Generalized limited sets with applications to spaces of type $\ell_\infty(X)$ and $c_0(X)$

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Abstract

A concept of limitedness of sets in one TVS relative to another TVS is discussed. Special attention is devoted to the limitedness relative to spaces without copies of ℓ_∞ . Applications to copies of ℓ_∞ in vector-valued c_0 -type spaces, and to the complementability of the latter in the containing them similar ℓ_∞ -type spaces are provided.

Keywords Topological vector space · Limited set · Copy of ℓ_∞ · ℓ_∞ -type space · c_0 -type space · Exhaustive measure · Uncomplemented subspace

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1 Introduction

Recall that a set A in a Banach space E is said to be *limited* if every weak*-null sequence in E^* converges uniformly on A . The reader may refer, for instance, to [6, 8], and in the locally convex case to [2, 10], for more information on this notion and the relevant *Gelfand–Phillips Property*, further references, as well as some historical comments. As for the origins of the research presented here, I would like to say this. First: For E and A as above, it is easy to verify that (*) If A is limited, then every sequence (u_k) of operators from E to another Banach space F converging pointwise to zero does actually converge uniformly on A . One is naturally tempted to express the assertion by saying that A is limited relative to F , or F -limited. Fact (*) along with the limitedness of the sequence (e_n) in ℓ_∞ was successfully used by Emmanuele [9]. Second: In a recent, as yet unfinished unrelated research I happened to prove or maybe rather reprove, following the ideas of [9], that (+) If X is a Banach space and $c_0(X)$ contains a copy of ℓ_∞ , so does X . I attributed this result to Emmanuele,

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although there was no clear mentioning of it in [9]. Quite possibly it is ‘folklore’. But in reality I desired to have something like (+) for a general TVS X , and that prompted me to think about a more general concept of limitedness. I have followed this urge, and the output is now presented here. Its essentials are pointed out in the abstract. Expressed in a somewhat different manner the main results show that the ℓ_∞ and c_0 functors (see the next section) behave quite well with respect to properties of a TVS X such as ‘ensuring X -limitedness’, ‘noncontainment of a copy of ℓ_∞ ’, and ‘uncomplementability of $c_0(X)$ in $\ell_\infty(X)$ ’.

2 Basic TVS terminology, notation, conventions, and two important theorems

The acronyms TVS and LCS will stand for *topological vector space* and *locally convex space*. All spaces considered will be over the same field of scalars $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and always tacitly assumed nonzero and Hausdorff. In addition, *operators*, in particular *projections*, are always meant to be linear and continuous. A ‘copy’ of one space in another means isomorphic copy. $X \simeq Y$ will indicate that the spaces X and Y are isomorphic, and a space that for instance contains no copy of ℓ_∞ will sometimes be called ℓ_∞ -free. Throughout the paper S will stand for an arbitrary infinite set, possibly different in different contexts, but considered the same when the contexts are connected. The ℓ_∞ -functor $X \rightsquigarrow \ell_\infty(S, X)$ and the c_0 -functor $X \rightsquigarrow c_0(S, X)$, as well as their simpler variants $X \rightsquigarrow \ell_\infty(X)$ and $X \rightsquigarrow c_0(X)$ when $S = \mathbb{N}$, acting in the category of TVS’s, will be present, mostly implicitly, in all of this work, and the obvious question being considered: “How these functors behave with respect to various properties of interest to us that a TVS may have?” Some details concerning ℓ_∞ and c_0 type spaces are given below: If X is a TVS, $\ell_\infty(S, X)$ and its closed subspace $c_0(S, X)$ are the familiar spaces of functions f from S to X that are bounded, resp., converge to 0 along the filterbase of cofinite subsets of S , both equipped with the topology of uniform convergence on S . Thus if $\mathcal{U}$ is a 0-base in X , then the sets $U^\bullet = \{f \in \ell_\infty(S, X) : f(S) \subset U\}$, $U \in \mathcal{U}$, form a 0-base in $\ell_\infty(S, X)$. As is well known, when $X = (X, \|\cdot\|)$ is a normed or F -normed space, this topology may also be defined by the associated supnorm or sup F -norm $\|\cdot\|_\infty$. The notation will be simplified to $\ell_\infty(S)$ and $c_0(S)$ when $E = \mathbb{K}$; to $\ell_\infty(X)$ and $c_0(X)$ when $S = \mathbb{N}$; and to ℓ_∞ and c_0 (the classical Banach sequence spaces) when $X = \mathbb{K}$ and $S = \mathbb{N}$. Moreover, in the general case, for any set $T \subset S$, 1_T will denote the characteristic function of T , P_T the characteristic projection $f \rightarrow 1_T f$ in each of the spaces, and $\ell_\infty(S|^\circ T, X)$, $c_0(S|^\circ T, X)$ their subspaces consisting of functions vanishing outside of T ; the notation $[e_s : s \in S]_\infty$ and $[(e_n)]_\infty$ will be used for the family of standard unit vectors $e_s = 1_{\{s\}}$ in $\ell_\infty(S)$, and the sequence of standard unit vectors $e_n = 1_{\{n\}}$ in ℓ_∞ , respectively. For the reader’s convenience, two theorems of crucial importance for what follows will yet be included.

