Kähler packings of projective complex manifolds

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Abstract

We show that the multipoint Seshadri constant determines the maximum possible radii of embeddings of Kähler balls and vice versa.

Keywords Multipoint Seshadri constant · Kähler packings · Nagata's conjecture

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1 Introduction

Seshadri constants were first introduced by Demailly in [3] as a way of studying local positivity of ample line bundles at a given point of a variety. Since then, Seshadri constants have attracted substantial attention in the field of algebraic geometry and can be used to reformulate many classical ideas. A nice example of this is the famous conjecture of Nagata on plane algebraic curves, which in its original form is as follows:

Conjecture 1.1 *Let $P_1, \dots, P_k$ be points of $\mathbb{P}^2$ in general position and $m_1, \dots, m_k$ be positive integers. Then for $k \geq 9$ any curve $C \in \mathbb{P}^2$ of degree d passing through each point P_i with multiplicity m_i must satisfy*

$$d \geq \frac{1}{\sqrt{k}} \sum_{i=1}^k m_i.$$

Using multipoint Seshadri constants this can be rewritten as:

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Conjecture 1.2 *Let $P_1, \dots, P_k$ be points of $\mathbb{P}^2$ in general position then for $k \geq 9$ the multipoint Seshadri constant*

$$\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), P_1, \dots, P_k) = \frac{1}{\sqrt{k}}.$$

Motivated by work of Biran [2], who proved a symplectic analagon between Nagata's conjecture formulated as a packing problem, Eckl proved in [4] that Nagata's conjecture is in fact equivalent to a more restricted packing problem, namely a Kähler packing problem. More generally in dimension 2 there is a direct correspondence between sizes of multi-ball Kähler packings and multipoint Seshadri constants. A similar result was obtained by Witt Nyström in [9] but this time for a variety of any dimension blown up at a single point. The aim of this note is to generalise these results to projective complex manifolds of arbitrary dimension blown up at any number of points (in general position or not). During the writing of this paper a similar result was achieved by Trusiani in [8] however there are some differences in the formulation of the statement and its proof. In more details:

Definition 1.3 Let (X, ω) be an n -dimensional Kähler manifold with Kähler form ω . Then a holomorphic embedding

$$\phi = \coprod_{q=1}^k \phi_q : \coprod_{q=1}^k B_0(r_q) \rightarrow X$$

is called a *Kähler embedding* of k disjoint complex flat balls in $\mathbb{C}^n$ centered in 0, of radius r_q , if there exists a Kähler form ω' such that $[\omega'] = c_1(L)$ and $\phi_q^*(\omega') = \omega_{\text{std}}$ is the standard Kähler form on $\mathbb{C}^n$ restricted to $B_0(r_q)$. Let

$$\begin{aligned} \gamma_k(X, \omega; P_1, \dots, P_n) \\ = \sup \{ r > 0 : \text{there exists a Kähler packing with } \phi_q(0) = P_q \}. \end{aligned}$$

We call γ_k the *k-ball packing constant*.

Since curvature is an invariant under open holomorphic embeddings we cannot define the packing condition $\phi_q^*(\omega) = \omega_{\text{std}}$ using the original Kähler form ω on X as long as that form is not flat enough around q . However the following theorem shows that under suitable conditions it is possible to find a Kähler form flat enough around q in the cohomology class of ω as requested in the definition.

Theorem 1.4 *With the notation as above we have that the square of the k-ball packing constant is equal to the multipoint Seshadri constant:*

$$\gamma_k(X, \omega; P_1, \dots, P_k) = \sqrt{\epsilon(X, L; P_1, \dots, P_k)}.$$

2 Multipoint Seshadri constants

In this section we start by recalling some facts and definitions related to Seshadri constants. Single point Seshadri constants have been quite well studied and a good introduction to them can be found in [6, Chapter 5] and [1]. As we are interested in multipoint Seshadri constants we will introduce the notation and some ideas needed later.

Let X be a complex projective manifold of dimension n , with $P_1, \dots, P_k$ distinct points of X . Let L be an ample divisor and $\tilde{X} := \text{Bl}_{P_1, \dots, P_k}(X) \xrightarrow{\pi} X$ be the blow-up of X at the points $P_1, \dots, P_k$. Let $\pi^{-1}(P_i) = E_i$ denote the exceptional divisor corresponding to the point P_i for all $i = 1, \dots, k$ and for $\epsilon_i \in \mathbb{Q}_{>0}$ set

$$\tilde{L} := \pi^*L - \sum_{i=1}^k \epsilon_i E_i \quad \text{for } \epsilon_i \in \mathbb{Q}_{>0}.$$

If L is a nef $\mathbb{Q}$ -Cartier divisor then [6] defines the *multipoint Seshadri constant*

$$\epsilon(X, L; P_1, \dots, P_k) = \max \left\{ \epsilon \geq 0 : \pi^*L - \sum_{i=1}^k \epsilon_i E_i \text{ is nef} \right\}.$$

Since we require L ample to achieve the desired Kähler packing we show that there is an equivalent definition where we replace $\max$ with $\sup$ and ≥ 0 with > 0 . The following lemmas prove that the two definitions are equivalent.

Lemma 2.1 *Assume that L is ample on X . Then $\pi^*L - \epsilon' E$ is ample on $\tilde{X} \xrightarrow{\pi} X$ if $0 < \epsilon' < \epsilon(X, L; P)$.*

Proof By Seshadri's Criterion [6, Theorem 1.4.13] if L is ample on X then there exists $\epsilon_L > 0$ such that for all points $Q \in X$ and all irreducible curves C containing Q we have that $\frac{L \cdot C}{\text{mult}_Q C} > \epsilon_L$. Let $\tilde{Q} \in \tilde{X}$ be such that $\pi(\tilde{Q}) = Q$ and let $\tilde{C} \subset \tilde{X}$ be an irreducible curve with $\tilde{Q} \in \tilde{C}$. Then there are the following cases. If $\tilde{Q} \notin E$ then

$$\frac{(\pi^*L - \epsilon' E) \cdot \tilde{C}}{\text{mult}_{\tilde{Q}} \tilde{C}} = \frac{L \cdot C}{\text{mult}_Q C} - \epsilon' \frac{\text{mult}_P C}{\text{mult}_Q C} \geq \frac{\epsilon_L}{2} \quad \text{if } \frac{\text{mult}_P C}{\text{mult}_Q C} < \frac{\epsilon_L}{2\epsilon'}.$$

On the other hand, if $\frac{\text{mult}_P C}{\text{mult}_Q C} \geq \frac{\epsilon_L}{2\epsilon'}$ then

$$\begin{aligned} \frac{(\pi^*L - \epsilon' E) \cdot \tilde{C}}{\text{mult}_{\tilde{Q}} \tilde{C}} &= \frac{(\pi^*L - \epsilon_P E) \cdot \tilde{C}}{\text{mult}_{\tilde{Q}} \tilde{C}} + (\epsilon_P - \epsilon') \frac{E \cdot \tilde{C}}{\text{mult}_{\tilde{Q}} \tilde{C}} \\ &\geq (\epsilon_P - \epsilon') \frac{\text{mult}_P C}{\text{mult}_Q C} \geq (\epsilon_P - \epsilon') \frac{\epsilon_L}{2\epsilon'}. \end{aligned}$$

If $\tilde{Q} \in E$ with $\tilde{C} \not\subset E$ then $\text{mult}_{\tilde{Q}} \tilde{C} \leq \text{mult}_P \pi(\tilde{C})$ and

$$\frac{(\pi^*L - \epsilon'E) \cdot \tilde{C}}{\text{mult}_{\tilde{Q}} \tilde{C}} \geq \frac{(\pi^*L - \epsilon'E) \cdot \tilde{C}}{\text{mult}_P \pi(\tilde{C})} \geq \epsilon_P - \epsilon' > 0.$$

Finally if $\tilde{Q} \in E$ and $\tilde{C} \subset E$ then

$$\frac{(\pi^*L - \epsilon'E) \cdot \tilde{C}}{\text{mult}_{\tilde{Q}} \tilde{C}} \geq -\frac{\epsilon'E \cdot \tilde{C}}{\text{mult}_{\tilde{Q}} \tilde{C}} \geq \epsilon'.$$

