials F, G, H in $S_1, T_1, \dots, S_n, T_n$ of multi-degree $(2, \dots, 2)$, where $F = 0$ defines $X_{n-1} \subset (\mathbb{P}^1)^n$. If we moreover write F as

$$F = S_n^2 F_1 + S_n T_n F_2 + T_n^2 F_3$$

for some multi-homogeneous polynomials F_1, F_2, F_3 in $S_1, T_1, \dots, S_{n-1}, T_{n-1}$ of multi-degree $(2, \dots, 2)$, the set $\Sigma \cap X_{n-1}$ is defined by

$$T_{n+1} = 2S_n F_1 + T_n F_2 = S_n F_2 + 2T_n F_3 = G = 0.$$

The first three equations define the ramification locus, that is, the set of critical points of the composition $X_{n-1} \hookrightarrow X_n \xrightarrow{\text{pr}_{1,\dots,n-1}} (\mathbb{P}^1)^{n-1}$, which is of dimension $n - 2$, thus the four equations together define a closed subset of dimension $n - 3$ by generality, as we wanted. $\square$

Now Theorem 3.2 implies:

Theorem 3.4 *The set $X_n(\mathbb{Q})$ is Zariski dense in X_n for any $n \geq 1$.*

Remark 3.5 For a smooth hypersurface X in $(\mathbb{P}^1)^{n+1}$ of multi-degree $(2, \dots, 2)$, the birational automorphism group $\text{Bir}(X_{\mathbb{C}})$ is infinite by Cantat–Oguiso [8, Remark 3.4]. It would be possible to give a proof of the density of rational points by using the action of $\text{Bir}(X_{\mathbb{C}})$.

4 Calabi–Yau threefolds containing an Enriques surface

In this section, we work over the complex numbers. We construct a Calabi–Yau threefold containing an Enriques surface and prove basic properties of the threefold, which will be used in Sect. 5.

Let $\Pi = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1))$. On the projective bundle Π , we consider a map of vector bundles

$$u: \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(2H_1) \oplus \mathcal{O}(2H),$$

where H_1 (resp. H) is the pull-back of the hyperplane section class on $\mathbb{P}^2$ (resp. the tautological class on Π). Let X be the rank one degeneracy locus of u .

Lemma 4.1 *If u is general, X is a Calabi–Yau threefold. We have the topological Euler characteristic $\chi_{\text{top}}(X) = c_3(T_X) = -84$ and the Hodge numbers $h^{1,1}(X) = 2$ and $h^{1,2}(X) = 44$.*

Proof Since the vector bundle $\mathcal{O}(2H_1) \oplus \mathcal{O}(2H)$ is globally generated, X is a smooth threefold by the Bertini theorem for degeneracy loci. Another projective model of X is defined as the zero set of a naturally defined section of $\mathcal{O}(1)^{\oplus 3}$ on the projective bundle $\tilde{\Pi} = \mathbb{P}_{\Pi}(\mathcal{O}(2H_1) \oplus \mathcal{O}(2H))$. Let $\tilde{H}$ be the tautological class on $\tilde{\Pi}$. The adjunction formula gives $\omega_X = \mathcal{O}_X(\tilde{H} - 2H)$. On the other hand, $\tilde{H} - 2H$ is the class of the intersection $X \cap \mathbb{P}_{\Pi}(\mathcal{O}(2H_1))$, which is empty, thus we have $\mathcal{O}_X(\tilde{H} - 2H) = \mathcal{O}_X$. It follows that $\omega_X = \mathcal{O}_X$. The rest of the statement is a consequence of a direct computation using the conormal exact sequence and the Koszul resolution of the ideal sheaf of X in $\tilde{\Pi}$. $\square$

We assume that u is general in what follows.

Lemma 4.2 *The threefold X admits an elliptic fibration $\phi: X \rightarrow \mathbb{P}^2$. Moreover, X contains an Enriques surface S and the linear system $|2S|$ defines a K3 fibration $\psi: X \rightarrow \mathbb{P}^1$.*

Remark 4.3 There are several examples of Calabi–Yau threefolds containing an Enriques surface. For instance, see Borisov–Nuer [5].

