A Montgomery–Hooley Theorem for sums of two cubes

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Abstract

Asymptotic formulae are obtained for the variance associated with the distribution in arithmetic progressions of the numbers that are the sum of two positive cubes of integers.

Keywords Variance · Arithmetic progressions · Sums of cubes · Hardy–Littlewood method

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1 Introduction

The sequence of numbers that are the sum of two positive cubes of integers, with or without multiplicities, has been studied from various perspectives. Hooley [5,7] showed that high multiplicities are rare, and his theme was taken up by Wooley [14] and Heath-Brown [4]. The distribution of the gaps between numbers that are sums of two cubes is the subject of recent work by Brüdern and Wooley [1]. Here we discuss the distribution in arithmetic progressions to provide an asymptotic formula for a certain variance that is analogous to the theorems of Montgomery [9] and Hooley [6] in the distribution of primes.

We require some notation to formulate our results. Let $r(n)$ denote the number of solutions of $x^3 + y^3 = n$ in natural numbers x, y , and let $\rho(q, a)$ denote the

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number of incongruent solutions of the congruence $x^3 + y^3 \equiv a \pmod{q}$. Finally, let $C = \Gamma(\frac{4}{3})^2 / \Gamma(\frac{5}{3})$ denote the area of the domain

$$\{(\xi, \eta) \in \mathbb{R}^2 : \xi > 0, \eta > 0, \xi^3 + \eta^3 < 1\}.$$

For $N \geq 1$, $Q \geq 1$ we are interested in establishing an asymptotic formula for

$$V(N, Q) = \sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} r(n) - \frac{\rho(q, a)}{q^2} C N^{2/3} \right|^2. \quad (1.1)$$

The average spacing between sums of two cubes suggests that such a formula should be available when Q is not too far from $N^{2/3}$.

Theorem 1.1 *Suppose that $Q \leq N$. Then*

$$V(N, Q) = 2CQ N^{2/3} + O(Q^{1/2} N^{29/30+\varepsilon} + Q^{13/18} N^{7/9+\varepsilon} + N^{19/15+\varepsilon}).$$

In stating this theorem, and elsewhere in this memoir, we assert that whenever the letter ε occurs in a statement, then this statement is true for any positive real number assigned to ε . Constants implicit in the familiar symbols of Landau and Vinogradov depend on the particular choice of ε .

A result similar to Theorem 1.1 is available when sums of two cubes are counted without multiplicity. Thus, we define $r_0(n) = 1$ whenever $r(n) \geq 1$ and let $r_0(n) = 0$ otherwise. As a consequence of the work of Hooley, we know that for a typical natural number n one has $r(n) = 2r_0(n)$. It is then natural to examine

$$V_0(N, Q) = \sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} r_0(n) - \frac{\rho(q, a)}{2q^2} C N^{2/3} \right|^2. \quad (1.2)$$

Theorem 1.2 *Suppose that $Q \leq N$. Then*

$$V_0(N, Q) = \frac{1}{2} C Q (N^{2/3} + O(N^{5/9+\varepsilon})) \\ + O(Q^{1/2} N^{29/30+\varepsilon} + Q^{13/18} N^{7/9+\varepsilon} + N^{19/15+\varepsilon}).$$

When studied on its own right, the ordering of sums of two cubes by size is certainly the most natural one. However, in the additive theory of numbers variables typically range over intervals independently, and it is then desirable to have at hand variance estimates that cover such situations. The following variant of Theorem 1.1 is directly applicable to the treatment of major arcs for additive problems involving cubes. Note that the range for Q where a valid asymptotic formula is obtained is larger than in Theorem 1.1. Let

$$V^*(X, Q) = \sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{\substack{1 \leq x, y \leq X \\ x^3 + y^3 \equiv a \pmod{q}}} 1 - \frac{\rho(q, a)}{q^2} X^2 \right|^2.$$

Theorem 1.3 *Suppose that $Q \leq N$. Then*

$$V^*(X, Q) = 2QX^2 + O(Q^{1/2}X^{17/6+\varepsilon} + Q^{13/18}X^{7/3+\varepsilon} + X^{11/3+\varepsilon}).$$

There is a rich literature on analogues of the Montgomery–Hooley formula for arithmetical functions. For example, the k -free numbers and other sifted sequences have been studied from this perspective (most recently Vaughan [13]) as well as other sequences that are reasonably well-behaved in arithmetic progressions (e.g. Hooley [8], Vaughan [12]). A common feature of existing work seems to be that the underlying sequence is fairly dense. In our situation, however, the expectation of $r(n)$ grows like $N^{2/3}$. We are not aware of other examples where a Montgomery–Hooley theorem is known and the growth rate for the expectation is as small as N^θ with some $\theta < 1$. Our method requires, at the very least, that the expectation grows faster than $\sqrt{N}$. In fact, as we shall demonstrate in a sequel to this paper, it is possible to establish results of the type considered here for the sequence of numbers that are representable as the sum of a square and a positive k -th power. Yet, a Montgomery–Hooley theorem for sums of two biquadrates seems to require new ideas.

The core argument of this memoir is directed at an imperfect version of V where we withdraw from $r(n)$ the Hardy–Littlewood expectation. To define the latter, let $e(\alpha) = \exp(2\pi i\alpha)$, write

$$S(q, a) = \sum_{x=1}^q e(ax^3/q)$$

and, whenever $T \geq 1$, put

$$\mathfrak{s}(n; T) = \sum_{t \leq T} \sum_{\substack{c=1 \\ (c, t)=1}}^t t^{-2} S(t, c)^2 e(-cn/t). \quad (1.3)$$

We consider

$$U(N, Q, T) = \sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \left(r(n) - \frac{2}{3} Cn^{-1/3} \mathfrak{s}(n; T) \right) \right|^2. \quad (1.4)$$

Theorem 1.4 *Suppose that $1 \leq T \leq N^{3/5}$ and $Q \geq 1$. Then*

$$U(N, Q, T) = Q \sum_{n \leq N} \left(r(n) - \frac{2}{3} C n^{-1/3} \mathfrak{s}(n; T) \right)^2 + O(N^\varepsilon (N^{19/15} + N^{3/5} T^2 + N^{4/3} T^{-1})).$$

In Sect. 3, we establish Theorem 1.4 by the Hardy–Littlewood method. This approach to Montgomery–Hooley formulæ originates in the work of Goldston and Vaughan [3]. Our application prominently features the exponential sum

$$g(\alpha) = \sum_{n \leq N} r(n) e(\alpha n) \quad (1.5)$$

that is less common than the cognate Weyl sum

$$f(\alpha, X) = \sum_{x \leq X} e(\alpha x^3). \quad (1.6)$$

A discussion of these exponential sums as well as some other auxiliary estimates can be found in Sect. 3. In Sect. 4 we deduce Theorem 1.1 from Theorem 1.4. In Sect. 5 we briefly describe alterations required to prove Theorem 1.3. Finally, in Sect. 6 we deduce Theorem 1.2 from Theorem 1.1.

2 Lemmata

We begin with some results concerning the function $r(n)$.

Lemma 2.1 *One has*

$$r(n) \ll n^\varepsilon$$

and the number $\mathcal{N}(N)$ of natural numbers n not exceeding N where $r(n) \neq 2r_0(n)$ satisfies

$$\mathcal{N}(N) \ll N^{4/9+\varepsilon}. \quad (2.1)$$

Further,

$$\sum_{n \leq N} r(n) = C N^{2/3} + O(N^{1/3}), \quad \sum_{n \leq N} r(n)^2 = 2C N^{2/3} + O(N^{4/9+\varepsilon}).$$

Proof The first claim is a trivial consequence of a familiar bound for the divisor function. It follows from Theorem 1 of Heath-Brown [4] that there are no more than $O(N^{4/9+\varepsilon})$ natural numbers $n \leq N$ with $r(n) > 2$. Moreover, $r(n) = 1$ gives $n = 2m^3$ for some $m \in \mathbb{N}$, and there are no more than $O(N^{1/3})$ such n . This confirms the second claim. The linear average of $r(n)$ is an instance of the Lipschitz principle

(Davenport [2]), and the formula for the quadratic average follows from the assertions of the lemma that we have already confirmed. $\square$

Our next lemma is the dual of the large sieve, and is a simple application of Theorem 1 of Montgomery and Vaughan [10].

