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RESEARCH ARTICLE

Nonproper intersection products and generalized cycles

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Abstract

We develop intersection theory in terms of the $\mathcal{B}$-group of a reduced analytic space. This group was introduced in a previous work as an analogue of the Chow group; it is generated by currents that are direct images of Chern forms and it contains all usual cycles. However, contrary to Chow classes, the $\mathcal{B}$-classes have well-defined multiplicities at each point. We focus on a $\mathcal{B}$-analogue of the intersection theory based on the Stückrad–Vogel procedure and the join construction in projective space. Our approach provides global $\mathcal{B}$-classes which satisfy a Bézout theorem and have the expected local intersection numbers. We also introduce $\mathcal{B}$-analogues of more classical constructions of intersections using the Gysin map of the diagonal. These constructions are connected via a $\mathcal{B}$-variant of van Gastel's formulas. Furthermore, we prove that our intersections coincide with the classical ones on cohomology level.

Keywords Analytic cycles · Currents · Nonproper intersections · Stückrad–Vogel procedure · Monge–Ampère type product

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1 Introduction

Let Y be a smooth manifold of dimension n. Assume that $\mu_1, \dots, \mu_r$ are cycles on Y of pure codimensions $\kappa_1, \dots, \kappa_r$, respectively, let $\kappa := \kappa_1 + \cdots + \kappa_r$, and let V be the set-theoretic intersection $V = |\mu_1| \cap \cdots \cap |\mu_r|$. If μ_j intersect properly, that is, if codim $V = \kappa$, then there is a well-defined intersection cycle

$$\mu_1 \cdot_Y \dots \cdot_Y \mu_r = \sum m_j V_j,$$

where V_j are the irreducible components of V and m_j are integers. In the nonproper case there is no canonical intersection cycle. However, following Fulton–MacPherson, see [12], there is an intersection product $\mu_1 \cdot_Y \cdots \cdot_Y \mu_r$, which is an element in the Chow group $\mathcal{A}_{n-\kappa}(V)$; that is, the product is represented by a cycle on V of dimension $n - \kappa$ that is determined up to rational equivalence. For instance, the self-intersection of a line L in $\mathbb{P}^n$ is obtained by intersecting L with a perturbation of L. If $n = 2$ one gets an arbitrary point on L, whereas if $n \geqslant 3$ the intersection is empty.

In case $Y = \mathbb{P}^n$ there is an intersection product due to Stückrad and Vogel [16,18], that in general consists of components of various dimensions. For instance the self-intersection of a line is actually the line itself independently of n. However, in general a nonproper intersection has so-called moving components, that are only determined up to rational equivalence. There is a relation to the classical (Fulton–MacPherson) intersection product via van Gastel's formulas [14], see also [11].

Tworzewski [17], introduced, for $x \in V$, *local intersection numbers*

$$\epsilon_\ell(\mu_1, \dots, \mu_r, x), \quad \ell = 0, \dots, \dim V, \tag{1.1}$$

see also [1,2,13] and Sect. 3 below. In the proper case $\epsilon_\ell(\mu_1, \dots, \mu_r, x)$ is precisely the multiplicity at x of the proper intersection $\mu_1 \cdot_Y \cdots \cdot_Y \mu_r$ for $\ell =$dim V and 0 otherwise. In the nonproper case the intersection numbers may be nonzero also for $\ell <$ dim V. In general no representative of the classical intersection product, cf. [6, Remark 1.4], or representative of the Stückrad–Vogel product, can represent these numbers at all points.

The main objective of this paper is to introduce a product of cycles in $\mathbb{P}^n$ that at each point carries the local intersection numbers and at the same time have reasonable global properties, such as respecting the Bézout formula. To this end we must extend the class of cycles, and our construction is based on the $\mathbb{Z}$-module $\mathcal{GZ}(X)$ of *generalized cycles* on a (reduced) analytic space X introduced in [4]. It is the smallest class of currents on analytic spaces that is closed under multiplication by components of Chern forms and under direct images under proper holomorphic mappings. It turns out that generalized cycles inherit a lot of geometric properties and preferably can be thought of as geometric objects. Actually we are primarily interested in a certain natural quotient group $\mathcal{B}(X)$ of $\mathcal{GZ}(X)$. Each μ in $\mathcal{GZ}(X)$ has a well-defined Zariski support $|\mu| \subset X$ that only depends on its class in $\mathcal{B}(X)$. For a subvariety $V \hookrightarrow X$ there is a natural identification of $\mathcal{B}(V)$ with elements in $\mathcal{B}(X)$ that have Zariski support on V. The group of cycles $\mathcal{Z}(X)$ is naturally embedded in $\mathcal{B}(X)$. Given $\mu \in \mathcal{B}(X)$ also its restriction $\mathbf{1}_V \mu$ to the subvariety V is an element in $\mathcal{B}(X)$. Each element in $\mathcal{GZ}(X)$, and in $\mathcal{B}(X)$, has a unique decomposition into sums of irreducible components. Each irreducible element has in turn a unique decomposition into components of various dimensions. We let $\mathcal{B}_k(X)$ denote the elements in $\mathcal{B}(X)$ of pure dimension k. We also introduce a notion of *effective* generalized cycle μ in $\mathcal{GZ}(X)$, and class in $\mathcal{B}(X)$, generalizing the notion of effective cycle. Each μ in $\mathcal{GZ}(X)$, and in $\mathcal{B}(X)$, has a well-defined multiplicity, $\text{mult}_x\, \mu$, at each point $x \in X$, that is an integer and nonnegative if μ is effective. Moreover, for each μ in $\mathcal{GZ}_k(X)$, or in $\mathcal{B}_k(X)$, there is a unique decomposition

$$\mu = \mu_{\text{fix}} + \mu_{\text{mov}}, \tag{1.2}$$

where μ_{fix} is an ordinary cycle of dimension k, whose irreducible components are called the *fixed* components of μ, and μ_{mov}, whose irreducible components, the *moving* components, have Zariski support on varieties of dimension strictly larger than k.