The final inequality appears, since $E \cdot \tilde{C} = \deg \tilde{C}$ on $E \cong \mathbb{P}^{n-1}$ and $\deg \tilde{C} \geq \text{mult}_{\tilde{Q}} \tilde{C}$. Hence we have shown that if $0 < \epsilon' < \epsilon_P$ then $\pi^*L - \epsilon'E$ is ample. $\square$

Corollary 2.2 $\max\{\epsilon \geq 0 : \pi^*L - \epsilon E \text{ is nef}\} = \sup\{\epsilon > 0 : \pi^*L - \epsilon E \text{ is } \mathbb{Q}\text{-ample}\}.$

Lemma 2.3 *There exist $\epsilon_1, \dots, \epsilon_k > 0$ such that $\tilde{L}$ is $\mathbb{Q}$ -ample for all $i = 1, \dots, k$.*

Proof By Seshadri's Criterion and Lemma 2.1 we know there exists some $\epsilon_1 > 0$ such that $\pi^*L - \epsilon_1 E$ is $\mathbb{Q}$ -ample on $\tilde{X}_1 = \text{Bl}_{P_1}(X)$. Choosing a second point $P_2 \in X$ Seshadri's Criterion and Lemma 2.1 again ensure the existence of some $\epsilon_2 > 0$ such that $\pi^*L - \epsilon_1 E_1 - \epsilon_2 E_2$ is $\mathbb{Q}$ -ample on $\tilde{X}_2 = \text{Bl}_{P_1, P_2}(X)$. Arguing iteratively for k points of X we obtain the claim of the lemma. $\square$

From the statement of the above lemma it is not obvious that we can choose all the ϵ_i to be equal. This follows from the next lemma.

Lemma 2.4 *If $\pi_k^*L - \sum_{i=1}^k \epsilon_i E_i$ is $\mathbb{Q}$ -ample then $\pi^*L - \sum_{i=1}^k \epsilon'_i E_i$ is ample if $0 < \epsilon'_i \leq \epsilon_i$, for all $i = 1, \dots, k$.*

Proof Lemma 2.1 proves the case when $k = 1$. If we show that $\pi^*L - \sum_{i=1}^k \epsilon_i E_i$ ample implies $\pi_{k-1}^*L - \sum_{i=1}^{k-1} \epsilon_i E_i$ is ample, where $\pi_{k-1}: \tilde{X}_{k-1} \rightarrow X$ denotes the blow-up of X at the first $k-1$ points, then we can simply apply Lemma 2.1 on the last blow-up. Let $C \subset \tilde{X}_{k-1}$ be an irreducible curve on $\tilde{X}_{k-1}$ and $\bar{C} \subset \tilde{X}_k$ be the strict transform of C on $\tilde{X}_k$. Let Q be a point on C . Then

$$\begin{aligned} \frac{(\pi_{k-1}^*L - \sum_{i=1}^{k-1} \epsilon_i E_i) \cdot C}{\text{mult}_Q C} &= \frac{(\pi_k^*L - \sum_{i=1}^k \epsilon_i E_i) \cdot \bar{C}}{\text{mult}_Q \bar{C}} + \frac{\epsilon_k E_k \cdot \bar{C}}{\text{mult}_Q \bar{C}} \\ &= \frac{(\pi_k^*L - \sum_{i=1}^k \epsilon_i E_i) \cdot \bar{C}}{\text{mult}_Q \bar{C}} + \epsilon_k \cdot \frac{\text{mult}_{P_k} C}{\text{mult}_Q C}. \end{aligned}$$

If $Q \neq P_k$ we have $\text{mult}_Q \bar{C} = \text{mult}_Q C$, so the sum above is $\geq \epsilon_{\pi^*L - \sum_{i=1}^k \epsilon_i E_i}$, the global Seshadri constant of $\pi_k^*L - \sum_{i=1}^k \epsilon_i E_i$. If $Q = P_k$, the sum above is $\geq \epsilon_k$.

Thus, Seshadri's Criterion implies the ampleness of $\pi_{k-1}^*L - \sum_{i=1}^{k-1} \epsilon_i E_i$. $\square$

Definition 2.5 For X and L defined as above the multipoint Seshadri constant associated to points $P_1, \dots, P_k$ of X is

$$\epsilon(X, L; P_1, \dots, P_k) = \sup \left\{ \epsilon \in \mathbb{Q}_{>0} : \pi^* L - \epsilon \sum_{i=1}^k E_i \text{ is } \mathbb{Q}\text{-ample} \right\}.$$

The multipoint Seshadri constant associated to an ample divisor is always > 0 by Lemma 2.4.

3 Degeneration of complex projective manifolds to multipoint blow-ups

First we fix some general notation and introduce some constructions that will be used later. The notation fixed in this section will be used for the remainder of the report unless otherwise stated.

Let X and L be defined as above and consider the product manifold $\mathcal{X}$.

$$\begin{array}{ccc} \mathcal{X} := X \times \mathbb{A}_{\mathbb{C}}^k & \xrightarrow{p} & X \\ \downarrow q & & \\ \mathbb{A}_{\mathbb{C}}^k & & \end{array}$$

Then if $t_1, \dots, t_k$ represent coordinates of $\mathbb{A}_{\mathbb{C}}^k$, set $Z_i := \{P_i\} \times \{t_i = 0\}$. These are exactly the coordinate hyperplanes cut out by $t_i = 0$ over the points P_i .

Remark 3.1 For all $1 \leq i < j \leq k$ we have $Z_i \cap Z_j = \emptyset$ since the points $P_i \in X$ are distinct.

By blowing up the family $\mathcal{X}$ over the union of all the centres Z_i we obtain a new algebraic family over $\mathbb{A}_{\mathbb{C}}^k$

$$\begin{array}{ccccc} \tilde{\mathcal{X}} := \text{Bl}_{\bigcup_{i=1}^k Z_i}(\mathcal{X}) & \xrightarrow{\Pi} & \mathcal{X} & \xrightarrow{p} & X \\ & \searrow (q \circ \Pi) & \downarrow q & & \\ & & \mathbb{A}_{\mathbb{C}}^k & & \end{array}$$

such that the exceptional divisor corresponding to the centre Z_i is $\mathcal{E}_i := \Pi^{-1}(Z_i)$. Set $\tilde{\mathcal{L}}_{dm_1, \dots, m_k} = d\Pi^* p_1^* L - \sum_{i=1}^k m_i \mathcal{E}_i$ a divisor on $\tilde{\mathcal{X}}$.

Remark 3.2 • $\tilde{\mathcal{X}}$ is a well-defined complex manifold projective over $\mathbb{A}_{\mathbb{C}}^k$.

- $\mathcal{E}_i \cong Z_i \times \mathbb{P}_{\mathbb{C}}^n$.
- $(q \circ \Pi)^{-1}((t_1, \dots, t_k))$ contains the blow-up of X in the points P_i where $t_i = 0$. When $k = 1$ the construction is the deformation to the normal cone and the central fibre has two components: the blow-up of X and the exceptional divisor $\mathcal{E}$.

Fig. 1 Illustration of parameter spaces for $k = 2$ and $k = 3$.

Running through an appropriate path through the parameter space $\mathbb{A}_{\mathbb{C}}^k$ these blow-ups accumulate iteratively. Let $\delta_1, \dots, \delta_k$ be positive real numbers and define lines in $\mathbb{A}_{\mathbb{C}}^k$,

$$l_i := \{(0, \dots, 0, t \cdot \delta_i, \delta_{i+1}, \dots, \delta_k) : t \in \mathbb{R}\}$$

consisting of points whose first $i - 1$ coordinates are zero, the i th coordinate is of the form $t \cdot \delta_i$ and the remaining $i + 1$ coordinates are constant. For the point $t^{(i)} = (0, \dots, 0, \delta_i, \delta_{i+1}, \dots, \delta_k) \in l_i$ for $i = 1, \dots, k$, the preimage is

$$(q \circ \Pi)^{-1}(t^{(i)}) = \text{Bl}_{P_1, \dots, P_{i-1}}(X) \cup \bigcup_{j=1}^{i-1} \mathcal{E}_j \cap (q \circ \Pi)^{-1}(t^{(i)}).$$

Following the line from l_i from $t^{(i)}$ to $t^{(i+1)}$ we trace a path through the parameter space such that the preimage of $t^{(1)}$ is simply X , the preimage of $t^{(2)}$ is simply the blow-up of X at P_1 with some contributions from the exceptional divisor and so on until we get that the preimage of $t^{(k+1)} = (0, \dots, 0)$ is the blow-up of X at $P_1, \dots, P_k$. See Fig. 1 for a visualisation.