Lemma 2.2 *Suppose that $M \in \mathbb{Z}$, $N \in \mathbb{N}$ and the $\eta(t, c)$ are arbitrary complex numbers. Then*

$$\sum_{M \leq n \leq M+N} \left| \sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1}}^t \eta(t, c) e(nc/t) \right|^2 \ll (N + T^2) \sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1}}^t |\eta(t, c)|^2.$$

Let κ be the multiplicative function defined for $v \geq 0$ and primes p by

$$\kappa(p^{3v}) = p^{-v}, \quad \kappa(p^{3v+1}) = 2p^{-v-1/2}, \quad \kappa(p^{3v+2}) = p^{-v-1}.$$

One routinely confirms the estimates

$$\sum_{t \leq T} \kappa(t)^2 \ll T^\varepsilon, \quad \sum_{t \leq T} t \kappa(t)^4 \ll T^\varepsilon. \quad (2.2)$$

Lemma 2.3 *Whenever $(c, t) = 1$ we have*

$$t^{-1} S(t, c) \ll \kappa(t) \quad \text{and} \quad t^{-1} S(t, c) \ll t^{-1/3}.$$

Proof This follows from Lemmas 4.2, 4.3, 4.4 and Theorem 4.2 of Vaughan [11]. $\square$

Let

$$v_0(\beta, X) = \int_0^X e(\beta t^3) \, d\beta, \quad \text{and} \quad v_1(\beta, X) = \frac{1}{3} \sum_{x \leq X^3} n^{-2/3} e(\beta n). \quad (2.3)$$

Then, by Lemma 2.8 of Vaughan [11], whenever $|\beta| \leq 1/2$, we have

$$v_j(\beta, X) \ll \frac{X}{(1 + X^3 |\beta|)^{1/3}} \quad (j = 0, 1). \quad (2.4)$$

We note that if $j = 0$ then this bound holds for all real numbers β .

It is convenient to introduce the notation

$$V_j(\alpha; t, c) = t^{-1} S(t, c) v_j(\alpha - c/t, X).$$

The next lemma is Theorem 4.1 of Vaughan [11].

Lemma 2.4 *Suppose $(t, c) = 1$. Then, for $j = 0$ and 1 one has*

$$f(\alpha, X) - V_j(\alpha; t, c) \ll t^{1/2+\varepsilon} (1 + X^3 |\alpha - c/t|)^{1/2}.$$

We are ready to investigate the exponential sum $g(\alpha)$, and begin by noting that by orthogonality and Lemma 2.1, one has

$$\int_0^1 |g(\alpha)|^2 d\alpha \ll N^{2/3}. \quad (2.5)$$

Next, we turn to an analogue of Lemma 2.4 for the exponential sum $g(\alpha)$. Our result is relatively weak but suffices for our immediate needs. Improvements in the q -aspect would allow wider ranges for Q in Theorems 1.1 and 1.2. We define the sum

$$w(\beta) = \frac{2}{3} C \sum_{n \leq N} n^{-1/3} e(n\beta)$$

and observe that the proof of Lemma 2.8 of Vaughan [11] provides the estimate

$$w(\beta) \ll N^{2/3} (1 + N\|\beta\|)^{-2/3}, \quad (2.6)$$

where $\|\beta\|$ denotes the distance of the real number β to the nearest integer. Further, we write

$$W(\alpha; q, a) = q^{-2} S(q, a)^2 w(\alpha - a/q). \quad (2.7)$$

Lemma 2.5 *Let $\alpha \in \mathbb{R}$, $q \in \mathbb{N}$ and $a \in \mathbb{Z}$. Then*

$$g(\alpha) = W(\alpha; q, a) + O(N^{1/3} q^{1/2+\varepsilon} (1 + N|\alpha - a/q|)^{1/2}).$$

Proof By (1.5) we have

$$g(\alpha) = \sum_{x^3+y^3 \leq N} e(\alpha(x^3 + y^3)).$$

We write $\beta = \alpha - a/q$ and apply Lemma 2.4 to the sum over y to conclude that

$$g(\alpha) = \sum_{x^3 \leq N} e(\alpha x^3) \left(\frac{S(q, a)}{q} v_0(\beta, (N - x^3)^{1/3}) + O(q^{1/2+\varepsilon} (1 + N\|\beta\|)^{1/2}) \right).$$

By (2.3),

$$v_0(\beta, (N - x^3)^{1/3}) = \frac{1}{3} \int_0^{N-x^3} t^{-2/3} e(\beta t) dt,$$

whence, on exchanging sum and integral, we infer

$$g(\alpha) = \frac{S(q, a)}{3q} \int_0^N t^{-2/3} e(\beta t) \sum_{x^3 \leq N-t} e(\alpha x^3) dt + O(N^{1/3} q^{1/2+\varepsilon} (1 + N|\beta|)^{1/2}).$$

We apply Lemma 2.4 again, this time to the sum over x . This yields

$$\sum_{x^3 \leq N-t} e(\alpha x^3) = \frac{S(q, a)}{3q} \int_0^{N-t} t'^{-2/3} e(\beta t') dt' + O(q^{1/2+\varepsilon} (1 + N|\beta|)^{1/2}),$$

and we find that

$$g(\alpha) = \frac{S(q, a)^2}{9q^2} I(\beta) + O(N^{1/3} q^{1/2+\varepsilon} (1 + N|\beta|)^{1/2})$$

where

$$\begin{aligned} I(\beta) &= \int_0^N t^{-2/3} \int_0^{N-t} t'^{-2/3} e(\beta(t+t')) dt' dt \\ &= \int_0^N e(\beta u) \int_0^u t^{-2/3} (u-t)^{-2/3} dt du. \end{aligned}$$

Next, the substitution $t = us$ in the inner integral produces

$$I(\beta) = B\left(\frac{1}{3}, \frac{1}{3}\right) \int_0^N u^{-1/3} e(\beta u) du$$

where

$$B\left(\frac{1}{3}, \frac{1}{3}\right) = \int_0^1 s^{-2/3} (1-s)^{-2/3} ds$$

is a special value of Euler's Beta function. In particular, we see that $B(\frac{1}{3}, \frac{1}{3}) = 6C$. Further, by Euler's summation formula,

$$\sum_{n \leq N} n^{-1/3} e(\beta n) = \int_0^N u^{-1/3} e(\beta u) du + O(1 + |\beta| N^{2/3}).$$

On collecting together, the lemma now follows. $\square$

It is convenient to estimate the mean square of the exponential sum

$$h(\alpha) = \sum_{n \leq N} n^{-1/3} \mathfrak{s}(n; T) e(\alpha n). \quad (2.8)$$

Lemma 2.6 *Let $T \geq 1$ and $N \geq 1$. Then*

$$\int_0^1 |h(\alpha)|^2 d\alpha \ll (N^{1/3} + T^2) T^\varepsilon.$$

Proof By orthogonality,

$$\int_0^1 |h(\alpha)|^2 d\alpha = \sum_{n \leq N} n^{-2/3} |\mathfrak{s}(n; T)|^2. \quad (2.9)$$

Now let $1 \leq M \leq N$. Then, by Lemmata 2.2 and 2.3,

$$\begin{aligned} \sum_{M \leq n \leq 2M} n^{-2/3} |\mathfrak{s}(n; T)|^2 &\leq M^{-2/3} \sum_{n \leq 2M} |\mathfrak{s}(n; T)|^2 \\ &\ll M^{-2/3} (M + T^2) \sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1}}^t \left| \frac{S(t, c)}{t} \right|^4 \\ &\ll M^{-2/3} (M + T^2) \sum_{t \leq T} t \kappa(t)^4 \ll M^{-2/3} (M + T^2) T^\varepsilon. \end{aligned}$$