Each μ in $\mathcal{GZ}_k(\mathbb{P}^n)$, or in $\mathcal{B}_k(\mathbb{P}^n)$, has the *degree*

$$\deg \mu := \int_{\mathbb{P}^n} \omega^k \wedge \mu, \tag{1.3}$$

where ω is the first Chern class of $\mathcal{O}(1) \to \mathbb{P}^N$, for instance represented by the Fubini–Study metric form. If $\mu = \mu_0 + \mu_1 + \cdots$, where μ_k has pure dimension k, then

$$\deg \mu := \deg \mu_0 + \deg \mu_1 + \cdots$$

For each point $x \in \mathbb{P}^n$ and $\mu_1, \ldots, \mu_r \in \mathcal{B}(\mathcal{U})$ for some open subset $\mathcal{U} \subset \mathbb{P}^n$ there are $\mathbb{Z}$-valued functions $\epsilon_\ell(\mu_1, \ldots, \mu_r, x)$, $\ell = 0, 1, \ldots$, that are $\mathbb{Z}$-multilinear in μ_j, only depend on the germs of μ_j at x, and which coincide with the local intersection numbers (1.1) if μ_j are cycles. We say that $\epsilon_\ell(\mu_1, \ldots, \mu_r, x)$ are the *local intersection numbers* of $\mu_1, \ldots, \mu_r$ at x. If μ_j are effective, then these numbers are nonnegative.

Our main result concerns a $\mathbb{Z}$-multilinear mapping

$$\mathcal{B}(\mathbb{P}^n) \times \cdots \times \mathcal{B}(\mathbb{P}^n) \to \mathcal{B}(\mathbb{P}^n), \quad \mu_1, \ldots, \mu_r \mapsto \mu_1 \bullet \cdots \bullet \mu_r. \tag{1.4}$$

We say that the image is the $\bullet$-product of $\mu_1, \ldots, \mu_r$. It is obtained, roughly speaking, in the following way: We first choose representatives for the $\mathcal{B}$-classes $\mu_1, \ldots, \mu_r$, then form a Stückrad–Vogel-type product of them. Even for cycles, this product depends on several choices. Taking a suitable mean value, we get a generalized cycle that turns out to define an element in $\mathcal{B}(\mathbb{P}^n)$ that is independent of all choices. If μ_j are cycles, then the fixed components in the Stückrad–Vogel product appear as fixed components of $\mu_1 \bullet \cdots \bullet \mu_r$. The formal definition, Definition 6.6, is expressed in terms of a certain Monge–Ampère type product, that can be obtained as a limit of quite explicit expressions, see Sect. 6. Here is our main result.

Theorem 1.1 *The $\mathbb{Z}$-multilinear mapping (1.4) has the following properties. To begin with, $|\mu_1 \bullet \cdots \bullet \mu_r| \subset \bigcap_{j=1}^r |\mu_j|$, $\mu_1 \bullet \cdots \bullet \mu_r$ is commutative, and*

$$\mathrm{mult}_x(\mu_1 \bullet \cdots \bullet \mu_r)_\ell = \epsilon_\ell(\mu_1, \ldots, \mu_r, x), \quad x \in \mathbb{P}^n, \quad \ell = 0, 1, \ldots, \tag{1.5}$$

where $(\,)_\ell$ denotes the component of dimension ℓ. If μ_j have pure dimensions and

$$\rho := \sum_1^r \dim \mu_j - (r-1)n \geqslant 0, \tag{1.6}$$

then

$$\deg(\mu_1 \bullet \cdots \bullet \mu_r) = \textstyle\prod_1^r \deg \mu_j. \tag{1.7}$$

If μ_j are effective, then $\mu_1 \bullet \cdots \bullet \mu_r$ is effective and

$$\deg(\mu_1 \bullet \cdots \bullet \mu_r) \leqslant \textstyle\prod_1^r \deg \mu_j. \tag{1.8}$$

If $\mu_1, \ldots, \mu_r$ are cycles that intersect properly, then

$$\mu_1 \bullet \cdots \bullet \mu_r = \mu_1 \cdot_{\mathbb{P}^n} \cdots \cdot_{\mathbb{P}^n} \mu_r. \tag{1.9}$$

One should keep in mind that the $\bullet$-product of r factors is not a repeated $\bullet$-product of two factors. In general, the $\bullet$-product of two factors is not associative, see Example 8.10.

Notice that ρ equals $n - (n - \dim \mu_1 + \cdots + n - \dim \mu_r)$, which is the "expected dimension" of the intersection. The Bézout formula (1.7) may hold even if $\rho < 0$: For instance, if μ_j are different lines through the point a, then their $\bullet$-product is a so that both sides of (1.8) are 1, see Example 8.8. Moreover, if we take a linear embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n'}$, $n' > n$, and consider μ_j as elements in $\mathcal{B}(\mathbb{P}^{n'})$, then the product is unchanged. In particular, the $\bullet$-self-intersection of a k-plane is always the k-plane itself.

The $\bullet$-self-intersection of the cuspidal curve $Z = \{x_1^3 - x_0 x_2^2 = 0\}$ in $\mathbb{P}^2$ is in the classical sense represented by nine points on Z obtained as the divisor of a generic meromorphic section of $\mathcal{O}_{\mathbb{P}^2}(3)$ restricted to Z. The fixed part of the self-intersection in the Stückrad–Vogel sense is the curve itself plus 3 times the point $a = [1, 0, 0]$, whereas the moving part consists of another three points on Z that are determined up to rational equivalence on Z. Our product $Z \bullet Z$ consists of the the fixed part $Z + 3a$ of the Stückrad–Vogel (SV)-product and a moving component μ of dimension zero and degree 3; we think of μ as three points "moving around" on Z, cf. Example 8.13. In this case the local intersection numbers are carried by the fixed components. In general also moving components can contribute, see, e.g., Example 8.6.

We also consider another intersection product that is a $\mathcal{B}$-variant of the classical nonproper intersection product in [12]: For any regular embedding i, in [4] we introduced a $\mathcal{B}$-analogue of the Gysin mapping $i^!$ used in [12], see Sect. 2.7 below. Let

$i : \mathbb{P}^n \to \mathbb{P}^n \times \cdots \times \mathbb{P}^n = (\mathbb{P}^n)^r$ be the diagonal embedding in $(\mathbb{P}^n)^r$. In analogy with the classical intersection product in [12] we define, for pure-dimensional μ_j,

$$\mu_1 \cdot_{\mathcal{B}(\mathbb{P}^n)} \cdots \cdot_{\mathcal{B}(\mathbb{P}^n)} \mu_r := i^!(\mu_1 \times \cdots \times \mu_r)$$

in $\mathcal{B}(\mathbb{P}^n)$. We have the following relation to the $\bullet$-product.