Similarly for $l_i \cong \mathbb{A}_{\mathbb{C}}^1$ we have

$$(q \circ \Pi)^{-1}(l_i) = \left(\bigcup_{j=1}^{i-1} \mathcal{E}_j \cap (q \circ \Pi)^{-1}(l_i) \right) \cup \tilde{\mathcal{X}}_i.$$

Where $\tilde{\mathcal{X}}_i$ is the blow-up of $\tilde{\mathcal{X}}_{i-1} \times l_i$ in the point $(P_i, 0)$.

$$\begin{array}{c} \tilde{\mathcal{X}}_i \xrightarrow{\Pi_i} \mathcal{X}_i = \tilde{\mathcal{X}}_{i-1} \times l_i \xrightarrow{q_{i-1}} l_i \cong \mathbb{A}_{\mathbb{C}}^1 \\ \downarrow p_{i-1} \\ \tilde{\mathcal{X}}_{i-1} \end{array}$$

On $\tilde{\mathcal{X}}_i$ there exists a family of divisors

$$\tilde{\mathcal{L}}_{d,m_1,\dots,m_i}^{(i)} := d\Pi_i^* p_{i-1}^* \tilde{\mathcal{L}}_{d;m_1,\dots,m_{i-1}}^{i-1} - m_i \mathcal{E}_i^{(i)}$$

where $\mathcal{E}_i^{(i)} \cong \mathbb{P}_{\mathbb{C}}^n$ is the exceptional divisor of the blow-up Π_i . Specific divisors associated to a particular fiber are denoted $\tilde{\mathcal{L}}_{d,\underline{m},t}^{(i)}$, where t denotes the parameter on $\mathbb{A}_{\mathbb{C}}^1$ and $d, \underline{m}_i = (m_1, \dots, m_i)$ record the degree and multiplicity (we will often just write $\tilde{\mathcal{L}}_t^{(i)}$ for short if the multiplicity and degree are fixed).

Remark 3.3 $\tilde{\mathcal{X}}_{i,0} = \tilde{X}_i \cup \mathcal{E}_i^{(i)}$, where $\mathcal{E}_i^{(i)} \cong \mathbb{P}_{\mathbb{C}}^n$ and $\tilde{\mathcal{X}}_{i,t} = \Pi^{(i)-1}(t)$ is the fiber over $t \in \mathbb{A}_{\mathbb{C}}^1$.

Now that we have defined an algebraic family $\tilde{\mathcal{X}}_i$ and a family of divisors $\tilde{\mathcal{L}}^{(i)} = \tilde{\mathcal{L}}_{d;\underline{m}_i}^{(i)}$ we would like to investigate the global sections of these divisors restricted to the fibers of $\Pi^{(i)}$, and how these global sections are constructed for different i . This is shown by the following theorem.

Theorem 3.4 *For all $i = 1, \dots, k$ there exist global sections $\sigma_0^{(i-1)}, \dots, \sigma_{N_i}^{(i-1)}$ of $\Pi_*^{(i)} \tilde{\mathcal{L}}^{(i)}$ such that:*

- (1) $\sigma_0^{(i-1)}, \dots, \sigma_{N_i}^{(i-1)}$ trivialise $\Pi_*^{(i)} \tilde{\mathcal{L}}^{(i)}$.
- (2) If 0_i denotes the zero of the line l_i then the sections

$$\sigma_{0,0_i}^{(i-1)}|_{\mathcal{E}_i^{(i)}}, \dots, \sigma_{N_i,0_i}^{(i-1)}|_{\mathcal{E}_i^{(i)}}$$

generate the Kähler form $m_i \cdot \omega_{\text{FS}}$ on $\mathcal{E}_i^{(i)} \cong \mathbb{P}_{\mathbb{C}}^n$.

- (3) If $i \geq 2$ then the restricted sections $\sigma_{0,\delta_i}^{(i-1)}, \dots, \sigma_{N_i,\delta_i}^{(i-1)}$ generate the same Kähler form on $\tilde{\mathcal{X}}_{i,\delta_i} \cong \tilde{X}_{i-1}$ as $\sigma_{0,0_{i-1}}^{(i-2)}|_{\tilde{X}_{i-1}}, \dots, \sigma_{N_{i-2},0_{i-1}}^{(i-2)}|_{\tilde{X}_{i-1}}$ from the previous family $\tilde{\mathcal{X}}_{i-1}$.

Proof First we illustrate the special case when $i = k = 1$. Let X be a complex projective manifold equipped with an ample line bundle L and fix the point P_1 . Let $\pi: \tilde{X}_1 = \text{Bl}_{P_1}(X) \rightarrow X$ be the blow-up of X at P_1 , with $E_1 = \pi^{-1}(P_1)$ the exceptional divisor and choose positive integers m and d such that $\tilde{L}^{(1)} = d\pi^*L - mE$ is ample (in particular, $m < d$). Now construct the families $\mathcal{X}$ and $\tilde{\mathcal{X}}_1$ using the strategy detailed above (see Fig. 2).

Recall that on $\tilde{\mathcal{X}}_1$ there exists a family of ample line bundles $\tilde{L}^{(1)}$ and the central fiber $\tilde{\mathcal{X}}_0 \cong \tilde{X}_1 \cup \mathcal{E}_1$ where $\mathcal{E}_1 \cong \mathbb{P}^n$. In this setting we can determine a basis of sections of $H^0(\tilde{\mathcal{X}}_0, \tilde{L}^{(1)}|_{\tilde{\mathcal{X}}_0})$ characterised by vanishing behaviour at the point P_1 and show that they satisfy the requirements of the theorem. First we compute sections of $H^0(\tilde{X}_1, \mathcal{O}_{\tilde{X}}(\tilde{L}^{(1)}))$ vanishing to order greater or equal m in P_1 . These sections are homogeneous polynomials of degree less than or equal m described in local coordinates around P_1 . Sections of $H^0(\mathcal{E}_1, \mathcal{O}_{\mathcal{E}_1}(-mE_1))$ are simply homogeneous monomials

Fig. 2 Product manifold and blow-up for $i = k = 1$.

of degree m on $\mathbb{P}^n$ under the isomorphism $\mathcal{E}_1 \cong \mathbb{P}^n$. These monomials restrict to monomials on E_1 if they vanish with multiplicity greater than m and to 0 otherwise. Similarly the section of $\tilde{\mathcal{L}}^{(1)}$ restricts to a homogeneous monomial on E_1 if it vanishes with multiplicity greater than m and 0 otherwise. These sections form a basis of $H^0(\tilde{X}_1, \mathcal{O}_{\tilde{X}_1}(\tilde{\mathcal{L}}^{(1)}))$ and to change between a basis of sections on $\tilde{X}_1$ and on X we simply skip those sections which vanish to order less than m at P_1 .

To obtain sections of $\mathcal{L}^{(1)}$ we pull back a section $s \in H^0(\tilde{X}_1, \mathcal{O}_{\tilde{X}_1}(\tilde{\mathcal{L}}^{(1)}))$ along p to get a section of $p^*\tilde{\mathcal{L}}^{(1)} = \mathcal{L}^{(1)}$ on $\tilde{\mathcal{X}}_1 = \tilde{X}_1 \times \mathbb{A}_{\mathbb{C}}^1$. If $\text{mult}_{P_1}s < m$ then the section of $H^0(\tilde{\mathcal{X}}_1, \tilde{\mathcal{L}}^{(1)})$ has multiplicity m in $(P_1, 0)$ and if $\text{mult}_{P_1}s \geq m$ then $\text{mult}_{(P_1, 0)}p^*s \geq m$. In both cases, we can subtract m copies of the exceptional divisor $\mathcal{E}_1$ from the pullback of this section along Π , thus obtaining sections of $H^0(\tilde{\mathcal{X}}_1, \tilde{\mathcal{L}}^{(1)})$.