The lemma follows immediately, on summing over $M = 2^v \leq N$. $\square$

Lemma 2.7 *Let $T \geq 1$. Then*

$$\begin{aligned} \sum_{n \leq N} \left(r(n) - \frac{2}{3} C n^{-1/3} \mathfrak{s}(n; T) \right)^2 \\ = \sum_{n \leq N} r(n)^2 + O(N^{1/2+\varepsilon} + T N^{1/3+\varepsilon} + T^2 N^\varepsilon). \end{aligned}$$

Proof We square out and recall the bound

$$\sum_{n \leq N} n^{-2/3} \mathfrak{s}(n; T)^2 \ll (N^{1/3} + T^2) N^\varepsilon$$

that is available from Lemma 2.6 and (2.9). For the cross product term, Cauchy's inequality together with Lemma 2.1 yields

$$\sum_{n \leq N} r(n) n^{-1/3} \mathfrak{s}(n; T) \ll N^{1/3} (N^{1/3} + T^2)^{1/2} N^\varepsilon,$$

and the lemma follows. $\square$

3 The proof of Theorem 1.4

We begin by squaring out (1.4). This yields

$$U(N, Q, T) = \lfloor Q \rfloor \sum_{n \leq N} \left(r(n) - \frac{2}{3} C\mathfrak{s}(n; T) n^{-1/3} \right)^2 + 2S \quad (3.1)$$

where

$$S = \sum_{q \leq Q} \sum_{\substack{1 \leq m < n \leq N \\ n \equiv m \pmod{q}}} \left(r(m) - \frac{2}{3} C\mathfrak{s}(m; T) m^{-1/3} \right) \left(r(n) - \frac{2}{3} C\mathfrak{s}(n; T) n^{-1/3} \right).$$

Now form the exponential sums

$$G(\alpha) = \sum_{n \leq N} \left(r(n) - \frac{2}{3} C\mathfrak{s}(n; T) n^{-1/3} \right) e(\alpha n)$$

and

$$F(\alpha) = \sum_{q \leq Q} \sum_{r \leq N/q} e(\alpha q r).$$

Then

$$S = \int_0^1 F(\alpha) |G(\alpha)|^2 d\alpha.$$

For later use, we note that

$$G(\alpha) = g(\alpha) - \frac{2}{3} Ch(\alpha) \quad (3.2)$$

so that

$$|G(\alpha)|^2 \ll |g(\alpha)|^2 + |h(\alpha)|^2.$$

By (2.5) and Lemma 2.6, it follows that

$$\int_0^1 |G(\alpha)|^2 d\alpha \ll N^{2/3} + T^{2+\varepsilon}. \quad (3.3)$$

We now follow Section 3 of Goldston and Vaughan [3]. By Dirichlet's method of the hyperbola,

$$\sum_{q \leq Q} \sum_{m \leq N/q} e(\alpha q m) = \sum_{q \leq \sqrt{N}} \sum_{m \leq N/q} e(\alpha q m) + \sum_{q \leq \sqrt{N}} \sum_{\sqrt{N} < m \leq \min(Q, N/q)} e(\alpha q m).$$

Given a number r , on the right-hand side we separate terms with $r \mid q$ from terms where $r \nmid q$. Then

$$F(\alpha) = F_r(\alpha) + H_r(\alpha)$$

where

$$F_r(\alpha) = \sum_{\substack{q \leq \sqrt{N} \\ r \mid q}} \sum_{m \leq N/q} e(\alpha q m) + \sum_{\substack{q \leq \sqrt{N} \\ r \nmid q}} \sum_{\sqrt{N} < m \leq \min(Q, N/q)} e(\alpha q m) \quad (3.4)$$

and $H_r(\alpha)$ is the corresponding multiple sum with $r \nmid q$. On performing the inner summation in $H_r(\alpha)$ we have

$$H_r(\alpha) \ll \sum_{\substack{q \leq \sqrt{N} \\ r \nmid q}} \min(Nq^{-1}, \|\alpha q\|^{-1}).$$

For given b and r with $(b, r) = 1$ we put

$$\beta = \alpha - b/r.$$

Then $\|\alpha q\| \geq \|bq/r\| - |\beta|q$ and so when

$$q \leq \sqrt{N}, \quad r \nmid q \quad \text{and} \quad |\beta| \leq \frac{1}{2} r^{-1} N^{-1/2}$$

we have

$$H_r(\alpha) \ll \sum_{\substack{q \leq \sqrt{N} \\ r \nmid q}} \|bq/r\|^{-1} \ll (\sqrt{N}r^{-1} + 1)r \log 2r$$

and so

$$H_r(\alpha) \ll (\sqrt{N} + r) \log 2r \quad \text{when} \quad |\beta| \leq \frac{1}{2} r^{-1} N^{-1/2}.$$

We suppose that R and T satisfy

$$2N^{1/2} \leq R \leq N^{2/3}, \quad 1 \leq T \leq R, \quad (3.5)$$

and consider a typical interval $\mathfrak{M}(r, b)$ associated with the element b/r of the Farey dissection of order R , namely, when $1 \leq b \leq r \leq R$ and $(b, r) = 1$,

$$\mathfrak{M}(r, b) = \left(\frac{b + b_-}{r + r_-}, \frac{b + b_+}{r + r_+} \right]$$

where $r_{\pm}$ is defined by $br_{\pm} \equiv \mp 1 \pmod{r}$ and $R - r < r_{\pm} \leq R$ and $b_{\pm}$ is defined by $b_{\pm} = (br_{\pm} \pm 1)/r$. We observe that

$$\left| \frac{b + b_{\pm}}{r + r_{\pm}} - \frac{b}{r} \right| = \frac{1}{r(r + r_{\pm})}$$

lies in $[1/(2rR), 1/(rR))$.

We have

$$S = \sum_{r \leq R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{M}(r,b)} F(\alpha) |G(\alpha)|^2 d\alpha$$

and

$$\sum_{r \leq R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{M}(r,b)} H_r(\alpha) |G(\alpha)|^2 d\alpha \ll R(\log N) \int_0^1 |G(\alpha)|^2 d\alpha.$$

We note also that $F_r(\alpha) = 0$ when $r > N^{1/2}$. Hence, by (3.3),

$$S = S_1 + O(R(N^{2/3+\varepsilon} + T^{2+\varepsilon})) \quad (3.6)$$

where

$$S_1 = \sum_{r \leq \sqrt{N}} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{M}(r,b)} F_r(\alpha) |G(\alpha)|^2 d\alpha.$$

Recalling that $\beta = \alpha - b/r$, we infer from (3.4) that

$$F_r(\alpha) \ll \sum_{m \leq \sqrt{N}/r} \min(Nr^{-1}m^{-1}, \|\beta rm\|^{-1}).$$

Hence

$$F_r(\alpha) \ll \frac{N \log(2\sqrt{N}/r)}{r + rN|\beta|} \left(r \leq \sqrt{N}, \quad |\beta| \leq \frac{1}{2} r^{-1} N^{-1/2} \right). \quad (3.7)$$

Suppose that $r \leq \sqrt{N}$ and define the major arc $\mathfrak{N}(r, b)$ by

$$\mathfrak{N}(r, b) = \left[\frac{b}{r} - \frac{1}{2rR}, \frac{b}{r} + \frac{1}{2rR} \right].$$

Then $\mathfrak{N}(r, b) \subset \mathfrak{M}(r, b)$ and for $\alpha \in \mathfrak{M}(r, b) \setminus \mathfrak{N}(r, b)$ we have

$$F_r(\alpha) \ll R \log N.$$

Moreover, the same conclusion holds when $\alpha \in \mathfrak{N}(r, b)$ and $r > N/R$. Thus, by (3.3) and (3.6),

$$S = S_2 + O(R(N^{2/3+\varepsilon} + T^{2+\varepsilon})) \quad (3.8)$$

where

$$S_2 = \sum_{r \leq N/R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} F_r(\alpha) |G(\alpha)|^2 d\alpha. \quad (3.9)$$