Theorem 1.2 *Assume that $\mu_1, \ldots, \mu_r \in \mathcal{B}(\mathbb{P}^n)$ have pure dimensions. Let $V = \bigcap_j |\mu_j|$ and let ρ be as in (1.6). Then*

$$\mu_1 \cdot_{\mathcal{B}(\mathbb{P}^n)} \cdots \cdot_{\mathcal{B}(\mathbb{P}^n)} \mu_r = \sum_{\ell=\max(\rho,0)}^{\dim V} \omega^{\ell-\rho} \wedge (\mu_1 \bullet \cdots \bullet \mu_r)_\ell.$$

In particular, $\mu_1 \cdot_{\mathcal{B}(\mathbb{P}^n)} \cdots \cdot_{\mathcal{B}(\mathbb{P}^n)} \mu_r = \mu_1 \cdot_{\mathbb{P}^n} \cdots \cdot_{\mathbb{P}^n} \mu_r$ if $\mu_1, \ldots, \mu_r$ are cycles that intersect properly, see (1.9).

In [4, Section 10] we introduced cohomology groups $\widehat{H}^{*,*}(V)$ for a reduced subvariety $V \hookrightarrow \mathbb{P}^n$ of pure dimension d that coincide with usual de Rham cohomology $H^{*,*}(V)$ when V is smooth. There are natural mappings $\mathcal{A}_k(V) \to \widehat{H}^{d-k,d-k}(V)$ and $\mathcal{B}_k(V) \to \widehat{H}^{d-k,d-k}(V)$.

Theorem 1.3 *Assume that $Z_1, \ldots, Z_r$ are cycles in $\mathbb{P}^N$ and let $V = \bigcap_j |Z_j|$. The images in $\widehat{H}^{*,*}(V)$ of the Chow class $Z_1 \cdot_{\mathbb{P}^n} \cdots \cdot_{\mathbb{P}^n} Z_r$ and the $\mathcal{B}$-class $Z_1 \cdot_{\mathcal{B}(\mathbb{P}^n)} \cdots \cdot_{\mathcal{B}(\mathbb{P}^n)} Z_r$ coincide.*

The plan of the paper is as follows. Sections 2 through 4 mainly contain material from [4] and well-known facts from [12], as well as the definition of local intersection numbers and of the notion of an effective generalized cycle. The product $\cdot_{\mathcal{B}(Y)}$ is introduced in Sect. 5. In Sect. 6 we define the $\bullet$-product and prove Theorem 1.1, whereas the connection to the $\cdot_{\mathcal{B}(\mathbb{P}^n)}$-product is worked out in Sect. 7. Finally we have collected several examples in Sect. 8.

2 Preliminaries

Throughout this section X is a reduced analytic space of pure dimension n. We will recall some basic notions from intersection theory that can be found in [12], and some notions and results from [4]; however the material in Sect. 2.4 and Lemma 2.1 are new. We formulate statements in terms of coherent sheaves, rather than schemes.

2.1 Currents and cycles

We say that a current μ on X of bidegree $(n-k, n-k)$ has (*complex*) *dimension* k. If $f : X' \to X$ is a proper mapping of analytic spaces, then f^* is well-defined on smooth forms, and f_* is well-defined on currents and preserves dimension, see [7, III Corollary 2.4.11] or [5]. If μ is a current on X' and η is a smooth form on X, then

$$\eta \wedge f_*\mu = f_*(f^*\eta \wedge \mu). \tag{2.1}$$

If μ has order zero then $f_*\mu$ has order zero. If $V \hookrightarrow X$ is a subvariety, then

$$\mathbf{1}_V f_*\mu = f_*\big(\mathbf{1}_{f^{-1}V}\mu\big). \tag{2.2}$$

If $V \hookrightarrow X$ has dimension k, then its associated Lelong current (current of integration) $[V]$ has dimension k. We will often identify V and $[V]$. An analytic k-cycle μ on X is a formal locally finite linear combination $\sum a_j V_j$, where $a_j \in \mathbb{Z}$ and $V_j \subset X$ are irreducible analytic sets of dimension k. We denote the $\mathbb{Z}$-module of analytic k-cycles on X by $\mathcal{Z}_k(X)$. The support $|\mu|$ of $\mu \in \mathcal{Z}_k(X)$ coincides with the support of its associated Lelong current. Recall that $\mathrm{mult}_x\mu = \ell_x\mu$, where $\ell_x\mu$ denotes the Lelong number (of the Lelong current) of $\mu \in \mathcal{Z}_k(X)$ at x, and $\mathrm{mult}_x\mu$ is the multiplicity of μ at x, see [9, Chapter 2.11.1].

If $f : X' \to X$ is a proper mapping, then we have a mapping

$$f_* : \mathcal{Z}_k(X') \to \mathcal{Z}_k(X), \tag{2.3}$$

and the Lelong current of the direct image $f_*\mu$ is the direct image of the Lelong current of μ. If $i : V \hookrightarrow X$ is a subvariety, then $\mu \in \mathcal{Z}_k(V)$ can be identified with the cycle $i_*\mu \in \mathcal{Z}_k(X)$. The cycle $\mu \in \mathcal{Z}_k(X)$ is *rationally equivalent* to 0 on X, $\mu \sim 0$, if there are finitely many subvarieties $i_j : V_j \hookrightarrow X$ of dimension $k+1$ and non-trivial meromorphic functions g_j on V_j such that[1]

$$\mu = \sum_j (i_j)_*[\mathrm{div}\, g_j] = \sum_j (i_j)_* dd^c \log|g_j|^2 = \sum_j dd^c(\log|g_j|^2[V_j]).$$

We denote the *Chow group* of cycles $\mathcal{Z}_k(X)$ modulo rational equivalence by $\mathcal{A}_k(X)$. If $f : X' \to X$ is a proper morphism and $\mu \sim 0$ in $\mathcal{A}_k(X')$, then $f_*\mu \sim 0$ in $\mathcal{A}_k(X)$ and there is an induced mapping $f_* : \mathcal{A}_k(X') \to \mathcal{A}_k(X)$.