Theorem 2.1 (Brooks–Jewett Theorem) *Let X be a TVS and $(\mathbf{m}_n)$ a sequence of exhaustive finitely additive (f.a.) X -valued measures on a σ -algebra Σ in S . If the sequence $(\mathbf{m}_n)$ converges setwise on Σ , then it is equi-exhaustive, converges uniformly on every disjoint sequence of sets in Σ , and the limit f.a. measure is exhaustive.*

Let us clarify that an f.a. measure $\mathbf{m}: \Sigma \rightarrow X$ is called *exhaustive* if, for every disjoint sequence (A_n) in Σ , $\mathbf{m}(A_n) \rightarrow 0$ (as $n \rightarrow \infty$). If the latter holds uniformly for $\mathbf{m} \in M$, a family of such measures, then M is said to be *equi-exhaustive*. See [3, Theorem (BJ)].

The next result extended an earlier original ℓ_∞ -theorem of Haskell P. Rosenthal for X a Banach space.

Theorem 2.2 (Rosenthal ℓ_∞ -Theorem) *If T is an operator from ℓ_∞ into a TVS X such that $Te_n \not\rightarrow 0$, where (e_n) are the unit vectors in ℓ_∞ , then there exists an infinite subset M of $\mathbb{N}$ such that $T|_{\ell_\infty(M)}$ is an isomorphic embedding. In consequence, X contains a copy of ℓ_∞ .*

See [4,5] for this, references, and a more general version. For the most recent (and much more general) results in this spirit see [7]. Their applicability to the problems discussed in this paper is yet to be seen.

3 Limited sets and sequences

Definition 3.1 A set A in a TVS E will be called *limited* relative to another TVS H , or shortly *H-limited*, if every sequence $\mathbf{u} = (u_k)$ of operators $u_k: E \rightarrow H$ converging to 0 pointwise on E converges to 0 uniformly on A . Likewise in the case of a sequence $\mathbf{x} = (x_n)$ in E in place of the set A .

Clearly, for the choice $H = \mathbb{K}$, the sequences $\mathbf{u}$ above are weak*-null sequences in the dual space E^* so that $\mathbb{K}$ -*limited* is the same as *limited*.

Remark 3.2 A weaker notion of an *H-equilimited* set will be used at a few occasions in this paper. In its definition the sequences (u_k) as above are additionally required to be equicontinuous. When $H = \mathbb{K}$ one may simply say *equilimited*. (Cf. the definition of limited sets adopted in [10, p. 65].) Of course, these two notions coincide when the Banach–Steinhaus theorem for pointwise convergent sequences of operators is available, for instance when E is an F -space.

Some elementary facts are now being collected below. Of the utmost value for us will be (4).

Fact 3.3 *Let $A \subset E$.*

- (1) *If A is H -limited, then it is limited.*
- (2) *If A is H -limited, then so is its any operator image, i.e., the image $f(A) \subset F$ by any operator $f: E \rightarrow F$, F a TVS; moreover A is also G -limited for any subspace G of H .*
- (3) *If every countable subset of A (alternatively, every sequence in A) is H -limited, so is the set A .*
- (4) *If A is H -limited, then it is also $l_\infty(S, H)$ -limited.*
- (5) *If A is limited, then it is Z -limited for every LCS Z .*

Likewise when ‘limited’ is replaced by ‘equilimited’.