By construction the general fiber $\tilde{\mathcal{X}}_{1,t}$ is isomorphic to X for $t \neq 0$, so it is clear that away from the central fiber the sections described above trivialise $\Pi_*\tilde{\mathcal{L}}^{(1)}$. Considering the central fiber we find that global sections of $\tilde{\mathcal{L}}^{(1)}|_{\tilde{\mathcal{X}}_{1,0}}$ consist of a section of $\tilde{\mathcal{L}}^{(1)}|_{\tilde{X}_1}$ and a section of $\tilde{\mathcal{L}}^{(1)}|_{\mathcal{E}_1}$ coinciding when restricted to E_1 . These sections are comprised of:

- pairs of basis sections on $\tilde{X}_1$ vanishing on E_1 and the zero section on $\mathcal{E}_1$,
- pairs of the zero section on $\tilde{X}_1$ and a basis section on $\mathcal{E}_1$ vanishing on E_1 ,
- pairs of basis sections of $\tilde{X}_1$ and $\mathcal{E}_1$ that restrict to the same non-zero homogeneous polynomial of degree m on E_1 .

These sections all coincide to basis sections of dL which either: vanish to multiplicity greater than m at P_1 , vanish to exactly order m at P_1 , or vanish to order less than m at P_1 . Hence these sections restrict to a basis of $\tilde{\mathcal{X}}_{1,0}$ and therefore trivialise $\Pi_*\tilde{\mathcal{L}}^{(1)}$ over all fibers of the family.

To prove the general case we show that the dimension of the space of global sections of $\tilde{\mathcal{L}}^{(i)}$ restricted to each fiber of $\Pi^{(i)}$ is the same, Grauert's semi-continuity theorem

[5, III.12] implies immediately that $\Pi_*^{(i)} \tilde{\mathcal{L}}^{(i)}$ is locally free, hence it is free by the Quillen–Suslin Theorem. Thus there exist trivialising global sections of $\Pi^* \tilde{\mathcal{L}}^{(i)}$. But instead of using these powerful theorems we describe the trivialising sections directly because they allow us to prove the properties of the theorem more easily. For the sections to be trivialising it is enough to show that they form a basis of $H^0(\tilde{\mathcal{X}}_{i,t}, \tilde{\mathcal{L}}_t^{(i)})$ then restricted to $\tilde{\mathcal{X}}_{i,t}$ for all $t \in \mathbb{A}_{\mathbb{C}}^1$.

We start by calculating a basis of sections of $H^0(X, \mathcal{O}_X(dL))$ characterised by their vanishing behaviour at the points $P_1, \dots, P_k$ so that subsets of this basis can be interpreted as generating sections of $H^0(\tilde{X}, \tilde{\mathcal{L}}^{(i)})$. Let $\mathfrak{m}_{X, P_i}$ denote the maximal ideal of X at P_i and consider the short exact sequence

$$0 \rightarrow \bigcap_{i=1}^k \mathfrak{m}_{X, P_i}^{m_i+1} \otimes \mathcal{O}_X(dL) \rightarrow \mathcal{O}_X(dL) \rightarrow \bigoplus_{i=1}^k \mathcal{O}_X/\mathfrak{m}_{X, P_i}^{m_i+1} \rightarrow 0.$$

Taking the long exact sequence of cohomology w.r.t. to the above sequence gives

$$\begin{array}{ccccccc} 0 \rightarrow H^0(X, \bigcap_{i=1}^k \mathfrak{m}_{X, P_i}^{m_i+1} \otimes \mathcal{O}_X(dL)) & \rightarrow & H^0(X, \mathcal{O}_X(dL)) & \rightarrow & H^0(X, \bigoplus_{i=1}^k \mathcal{O}_X/\mathfrak{m}_{X, P_i}^{m_i+1}) & \rightarrow & \cdots \\ \leftarrow & & & & & & \\ & \rightarrow & H^1(X, \bigcap_{i=1}^k \mathfrak{m}_{X, P_i}^{m_i+1} \otimes \mathcal{O}_X(dL)) & \rightarrow & H^1(X, \mathcal{O}_X(dL)) & \rightarrow & \cdots \end{array}$$

By Serre Vanishing $H^1(X, \bigcap_{i=1}^k \mathfrak{m}_{X, P_i}^{m_i+1} \otimes \mathcal{O}_X(dL)) = 0$, since applying the projection formula yields

$$H^1\left(X, \bigcap_{i=1}^k \mathfrak{m}_{X, P_i}^{m_i+1} \otimes \mathcal{O}_X(dL)\right) = H^1(\tilde{X}, \mathcal{O}_X(\tilde{\mathcal{L}}_{d, m_i}))$$

and we assume $\tilde{\mathcal{L}}_{d, m_i}$ is ample and $d, \underline{m}_k \gg 0$. Hence $\phi: H^0(X, \mathcal{O}_X(dL)) \rightarrow H^0(X, \bigoplus_{i=1}^k \mathcal{O}_X/\mathfrak{m}_{X, P_i}^{m_i+1})$ is surjective. This gives a basis of sections of $H^0(X, \mathcal{O}_X(dL))$ which is the union of:

- A basis B_0 of $H^0(X, \bigcap_{i=1}^k \mathfrak{m}_{X, P_i}^{m_i+1} \otimes \mathcal{O}_X(dL))$, which are sections of $\mathcal{O}_X(dL)$ vanishing to multiplicity at least $m_i + 1$ in P_i for $i = 1, \dots, k$.
- A set B_i of sections of $\mathcal{O}_X(dL)$ mapped to 0 in $\mathcal{O}_X/\mathfrak{m}_{X, P_j}^{m_j+1}$ for $j \neq i$ and to a basis of homogeneous polynomials of degree $\leq m_i$ in $\mathcal{O}_X/\mathfrak{m}_{X, P_i}^{m_i+1}$, in variables given by local coordinates around P_i .
- A set of sections $\tilde{B}_i$ of $\mathcal{O}_X(dL)$ as above, but with homogeneous polynomials of degree $= m_i$, and that $\tilde{B}_i \subset B_i$.

Hence a basis of $H^0(\tilde{X}_i, \mathcal{O}_{\tilde{X}_i}(\tilde{\mathcal{L}}^{(i)}))$ is given by

$$B^{(i)} := B_0 \cup \bigcup_{j=1}^i \tilde{B}_j \cup \bigcup_{j=i+1}^k B_j.$$

To change between a basis of sections on $\tilde{X}_{i-1}$ and on $\tilde{X}_i$ we simply skip those sections which vanish to order less than m_i at P_i .

Since $\mathcal{X}_i = \tilde{X}_{i-1} \times \mathbb{A}_{\mathbb{C}}^1$, we can use these sections to construct the sections of $H^0(\tilde{\mathcal{X}}_i, \tilde{\mathcal{L}}^{(i)})$ using the following procedure:

Step 1: Pull back a section $s \in H^0(\tilde{X}_{i-1}, \mathcal{O}_{\tilde{X}_{i-1}}(\tilde{\mathcal{L}}^{(i-1)}))$ along p_{i-1} to get a section of $p_{i-1}^* \tilde{\mathcal{L}}^{(i-1)} = \mathcal{L}^{(i-1)}$ on $\mathcal{X}_i = \tilde{X}_{i-1} \times \mathbb{A}_{\mathbb{C}}^1$.

Step 2: If $\text{mult}_{P_i} s < m_i$ then the section $t^{m_i - \text{mult}_{P_i} s} p_{i-1}^* s \in H^0(\tilde{\mathcal{X}}_i, \tilde{\mathcal{L}}^{(i-1)})$ has multiplicity m_i in $(P_i, 0)$. If $\text{mult}_{P_i} s \geq m_i$ then $\text{mult}_{(P_i, 0)} p_{i-1}^* s \geq m_i$.

Step 3: In both cases, we can subtract m_i copies of the exceptional divisor $\mathcal{E}_i$ from the pullback of this section along Π_i , thus obtaining sections of $H^0(\tilde{\mathcal{X}}_i, \tilde{\mathcal{L}}^{(i)})$.