2.2 Chern and Segre forms

Recall that to any Hermitian line bundle[2] $L \to X$ there is an associated (total) Chern form[3] $\widehat{c}(L) = 1 + \widehat{c}_1(L)$ and that two Hermitian metrics give rise to Chern forms whose difference is $dd^c\gamma$ for a smooth form γ on X. We let $c(L)$ denote the associated cohomology class.

Assume that $E \to X$ is a Hermitian vector bundle, and let $\pi : \mathbb{P}(E) \to X$ be the projectivization of E, i.e., the projective bundle of lines in E. Let $L = \mathcal{O}(-1)$ be the tautological line bundle in the pullback $\pi^*E \to \mathbb{P}(E)$, and let $\widehat{c}(L)$ be the induced Chern form on $\mathbb{P}(E)$. Since π is a submersion, $\widehat{s}(E) := \pi_*(1/\widehat{c}(L))$ is a smooth form

[1] Here $d^c = (\partial - \bar{\partial})/4\pi i$ so that $dd^c \log|z|^2 = [0]$ in $\mathbb{C}$, writing $[0]$ rather than $[\{0\}]$ for the point mass at 0.

[2] All line bundles and vector bundles and morphism between them are assumed to be holomorphic.

[3] For Chern and Segre forms (and classes), the index k denotes the component of bidegree (k, k), i.e., of (complex) dimension $n-k$.

on X called the Segre form of E. If E is a line bundle, then $\mathbb{P}(E) \simeq X$ and hence

$$\widehat{c}(E) = 1/\widehat{s}(E). \tag{2.4}$$

For a general Hermitian $E \to X$ we take (2.4) as the definition of its associated Chern form. If $f : X' \to X$ is a proper mapping, then

$$\widehat{c}_\ell(f^*E) = f^*\widehat{c}_\ell(E). \tag{2.5}$$

Since π is a submersion two different metrics on E give rise to Segre forms and Chern forms that differ by $dd^c\gamma$ for a smooth form γ on X. The induced cohomology classes are denoted by $s(E)$ and $c(E)$, respectively. There are induced mappings

$$\mathcal{A}_k(X) \to \mathcal{A}_{k-\ell}(X), \quad \alpha \mapsto c_\ell(E) \cap \alpha.$$

2.3 Generalized cycles

Generalized cycles on X were introduced in [4] and all statements in this subsection except Lemma 2.1 are proved in [4, Sections 3 and 4]. We say that a current μ is a *generalized cycle* if it is a locally finite linear combination over $\mathbb{Z}$ of currents of the form $\tau_*\alpha$, where $\tau : W \to X$ is a proper map, W is smooth and connected, and α is a product of components of Chern forms for various Hermitian vector bundles over W, i.e.,

$$\alpha = \widehat{c}_{\ell_1}(E_1) \wedge \cdots \wedge \widehat{c}_{\ell_r}(E_r), \tag{2.6}$$

where E_j are Hermitian vector bundles over W. One can just as well use components of Segre forms, and one can in fact assume that all E_j are line bundles.

Notice that a generalized cycle is a real closed current of order zero with components of bidegree $(*, *)$. We let $\mathcal{GZ}_k(X)$ denote the $\mathbb{Z}$-module of generalized cycles of (complex) *dimension* k (i.e., of bidegree $(n-k, n-k)$) and we let $\mathcal{GZ}(X) = \bigoplus \mathcal{GZ}_k(X)$. If $\mu \in \mathcal{GZ}(X)$ and γ is a component of a Chern form on X, then $\gamma \wedge \mu \in \mathcal{GZ}(X)$. If $E \to X$ is a Hermitian vector bundle we thus have mappings $\mathcal{GZ}_k(X) \to \mathcal{GZ}_{k-\ell}(X)$ defined by $\mu \mapsto \widehat{c}_\ell(E) \wedge \mu$.

If $i : V \hookrightarrow X$ is a subvariety and $\mu \in \mathcal{GZ}(X)$, then $\mathbf{1}_V \mu \in \mathcal{GZ}(X)$. More precisely, if

$$\mu = \sum_j (\tau_j)_* \alpha_j, \tag{2.7}$$

where $\tau_j : W_j \to X$, then

$$\mathbf{1}_V \mu = \sum_{\tau_j(W_j) \subset V} (\tau_j)_* \alpha_j.$$

Each subvariety of X is a generalized cycle so we have an embedding

$$\mathcal{Z}_k(X) \to \mathcal{GZ}_k(X).$$

Given $\mu \in \mathcal{GZ}(X)$ there is a smallest variety $|\mu| \subset X$, the *Zariski support* of μ, such that μ vanishes outside $|\mu|$. If $f: X' \to X$ is proper, then we have a natural mapping

$$f_*: \mathcal{GZ}_k(X') \to \mathcal{GZ}_k(X)$$

that coincides with (2.3) on $\mathcal{Z}_k$. If $i: V \hookrightarrow X$ is a subvariety, then

$$i_*: \mathcal{GZ}_k(V) \to \mathcal{GZ}_k(X) \tag{2.8}$$

is an injective mapping whose image is precisely those $\mu \in \mathcal{GZ}_k(X)$ such that $|\mu| \subset V$. Thus we can identify $\mathcal{GZ}(V)$ with generalized cycles in X that have Zariski support on Z. We have

Dimension principle *Assume that $\mu \in \mathcal{GZ}_k(X)$ has Zariski support on a variety V. If* $\dim V = k$*, then* $\mu \in \mathcal{Z}_k(X)$*. If* $\dim V < k$*, then* $\mu = 0$.

A nonzero generalized cycle $\mu \in \mathcal{GZ}(X)$ is *irreducible* if $|\mu|$ is irreducible and $\mathbf{1}_V \mu = 0$ for any proper analytic subvariety $V \hookrightarrow |\mu|$. If μ has Zariski support $V \subset X$ it is irreducible if and only if V is irreducible and μ has a representation (2.7) where $\tau_j(W_j) = V$ for each j. An irreducible $\mu \in \mathcal{GZ}(X)$ has the decomposition $\mu = \mu_p + \cdots + \mu_1 + \mu_0$, $\mu_k \in \mathcal{GZ}_k(X)$, where p is the dimension of $|\mu|$. Each $\mu \in \mathcal{GZ}(X)$ has a unique decomposition

$$\mu = \sum_\ell \mu_\ell,$$

where μ_ℓ are irreducible with different Zariski supports.