Proof Let $\mathcal{W}$ be a 0-base in H .

(1): Use operators u_k with values in a one-dimensional subspace of H .

(2): Obvious.

(3): Suppose A is not H -limited. Then there exists $W \in \mathcal{W}$ such that for each $l \in \mathbb{N}$ one can find $k > l$ with $u_k(A) \not\subset W$ so that $u_k(a_k) \notin W$ for some $a_k \in A$. Proceeding by induction, we then find a subsequence (v_m) of the sequence (u_k) , and a sequence (a_m) in A such that $v_m(a_m) \notin W$ for all m . Thus although $v_m \rightarrow 0$ pointwise on E , $v_m \not\rightarrow 0$ uniformly on the countable set $\{a_m : m \in \mathbb{N}\} \subset A$, contrary to the assumption.

(4): Suppose it is not so. Hence there exists a sequence (U_k) of operators from E to $l_\infty(S, H)$ that converges pointwise to 0 on E , but does not converge to 0 uniformly on A . Hence there must exist $W \in \mathcal{W}$ such that for each $l \in \mathbb{N}$ one can find $k > l$ with $U_k(A) \not\subset W^\bullet$. Thus, for some $a_k \in A$ and some $s_k \in S$ one has $U_k(a_k)(s_k) \notin W$. It follows that there are a strictly increasing sequence (k_m) in $\mathbb{N}$, a sequence (b_m) in A , and a sequence (t_m) in S with $U_{k_m}(b_m)(t_m) \notin W$ for all m . Now, define operators $u_m: E \rightarrow H$ by

$$u_m(x) = U_{k_m}(x)(t_m), \quad x \in E.$$

Since for every $t \in S$, $P_t: f \mapsto f(t)$ is a continuous! operator from $l_\infty(S, H)$ into H , each u_m is an operator. Moreover, again for each $x \in E$, $U_{k_m}(x) \rightarrow 0$ uniformly on S . Therefore, $u_m(x) \rightarrow 0$ in H . But we also have that $u_m(b_m) \notin W$ for all m , which is impossible because (u_m) ought to converge uniformly to 0 on A .

(5): This is ‘folklore’. A proof is given just in case. Assume first that $Z = (Z, \|\cdot\|)$ is a normed space (cf. [8, fact (D), p. 379], take any sequence of operators $v_k: E \rightarrow Z$ converging pointwise to 0 on E but not uniformly on A so that there are an $\varepsilon > 0$, a subsequence (w_l) of (v_k) and a sequence (a_l) in A with $\|w_l(a_l)\| > \varepsilon$. Then for each l choose $z_l^* \in Z^*$ such that $\|z_l^*\| = 1$ and $z_l^*(w_l(a_l)) = \|w_l(a_l)\|$. Define $f_l \in E^*$ by $f_l = z_l^* \circ w_l$. Then for every $x \in X$: $|f_l(x)| \leq \|w_l(x)\| \rightarrow 0$ (as $l \rightarrow \infty$). However, $f_l(a_l) = \|w_l(a_l)\| > \varepsilon$ for all l and so the sequence (f_l) does not converge to 0 uniformly on A , contradicting the limitedness of A . In the general case start as above with a ‘bad’ sequence of operators (v_k) , choose a suitable ‘bad’ continuous seminorm p in Z , then pass to the associated normed space $Z/\ker p$ and the relevant quotient map q , and apply the first part to the sequence of operators (qv_k) . $\square$

Remark 3.4 One of the referees came up with the following simple but clever example showing that Fact (5) above is no longer valid for nonlocally spaces Z . I repeat it here *verbatim*: Consider the F -space $L^p[0, 1]$ for any $0 < p < 1$ and the sequence $A = (x_n)$ in it, with $x_n = 2^n 1_{[0, 1]}$, $n \in \mathbb{N}$. Since E has a trivial dual, A is clearly limited. However, it is not Z -limited for $Z = E$. Indeed, the sequence of operators $U_k: E \rightarrow E$ given by $U_k = 2^{-k}I$ (I the identity operator in E) converges to 0 pointwise on E , but it does not converge to 0 uniformly on A , because $U_k(x_k) = 1_{[0, 1]}$ for all k .