When starting with the basis sections in $B^{(i-1)}$, restricting to the general fibers $\tilde{\mathcal{X}}_{i,t} \cong \tilde{X}_{i-1}$ of $\Pi^{(i)}$ (that is $t \neq 0$) we get back the sections in $B^{(i-1)}$ possibly multiplied with some power of t . Thus all the sections $\sigma_1^{(i-1)}, \dots, \sigma_{N_i}^{(i-1)}$ obtained from $B^{(i-1)}$ in the way described above are trivializing $\Pi_*^{(i)} \tilde{\mathcal{L}}^{(i)}$ outside the central fiber. To understand the restriction of these sections to the central fiber we need to analyse $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$ and its global sections. Note, $\tilde{\mathcal{X}}_{i,0} = \tilde{X}_i \cup \mathcal{E}_i^{(i)}$ and $\tilde{X}_i \cap \mathcal{E}_i^{(i)} = E_i$. Furthermore $\tilde{\mathcal{L}}^{(i)}|_{\tilde{X}_i} = \tilde{\mathcal{L}}^{(i)}$, and $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}} = \mathcal{O}_{\mathcal{E}_i^{(i)}}(-m_i \mathcal{E}_i^{(i)}) \cong \mathcal{O}_{\mathbb{P}^n}(m_i)$ via the isomorphism $\mathcal{E}_i^{(i)} \cong \mathbb{P}_{\mathbb{C}}^n$. This means that global sections of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$ consist of a section of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{X}_i}$ and a section of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ coinciding when restricted to E_i .

- Global sections of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{X}_i}$ and their restriction to E_i :

These are all sections of dL on X which vanish to multiplicity $\geq m_j$ in P_j for $j = 1, \dots, i$. Equivalently these are global sections of $\tilde{\mathcal{L}}^{(i-1)}$ on $\tilde{X}_{i-1}$ vanishing in multiplicity $\geq m_i$ in P_i . To obtain a section of $\tilde{\mathcal{L}}^{(i)}$ from a section of $\tilde{\mathcal{L}}^{(i-1)}$ we pull back the section along the map $\pi^{(i-1)}: \tilde{X}_i \rightarrow \tilde{X}_{i-1}$ then subtract m_i copies of the exceptional divisor. If a section of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{X}_i} = \tilde{\mathcal{L}}^{(i)}$ corresponds to a section of $\tilde{\mathcal{L}}^{(i-1)}$ with multiplicity greater than m_i at P_i the restriction of this section to E_i is zero. If the multiplicity is exactly m_i at P_i then $\tilde{\mathcal{L}}^{(i)}|_{E_i} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(m_i)$ hence the restriction of such a section of $\tilde{\mathcal{L}}^{(i)}$ to $E_i \cong \mathbb{P}_{\mathbb{C}}^n$ will be described by a non-zero homogeneous polynomials of degree m_i in $Y_1, \dots, Y_n$.

- Global sections of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ and their restriction to E_i :

To describe these we introduce homogeneous coordinates $[T : Y_1 : \dots : Y_n]$ on $\mathcal{E}_i^{(i)}$. The coordinates $Y_1, \dots, Y_n$ come from local coordinates $y_1, \dots, y_n$ around P_i , the T coordinate comes from the affine base parameter t , and $T = 0$ describes $E_i \subset \mathcal{E}_i^{(i)}$. Hence the restriction of a section of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ to E_i is obtained by setting $T = 0$. Since $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}} \cong \mathcal{O}_{\mathbb{P}^n}(m_i)$, sections of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ are non-zero homogeneous polynomials of degree m_i in $T, Y_1, \dots, Y_n$.

Sections of $\tilde{\mathcal{L}}^{(i)}$ are therefore described by pairs of sections of $\tilde{X}_i$ and $\mathcal{E}_i^{(i)}$ both vanishing on E_i and pairs of sections of $\tilde{X}_i$ and $\mathcal{E}_i^{(i)}$ restricting to the same non-zero

homogeneous polynomial in $Y_i, \dots, Y_n$ of degree m_i on E_i . A basis of global sections of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$ can therefore be built from the basis of $H^0(\tilde{X}_i, \tilde{\mathcal{L}}^{(i)})$ and $H^0(\tilde{\mathcal{E}}_i^{(i)}, \tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}})$ as the set of

- Pairs of basis sections on $\tilde{X}_i$ vanishing on E_i and the zero section on $\mathcal{E}_i^{(i)}$.
- Pairs of the zero section on $\tilde{X}_i$ and a basis section on $\mathcal{E}_i^{(i)}$ vanishing on E_i .
- Pairs of basis sections of $\tilde{X}_i$ and $\mathcal{E}_i^{(i)}$ that restrict to the same non-zero homogeneous polynomial of degree m_i on E_i .

Now we want to show that the restriction of the $\sigma_0^{(i-1)} \dots \sigma_{N_i}^{(i-1)}$ to $\tilde{\mathcal{X}}_{i,0}$ yields a basis of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$. To this purpose, we follow Steps 1–3 to construct $\sigma_j^{(i)}$ from a section of $\tilde{\mathcal{L}}^{(i-1)}$ and then identify the pair of sections describing the restriction to $\tilde{\mathcal{X}}_{i,0}$. If a section of $\tilde{\mathcal{L}}^{(i-1)}$ vanishes to multiplicity strictly greater than m_i at P_i then it corresponds to a pair of sections on $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$ consisting of a section of $\tilde{\mathcal{L}}^{(i)}$ that restricts to zero on E_i , and the zero section on $\mathcal{E}_i^{(i)}$. If the section of $\tilde{\mathcal{L}}^{(i-1)}$ on $\tilde{X}_{i-1}$ vanishes to exactly multiplicity m_i at P_i then pulling back and subtracting m_i copies of the exceptional divisor we obtain non-zero sections on $\tilde{X}_i$ and on $\mathcal{E}_i^{(i)}$. Such a section of $\tilde{\mathcal{L}}^{(i-1)}$ corresponds to a pair of sections on $\tilde{\mathcal{X}}_{i,0}$ that restrict to the same non-zero homogeneous monomial on E_i . Finally if a section s of $\tilde{\mathcal{L}}^{(i-1)}$ on $\tilde{X}_{i-1}$ vanishes to multiplicity strictly less than m_i at P_i then after pulling back along p_{i-1} we must multiply by $t^{m_i - \text{mult}_{P_i} s}$ before we can subtract copies of the exceptional divisor. This type of section corresponds to a pair of sections on $\tilde{\mathcal{X}}_{i,0}$ consisting of the zero section on $\tilde{X}_i$ and a section on $\mathcal{E}_i^{(i)}$ that restricts to zero on E_i .

In terms of the sets B_0 , B_j and $\tilde{B}_j$, making up the basis $B^{(i)}$ of $\tilde{\mathcal{L}}^{(i)}$ we find that a basis of global sections of $\tilde{\mathcal{L}}^{(i-1)}$ vanishing to multiplicity $> m_i$ at P_i can be written as $B_0 \cup \bigcup_{j=1}^{i-1} \tilde{B}_j \cup \bigcup_{i+1}^k B_j$. A basis of global sections vanishing to multiplicity exactly m_i at P_i is $\tilde{B}_i$ and finally a basis of global sections vanishing to multiplicity less than m_i at P_i is given by $B_i - \tilde{B}_i$. The union of all three bases provides a basis $B^{(i-1)}$ for $\tilde{\mathcal{X}}_{i,t} \cong \tilde{X}_{i-1}$. The union of the basis corresponding to all sections of $\tilde{\mathcal{L}}^{(i-1)}$ vanishing to multiplicity less than or equal to m_i at P_i is isomorphic to a basis of global sections of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ where the identification is given by the correspondence between homogeneous polynomials in local coordinates around P_i and homogeneous coordinates of $\mathcal{E}_i^{(i)} \cong \mathbb{P}_{\mathbb{C}}^n$.

Thus, the sections $\sigma_0^{(i-1)}, \dots, \sigma_{N_i}^{(i-1)}$ also restrict to a basis of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$, hence trivialize $\Pi_*^{(i)} \tilde{\mathcal{L}}^{(i)}$ over all fibers of the family.