If $0 \to S \to E \to Q \to 0$ is a short exact sequence of Hermitian vector bundles over X, then we say that $\widehat{c}(E) - \widehat{c}(S) \wedge \widehat{c}(Q)$ is a B-form. If β is a component of a B-form, then there is a smooth form γ on X such that $dd^c\gamma = \beta$. We say that $\mu \in \mathcal{GZ}_k(X)$ is equivalent to 0 in X, $\mu \sim 0$, if μ is a locally finite sum of currents of the form

$$\rho = \tau_*(\beta \wedge \alpha) = dd^c \tau_*(\gamma \wedge \alpha), \tag{2.9}$$

where $\tau: W \to X$ is proper, β is a component of a B-form, α is a product of components of Chern or Segre forms, and γ is a smooth form on W. If $\mu = \mu_0 + \cdots + \mu_n$, where $\mu_k \in \mathcal{GZ}_k(X)$, we say that $\mu \sim 0$ if $\mu_k \sim 0$ for each k. Let $\mathcal{B}(X)$ denote the $\mathbb{Z}$-module of generalized cycles on X modulo this equivalence. A class $\mu \in \mathcal{B}(X)$ has *pure dimension* k, $\mu \in \mathcal{B}_k(X)$, if μ has a representative in $\mathcal{GZ}_k(X)$. Thus $\mathcal{B}(X) = \bigoplus_k \mathcal{B}_k(X)$. The mapping $\mathcal{Z}(X) \to \mathcal{B}(X)$ is injective so we can consider $\mathcal{Z}(X)$ as a subgroup of $\mathcal{B}(X)$.

If $\mu \in \mathcal{B}(X)$ and $\widehat{\mu} \in \mathcal{GZ}(X)$ is a representative for μ, then the *Zariski support* $|\mu| \subset X$ of μ is the union of the Zariski supports of the irreducible components of $\widehat{\mu}$ that are nonzero in $\mathcal{B}(X)$. Moreover, $\mu \in \mathcal{B}(X)$ is *irreducible* if there is a representative $\widehat{\mu} \in \mathcal{GZ}(X)$ that is irreducible. The decomposition into irreducible components, as well as the decomposition into components of different dimensions, extend from $\mathcal{GZ}(X)$ to $\mathcal{B}(X)$.

If $0 \to S \to E \to Q \to 0$ is a short exact sequence of Hermitian vector bundles and $\widehat{\mu} \in \mathcal{GZ}(X)$, then

$$\widehat{c}_\ell(E) \wedge \widehat{\mu} \sim (\widehat{c}(S) \wedge \widehat{c}(Q))_\ell \wedge \widehat{\mu}. \tag{2.10}$$

In particular, if E and E' are the same vector bundle with two different Hermitian metrics, then $\widehat{c}_\ell(E) \wedge \widehat{\mu} \sim \widehat{c}_\ell(E') \wedge \widehat{\mu}$ so we have mappings

$$\mathcal{B}_k(X) \to \mathcal{B}_{k-\ell}(X), \quad \mu \mapsto c_\ell(E) \wedge \mu.$$

If $f: X' \to X$ is a proper mapping, then we have a natural mapping

$$f_*: \mathcal{B}(X') \to \mathcal{B}(X).$$

If $i: V \hookrightarrow X$ is a subvariety, then

$$i_*: \mathcal{B}(V) \to \mathcal{B}(X) \tag{2.11}$$

is injective, and we can identify its image with the elements in $\mathcal{B}(X)$ that have Zariski support on V.

Each $\mu \in \mathcal{B}_k(X)$ (and $\mu \in \mathcal{GZ}_k(X)$) has a unique decomposition (1.2) where $\mu_{\rm fix}$ is a cycle of pure dimension k and the irreducible components of $\mu_{\rm mov}$ have Zariski supports of dimension strictly larger than k. We say that the irreducible components of $\mu_{\rm fix}$ are *fixed* and that the irreducible components of $\mu_{\rm mov}$ are *moving*.

We will need the following simple lemma.

Lemma 2.1 *Assume that $\mu_1, \ldots, \mu_r$ are generalized cycles on reduced analytic spaces $X_1, \ldots, X_r$. Let $p_j: X_1 \times \cdots \times X_r \to X_j$ be the natural projections. Then*

$$\mu_1 \times \cdots \times \mu_r := p_1^* \mu_1 \wedge \cdots \wedge p_r^* \mu_r$$

is a generalized cycle on $X_1 \times \cdots \times X_r$. If $\mu_j \sim 0$ in X_j for some j, then $\mu_1 \times \cdots \times \mu_r \sim 0$ in $X_1 \times \cdots \times X_r$.

In particular, for $\mu_j \in \mathcal{B}(X_j)$, $j = 1, \ldots, r$, there is a well-defined $\mu_1 \times \cdots \times \mu_r \in \mathcal{B}(X_1 \times \cdots \times X_r)$.

Proof Assume that $\mu_j = (\tau_j)_* \alpha_j$, where $\tau_j: W_j \to X_j$ are proper and α_j are products of components of Chern forms. Let $\pi_j: W_1 \times \cdots \times W_r \to W_j$ be the natural projections. Then $\pi_1^*\alpha_1\wedge\cdots\wedge\pi_r^*\alpha_r$ is a product of components of Chern forms on $W_1\times\cdots\times W_r$ and

$$\mu_1\times\cdots\times\mu_r = p_1^*\mu_1\wedge\cdots\wedge p_r^*\mu_r = (\tau_1\times\cdots\times\tau_r)_*(\pi_1^*\alpha_1\wedge\cdots\wedge\pi_r^*\alpha_r),$$

and hence it is a generalized cycle on $X_1\times\cdots\times X_r$. If, say, $\mu_1 \sim 0$, we may assume, cf. (2.9), that $\mu_1 = (\tau_1)_*(\beta\wedge\alpha_1)$, where β is a component of a B-form. Then also $\pi_1^*\beta$ is a component of a B-form. Now $\mu_1\times\cdots\times\mu_r$ is the push-forward of $\pi_1^*\beta\wedge\pi_1^*\alpha_1\wedge\pi_2^*\alpha_2\wedge\cdots\wedge\pi_r^*\alpha_r$ and therefore it is equivalent to 0 in $X_1\times\cdots\times X_r$ by definition. □

2.4 Effective generalized cycles

We say that a generalized cycle μ is *effective* if it is a positive current, see, e.g., [10, Chapter III, Definition 1.13]. Clearly effectivity is preserved under direct images.