We now apply major arcs arguments to the function $G(\alpha)$. By (2.8) and (1.3),

$$h(\alpha) = \sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1}}^t \left(\frac{S(t, c)}{t} \right)^2 \sum_{n \leq N} n^{-1/3} e((\alpha - c/t)n),$$

whence by (3.2) and (2.7),

$$G(\alpha) = g(\alpha) - \sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1}}^t W(\alpha; t, c). \quad (3.10)$$

We wish to use this within (3.9). Hence, suppose that $\alpha \in \mathfrak{N}(r, b)$ where $1 \leq b \leq r \leq N/R$ and $(b, r) = 1$. Then

$$G(\alpha) = g(\alpha) - W(\alpha; r, b) + D(\alpha; r, b) \quad (3.11)$$

where in view of (3.10) we have

$$D(\alpha; r, b) = W(\alpha; r, b) - \sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1}}^t W(\alpha; t, c). \quad (3.12)$$

Here we estimate a typical summand when $c/t \neq b/r$. If $\alpha \in \mathfrak{N}(r, b)$, then $|\alpha - b/r| \leq (2rR)^{-1}$, and therefore, by (3.5)

$$\left\| \alpha - \frac{c}{t} \right\| \geq \left\| \frac{b}{r} - \frac{c}{t} \right\| - \frac{1}{2rR} \geq \frac{1}{2} \left\| \frac{b}{r} - \frac{c}{t} \right\|.$$

Hence, by Lemma 2.3, (2.6) and (2.7), we have the bounds

$$W(\alpha; t, c) \ll \min(\kappa(t)^2, t^{-2/3}) \left\| \frac{b}{r} - \frac{c}{t} \right\|^{-2/3}. \quad (3.13)$$

We wish to sum this over all c and t with $c/t \neq b/r$. For a given t there is at most one c with $1 \leq c \leq t$, $(c, t) = 1$ and

$$\left\| \frac{b}{r} - \frac{c}{t} \right\| < \frac{1}{2t}. \quad (3.14)$$

If such a c exists we denote it by $c_0(t)$. The fractions c/t are spaced at least $1/t$ apart, modulo 1. Hence, by (3.13) and (3.14),

$$\sum_{\substack{c=1 \\ (c,t)=1 \\ c \neq c_0(t)}}^t W(\alpha; t, c) \ll \kappa(t)^2 \sum_{l=1}^t \left(\frac{l}{t} \right)^{-2/3} \ll \kappa(t)^2 t.$$

By (2.2) it follows that

$$\sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1 \\ c \neq c_0(t)}}^t W(\alpha; t, c) \ll \sum_{t \leq T} t \kappa(t)^2 \ll T^{1+\varepsilon}. \quad (3.15)$$

Let $\mathcal{T}$ be the set of all t with $1 \leq t \leq T$ where $c_0(t)$ exists and satisfies $c_0(t)/t \neq b/r$. Then

$$\left\| \frac{b}{r} - \frac{c_0(t)}{t} \right\| \geq \frac{1}{rt}.$$

We split into dyadic ranges. For $1 \leq Z \leq T$ let $\mathcal{T}(Z) = \mathcal{T} \cap [Z, 2Z)$. Then the fractions $c_0(t)/t$ are spaced $(2Z)^{-2}$ apart as t varies over $\mathcal{T}(Z)$, and the $c_0(t)/t$ stay away from b/r at least $(2rZ)^{-1}$. By (3.13), we infer that

$$\sum_{t \in \mathcal{T}(Z)} W(\alpha; t, c_0(t)) \ll Z^{-2/3} \sum_{0 \leq l \leq Z} \left(\frac{1}{rZ} + \frac{l}{Z^2} \right)^{-2/3} \ll r^{2/3} + Z.$$

We sum over $Z = 2^v \leq T$ and combine the result with (3.15). This yields

$$\sum_{t \leq T} \sum_{\substack{c=1 \\ (c,t)=1 \\ c/t \neq b/r}}^t W(\alpha; t, c) \ll (r^{2/3} + T) T^\varepsilon. \quad (3.16)$$

Now, inspecting (3.12), we see that if $r \leq T$, then the sum on the left-hand side of (3.16) is $-D(\alpha; r, b)$ while in the case $r > T$ the condition $c/t \neq b/r$ is always satisfied. Consequently, by (3.16), whenever $\alpha \in \mathfrak{N}(r, b)$ we have the estimates

$$D(\alpha; r, b) \ll (r^{2/3} + T) T^\varepsilon \quad (r \leq T) \quad (3.17)$$

and

$$D(\alpha; r, b) \ll |W(\alpha; r, b)| + (r^{2/3} + T)T^\varepsilon \quad (r > T). \quad (3.18)$$

Lemma 3.1 Suppose that (3.5) holds. Then

$$\begin{aligned} \sum_{r \leq N/R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} |F_r(\alpha)| |D(\alpha; r, b)|^2 d\alpha \\ \ll \left(\left(\frac{N}{R} \right)^{7/3} + \frac{NT^2}{R} \right) N^\varepsilon + N^{4/3} T^{\varepsilon-1}. \end{aligned}$$

Proof We begin with the set of all α where $D(\alpha; r, b) \ll (r^{2/3} + T)T^\varepsilon$ holds. This set makes a contribution to the integral in question that in view of (3.7) does not exceed

$$\begin{aligned} &\ll T^\varepsilon \sum_{r \leq N/R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} |F_r(\alpha)| (r^{4/3} + T^2) d\alpha \\ &\ll T^\varepsilon \sum_{r \leq N/R} \int_0^1 \frac{N \log N}{1 + N|\beta|} (r^{4/3} + T^2) d\beta \\ &\ll \left(\left(\frac{N}{R} \right)^{7/3} + \frac{NT^2}{R} \right) N^\varepsilon. \end{aligned}$$

By (3.17) and (3.18), on the remaining set we have $r > T$ and $D(\alpha; r, b) \ll |W(\alpha; r, b)|$. Then, by (3.14) and (3.13), together with (2.2), (2.6), (2.7) and Lemma 2.3 we see that

$$\begin{aligned} \sum_{T < r \leq N/R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} |F_r(\alpha)| |W(\alpha; r, b)|^2 d\alpha \\ \ll N^{7/3+\varepsilon} \sum_{r > T} \kappa(r)^4 \int_{-\infty}^{\infty} (1 + N|\beta|)^{-2} d\beta \ll N^{4/3+\varepsilon} T^{-1}. \end{aligned} \quad (3.19)$$

This confirms the estimate proposed in Lemma 3.1. $\square$

Lemma 3.2 Suppose that (3.5) holds. Then

$$\sum_{r \leq N/R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} |F_r(\alpha) W(\alpha; r, b)| d\alpha \ll N^{2/3+\varepsilon}.$$

Proof The reasoning leading to (3.19) applies here as well, and estimates the integral in question by

$$\ll N^{5/3+\varepsilon} \sum_{r \leq N/R} \kappa(r)^2 \int_0^1 (1 + N|\beta|)^{-5/3} d\beta \ll N^{2/3+\varepsilon}.$$

□

Lemma 3.3 Suppose that (3.5) holds. Then

$$\sum_{r \leq N/R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} |F_r(\alpha) g(\alpha)| d\alpha \ll N^{4/3+\varepsilon} R^{-1}.$$

Proof Within this proof, we abbreviate $f(\alpha, N^{1/3})$ to $f(\alpha)$. The main observation is that in a suitable averaged sense one may replace $|g(\alpha)|$ by $|f(\alpha)|^2$. Let

$$K(\alpha) = \sum_{n \leq N} e(-\alpha n).$$

By orthogonality,

$$g(\alpha) = \int_0^1 f(\alpha + \gamma)^2 K(\gamma) d\gamma. \quad (3.20)$$

Let J be the expression on the left-hand side of (3.20). Then, by (3.20),

$$J \leq \int_0^1 |K(\gamma)| \sum_{r \leq X/R} \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} |F_r(\alpha)| |f(\alpha + \gamma)|^2 d\alpha d\gamma. \quad (3.21)$$