Now let us construct a Kähler form using the trivializing sections of $\tilde{\mathcal{L}}^{(i)}$ restricted to $\mathcal{E}_i^{(i)} \cong \mathbb{P}_{\mathbb{C}}^n$. On this exceptional divisor we can choose homogeneous coordinates

$T, Y_1, \dots, Y_n$ as above. Then

$$\begin{aligned} m_i \cdot \omega_{\text{FS}} &= \frac{i}{2\pi} \partial \bar{\partial} m_i \cdot \log(Y_1 \bar{Y}_1 + \dots + Y_n \bar{Y}_n + T \bar{T}) \\ &= \frac{i}{2\pi} \partial \bar{\partial} \log(Y_1 \bar{Y}_1 + \dots + Y_n \bar{Y}_n + T \bar{T})^{m_i} \\ &= \frac{i}{2\pi} \partial \bar{\partial} \log(Y_1^{m_i} \bar{Y}_1^{m_i} + m_i Y_1^{m_i-1} \bar{Y}_1^{m_i-1} Y_2 \bar{Y}_2 \dots) \\ &= m_i \cdot \omega_{\text{FS}} = \frac{i}{2\pi} \partial \bar{\partial} \log \left(\sum_{\alpha+\beta=m_i} c_{\alpha,\beta} Y^\alpha \bar{Y}^\beta T^\beta \bar{T}^\beta \right), \end{aligned}$$

where $c_{\alpha,\beta}$ are positive integers.

If we choose the sections $\sigma_j^{(i-1)}$ that do not vanish on the exceptional divisor (i.e., coming from sections of $\tilde{L}^{(i-1)}$ vanishing with multiplicity $\leq m_i$ in P_i) such that they restrict to the basis of monomials $\sqrt{c_{\alpha,\beta}} Y^\alpha T^\beta$ of the homogeneous polynomials of degree m_i , then $m_i \cdot \omega_{\text{FS}}$ is the Kähler form on $\mathcal{E}_i^{(i)}$ generated by the restriction of these sections $\sigma_j^{(i-1)}$. Note that $c_{\alpha,\beta}$ is a positive integer, so $\sqrt{c_{\alpha,\beta}}$ is just the usual real square root.

To achieve property (3) we have to construct the trivializing sections of $\tilde{L}^{(i)}$ on $\tilde{\mathcal{X}}_i$ iteratively, starting with $i = k$. On $\tilde{\mathcal{X}}_k$ we construct sections $\sigma_0^{(k-1)}, \dots, \sigma_{N_k}^{(k-1)}$ as above, which restricted to $\tilde{\mathcal{X}}_{k,\delta_k}$ provides a basis of sections of $\tilde{L}^{(k-1)}$ on $\tilde{\mathcal{X}}_{k-1} = \tilde{\mathcal{X}}_{k,\delta_k}$. We can interpret these sections as sections of $\tilde{L}^{(k-2)}$ on $\tilde{\mathcal{X}}_{k-2}$ vanishing with multiplicity $\geq m_{k-1}$ in P_{k-1} , and we can complete these sections to a basis of all sections of $\tilde{L}^{(k-2)}$ by adding sections which vanish with multiplicity $< m_{k-1}$ in P_{k-1} . This basis can be used to construct trivializing sections of $\tilde{L}^{(k-1)}$ on $\tilde{\mathcal{X}}_{k-1}$ because by its construction it can be split up into the subsets $B_0, B_{k-1}, \tilde{B}_{k-1}$ and $B^{(k-1)}$. The Kähler form on $\mathcal{E}_{k-1}^{(k-1)}$ is also as requested since we can choose the completing basis sections of $\tilde{L}^{(k-2)}$ as required above. Iterating this process for $\tilde{\mathcal{X}}_{k-2}, \dots, \tilde{\mathcal{X}}_1$ we deduce property (3) for each $i = 1 \dots, k$. $\square$

4 Symplectic blow-up of Kähler manifolds

Before we prove the main result of this paper, for the convenience of the reader we present the construction of symplectic blow-ups in the Kähler setting in more details. It immediately yields one half of Theorem 5.1, by relating Kähler packings to the ampleness of line bundles on the blown-up varieties. We follow McDuff's and Polterovich's original construction in [7], especially emphasizing how to keep track of the Kähler condition.

Consider the triples $(\mathbb{C}^n, J_0, \omega_{\text{std}})$ and $(\mathbb{CP}^{n-1}, J_0, \omega_{\text{FS}})$, where ω_{std} is the standard Euclidean form on $\mathbb{C}^n$, J_0 is the standard complex structure on both $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$, and ω_{FS} is the Fubini–Study form on $\mathbb{CP}^{n-1}$. Let $l \subset \mathbb{CP}^{n-1}$ be a line and fix a point

$z \in \mathbb{C}^n$. Then the complex manifold $\mathcal{M}$

$$\begin{array}{ccc} \mathcal{M} = \{(z, l) \mid z \in l\} \subset \mathbb{C}^n \times \mathbb{CP}^{n-1} & \xrightarrow{q} & \mathbb{CP}^{n-1} \\ \downarrow \pi & & \\ \mathbb{C}^n & & \end{array}$$

is the blow-up of $\mathbb{C}^n$ at the origin. To help the reader visualise this one can imagine replacing the origin of $\mathbb{C}^n$ by the set of all complex lines passing through the origin.

Let $B_\lambda^{2n}(0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \dots + |z_n|^2 \leq \lambda^2\}$ denote an n -dimensional complex ball of radius λ and centred at the origin. For $\lambda > 0$ the set $\mathcal{M}(\lambda) := \pi^{-1}(B_\lambda^{2n}(0))$ is the sub-manifold obtained by taking the inverse image of $B_\lambda^{2n}(0)$ under the projection π . We call the fiber $\pi^{-1}(0)$ over 0 the exceptional divisor and the projection π the blow-up map. It is worth noting that away from the exceptional divisor π is biholomorphic onto $\mathbb{C}^n$.

Proposition 4.1 *On $\mathcal{M}$ there exists a Kähler form*

$$\rho(1, \lambda) := \pi^* \omega_{\text{std}} + \lambda^2 q^* \omega_{\text{FS}} \quad \text{for } \lambda > 0.$$

Proof To prove that the bilinear form $\rho(1, \lambda)$ is a Kähler form we must show that it is closed, skew symmetric, that it is compatible with the complex structure on $\mathcal{M}$ and the associated Riemannian metric is positive definite. First we note $\rho(1, \lambda)$ is closed and skew symmetric, as pullbacks of closed and skew symmetric forms are closed and skew symmetric. Since ω_{std} and ω_{FS} are compatible with the complex structure on $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$ and the projections π and q are holomorphic, we can deduce that the associated bilinear form is compatible with the complex structure on $\mathcal{M}$. Finally the associated Riemannian metric is positive definite as the pullbacks are positive-semi definite and furthermore $\pi^* \omega_{\text{std}}$ is positive definite away from the exceptional divisor, and positive definite on the tangent vectors mapped to 0 by the differential of q . Moreover on the other tangent vectors of $\mathcal{M}$, $q^* \omega_{\text{FS}}$ is positive definite. $\square$

Corollary 4.2 *On the exceptional divisor $\pi^{-1}(0)$ the $(1, 1)$ -form $\rho(1, \lambda)|_{\pi^{-1}(0)} = \lambda^2 q^* \omega_{\text{FS}}$.*

Definition 4.3 An embedding $F: \mathbb{C}^n - \{0\} \rightarrow \mathbb{C}^n$ is called *monotone* if in spherical coordinates $(u, r) \in S^{2n-1} \times (0, \infty) \cong \mathbb{C}^n - \{0\}$ it can be written as $(u, r) \mapsto (u, f(r))$, where f is a strictly increasing function.

Lemma 4.4 *There exists a smooth family of monotone embeddings*

$$h_\lambda: \mathbb{C}^n - \{0\} \rightarrow \mathbb{C}^n - B(\lambda)$$

such that $\pi^* h_\lambda^* \omega_{\text{std}} = \rho(1, \lambda)$ on $\mathcal{M} - \pi^{-1}(0)$.