Lemma 2.2 *Let $\mu = \mu_1+\mu_2+\cdots$ be the decomposition of $\mu\in\mathcal{GZ}(X)$ into its irreducible components. Then μ is effective if and only if each μ_j is effective.*

Proof The if-part is clear. For the converse, let V be an irreducible subvariety of X. We already know that $\mathbf{1}_V\mu$ is a generalized cycle. It is not hard to see that it is positive if μ is positive. It is also part of the Skoda–El Mir theorem, see, e.g., [10, Chapter III, Theorem 2.3]. Now let V_j be the Zariski supports of the various μ_j and assume that V_k has minimal dimension. Then $V_k\cap V_j$ has positive codimension in V_j for each $j\neq k$. By the definition of irreducibility it follows that $\mathbf{1}_{V_k}\mu = \mathbf{1}_{V_k}\mu_k = \mu_k$. We conclude that μ_k is positive for each k such that V_k has minimal dimension. Let V' be the union of these V_k and let μ' be the sum of the remaining irreducible components. Clearly μ' is positive in $X\setminus V'$. Let $A = ia_1\wedge\bar{a}_1\wedge\cdots\wedge ia_r\wedge\bar{a}_r$ for smooth $(1,0)$-forms a_j and some r. It follows that $A\wedge\mu'$ is positive outside V' by definition. However, $\mathbf{1}_{V'}\mu' = 0$ and so $A\wedge\mu' = A\wedge\mathbf{1}_{X\setminus V'}\mu'$ is positive. Since A is arbitrary, we conclude that μ' is positive. Now the lemma follows by induction. □

We say that $\mu\in\mathcal{B}(X)$ is *effective* if it has a representative $\widehat{\mu}\in\mathcal{GZ}(X)$ that is effective. It follows that μ is effective if and only each of its irreducible components is effective. Moreover, the multiplicities of an effective $\mu\in\mathcal{B}(X)$ are nonnegative.

2.5 The Segre and $\mathcal{B}$-Segre class

The material in this subsection is found in [4, Section 5] or in [12]. Let $\mathcal{J}\to X$ be a coherent ideal sheaf over X with zero set Z. First assume that X is irreducible. If $\mathcal{J}=0$ on X, then we define the Segre class $s(\mathcal{J},X) = s_0(\mathcal{J},X) = \mathbf{1}_X\in\mathcal{A}_n(X)$. Otherwise, let $\pi: X'\to X$ be a modification such that $\pi^*\mathcal{J}$ is principal[4]. For instance X' can be the blow-up of X along $\mathcal{J}$, or its normalization. Let D be the exceptional divisor, and let L_D be the associated line bundle that has a section σ^0 that defines D

[4] In this paper, $\pi^*\mathcal{J}$ denotes the ideal sheaf on X generated by the pullback of local generators of $\mathcal{J}$.

and hence generates $\pi^*\mathcal{J}$. Then

$$s(\mathcal{J}, X) := \sum_{j \geqslant 0} (-1)^j \pi_*\big(c_1(L_D)^j \cap [D]\big) = \pi_*\Big(\frac{1}{1+c_1(L_D)} \cap [D]\Big);$$

it is a well-defined element in $\mathcal{A}_*(X)$. If X has irreducible components $X_1, X_2, \ldots$, then $s(\mathcal{J}, X) = s(\mathcal{J}, X_1) + s(\mathcal{J}, X_2) + \cdots$. Notice that $s(\mathcal{J}, X)$ has support in Z so that it can be identified with an element $s(\mathcal{J}, X)$ in $\mathcal{A}_*(Z)$. If $\mathcal{J}$ is the sheaf associated with the subscheme V of X, then $s(\mathcal{J}, X)$ coincides with the classical Segre class $s(V, X)$, cf. [12, Corollary 4.2.2].

We can define the $\mathcal{B}$-Segre class $S(\mathcal{J}, X)$ in an analogous way by just interpreting $\cap$ as the ordinary wedge product. However, we are interested in more explicit representations and also in a definition of a $\mathcal{B}$-Segre class on $\mu \in \mathcal{B}(X)$. To this end we assume that the ideal sheaf $\mathcal{J} \to X$ is generated by a holomorphic section σ of a Hermitian vector bundle $E \to X$. If X is projective one can always find such a σ for any coherent ideal sheaf $\mathcal{J} \to X$. We shall consider Monge–Ampère products on a generalized cycle μ.

Theorem 2.3 *Assume that σ is a holomorphic section of $E \to X$ and let $\mathcal{J}$ be the associated coherent ideal sheaf with zero set Z. For each $\mu \in \mathcal{GZ}(X)$ the limits*

$$(dd^c \log|\sigma|^2)^k \wedge \mu := \lim_{\epsilon \to 0} \big(dd^c \log(|\sigma|^2 + \epsilon)\big)^k \wedge \mu, \quad k = 0, 1, 2, \ldots,$$

exist and are generalized cycles with Zariski support on $|\mu|$. The generalized cycles

$$M_k^\sigma \wedge \mu := \mathbf{1}_Z\big((dd^c \log|\sigma|^2)^k \wedge \mu\big), \quad k = 0, 1, 2, \ldots,$$

have Zariski support on $Z \cap |\mu|$. If $\mu \sim 0$, then $M_k^\sigma \wedge \mu \sim 0$. If g is a holomorphic section of another vector bundle that also defines $\mathcal{J}$, then $M_k^\sigma \wedge \mu \sim M_k^g \wedge \mu$.

In case $\mu = \mathbf{1}_X$ we write M_k^σ rather than $M_k^\sigma \mathbf{1}_X$. We let $M^\sigma \wedge \mu = M_0^\sigma \wedge \mu + M_1^\sigma \wedge \mu + \cdots$.