In the rest of this section the limitedness of the sequence $[(e_n)]_\infty$, of essential importance for us, will be discussed. Recall that the fact that it is limited goes back to Ralph S. Phillips (1940).

Theorem 3.5 *If the TVS H contains no copy of ℓ_∞ , then the sequence $[(e_n)]_\infty$ is H -limited, hence also $\ell_\infty(H)$ -limited. In fact, it is so also for the set $[e_s : s \in S]_\infty$.*

Proof For every operator $u : \ell_\infty \rightarrow H$ the associated f.a. measure $\mathbf{m}_u : \mathcal{P}(\mathbb{N}) \rightarrow H$ is defined by $\mathbf{m}_u(A) = u(1_A)$ for $A \subset \mathbb{N}$. Since for any disjoint sequence (A_n) of nonempty subsets of $\mathbb{N}$ the sequence (1_{A_n}) in ℓ_∞ is an isomorphic copy of $[(e_n)]_\infty$, and since $H \not\supset \ell_\infty$, from Theorem 2.2 it now follows that $\mathbf{m}_u(A_n) \rightarrow 0$. Thus the measure $\mathbf{m}_u$ is exhaustive. Now, consider any sequence of operators $u_k : \ell_\infty \rightarrow H$ converging to 0 pointwise on ℓ_∞ . To conclude the proof it suffices to apply Theorem 2.1 to the sequence of associated exhaustive f.a. measures $(\mathbf{m}_{u_k})$ and the sequence of singletons $(\{n\})$. Having done that, it remains to make use of Fact 3.3(4). An appeal to Fact 3.3(3) justifies the final assertion. $\square$

Two weaker variants of the theorem above will yet be presented, the main reason for doing so being that their proofs avoid using the Brooks–Jewett theorem. After that the necessity of the condition that $c_0(H)$ be ℓ_∞ -free is revealed.

Proposition 3.6 (1) *Let $H = (H, \|\cdot\|)$ be an F -space such that the F -space $c_0(H)$ with its sup F -norm $\|\cdot\|_\infty$ contains no copy of ℓ_∞ . Then the sequence $[(e_n)]_\infty$ is H -limited.*

(2) *Let H be a general TVS such that the space $c_0(H)$ contains no copy of ℓ_∞ . Then the sequence $[(e_n)]_\infty$ is H -equilimited.*

Proof (1): Suppose it is not so. Thus there exists a sequence (u_k) of operators from ℓ_∞ to H that converges on ℓ_∞ pointwise to 0, but the convergence is not uniform on (e_n) . Hence for some $\varepsilon > 0$ there are a strictly increasing sequence (k_n) in $\mathbb{N}$ and a sequence (m_n) also in $\mathbb{N}$ such that $\|u_{k_n}(e_{m_n})\| > \varepsilon$ for all n . Define a linear map $T : \ell_\infty \rightarrow c_0(H)$ by $T(\mathbf{a}) = (u_{k_n}(\mathbf{a}))$, $\mathbf{a} \in \ell_\infty$. It is obviously continuous when $c_0(H)$ is considered with the topology of coordinate-wise topology. Hence, by the closed graph theorem, it is also continuous for the F -norm $\|\cdot\|_\infty$ in $c_0(X)$. But we also have $\|T(e_{m_n})\|_\infty > \varepsilon$ for every n . By Theorem 2.2, $c_0(H)$ contains a copy of ℓ_∞ , a contradiction.