Proof Using spherical coordinates we can choose $h_\lambda: (u, v) \mapsto (u, (r^2 + \lambda^2)^{1/2})$. Fix a point $P = ((c, 0, \dots, 0), [1:0:\dots:0]) \in \mathcal{M}$ for $c \in \mathbb{R}_{>0}$. Recall that around

the point $(c, 0, \dots, 0)$ if we take local coordinates $z_j = x_j + iy_j$ of $\mathbb{C}^n$ then $\omega_{\text{std}} = \sum_{j=1}^n dx_j \wedge dy_j$. Furthermore if $t_j = u_j + iv_j$ are local coordinates of $\mathbb{CP}^{n-1}$ around $[1:0:\dots:0]$ then the Fubini–Study form in P is given as $\omega_{\text{FS}}(P) = \sum_{j=2}^n du_j \wedge dv_j$. Since $t_j = z_j/z_1$ on $\mathcal{M}$ around P , the z -coordinates provide a chart of $\mathcal{M}$ around P . Calculations show that in these coordinates

$$\begin{aligned} \pi^* \omega_{\text{std}} + \lambda^2 q^* \omega_{\text{FS}}(P) &= \sum_{j=1}^n dx_j \wedge dy_j + \lambda^2 \sum_{j=2}^n du_j \wedge dv_j \\ &= \sum_{j=1}^n dx_j \wedge dy_j + \frac{\lambda^2}{c^2} \left(\sum_{j=2}^n dx_j \wedge dy_j \right) \\ &= dx_1 \wedge dy_1 + \sum_{j=1}^n d \frac{(r^2 + \lambda^2)^{1/2}}{r} x_j \wedge d \frac{(r^2 + \lambda^2)^{1/2}}{r} y_j(P) \\ &= (\pi^* h_\lambda^* \omega_{\text{std}})(P). \end{aligned}$$

Then in P we have that $\rho(1, \lambda)(P) = \pi^* h_\lambda^* \omega_{\text{std}}(P)$. Now since h_λ and π are compatible under the natural $U(n)$ -action and ω_{std} and $\rho(1, \lambda)$ are invariant under the same action we can deduce that $\pi^* h_\lambda^* \omega_{\text{std}} = \rho(1, \lambda)$ everywhere on $\mathcal{M}$ away from $\pi^{-1}(0)$. $\square$

Lemma 4.5 *For every monotone embedding $F: \mathbb{C}^n - \{0\} \rightarrow \mathbb{C}^n$ the form $F^* \omega_{\text{std}}$ is Kähler.*

Proof Since F is monotone, it is once again compatible with the natural $U(n)$ -actions on $\mathbb{C}^n$ and $\mathbb{C}^n - \{0\}$. Furthermore, since

$$(dF)|_{(F(1), 0, \dots, 0)} \left(\frac{\partial}{\partial x_1} \right) = \frac{dF}{dr}(F(1), 0, \dots, 0) \cdot \left(\frac{\partial}{\partial x_1} \right),$$

and $\frac{dF}{dr}(F(1), 0, \dots, 0) > 0$ as F is strictly increasing and $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_k}$ are left invariant by F at $(F(1), 0, \dots, 0)$, for $j = 2, \dots, n$ and $k = 1, \dots, n$, we can deduce that $F^* \omega_{\text{std}}(F(1), 0, \dots, 0)$ is compatible with the complex structure. $\square$

Proposition 4.6 ([7, Proposition 5.1A]) *For every $\eta, \lambda > 0$ there exists a Kähler form $\bar{\tau} = \bar{\tau}(\eta, \lambda)$ on $\mathcal{M}$ such that:*

- (1) $\bar{\tau}|_{\mathcal{M} - \mathcal{M}(1+\eta)} = \pi^* \omega_{\text{std}}$.
- (2) $\bar{\tau}|_{\mathcal{M}(\delta)} = \rho(1, \lambda)$ for some $\delta > 0$.

Proof Take $\delta < 1 + \eta$ such that $h_\lambda(B_\delta^{2n}(0)) \subset B_{\lambda(1+\eta/2)}^{2n}(0)$. Then we can use a smoothing procedure to find a monotone embedding F , such that:

- (1) $F(z) = \lambda z$ for $|z| > 1 + \eta$.
- (2) $F(z) = h_\lambda(z)$ for $|z| < \delta$.

The form $\bar{\tau} = \pi^*(F^*(\omega_{\text{std}}))$ extended to $\pi^{-1}(0)$ has the required properties. $\square$

Construction 4.7 (Symplectic blow-up of a Kähler manifold) *Let (M, ω) be an n -dimensional Kähler manifold with a Kähler form ω and fix a point $P \in M$. Assume that*

$$\phi: (B_{(1+2\eta)}^{2n}(0), \omega_{\text{std}}) \hookrightarrow (M, \omega)$$

is a holomorphic embedding with $\phi(0) = P$ and $\phi^\omega = \lambda^2\omega_{\text{std}}$ for $\lambda \in \mathbb{R}_{>0}$.*

Since $\mathcal{M}(1+2\eta)$ and $B_{1+2\eta}^n(0)$ are holomorphic away from 0 and $\pi^{-1}(0)$ respectively, we can glue in $\mathcal{M}(1+2\eta)$ instead of $B_{1+2\eta}^n(0)$ and obtain the blown-up complex variety $\tilde{M}$, with the blow-up map $\Pi: \tilde{M} \rightarrow M$ and exceptional divisor $E = \pi^{-1}(0)$:

$$\begin{array}{ccc} E = \pi^{-1}(0) \subset \mathcal{M}(1+2\eta) & \xrightarrow{\tilde{\phi}} & \tilde{M} \\ \downarrow & \pi \downarrow & \downarrow \Pi \\ 0 & \in & B_{1+2\eta}^{2n}(0) \xrightarrow{\phi} M \end{array}$$

We can construct a Kähler form $\tilde{\omega}$ on $\tilde{M}$ with $\phi^\tilde{\omega} = \omega_{\text{std}}$ as follows:*

$$\tilde{\omega} = \begin{cases} \Pi^*\omega & \text{on } \tilde{M} - \tilde{\phi}(\mathcal{M}(1+2\eta)), \\ (\tilde{\phi}^{-1})^*\bar{\tau}(\eta, \lambda) & \text{on } \tilde{\phi}(\mathcal{M}(1+2\eta)) \end{cases}$$

where $\bar{\tau}(\eta, \lambda)$ is defined as above. This is possible since $\tilde{\phi}^\tilde{\omega}$ coincides with $\phi^*\omega = \omega_{\text{std}}$ on $\mathcal{M}(1+2\eta) - \mathcal{M}(1)$ by construction of $\bar{\tau}(\eta, \lambda)$. Moreover $\tilde{\omega}$ is a Kähler form as the two glued components are Kähler forms.*

In the setting of Construction 4.7 if M is a projective complex manifold and $[\tilde{\omega}]$ the ample class $[\pi^*\omega] - \lambda[E]$ on M , where $[E]$ is the cohomology class of (the Poincaré dual of) the exceptional divisor E then $\tilde{\omega}$ represents an ample class on $\tilde{M}$. By construction $[\tilde{\omega}]$ is in the interior of a cone generated by integral Kähler classes. Kodaira's embedding theorem says that Kähler forms representing such integral Kähler classes are curvature forms of Kähler metrics on ample line bundles.

From Construction 4.7 it is clear that to define a symplectic form on the blow-up of a symplectic manifold M at points $P_1, \dots, P_k$ of X is equivalent to specifying a holomorphic embedding

$$\phi: \bigsqcup_{i=1}^k (B_i(\lambda_i), \omega_{\text{std}}) \hookrightarrow (M, \omega)$$

which sends the center of the i th ball to P_i for every $i = 1, \dots, k$. Then one can cut out images of the embedded balls and collapse their boundaries to copies of $\mathbb{CP}^{n-1}$ called exceptional divisors. Conversely, given a symplectic form on the blow-up, by blowing down we obtain a symplectic form ω on M and a packing of (M, ω) by balls. Both constructions preserve Kähler forms. In terms of the blow-down this amounts

to determining if the cohomology class on the blow-up of M can be represented by symplectic forms.