Definition 2.4 Assume that $\mathcal{J} \to X$ is defined by the section σ of the Hermitian vector bundle $E \to X$. Given $\mu \in \mathcal{B}(X)$ and a representative $\widehat{\mu} \in \mathcal{GZ}(X)$, we let the $\mathcal{B}$-Segre class $S_k(\mathcal{J}, \mu)$ be the class in $\mathcal{B}(X)$ defined by $M_k^\sigma \wedge \widehat{\mu}$. We let $S(\mathcal{J}, \mu) = S_0(\mathcal{J}, \mu) + S_1(\mathcal{J}, \mu) + \cdots$.

Notice that $M_k^\sigma \wedge \widehat{\mu}$ has support in $Z \cap |\mu|$ so that we may identify $S(\mathcal{J}, \mu)$ with an element in $\mathcal{B}(Z \cap |\mu|)$, in $\mathcal{B}(Z)$, or in $\mathcal{B}(|\mu|)$. If $\mu = \mathbf{1}_X$ we denote $S(\mathcal{J}, \mu)$ by $S(\mathcal{J}, X)$.

Remark 2.5 If $\kappa = \max(0, \dim\mu - \dim(Z \cap |\mu|))$, then

$$S(\mathcal{J}, \mu) = S_\kappa(\mathcal{J}, \mu) + S_{\kappa+1}(\mathcal{J}, \mu) + \cdots + S_{\dim\mu}(\mathcal{J}, \mu).$$

Indeed, $S_\ell(\mathcal{J},\mu)$ has dimension $\dim\mu-\ell$ and Zariski support $Z\cap|\mu|$, so $S_\ell(\mathcal{J},\mu)=0$ if $\dim Z\cap\mu<\dim\mu-\ell$ by the dimension principle. Moreover, clearly $S_\ell(\mathcal{J},\mu)=0$ for degree reasons if $\ell>\dim\mu$.

If $\mathcal{J}$ vanishes identically on $|\mu|$, then it follows from the definition that $S(\mathcal{J},\mu)=\mu$.

One can define $M_k^\sigma\wedge\mu$ by a limit procedure without applying $\mathbf{1}_Z$, see [4, Proposition 5.7 and Remark 5.9].

Proposition 2.6 *Let σ be a holomorphic section of a Hermitian bundle $E\to X$ and let*

$$M^\sigma_{k,\epsilon}=\frac{\epsilon}{(|\sigma|^2+\epsilon)^{k+1}}(dd^c|\sigma|^2)^k,\quad k=0,1,2,\ldots$$

If $\mu\in\mathcal{GZ}(X)$, then

$$M^\sigma_k\wedge\mu=\lim_{\epsilon\to 0}M^\sigma_{k,\epsilon}\wedge\mu,\quad k=0,1,2,\ldots \tag{2.12}$$

Moreover, $M^\sigma\wedge\mu=\sum_k M^\sigma_k\wedge\mu$ is the value at $\lambda=0$, via analytic continuation from $\mathrm{Re}\,\lambda\gg 0$, *of*

$$M^{\sigma,\lambda}\wedge\mu=\left(1-|\sigma|^{2\lambda}+\sum_{k=1}\bar\partial|\sigma|^{2\lambda}\wedge\frac{\partial|\sigma|^2}{2\pi i|\sigma|^2}\wedge(dd^c\log|\sigma|^2)^{k-1}\right)\wedge\mu.$$

Example 2.7 If $\mu\in\mathcal{GZ}(X)$ and $\gamma\wedge\mu\in\mathcal{GZ}(\mathcal{U})$, where $\mathcal{U}\subset X$ is open and γ is a smooth form in $\mathcal{U}$, then by (2.12)

$$M^\sigma\wedge(\gamma\wedge\mu)=\gamma\wedge M^\sigma\wedge\mu \tag{2.13}$$

in $\mathcal{U}$.

Example 2.8 If $f\colon X'\to X$ is proper, $\mu'\in\mathcal{GZ}(X')$, and $\mu=f_*\mu'$, then (2.1) and (2.12) imply that

$$M^\sigma\wedge\mu=f_*(M^{f^*\sigma}\wedge\mu'). \tag{2.14}$$

Let ξ be a section of a vector bundle in a neighborhood $\mathcal{U}\subset X$ of x such that ξ defines the maximal ideal at x. Notice that if $\mu\in\mathcal{GZ}_k(X)$, then by Theorem 2.3, $M^\xi\wedge\mu$ is a generalized cycle with Zariski support at x and its image in $\mathcal{B}(X)$ is independent of the choice of section ξ defining the maximal ideal. In view of the dimension principle, see Sect. 2.3, $M^\xi\wedge\mu=M^\xi_k\wedge\mu=a[x]$ for some real number a. We say that a is the *multiplicity*, $\mathrm{mult}_x\mu$, of μ at x, i.e.,

$$\mathrm{mult}_x\mu=\int_{\mathcal{U}}M^\xi\wedge\mu. \tag{2.15}$$

It is an integer that is independent of the choice of neighborhood $\mathcal{U}$ and only depends on the class of μ in $\mathcal{B}(X)$. If μ is effective (i.e., represented by a positive current), then $\text{mult}_x\mu$ is the Lelong number of μ at x and hence nonnegative, see [4, Section 6].

Example 2.9 If $\mu \in \mathcal{GZ}(X)$ is of the form $\mu = \gamma \wedge \mu'$ in a neighborhood of x, where γ is a closed smooth form of positive degree and $\mu' \in \mathcal{GZ}(X)$, then $\text{mult}_x\mu = 0$. In fact, by (2.13), $M^\xi \wedge \mu = \gamma \wedge M^\xi \wedge \mu'$ which must vanish by the dimension principle, since $M^\xi \wedge \mu'$ has support at x and γ has positive degree.

2.6 Segre numbers

Let $\mathcal{J} \to X$ be a coherent ideal sheaf over X of codimension p. In [13,17] Tworzewski, and Gaffney and Gassler, independently introduced, at each point $x \in X$, a list of numbers $(e_p(\mathcal{J}, X, x), \ldots, e_n(\mathcal{J}, X, x))$, called Segre numbers in [13]. The Segre numbers generalize the Hilbert–Samuel multiplicity at x in the sense that if $\mathcal{J}$ has codimension n at x then $e_n(\mathcal{J}, X, x)$ is the Hilbert–Samuel multiplicity at x. The definitions in [13,17], though slightly different, are both of geometric nature. There is also a purely algebraic definition, [1,2]. In [6] were introduced semi-global currents whose Lelong numbers are precisely the Segre numbers. These currents are generalized cycles.