(2): Proceed as in part (1) above, this time, however, assuming the sequence (u_k) to be equicontinuous so that the linear map T defined later is automatically continuous. (No appeal to the closed graph theorem is needed.) $\square$

Proposition 3.7 *Let H be general TVS. If $c_0(H)$ contains a copy of ℓ_∞ , then the sequence $[(e_n)]_\infty$ is not H -equilimited, hence neither it is H -limited.*

Proof Let $j : \ell_\infty \rightarrow c_0(H)$ be an isomorphic embedding. Choose $U \in \mathcal{U}$ so that $j e_n \notin U^\bullet$ for all n . Next, for each n select $k_n \in \mathbb{N}$ with $j e_{k_n} \notin U$. Then consider the sequence (p_k) of coordinate projections $p_k : c_0(H) \rightarrow H$; $\mathbf{x} \mapsto \mathbf{x}(k)$. Observe that the operators $u_k = p_k j : \ell_\infty \rightarrow H$ converge to 0 pointwise on ℓ_∞ . Moreover, both sequences (p_k) and (u_k) are equicontinuous. However, $u_{k_n} e_n \notin U$ for every n and thus the convergence $u_k \rightarrow 0$ is not uniform on (e_n) , finishing the proof. $\square$

4 Applications to spaces $\ell_\infty(S, X)$ and $c_0(S, X)$

For the sake of clarity it is reasonable to start by quoting a known result [2, Corollary 6.2]:

Theorem 4.1 *If X is a (nonzero!) LCS, then $c_0(X)$ is uncomplemented in $\ell_\infty(X)$.*

Remark 4.2 The assertion remains valid for $c_0(S, X)$ and $\ell_\infty(S, X)$. (Use that part of the Proof of Theorem 4.4 (2) below, where the projection Q appears.

It seems to be unknown whether Theorem 4.1 is still valid when X is a TVS. Something in this direction can be proved, however, see Theorem 4.4 (2). But first comes the following.

Theorem 4.3 *For any TVS X the following are equivalent:*

- (1) X is ℓ_∞ -free.
- (2) $c_0(X)$ is ℓ_∞ -free.
- (3) $c_0(S, X)$ is ℓ_∞ -free for every infinite set S .
- (4) $c_0(S, X)$ is ℓ_∞ -free for some infinite set S .

Proof Fix a 0-base $\mathcal{U}$ in X .

(1) $\Rightarrow$ (2): Suppose it is not so and let $j: \ell_\infty \rightarrow c_0(X)$ be an isomorphic embedding. A decisive point will be finding an $m \in \mathbb{N}$ with $j e_n(m) \not\rightarrow 0$ in X as $n \rightarrow \infty$. For this, first note that the sequence $(j e_n)$ in $c_0(X)$ is $\ell_\infty(X)$ -limited, by Theorem 3.5 and Fact 3.3 (2). Next, for every $k \in \mathbb{N}$, define an operator $R_k: c_0(X) \rightarrow c_0(X)$ by $R_k x = x - 1_{[k]}x$, where $[k] = \{1, \dots, k\}$. Obviously, $R_k x \rightarrow 0$ as $k \rightarrow \infty$, for each $x \in c_0(X)$. Therefore, by the $\ell_\infty(X)$ -limitedness, the convergence is uniform on the sequence $(j e_n)$. Since j is an isomorphism, there is $U \in \mathcal{U}$ such that $j e_n \notin U^\bullet$ for all n . Choose $V \in \mathcal{U}$ with $V + V \subset U$. By the uniform convergence just noted, there is k such that $R_k j e_n \in V^\bullet$ for all n . In consequence, $1_{[k]} j e_n \notin V^\bullet$ for all n . Hence there is $m \in [k]$ such that $j e_n(m) \notin V$ for infinitely many n 's. Finally, denoting by p_m the natural projection of $c_0(X)$ onto its m th coordinate space, which is X of course, we see that for the operator $T = p_m j: \ell_\infty \rightarrow X$ one has that $T e_n \not\rightarrow 0$. By Theorem 2.2, X must contain a copy of ℓ_∞ , contrary to (1).

(2) $\Rightarrow$ (3): Suppose there is an isomorphic embedding $j: \ell_\infty \rightarrow c_0(S, X)$. Choose $U \in \mathcal{U}$ so that $j e_n \notin U^\bullet$ for all n . Next, for each n select $t_n \in S$ with $j e_n(t_n) \notin U$. Define an operator $h: \ell_\infty \rightarrow c_0(X)$ by $h a(n) = j a(t_n)$, $n \in \mathbb{N}$, for all $a \in \ell_\infty$. Clearly, $h e_n(n) = j e_n(t_n) \notin U$ for each n . Therefore, by Theorem 2.2, h must be an isomorphism on $\ell_\infty(M)$ for some infinite set $M \subset \mathbb{N}$ and thus $c_0(X)$ contains a copy of ℓ_∞ , contradicting (2).