5 Kähler packings of projective, complex, manifolds

Theorem 5.1 *Let X be a projective complex manifold of dimension n , L an ample line bundle on X , and $P_1, \dots, P_k \in X$. Let $\epsilon_0 = \epsilon(X; L, P_1, \dots, P_k)$ denote the multipoint Seshadri constant of L on X in $P_1, \dots, P_k$. Then, for any radius $r < \sqrt{\epsilon_0}$ there exists a Kähler packing of k flat Kähler balls of radius r into X .*

Proof First we construct families $\tilde{\mathcal{X}}_i$ as in Sect. 3. Then we embed Fubini–Study Kähler balls of large enough volume on $\mathcal{E}_i^{(i)} - E_i$ provided with the Kähler metric $\omega_i^{(i)}$ induced by the global sections of $\tilde{\mathcal{L}}^{(i)}$ constructed in Theorem 3.4 on $\mathcal{E}_i^{(i)}$, for $i = k, k-1, \dots, 1$. These balls can be deformed to Kähler balls on non-central fibers $\tilde{\mathcal{X}}_{i,\delta_i} \cong \tilde{\mathcal{X}}_{i-1}$, and then iteratively to non-central fibers $\tilde{\mathcal{X}}_{j,\delta_j} \cong \tilde{\mathcal{X}}_{j-1}$ for $j = i-1, \dots, 1$ if the δ_j are chosen small enough. Doing this carefully the deformed balls will not intersect on $\tilde{\mathcal{X}}_{i,\delta_i}$ and descend to disjoint balls over X . Moreover, there exists a symplectomorphism between flat Kähler balls and Fubini–Study balls given by

$$\phi: (B_0^{2n}(1), \omega_{\text{std}}) \hookrightarrow (\mathbb{C}^n, \omega_{\text{FS}}) \subset (\mathbb{CP}^n, \omega_{\text{FS}})$$

such that

$$(z_1, \dots, z_n) \mapsto \frac{1}{(1 - \sum_{i=1}^n |z_i|^2)^{1/2}} \cdot (z_1, \dots, z_n).$$

This symplectomorphism along with its inverse allows us to glue Fubini–Study balls into flat Kähler balls and vice versa. Hence, gluing standard Kähler balls into the Fubini–Study Kähler balls we obtain the claim.

In more details:

1. Let $\Delta_\delta \subset \mathbb{A}_{\mathbb{C}}^1$ denote the open disk of radius δ , with affine parameter t . Choose local coordinates $y_1, \dots, y_n$ of X around P_i , so $t, y_1, \dots, y_n$ are local coordinates around $(0, P_i)$ in $\tilde{\mathcal{X}}_i$. Then over the open subset $\mathcal{U}_t \subset \tilde{\mathcal{X}}_i$ where these coordinates are defined there is a chart of the blow-up of $\mathcal{X}_i$ in $(P_i, 0)$ with coordinates $t, z_1, \dots, z_n$ such that the blow-up map to $\mathcal{U}_t$ is described by $(t, z_1, \dots, z_n) \mapsto (t, y_1, \dots, y_n) := (t, tz_1, \dots, tz_n)$. Because the coordinates y_i are bounded, the central fiber of the induced projection of $\mathcal{U}_t$ onto $\mathbb{A}_{\mathbb{C}}^1$ is $\mathcal{E}_i^{(i)} - E \cong \mathbb{A}_{\mathbb{C}}^n$. The non-central fibers are not quite isomorphic to $\mathbb{A}_{\mathbb{C}}^n$ but contain balls $B_R(0)$ with R arbitrarily large close to $t = 0$. Thus for R arbitrarily large we can find δ sufficiently small and an embedding $\iota_i: \Delta_\delta \times B_R(0) \hookrightarrow \tilde{\mathcal{X}}_i$. See Fig. 3.
2. Choosing in 1. R large enough and δ small enough implies that there exist embeddings of Fubini–Study Kähler balls $B_{R'}(0)$ with $R' \leq R$, of volume arbitrarily

Fig. 3 Illustration of embedded ball away from central fiber.

close to the volume of $(\mathcal{E}_i^{(i)}, \omega_i^{(i)})$ in all fibers of $\Delta_\delta \times B_R(0)$ over $t \in \Delta_\delta$ with respect to the same Fubini–Study Kähler form $\omega_i^{(i)}$.

- By continuity, for t small enough these Fubini–Study forms $\omega_i^{(i)}$ differ to an arbitrarily small amount from the Kähler form $\omega_{i,t}$ on $\tilde{\mathcal{X}}_{i,t}$ obtained from the trivializing sections of $\tilde{\mathcal{L}}^{(i)}$ pulled back via the embedding ι_i . This allows us to glue in the Fubini–Study Kähler balls of step 2. into non-central fibers $\tilde{\mathcal{X}}_{i,t}$ provided with the Kähler form $\omega_{i,t}$ for $t \ll 1$ small enough. Assume that $\omega_i^{(i)} = \frac{i}{2\pi} \partial \bar{\partial} \log(s_i^{(i)})$ and $\omega_{i,t} = \frac{i}{2\pi} \partial \bar{\partial} \log s_{i,t}$ on $\{t\} \times B_R(0)$, where $s_i^{(i)}, s_{i,t}$ are functions constructed in the usual way from the appropriate sections. Then choose a partition of unity (ρ_1, ρ_2) such that $\rho_1|_{B_{R'}(0)} \equiv 1$ and $\rho_2|_{B_{R'}(0)} \equiv 0$. The glued 2-form $\tilde{\omega}_{i,t}$ is given by $\frac{i}{2\pi} \partial \bar{\partial} \log(\rho_1 s_i^{(i)} + \rho_2 s_{i,t}) = \frac{i}{2\pi} \partial \bar{\partial} \log(\rho_1(s_i^{(i)} - s_{i,t}) + s_{i,t})$. This form is obviously closed, and it is non-degenerate because $s_i^{(i)} - s_{i,t}$ gets arbitrarily small for $t \ll 0$, thus $\rho_1(s_i^{(i)} - s_{i,t}) + s_{i,t}$ is arbitrarily close to $s_{i,t}$. Consequently, $\tilde{\omega}_{i,t}$ is a Kähler form. Furthermore, since the gluing is done on a function/metric level and $\omega_{i,t}$ is obtained by applying $i \partial \bar{\partial} \log$, the gluing factors vanish in the cohomology class. Hence $[\tilde{\omega}_{i,t}] = [\omega_{i,t}] = c_1(\tilde{\mathcal{L}}^{i-1})$.
- The i^{th} Fubini–Study Kähler ball on $\tilde{\mathcal{X}}_{i,\delta_i}$ does not intersect the $(i+1)^{\text{st}}, \dots, k^{\text{th}}$ Kähler ball constructed before: The i^{th} ball is deformed from a ball on the central fiber $\tilde{\mathcal{X}}_{i,0}$ lying in $\mathcal{E}_i^{(i)} - E_i$ whereas the other balls are deformed from balls in $\tilde{\mathcal{X}}_i - E_i \subset \tilde{\mathcal{X}}_{i,0}$. So these balls do not intersect on the central fiber, and this will not change when we deform the balls to $\tilde{\mathcal{X}}_{i,\delta_i}$ if we choose δ_i small enough.

5. Assume $\int \omega_{\text{FS}}^n = 1$, and $\int_{B_1(0)} \omega_{\text{std}} = 1$. Then there exists a Kähler embedding $(B_r(0), \omega_{\text{std}}) \hookrightarrow (\mathbb{CP}^n, \omega_{\text{FS}})$, for all $r < 1$. Hence for all $r < \sqrt{m_i}$ there exists a Kähler embedding $(B_r(0), \omega_{\text{std}}) \hookrightarrow (\mathbb{CP}^n, m_i \omega_{\text{FS}})$. Rescaling by d_i we obtain a Kähler embedding $(B_r(0), \omega_{\text{std}}) \hookrightarrow (\mathbb{CP}^n, \frac{m_i}{d_i} \omega_{\text{FS}})$ for all $r < \sqrt{m_i/d_i}$. Since $\omega_i^{(i)} = m_i \omega_{\text{FS}}$, the embeddings constructed above imply that $m_i/d_i < \epsilon_0$, but since m_i/d_i can be chosen arbitrarily close to ϵ_0 we can conclude that $\sqrt{\epsilon_0} \leq \gamma_k$, where γ_k is the k -ball packing constant. $\square$

Proof of Theorem 1.4 The claim that the k -point Seshadri constant is less or equal to the Kähler packing constant is a direct consequence of Theorem 5.1. The converse argument is a consequence of the symplectic blow-up construction detailed in Sect. 4. The embedded balls allow us to construct Kähler forms on the blow-up π of the centres whose curvature lies in the first Chern class of $\pi^*L - \sum \gamma E_i$. $\square$

Remark 5.2 The method to prove Theorem 5.1 also allows us to construct Kähler packings of balls with radius $r_1, \dots, r_k$ arbitrarily close to $\epsilon_1, \dots, \epsilon_k$ as long as $\pi^*L - \sum_{i=1}^k \epsilon_i E_i$ is nef.

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