Proof of Lemma 4.2 A natural projection $\Pi \rightarrow \mathbb{P}^2$ restricts to a surjection $\phi: X \rightarrow \mathbb{P}^2$ with the geometric generic fiber a complete intersection of two quadrics in $\mathbb{P}^3$, which is an elliptic curve. The morphism ϕ has equidimensional fibers, hence it is flat.

Moreover, the map u restricts to a map of vector bundles

$$v: \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2)$$

on $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 3}) = \mathbb{P}^2 \times \mathbb{P}^2$. The intersection $X \cap \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 3})$ is the rank one degeneracy locus of v , which is an Enriques surface S by generality (see [17, Lemma 2.1]). Then the linear system $|2S|$ defines a K3 fibration $\psi: X \rightarrow \mathbb{P}^1$ by [5, Proposition 8.1]. $\square$

Corollary 4.4 *The threefold X contains no abelian surface.*

Proof If X contains an abelian surface A , then it is easy to see that the linear system $|A|$ defines an abelian surface fibration $\eta: X \rightarrow \mathbb{P}^1$. Then pulling back $\text{NS}(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{R}}$ by $(\psi, \eta): X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ defines a two-dimensional linear subspace of $\text{NS}(X)_{\mathbb{R}}$ which does not contain an ample divisor. Therefore we should have $\rho(X) \geq 3$. This is impossible since we have $\rho(X) = h^{1,1}(X) = 2$ by Lemma 4.1. $\square$

Lemma 4.5 $\text{Nef}(X) = \overline{\text{Eff}(X)} = \mathbb{R}_{\geq 0}[H_1] \oplus \mathbb{R}_{\geq 0}[S]$.

Proof Since the linear systems $|H_1|$ and $|2S|$ define the fibrations $\phi: X \rightarrow \mathbb{P}^2$ and $\psi: X \rightarrow \mathbb{P}^1$ respectively, the divisors H_1 and S are semi-ample but not big, so their classes give extremal rays in $\overline{\text{Eff}(X)}$. $\square$

Corollary 4.6 *The threefold X is a unique minimal model in its birational equivalence class. Moreover, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ and these groups are finite.*

Proof Let $f: X \dashrightarrow X'$ be a birational map with X' a minimal model. Then f can be decomposed into a sequence of flops by a result of Kawamata [13, Theorem 1]. Note that any flopping contraction of a Calabi–Yau variety is given by a codimension one face of the nef cone (see [12, Theorem 5.7]). Since the codimension one faces $\mathbb{R}_{\geq 0}[H_1]$ and $\mathbb{R}_{\geq 0}[S]$ of $\mathrm{Nef}(X)$ give fibrations, it follows that f is in fact an isomorphism. For the last statement, we note that Oguiso proved that the automorphism group of any odd-dimensional Calabi–Yau variety in a wider sense with $\rho = 2$ is finite. $\square$

Proposition 4.7 *The threefold X is simply connected.*

Proof Applying [14, Lemma 5.2.2] to the K3 fibration $\psi: X \rightarrow \mathbb{P}^1$, whose smooth fibers are simply connected, we are reduced to showing that $2S$ is the only one multiple fiber of ψ . The K3 fibration $\psi: X \rightarrow \mathbb{P}^1$ and the morphism $X \rightarrow \mathbb{P}^5$ given by the linear system $|H|$ induce a morphism $X \rightarrow \mathbb{P}^1 \times \mathbb{P}^5$, which is the blow-up of a non-normal complete intersection X_0 in $\mathbb{P}^1 \times \mathbb{P}^5$ of three hypersurfaces of bi-degree $(1, 2)$ along the non-normal locus with the exceptional divisor S . Now we only need to prove that the first projection $\mathrm{pr}_1: X_0 \rightarrow \mathbb{P}^1$ admits no multiple fibers. This follows from the Lefschetz hyperplane section theorem. $\square$

5 Construction III

We recall a system of affine equations for an Enriques surface introduced by Colliot-Thélène–Skorobogatov–Swinnerton-Dyer [9].