We have

$$|f(\theta)|^2 = \sum_{h \in \mathbb{Z}} \psi_h e(\theta h) \quad (3.22)$$

where ψ_h is the number of $(x, y) \in \mathbb{Z}^2$ with $x^3 - y^3 = h$ and $1 \leq x, y \leq N^{1/3}$. By (3.7), we have

$$\begin{aligned} & \int_{\mathfrak{N}(r,b)} |F_r(\alpha)| |f(\alpha + \gamma)|^2 d\alpha \\ & \ll N^{1+\varepsilon} r^{-1} \int_{-1}^1 (1 + N|\beta|)^{-1} \left| f\left(\frac{b}{r} + \beta + \gamma\right) \right|^2 d\beta. \end{aligned}$$

We now use (3.22) and sum over b to bring in Ramanujan's sum

$$c_r(h) = \sum_{\substack{b=1 \\ (b,r)=1}}^r e\left(\frac{bh}{r}\right).$$

This produces

$$\begin{aligned} & \sum_{\substack{b=1 \\ (b,r)=1}}^r \int_{\mathfrak{N}(r,b)} |F_r(\alpha)| |f(\alpha + \gamma)|^2 d\alpha \\ & \ll N^{1+\varepsilon} r^{-1} \sum_h c_r(h) \psi_h \int_{-1}^1 \frac{e((\beta + \gamma)h)}{1 + N|\beta|} d\beta. \end{aligned}$$

Now note that $\psi_h \geq 0$. Hence, the classical bound $|c_r(h)| \leq (r, h)$ suffices to conclude that the above expression does not exceed

$$\ll N^{1+\varepsilon} r^{-1} \sum_h (r, h) \psi_h \int_{-1}^1 (1 + N|\beta|)^{-1} d\beta \ll N^{2\varepsilon} r^{-1} \sum_h (r, h) \psi_h.$$

Observe that this bound is uniform in γ . We sum over r and deduce from (3.21) that

$$J \ll \left(\int_0^1 |K(\gamma)| d\gamma \right) N^\varepsilon \sum_h \psi_h \sum_{r \leq N/R} \frac{(r, h)}{r}.$$

The elementary bound $K(\gamma) \ll \min(N, \|\gamma\|^{-1})$ shows that the first factor on the right is $O(\log N)$. This yields

$$J \ll \psi_0 N^{1+\varepsilon} R^{-1} + N^\varepsilon \sum_h \psi_h.$$

The lemma follows immediately. $\square$

The proof of Theorem 1.4 is now readily completed. We return to (3.11) and apply the elementary inequality $|\alpha + \beta|^2 \leq 2|\alpha|^2 + 2|\beta|^2$ to confirm that

$$\begin{aligned} |G(\alpha)|^2 & \leq 2|g(\alpha) - W(\alpha; r, b)|^2 + 2|D(\alpha; r, b)|^2 \\ & \leq 2|g(\alpha) - W(\alpha; r, b)|(|g(\alpha)| + |W(\alpha; r, b)|) + 2|D(\alpha; r, b)|^2. \end{aligned}$$

We multiply by $|F_r(\alpha)|$, integrate over $\mathfrak{N}(r, b)$ and sum over $1 \leq b \leq r$, $(b, r) = 1$ and $r \leq N/R$. The factor $g - W$ is bounded by $N^{5/6+\varepsilon} R^{-1/2}$, as can be seen from Lemma 2.5. Then, by (3.9) and Lemmata 3.1, 3.2 and 3.3, we find that

$$S_2 \ll N^\varepsilon (N^{13/6} R^{-3/2} + (N/R)^{7/3} + NR^{-1} T^2 + N^{4/3} T^{-1} + N^{3/2} R^{-1/2}).$$

This bound combines with (3.8) to an estimate for S . The choice $R = N^{3/5}$ balances the term $N^{13/6}R^{-3/2}$ in the preceding display with the term $RN^{2/3}$ in (3.8). The resulting bound for S in conjunction with (3.1) establishes Theorem 1.4 when $Q \leq N$. When $Q > N$ the extra terms compared with the case $Q = N$ contribute trivially

$$(\lfloor Q \rfloor - N) \sum_{n=1}^N \left(r(n) - \frac{2}{3} C \mathfrak{s}(n; T) n^{-1/3} \right)^2$$

and the proof is completed by an appeal to Lemma 2.7.

4 The proof of Theorem 1.1

In this section we compare $V(N, Q)$ with $U(N, Q, T)$, thereby establishing Theorem 1.1. At the core of our argument is a mean square estimate for the difference between the approximants to the expectation used in U and V . In U we encountered the term

$$\Xi(N, T; q, a) = \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} \mathfrak{s}(n; T) \quad (4.1)$$

while in V the arithmetically obvious expectation is used. Thus we are led to consider

$$\Delta(N, Q, T) = \sum_{q \leq Q} \sum_{a=1}^q \left| \Xi(N, T; q, a) - \frac{3\rho(q, a)}{2q^2} N^{2/3} \right|^2. \quad (4.2)$$

Lemma 4.1 *Let $1 \leq T \leq Q \leq N$. Then*

$$\Delta(N, Q, T) \ll (Q^{4/3}T^2 + N^{4/3}T^{-1})N^\varepsilon.$$

Proof The main observation underpinning the proof of Lemma 4.1 is the identity

$$\frac{\rho(q, a)}{q^2} = \frac{1}{q} \sum_{t|q} \sum_{\substack{c=1 \\ (c,t)=1}}^t \frac{S(t, c)^2}{t^2} e(-ca/t) \quad (4.3)$$

that is an instance of [11, Lemma 2.12]. This suggests to split the partial singular series $\mathfrak{s}(n; T)$ into

$$\mathfrak{s}_q^*(n; T) = \sum_{\substack{t \leq T \\ t|q}} \sum_{\substack{c=1 \\ (c,t)=1}}^t \frac{S(t, c)^2}{t^2} e(-cn/t) \quad (4.4)$$

and its complement

$$\mathfrak{s}_q^\dagger(n; T) = \sum_{\substack{t \leq T \\ t \nmid q}} \sum_{\substack{c=1 \\ (c,t)=1}}^t \frac{S(t, c)^2}{t^2} e(-cn/t). \quad (4.5)$$

Let $\Xi^\dagger(N, T; q, a)$ be the sum defined in (4.1), but with $\mathfrak{s}(n; T)$ replaced by $\mathfrak{s}_q^\dagger(n; T)$. Later, we shall use this notation also in situations where Ξ and $\mathfrak{s}$ are decorated by symbols other than $\dagger$. We expect $\Xi^\dagger(N, T; q, a)$ to be small on average. Temporarily we suppose that $1 \leq a \leq q$. Then

$$\begin{aligned} \Xi^\dagger(N, T; q, a) &= \sum_{0 \leq h \leq (N-a)/q} (a + hq)^{-1/3} \mathfrak{s}_q^\dagger(a + hq, T) \\ &= \sum_{\substack{t \leq T \\ t \nmid q}} \sum_{\substack{c=1 \\ (c,t)=1}}^t \frac{S(t, c)^2}{t^2} e\left(-\frac{ca}{t}\right) \sum_{0 \leq h \leq (N-a)/q} (a + hq)^{-1/3} e\left(-\frac{hqc}{t}\right). \end{aligned}$$

Note that $q \leq N$, so the sums over h always include the term with $h = 0$. Their total contribution to $\Xi^\dagger(N, T; q, a)$ is

$$a^{-1/3} \sum_{\substack{t \leq T \\ t \nmid q}} \sum_{\substack{c=1 \\ (c,t)=1}}^t \frac{S(t, c)^2}{t^2} e\left(-\frac{ca}{t}\right). \quad (4.6)$$