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are trivial. For the latter just observe that for any fixed $t \in S$ the map $x \mapsto 1_{\{t\}}x$ is an isomorphism from X into $c_0(S, X)$. $\square$

Theorem 4.4 *Let X be a TVS. Consider the following statements:*

- (1) X is ℓ_∞ -free.
- (2) $c_0(X)$ is uncomplemented in $\ell_\infty(X)$.

- (3) $c_0(S, X)$ is uncomplemented in $\ell_\infty(S, X)$ for every infinite set S . Then (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (2). In particular, (2) and (3) hold when X is separable.

Proof Again, Fix a 0-base $\mathcal{U}$ in X .

(1) $\Rightarrow$ (2): Suppose to the contrary that there exists a projection $P: \ell_\infty(X) \rightarrow c_0(X)$. Select any $0 \neq x \in X$ and consider the isomorphism into $i: \ell_\infty \rightarrow \ell_\infty(X)$ given by $ia(n) = a_n x, n \in \mathbb{N}$, for $a = (a_n) \in \ell_\infty$. Moreover, for the operator $R = Pi: \ell_\infty \rightarrow c_0(X)$ one sees that $Re_n \not\rightarrow 0$ as $n \rightarrow \infty$. Hence, by Theorem 2.2, $c_0(X)$ contains a copy of ℓ_∞ , contradicting (1) in virtue of Theorem 4.3.

(2) $\Rightarrow$ (3): Suppose to the contrary that there exists a projection $P: \ell_\infty(S, X) \rightarrow c_0(S, X)$. Select any $0 \neq x \in X$ and any infinite countable subset $T = \{t_n : n \in \mathbb{N}\}$ of S and denote by P_T the characteristic projection $f \rightarrow 1_T f: c_0(S, X) \rightarrow c_0(T, X)$. Note that its range is the closed subspace $c_0(S|^\circ T, X)$, obviously isomorphic to $c_0(T, X)$. Consider also the isomorphism into $j: \ell_\infty(X) \rightarrow \ell_\infty(S, X)$ defined for any $x = (x_n) \in \ell_\infty(X)$ by $jx(s) = x_n$ if $s = t_n, n \in \mathbb{N}$, and $= 0$ otherwise. Of course, $j(c_0(X)) \subset c_0(S|^\circ T, X)$. It is easily checked that the operator $Q = j^{-1}P_T Pj$ is a projection from $\ell_\infty(X)$ onto $c_0(X)$, contradicting (2).

(3) $\Rightarrow$ (2): Trivial.

The assertion after (3): For X separable just note that $\ell_\infty \subset c_0(X)$ is impossible because $c_0(X)$ is separable. $\square$

The next result has a somewhat amazing proof.

Theorem 4.5 *Let X be a TVS.*

- (1) *If $c_0(X)$ is not complemented in $\ell_\infty(X)$, then $c_0(\ell_\infty(X))$ is not complemented in $\ell_\infty(\ell_\infty(X))$.*
- (2) *If there is no operator from $\ell_\infty(X)$ onto $c_0(X)$, then there is also no operator from $\ell_\infty(\ell_\infty(X))$ onto $c_0(\ell_\infty(X))$.*