Proposition 5.1 ([9, Proposition 4.1, Example 4.1.2], [15, Proposition 1.1]) *Let k be a field of characteristic zero. Let $c, d, f \in k[t]$ be polynomials of degree 2 such that $c \cdot d \cdot (c - d) \cdot f$ is separable. Let S^0 be the affine variety in $\mathbb{A}^4 = \mathrm{Spec} k[t, u_1, u_2, u_3]$ defined by*

$$u_1^2 - c(t) = f(t)u_2^2, \quad u_1^2 - d(t) = f(t)u_3^2.$$

Then a minimal smooth projective model S of S^0 is an Enriques surface. The projection $S^0 \rightarrow \mathbb{A}^1 = \mathrm{Spec} k[t]$ induces an elliptic fibration $S \rightarrow \mathbb{P}^1$ with reduced discriminant $c \cdot d \cdot (c - d) \cdot f$ which admits double fibers over $f = 0$ whose reductions are smooth elliptic curves.

Now we apply Proposition 5.1 to $k = \mathbb{Q}$ and

$$c = -3t^2 - 2t + 8, \quad d = \frac{t^2 - 15t + 16}{2}, \quad f = \frac{t^2 + 1}{2}.$$

Let S be the corresponding Enriques surface. We prove:

Lemma 5.2 *The surface S has a Zariski dense set of $\mathbb{Q}$ -rational points.*

Proof We follow the strategy of the proof of [21, Proposition 5]. It will be easier to work on the K3 double cover $\tilde{S}$. By setting

$$w^2 = \frac{t^2 + 1}{2}, \quad v_2 = wu_2, \quad v_3 = wu_3,$$

we obtain the defining equations of its affine model $\tilde{S}^0$ in $\mathbb{A}^5 = \text{Spec } \mathbb{Q}[t, u_1, v_2, v_3, w]$:

$$u_1^2 - (-3t^2 - 2t + 8) = v_2^2, \quad u_1^2 - \frac{t^2 - 15t + 16}{2} = v_3^2, \quad \frac{t^2 + 1}{2} = w^2.$$

The projection $\tilde{S}^0 \rightarrow C^0 = \text{Spec } \mathbb{Q}[t, w]/(\frac{t^2+1}{2} - w^2)$ defines an elliptic fibration $\tilde{S} \rightarrow C$ with reduced discriminant $c \cdot d \cdot (c - d)$.

We consider the curve $E^0 \subset \tilde{S}^0$ cut out by

$$u_1 = t - 3, \quad v_2 = 2t - 1,$$

which is isomorphic to the affine curve in $\mathbb{A}^3 = \text{Spec } \mathbb{Q}[t, v_3, w]$ defined by

$$\frac{1}{2}(t+1)(t+2) = v_3^2, \quad \frac{t^2 + 1}{2} = w^2.$$

Then E^0 gives an elliptic curve E . We prove that $E(\mathbb{Q})$ is Zariski dense in E . Let $O \in E$ be the point given by

$$t = -1, \quad v_3 = 0, \quad w = -1$$

and $P \in E$ be the point given by

$$t = 7, \quad v_3 = -6, \quad w = 5.$$

We only need to prove that $P - O \in \text{Pic}^0(E)$ is of infinite order. It is a simple matter to check that the map

$$(t, v_3, w) \mapsto \left(\frac{2(3t - 4w - 1)}{t - 2w - 1}, \frac{8v_3}{t - 2w - 1} \right)$$

gives a transformation into the Weierstrass model

$$y^2 = x^3 - 52x + 144$$

and sends O (resp. P) to the point at infinity (resp. $(x, y) = (0, 12)$). Thus it is enough to verify that $(x, y) = (0, 12)$ defines a point of infinite order. This follows from a theorem of Lutz and Nagell [20, VIII. Corollary 7.2] since the y -value 12 is non-zero and 12^2 does not divide $4 \cdot (-52)^3 + 27 \cdot 144^2$.

Moreover, E is a saliently ramified multi-section of the elliptic fibration $\tilde{S} \rightarrow C$. Indeed, it is easy to verify that $E \rightarrow C$ is branched over

$$t = -1, \quad w = \pm 1,$$

while $t = -1$ is not a root of the reduced discriminant

$$c \cdot d \cdot (c - d) = (-3t^2 - 2t + 8) \left(\frac{t^2 - 15t + 16}{2} \right) \left(\frac{-7t^2 + 11t}{2} \right).$$

Now Theorem 3.2 shows that $\tilde{S}(\mathbb{Q})$ is Zariski dense in $\tilde{S}$. This in turn implies that $S(\mathbb{Q})$ is Zariski dense in S . $\square$