For $h \geq 1$ we have $a + hq \gg hq$, and so, by partial summation,

$$\sum_{1 \leq h \leq (N-a)/q} (a + hq)^{-1/3} e\left(-\frac{hqc}{t}\right) \ll q^{-1/3} \left\| \frac{qc}{t} \right\|^{-1}.$$

Hence the contribution to $\Xi^\dagger(N, T; q, a)$ from terms with $h \geq 1$ is bounded by

$$\begin{aligned} \sum_{\substack{t \leq T \\ t \nmid q}} \sum_{\substack{c=1 \\ (c,t)=1}}^t \left| \frac{S(t, c)}{t} \right|^2 q^{-1/3} \left\| \frac{qc}{t} \right\|^{-1} &\ll q^{-1/3} \sum_{\substack{t \leq T \\ t \nmid q}} \kappa(t)^2 \sum_{\substack{c=1 \\ (c,t)=1}}^t \left\| \frac{qc}{t} \right\|^{-1} \\ &\ll q^{-1/3} \sum_{t \leq T} t^{1+\varepsilon} \kappa(t)^2 \ll q^{-1/3} T^{1+2\varepsilon}. \end{aligned}$$

Applying similar estimates within (4.6), we first find

$$\Xi^\dagger(N, T; q, a) \ll (a^{-1/3} + q^{-1/3}) T^{1+\varepsilon}$$

and then see that

$$\sum_{q \leq Q} \sum_{a=1}^q |\Xi^{\dagger}(N, T; q, a)|^2 \ll Q^{4/3} T^{2+\varepsilon}. \quad (4.7)$$

A similar argument applies to a tail version of the sum in (4.4). This we define by

$$\mathfrak{s}_q^{\ddagger}(n; T) = \sum_{\substack{t > T \\ t|q}} \sum_{\substack{c=1 \\ (c,t)=1}}^t \frac{S(t, c)^2}{t^2} e(-cn/t). \quad (4.8)$$

This sum depends on n only modulo q . Therefore, with the convention concerning decorations of Ξ in mind,

$$\Xi^{\ddagger}(N, T; q, a) = \mathfrak{s}_q^{\ddagger}(a; T) \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3}.$$

Much as before, we find that

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} \ll a^{-1/3} + \sum_{1 \leq h \leq (N-a)/q} (a + hq)^{-1/3} \ll a^{-1/3} + N^{2/3} q^{-1},$$

and consequently,

$$\sum_{a=1}^q |\Xi^{\ddagger}(N, T; q, a)|^2 \ll N^{4/3} q^{-2} \sum_{a=1}^q |\mathfrak{s}_q^{\ddagger}(a; T)|^2 + \sum_{a=1}^q a^{-2/3} |\mathfrak{s}_q^{\ddagger}(a; T)|^2. \quad (4.9)$$

By (4.8) and Cauchy's inequality, and then by orthogonality,

$$\begin{aligned} \sum_{a=1}^q |\mathfrak{s}_q^{\ddagger}(a; T)|^2 &\ll q^{\varepsilon} \sum_{a=1}^q \sum_{\substack{t > T \\ t|q}} \left| \sum_{\substack{c=1 \\ (c,t)=1}}^t \frac{S(t, c)^2}{t^2} e\left(-\frac{ca}{t}\right) \right|^2 \\ &\ll q^{1+\varepsilon} \sum_{\substack{t > T \\ t|q}} \sum_{\substack{c=1 \\ (c,t)=1}}^t \left| \frac{S(t, c)}{t} \right|^4 \ll q^{1+\varepsilon} \sum_{\substack{t > T \\ t|q}} t \kappa(t)^4. \end{aligned}$$

Similarly, again via (4.8), we have

$$\mathfrak{s}_q^{\ddagger}(a; T) \ll \sum_{\substack{t > T \\ t|q}} t \kappa(t)^2,$$

implying that

$$\sum_{a=1}^q a^{-2/3} |\mathfrak{s}_q^\ddagger(a; T)|^2 \ll q^{1/3} \left(\sum_{\substack{t>T \\ t|q}} t \kappa(t)^2 \right)^2.$$

From (4.9) we now see

$$\sum_{q \leq Q} \sum_{a=1}^q |\Xi^\ddagger(N, T; q, a)|^2 \ll N^{4/3+\varepsilon} Y_1 + Y_2$$

where

$$Y_1 = \sum_{q \leq Q} q^{-1} \sum_{\substack{t>T \\ t|q}} t \kappa(t)^4, \quad Y_2 = \sum_{q \leq Q} q^{1/3} \left(\sum_{\substack{t>T \\ t|q}} t \kappa(t)^2 \right)^2.$$

To bound Y_1 , write $q = ts$ and reverse the order of summation. Via (2.2), we find that

$$Y_1 \ll \sum_{T < t \leq Q} \kappa(t)^4 \sum_{s \leq Q/t} s^{-1} \ll (\log Q) \sum_{T < t \leq Q} \kappa(t)^4 \ll N^\varepsilon T^{-1}.$$

The estimation of Y_2 begins with Cauchy's inequality, applied to the sum over t . Then, by the same procedure that was successful with Y_1 , we find that

$$\begin{aligned} Y_2 &\ll Q^\varepsilon \sum_{q \leq Q} q^{1/3} \sum_{\substack{t>T \\ t|q}} t^2 \kappa(t)^4 \\ &\ll Q^\varepsilon \sum_{T < t \leq Q} t^{7/3} \kappa(t)^4 \sum_{s \leq Q/t} s^{1/3} \ll Q^{4/3+\varepsilon} \sum_{t \leq Q} t \kappa(t)^4 \ll Q^{4/3+2\varepsilon}. \end{aligned}$$

This proves that

$$\sum_{q \leq Q} \sum_{a=1}^q |\Xi^\ddagger(N, T; q, a)|^2 \ll N^{4/3+\varepsilon} T^{-1} + Q^{4/3+\varepsilon}. \quad (4.10)$$

We collect the results obtained so far in a single estimate. By (4.3), (4.4), (4.5) and (4.8), we have

$$\mathfrak{s}(n; T) = \mathfrak{s}_q^\dagger(n; T) + \mathfrak{s}_q^*(n; T) = \mathfrak{s}_q^\dagger(n; T) - \mathfrak{s}_q^\ddagger(n; T) + \frac{\rho(q, a)}{q}.$$

It follows that

$$\Xi(N, T; q, a) - \frac{\rho(q, a)}{q} \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} = \Xi^\dagger(N, T; q, a) - \Xi^\ddagger(N, T; q, a),$$

and from (4.7) and (4.10) we then conclude

$$\begin{aligned} \sum_{q \leq Q} \sum_{a=1}^q \left| \Xi(N, T; q, a) - \frac{\rho(q, a)}{q} \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} \right|^2 \\ \ll (Q^{4/3} T^2 + N^{4/3} T^{-1}) N^\varepsilon. \end{aligned} \quad (4.11)$$

By (4.11) and (4.2), we are reduced to comparing

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} \quad \text{with} \quad \frac{1}{q} \sum_{n \leq N} n^{-1/3}.$$

By orthogonality, we see that

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} - \frac{1}{q} \sum_{n \leq N} n^{-1/3} = \frac{1}{q} \sum_{b=1}^{q-1} e\left(-\frac{ab}{q}\right) \sum_{n \leq N} n^{-1/3} e\left(\frac{bn}{q}\right).$$

Here, by (2.6), the sum on the far right over n is $O(\|b/q\|^{-2/3})$. Further, by (4.3) and Lemma 2.3,

$$\frac{\rho(q, a)}{q} \ll \sum_{t|q} t \kappa(t)^2. \quad (4.12)$$

It now follows that the sum

$$\sum_{q \leq Q} \sum_{a=1}^q \left(\frac{\rho(q, a)}{q} \right)^2 \left| \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} - \frac{1}{q} \sum_{n \leq N} n^{-1/3} \right|^2 \quad (4.13)$$

is bounded above by

$$\begin{aligned} &\ll \sum_{q \leq Q} \left(\sum_{t|q} t \kappa(t)^2 \right)^2 \frac{1}{q^2} \sum_{a=1}^q \left| \sum_{b=1}^{q-1} e\left(-\frac{ab}{q}\right) \sum_{n \leq N} n^{-1/3} e\left(\frac{bn}{q}\right) \right|^2 \\ &\ll \sum_{q \leq Q} \left(\sum_{t|q} t \kappa(t)^2 \right)^2 \frac{1}{q} \sum_{b=1}^{q-1} \left\| \frac{b}{q} \right\|^{-4/3} \ll \sum_{q \leq Q} q^{1/3} \left(\sum_{t|q} t \kappa(t)^2 \right)^2 \ll Q^{4/3+\varepsilon}, \end{aligned}$$

where the very last bound has been confirmed earlier, in our estimate of Y_2 above.