Proof (1): It is quite an obvious observation that $\ell_\infty(\ell_\infty(X))$ can be isomorphically identified with the space $\ell_\infty(\mathbb{N} \times \mathbb{N}, X)$ of bounded X -valued matrices (or double sequences) $\mathbf{m} = [x_{n,k} : n, k \in \mathbb{N}]$. Simply assign to each $x = (x_n) \in \ell_\infty(\ell_\infty(X))$ the matrix $\mathbf{m}$ whose n th row is $(x_n(k) : k \in \mathbb{N})$ so that $x_{n,k} = x_n(k)$ for all (n, k) . Denote the isomorphism thus described by h . On the other hand, $\ell_\infty(\mathbb{N} \times \mathbb{N}, X)$ is clearly isomorphic to $\ell_\infty(X)$ (via the composition operator $C_\beta: \ell_\infty(\mathbb{N} \times \mathbb{N}, X) \rightarrow \ell_\infty(X)$ induced by any bijection $\beta: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$; thus $C_\beta \mathbf{m}(i) = x_{\beta(i)}$ for $\mathbf{m}$ as above and all $i \in \mathbb{N}$). Returning to the isomorphism h , note that it maps $c_0(\ell_\infty(X))$ onto the subspace $c_0((\mathbb{N} \times \mathbb{N}; \xrightarrow[\mathcal{B}]{} 0), X)$ of $\ell_\infty(\mathbb{N} \times \mathbb{N}, X)$ consisting of those $\mathbf{m}$ that have limit 0 along the filterbase $\mathcal{B}$ in $\mathbb{N} \times \mathbb{N}$ formed by the sets $B_m = N_m \times \mathbb{N}$, where $N_m = \{n \in \mathbb{N} : n \geq m\}, m \in \mathbb{N}$. Beware not to confuse this subspace with $c_0(\mathbb{N} \times \mathbb{N}, X)$. Now, let us introduce an isomorphic embedding $j: \ell_\infty(X) \rightarrow \ell_\infty(\mathbb{N} \times \mathbb{N}, X)$ by setting for every $x = (x(n))_n: jx = \mathbf{m} = [x_{n,k}]$ with $x_{n,k} = x(n)$ for all $n, k \in \mathbb{N}$; thus x becomes the k th column of $\mathbf{m}$ for every $k \in \mathbb{N}$. Note that then j maps $c_0(X)$ into $c_0((\mathbb{N} \times \mathbb{N}; \xrightarrow[\mathcal{B}]{} 0), X)$. For future reference also observe that the map $\pi: c_0((\mathbb{N} \times \mathbb{N}; \xrightarrow[\mathcal{B}]{} 0), X) \rightarrow j(c_0(X))$ that assigns to every matrix $\mathbf{m} \in c_0((\mathbb{N} \times \mathbb{N}; \xrightarrow[\mathcal{B}]{} 0), X)$ the

matrix with all columns equal to the first column of $\mathbf{m}$ is a projection. Finally, suppose there exists a projection P from $\ell_\infty(\mathbb{N} \times \mathbb{N}, X)$ onto $c_0((\mathbb{N} \times \mathbb{N}; \xrightarrow{\mathcal{B}} 0), X)$. Then consider also the ‘diagonal’ operator $D: c_0((\mathbb{N} \times \mathbb{N}; \xrightarrow{\mathcal{B}} 0), X) \rightarrow c_0(X)$ defined for $\mathbf{m} = [x_{n,k}]$ by $D\mathbf{m} = (x_{n,n})$. As easily seen, $DPj: \ell_\infty(X) \rightarrow c_0(X)$ is a surjective operator; in fact, $DPj|_{c_0(X)}$ is the identity operator on $c_0(X)$. Thus DPj is a projection, contradicting the assumption.

(2): Proceed as in (1), replacing the projection P by a supposed surjective operator R from $\ell_\infty(\ell_\infty(X))$ onto $c_0(\ell_\infty(X))$. Then the operator $j^{-1}\pi Rj$ maps $\ell_\infty(X)$ onto $c_0(X)$, a contradiction. $\square$

Remark 4.6 As follows from the observation made in the proof above, for any TVS X the space $c_0(\ell_\infty(X))$ contains a complemented copy of $c_0(X)$. Thus, in particular (for $X = \mathbb{K}$), $c_0(\ell_\infty)$ contains a complemented copy of c_0 . In consequence, $c_0(\ell_\infty)$ is not a Grothendieck space, hence it is not isomorphic to ℓ_∞ . For basic information on Grothendieck spaces see [1, p. 179].

We finish with an awfully annoying problem.

Problem 4.7 Does there exist a necessarily nonlocally convex and containing a copy of ℓ_∞ TVS X such that $c_0(X)$ is complemented in $\ell_\infty(X)$?

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