We define a compactification of S^0 in $\mathbb{P}^2 \times \mathbb{P}^2$ as follows. We set

$$\mathbb{P}^2 \times \mathbb{P}^2 = \text{Proj } \mathbb{Q}[X_0, X_1, X_2] \times \text{Proj } \mathbb{Q}[Y_0, Y_1, Y_2].$$

On $\mathbb{P}^2 \times \mathbb{P}^2$, we consider the map of vector bundles

$$v: \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2)$$

given by the 2×3 matrix

$$\begin{pmatrix} X_0^2 & X_1^2 & X_2^2 \\ 2Y_0^2 + 6Y_1^2 + 4Y_1Y_2 - 16Y_2^2 & 2Y_0^2 - Y_1^2 + 15Y_1Y_2 - 16Y_2^2 & Y_1^2 + Y_2^2 \end{pmatrix}.$$

Let S' be the rank one degeneracy locus of v . It is straightforward to see that S' is indeed a compactification of S^0 . The surface S' is a local complete intersection, so in particular, Gorenstein. The surface S' has isolated singular points and blowing up along the points gives a crepant resolution $S \rightarrow S'$.

Finally, we give the construction of the Calabi–Yau threefold. On the projective bundle $\Pi = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1))$, we let

$$u: \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(2H_1) \oplus \mathcal{O}(2H)$$

be general among all maps defined over $\mathbb{Q}$ which restrict to v on $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 3}) = \mathbb{P}^2 \times \mathbb{P}^2$. We define X to be the rank one degeneracy locus of u .

Theorem 5.3 *The threefold X is a Calabi–Yau threefold in a strict sense defined over $\mathbb{Q}$ with a Zariski dense set of $\mathbb{Q}$ -rational points. Moreover, X satisfies the following geometric properties:*

- (i) $X_{\mathbb{C}}$ admits K3 and elliptic fibrations;
- (ii) $X_{\mathbb{C}}$ contains no abelian surface;
- (iii) $X_{\mathbb{C}}$ is a unique minimal model in its birational equivalence class;
- (iv) $\text{Bir}(X_{\mathbb{C}}) = \text{Aut}(X_{\mathbb{C}})$ and these groups are finite.

Proof One can check that X is smooth and the proofs in Sect. 4 still go through. It remains to show that $X(\mathbb{Q})$ is Zariski dense in X . By a similar argument to that in Lemma 3.3, S' is a saliently ramified multi-section of the elliptic fibration $\phi: X \rightarrow \mathbb{P}^2$. Since $S'(\mathbb{Q})$ is Zariski dense in S' by Lemma 5.2, the result follows from Theorem 3.2. $\square$

One can be more explicit. We fix a non-zero element $Z \in H^0(\Pi, \mathcal{O}_\Pi(H - H_1))$. Let u be given by the matrix

$$\begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{pmatrix},$$

where

$$P_1 = X_0^2, \quad Q_1 = X_1^2, \quad R_1 = X_2^2$$

and

$$\begin{aligned} P_2 &= (X_1^2 + X_2^2)Z^2 + (X_1Y_1 + X_2Y_2)Z + 2Y_0^2 + 6Y_1^2 + 4Y_1Y_2 - 16Y_2^2, \\ Q_2 &= (X_0^2 + X_2^2)Z^2 + (X_0Y_0 + X_2Y_2)Z + 2Y_0^2 - Y_1^2 + 15Y_1Y_2 - 16Y_2^2, \\ R_2 &= (X_0^2 + X_1^2)Z^2 + (X_0Y_0 + X_1Y_1)Z + Y_1^2 + Y_2^2. \end{aligned}$$

Then *Macaulay2* shows that the corresponding X is smooth and that the discriminant locus $\Delta \subset \mathbb{P}^2$ of $\phi: X \rightarrow \mathbb{P}^2$ and the branch locus $B \subset \mathbb{P}^2$ of $\phi|_S: S \rightarrow \mathbb{P}^2$ meet properly. Consequently, the set $X(\mathbb{Q})$ is Zariski dense in X .

Acknowledgements The author wishes to thank Lawrence Ein, Yohsuke Matsuzawa, John Christian Ottem, Ramin Takloo-Bighash, and Burt Totaro for interesting discussions. He also thanks the anonymous referee for useful suggestions.

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