We now invoke the elementary evaluation

$$\sum_{n \leq N} n^{-1/3} = \frac{3}{2} N^{2/3} + O(1) \quad (4.14)$$

together with (4.12) to confirm the bounds

$$\sum_{q \leq Q} \sum_{a=1}^q \frac{\rho(q, a)^2}{q^4} \left| \sum_{n \leq N} n^{-1/3} - \frac{3}{2} N^{2/3} \right|^2 \ll \sum_{q \leq Q} \frac{1}{q} \left(\sum_{t|q} t \kappa(t)^2 \right)^2 \ll Q^\varepsilon.$$

Equipped with the estimate obtained for the sum in (4.13), we deduce that

$$\sum_{q \leq Q} \sum_{a=1}^q \frac{\rho(q, a)^2}{q^2} \left| \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} - \frac{3N^{2/3}}{2q} \right|^2 \ll Q^{4/3+\varepsilon}.$$

This bound coupled with (4.11) implies Lemma 4.1. $\square$

The endgame begins by writing (1.1) as

$$V(N, Q) = \sum_{q \leq Q} \sum_{a=1}^q (A(N, q, a) + B(N, q, a))^2$$

where

$$\begin{aligned} A(N, q, a) &= \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \left(r(n) - \frac{2}{3} C n^{-1/3} \mathfrak{s}(n; T) \right), \\ B(N, q, a) &= \frac{2}{3} C \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} n^{-1/3} \mathfrak{s}(n; T) - \frac{\rho(q, a)}{q^2} C N^{2/3}. \end{aligned}$$

Then, squaring out and estimating the cross term by Cauchy's inequality, we obtain, by (1.4), (4.1) and (4.2),

$$V(N, Q) = U(N, Q, T) + \left(\frac{2}{3} C \right)^2 \Delta(N, Q, T) + O((U \Delta)^{1/2}).$$

First suppose that $N^{3/5} \leq Q \leq N^{17/20}$, and take $T = N^{1/15}$. Then, by Lemmata 2.7 and 2.1, we have

$$\sum_{n \leq N} \left(r(n) - \frac{2}{3} C n^{-1/3} \mathfrak{s}(n; T) \right)^2 = 2C N^{2/3} + O(N^{1/2+\varepsilon}),$$

and consequently, by Theorem 1.4,

$$U(N, Q, T) = Q(2CN^{2/3} + O(N^{1/2+\varepsilon})) + O(N^{19/15+\varepsilon}).$$

Lemma 4.1 yields $\Delta(N, Q, T) \ll N^{19/15+\varepsilon}$, and we see that

$$V(N, Q) = 2CN^{2/3} + O(QN^{1/2+\varepsilon} + N^{19/15+\varepsilon} + Q^{1/2}N^{29/30+\varepsilon}).$$

Here the first summand in the error term is dominated by the last summand, so this reduces to the estimate claimed in Theorem 1.1.

Next suppose that $N^{17/20} < Q \leq N$, and take $T = (N/Q)^{4/9}$. Then $1 \leq T \leq N^{1/15}$, Lemma 4.1 gives $\Delta \ll N^{8/9+\varepsilon}Q^{4/9}$ and as above, Theorem 1.4 delivers

$$U(N, Q, T) = Q(2CN^{2/3} + O(N^{1/2+\varepsilon})) + O(N^{8/9+\varepsilon}Q^{4/9}).$$

Proceeding as before, we obtain

$$V(N, Q) = 2CQN^{2/3} + O(QN^{1/2+\varepsilon} + Q^{4/9}N^{8/9+\varepsilon} + Q^{13/18}N^{7/9+\varepsilon}).$$

Here, in the error term, the last summand dominates the others, so this also reduces to the estimate claimed in Theorem 1.1.

In the range $Q \leq N^{3/5}$ the theorem only asserts that $V(N, Q) \ll N^{19/15+\varepsilon}$ which follows from the bound for $V(N, N^{3/5})$ that we have already established.

5 The proof of Theorem 1.3

The proof of Theorem 1.3 is in principle simpler than that of Theorem 1.1 as it does not require the use of the convolution device embedded in the proof of Lemma 3.3. One follows the proceedings of Sect. 3 and replaces $g(\alpha)$ with $f(\alpha)^2$ where $f(\alpha) = f(\alpha; X)$ is as in (1.6). Then, by Lemma 2.4, (2.4) and Lemma 2.3, the estimate given in Lemma 2.5 can be replaced by

$$f(\alpha)^2 = V_1(\alpha; q, a)^2 + O(Xq^{1/6+\varepsilon}(1 + X^3|\alpha - a/q|)^{1/6} + q^{1+\varepsilon}(1 + X^3|\alpha - a/q|))$$

in the final stages of the proof of Theorem 1.4. For easier comparison, we put $W^*(\alpha; q, a) = V_1(\alpha; q, a)^2$ and then have, in the context of the last paragraph of Sect. 3,

$$f(\alpha)^2 - W^*(\alpha; b, r) \ll X^{3/2+\varepsilon}R^{-1/6} + X^{3+\varepsilon}R^{-1} \quad (5.1)$$

which can be used in place of the bound $g - W \ll N^{5/6+\varepsilon}R^{-1/2}$ and results in the possibility of taking a smaller value for R .

To be more precise, to effect the translation let $N = 2X^3$, and take

$$\begin{aligned} R(n) &= \sum_{\substack{x, y \leq X \\ x^3 + y^3 = n}} 1, \\ \tau(n, Y) &= \sum_{\substack{x, y \leq Y \\ x^3 + y^3 = n}} \frac{1}{9} (xy)^{-2/3}, \\ U^*(X, Q, T) &= \sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{n \equiv a \pmod{q}} (R(n) - \tau(n, X^3) \mathfrak{s}(n; T)) \right|^2, \end{aligned}$$

and

$$h^*(\alpha) = \sum_{t \leq T} \sum_{\substack{c=1 \\ (t, c)=1}}^t W^*(\alpha; t, c).$$

Then following the argument of Sect. 3 we have

$$U^*(X, Q, T) = \lfloor Q \rfloor \sum_n (R(n) - \tau(n, X^3) \mathfrak{s}(n; T))^2 + 2S^*$$

where

$$S^* = \int_0^1 F(\alpha) |G^*(\alpha)|^2 d\alpha$$

and

$$G^*(\alpha) = f(\alpha)^2 - h^*(\alpha).$$

Then it follows that, cf. (3.8),

$$S^* = S_2^* + O(RX^{2+\varepsilon} + RT^{2+\varepsilon})$$

where

$$S_2^* = \sum_{r \leq N/R} \sum_{\substack{b=1 \\ (b, r)=1}}^r \int_{\mathfrak{N}(r, b)} F_r(\alpha) |G^*(\alpha)|^2 d\alpha.$$

Proceeding to use (5.1) in place of Lemma 2.5 we have

$$S_2^* \ll (X^{11/2} R^{-7/6} + X^7 R^{-2} + X^3 T^2 R^{-1} + X^4 T^{-1}) X^\varepsilon.$$

The choice $R = X^{5/3}$ gives

$$S^* \ll (X^{11/3} + X^4 T^{-1} + X^{5/3} T^2) X^\varepsilon.$$

The analogue of Lemma 2.7 gives

$$\sum_n (R(n) - \tau(n, X^3) \mathfrak{s}(n; T))^2 = \sum_n R(n)^2 + O(X^{3/2+\varepsilon} + T X^{1+\varepsilon} + T^2 X^\varepsilon).$$

Thus

$$\begin{aligned} U^*(X, Q, T) \\ = 2QX^2 + O((X^{11/3} + X^4 T^{-1} + X^{5/3} T^2 + QX^{3/2} + QTX + QT^2) X^\varepsilon). \end{aligned}$$

We define

$$\Delta^*(X, Q, T) = \sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{n \equiv a \pmod{q}} \tau(n, X^3) \mathfrak{s}(n; T) - \frac{\rho(q, a)}{q^2} X^2 \right|^2.$$

Then, for $1 \leq T \leq Q \leq X^3$, as in Lemma 4.1, one has

$$\Delta^*(X, Q, T) \ll (Q^{4/3} T^2 + X^4 T^{-1}) X^\varepsilon. \quad (5.2)$$

Some justification is in order at this stage. Indeed, for a proof of (5.2), in the arguments of Sect. 4, one needs to replace the sum (4.1) by

$$\sum_{n \equiv a \pmod{q}} \tau(n, X^3) \mathfrak{s}(n; T)$$

in which $\tau(n, X^3)$ substitutes the monotonic weight $n^{-1/3}$. While the function $\tau(n, X^3)$ is no longer initially decreasing, mundane asymptotic analysis reveals that

$$\tau(n, n) - \tau(n+1, n+1) = cn^{-4/3} + o(n^{-4/3})$$

where

$$9c = \frac{4}{3} \int_0^{1/2} s^{-2/3} (1-s)^{-5/3} ds - 2^{4/3}$$

is positive. Hence, there is a number n_0 such that $\tau(n, n)$ is decreasing for $n \geq n_0$. Furthermore, a rather more direct argument shows that $\tau(n, X^3)$ is also decreasing for $X^3 < n \leq N$, and one immediately has $\tau(n, X^3) \ll n^{-1/3}$. These properties suffice to

check that the initial phase of the proof of Lemma 4.1 still applies in the new set-up, and one confirms the bound

$$\sum_{q \leq Q} \sum_{a=1}^q \left| \sum_{n \equiv a \pmod{q}} \tau(n, X^3) \mathfrak{s}(n; T) - \frac{\rho(q, a)}{q} \sum_{n \equiv a \pmod{q}} \tau(n, X^3) \right|^2 \ll (Q^{4/3} T^2 + N^{4/3} T^{-1}) N^\varepsilon$$

as the appropriate counterpart of (4.11). The next step is to estimate

$$\begin{aligned} \sum_{n \equiv a \pmod{q}} \tau(n, X^3) - \frac{1}{q} \sum_n \tau(n, X^3) \\ = \frac{1}{q} \sum_{b=1}^{q-1} e\left(-\frac{ab}{q}\right) \sum_n \tau(n, X^3) e\left(\frac{bn}{q}\right), \end{aligned}$$

and progress then depends on the bound

$$\sum_n \tau(n, X^3) e\left(\frac{bn}{q}\right) \ll \left\| \frac{b}{q} \right\|^{-2/3}.$$

This is readily established by applying Abel summation on the interval $\|b/q\|^{-1} \leq n \leq N$, using the monotonicity of $\tau(n, X^3)$, while in the range $n \leq \|b/q\|^{-1}$ the trivial $\tau(n, X^3) \ll n^{-1/3}$ is enough. Finally, we note that

$$\sum_n \tau(n, X^3) = \frac{1}{9} \left(\sum_{x \leq X^3} x^{-2/3} \right)^2 = X^2 + O(X).$$

This is weaker than the bound (4.14) but suffices to mimic the work after (4.11) successfully. However, the error term in the previous display now introduces an extra term of size X^2 into the final bound for Δ^* . This can be absorbed into the expression $Q^{4/3} T^2 + X^4 T^{-1}$, and this proves (5.2).

Now choose $T = X^{1/3}$ for $Q \leq X^{9/4}$, $T = X^{4/3} Q^{-4/9}$ for $X^{9/4} \leq Q \leq 2X^3$. Then, for $Q \leq X^{9/4}$, we find that

$$U^*(X, Q, T) = 2QX^2 + O((X^{11/3} + QX^{3/2})X^\varepsilon)$$

and

$$\Delta^*(X, Q, T) \ll X^{11/3+\varepsilon}.$$

As in the proof of Theorem 1.2 we have

$$V^*(X, Q) = U^*(X, Q, T) + \Delta^*(X, Q, T) + O((U^* \Delta^*)^{1/2}),$$

and this reduces to

$$V^*(X, Q) = 2QX^2 + O(X^{11/3+\varepsilon} + Q^{1/2}X^{17/6+\varepsilon}),$$

and gives the first two cases of the theorem.

When $X^{9/4} \leq Q \leq 2X^3$ we instead have

$$U^*(X, Q, T) = 2QX^2 + O(QX^{3/2+\varepsilon}),$$

and

$$\Delta^*(X, Q, T) \ll Q^{4/9}X^{8/3+\varepsilon}$$

and the final case follows.

6 The proof of Theorem 1.2

The deduction of Theorem 1.2 from Theorem 1.1 is based on Lemma 2.1, and is straightforward. We define $\eta(n)$ through

$$2r_0(n) = r(n) + \eta(n).$$

Then $|\eta(n)| \leq r(n) \ll n^\varepsilon$, and (2.1) yields

$$\sum_{n \leq N} |\eta(n)|^v \ll N^{4/9+\varepsilon} \quad (v = 1, 2).$$

By (1.1) and (1.2),

$$4V_0(N, Q) = V(N, Q) + E_0 + 2E_1 \tag{6.1}$$

where

$$E_0 = \sum_{q \leq Q} \sum_{a=1}^q \left(\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \eta(n) \right)^2, \tag{6.2}$$

$$E_1 = \sum_{q \leq Q} \sum_{a=1}^q \left(\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} r(n) - \frac{\rho(q, a)}{q^2} CN^{2/3} \right) \sum_{\substack{m \leq N \\ m \equiv a \pmod{q}}} \eta(m). \tag{6.3}$$

To bound E_0 , one opens the square and finds that

$$E_0 = \sum_{q \leq Q} \sum_{\substack{n, m \leq N \\ n \equiv m \pmod{q}}} \eta(n) \eta(m).$$

Hence

$$E_0 \ll \sum_{n, m \leq N} |\eta(n)\eta(m)| \sum_{\substack{q|n-m \\ q \leq Q}} 1.$$

Here the terms with $n = m$ contribute

$$\ll Q \sum_{n \leq N} |\eta(n)|^2 \ll QN^{4/9+\varepsilon}$$

while the terms with $n \neq m$ contribute no more than

$$\ll Q^\varepsilon \left(\sum_{n \leq N} |\eta(n)| \right)^2 \ll N^{8/9+\varepsilon}.$$

This shows that

$$E_0 \ll QN^{4/9+\varepsilon} + N^{8/9+\varepsilon}.$$

Further, by (6.2), (6.3), (1.1) and Cauchy's inequality,

$$E_1^2 \ll V(N, Q)E_0.$$

We now temporarily suppose that $N^{3/5} \leq Q \leq N$. Then $E_0 \ll QN^{4/9+\varepsilon}$, and Theorem 1.1 gives $V(N, Q) \ll QN^{2/3}$. Hence $E_1 \ll QN^{5/9+\varepsilon}$. A short calculation now shows that the estimate in Theorem 1.2 follows from (6.1) and Theorem 1.1. As in the proof of Theorem 1.1, the case $Q \leq N^{3/5}$ follows by considering the upper bound implied by the estimate for $V_0(N, N^{3/5})$ that is already established.

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