We can define Segre numbers for $\mathcal{J}$ over a generalized cycle $\mu \in \mathcal{GZ}(X)$: In a neighborhood $\mathcal{U}$ of a given point x we can take a section σ of a trivial Hermitian bundle such that σ generates $\mathcal{J}$ and define the *Segre numbers*

$$e_k(\mathcal{J}, \mu, x) := \text{mult}_x(M_k^\sigma \wedge \mu), \quad k = \kappa, \ldots, \dim \mu,$$

where κ is as in Remark 2.5. In view of Theorem 2.3, these numbers are independent of the choice of neighborhood $\mathcal{U}$ and of section σ, and only depend on the class of μ in $\mathcal{B}(X)$. If $\mu = \mathbf{1}_X$, then $e_k(\mathcal{J}, \mu, x)$ coincides with $e_k(\mathcal{J}, X, x)$, see [6, Theorem 1.1].

2.7 Regular embeddings and Gysin mappings

Assume now that X is smooth and that $\mathcal{J} \to X$ is locally a complete intersection of codimension κ. This means that $\iota\colon Z_\mathcal{J} \hookrightarrow X$ is a *regular embedding*, where $Z_\mathcal{J}$ is the non-reduced space of codimension κ defined by $\mathcal{J}$. Then the normal cone $N_\mathcal{J}X$ is a vector bundle over the reduced space $i\colon Z \hookrightarrow X$ and hence there is a well-defined cohomology class $c(N_\mathcal{J}X)$ on Z. Therefore there is a well-defined mapping, the classical *Gysin mapping*[5]

$$\iota^!\colon \mathcal{A}_k(X) \to \mathcal{A}_{k-\kappa}(Z), \quad i_*\iota^!\mu = (c(N_\mathcal{J}X) \cap s(\mathcal{J}, \mu))_{k-\kappa}, \tag{2.16}$$

where the lower index $k-\kappa$ denotes the component of dimension $k-\kappa$. We have the analogous $\mathcal{B}$-*Gysin mapping*

$$\iota^!\colon \mathcal{B}_k(X) \to \mathcal{B}_{k-\kappa}(Z), \quad i_*\iota^!\mu = (c(N_\mathcal{J}X) \wedge S(\mathcal{J}, \mu))_{k-\kappa}. \tag{2.17}$$

[5] Since this is a map to $\mathcal{A}_{k-\kappa}(Z)$, to be formally correct, we must insert i_* in the formula defining $\iota^!$, cf. Sect. 2.5.

Our main interest is when $\mathcal{J}$ defines a submanifold; in this case $Z = Z_{\mathcal{J}}$ and $i = \iota$.

By suitable choices we can represent (2.17) by a mapping on $\mathcal{GZ}(X)$: Assume that $\mathcal{J}$ is defined by a section σ of a Hermitian vector bundle $E \to X$ and let E' be the pull-back to Z. There is a canonical holomorphic embedding $\varphi\colon N_{\mathcal{J}}X \to E'$, see [4, Section 7]. Let us equip $N_{\mathcal{J}}X$ with the induced Hermitian metric and let $\widehat{c}(N_{\mathcal{J}}X)$ be the associated Chern form, cf. Sect. 2.2. Then we have the concrete mapping

$$\iota^!\colon \mathcal{GZ}_k(X) \to \mathcal{GZ}_{k-\kappa}(Z), \quad i_*\iota^!\mu = (\widehat{c}(N_{\mathcal{J}}X)\wedge M^\sigma \wedge \mu)_{k-\kappa}$$

which induces the mapping (2.17). We recall [4, Propositions 1.4 and 1.5].

Proposition 2.10 *If $\mathcal{J} \to X$ defines a regular embedding, then*

$$S(\mathcal{J}, X) = s(N_{\mathcal{J}}X)\wedge [Z_{\mathcal{J}}], \quad S_k(\mathcal{J}, X) = s_{k-\kappa}(N_{\mathcal{J}}X)\wedge [Z_{\mathcal{J}}]$$

in $\mathcal{B}(X)$, where $[Z_{\mathcal{J}}]$ is (the Lelong current of) the fundamental cycle associated to $\mathcal{J}$. If σ defines $\mathcal{J}$, then

$$M^\sigma = \widehat{s}(N_{\mathcal{J}}X)\wedge [Z_{\mathcal{J}}], \quad M^\sigma_k = \widehat{s}_{k-\kappa}(N_{\mathcal{J}}X)\wedge [Z_{\mathcal{J}}]$$

in $\mathcal{GZ}(X)$.

Example 2.11 Let $i\colon Z \to X$ be the inclusion of a smooth submanifold of codimension κ and suppose that $\mu \in \mathcal{GZ}_k(X)$ is a smooth form. Then, in view of Proposition 2.10,

$$i_*i^!\mu = \big(\widehat{c}(N_Z X)\wedge \widehat{s}(N_Z X)\wedge [Z]\wedge \mu\big)_{k-\kappa} = [Z]\wedge \mu.$$

Thus, $i^!\mu = i^*\mu$ is the usual pullback.

2.8 Intersection with divisors and the Poincaré–Lelong formula on a generalized cycle

See [4, Section 8] for proofs of the statements in this subsection. Let h be a meromorphic section of a line bundle $L \to X$. We say that $\operatorname{div} h$ intersects the generalized cycle μ *properly* if h is generically holomorphic and nonvanishing on the Zariski support $|\mu_j|$ of each irreducible component μ_j of μ. If $\operatorname{div} h$ and μ intersect properly there is a generalized cycle $\operatorname{div} h \cdot \mu$ with Zariski support on $|\operatorname{div} h| \cap |\mu|$ that we call the *proper intersection* of $\operatorname{div} h$ and μ.

If $\tau\colon W \to X$ such that $\mu = \tau_*\alpha$, where α is a product of components of Chern or Segre forms, then $\operatorname{div} h \cdot \mu = \tau_*([\operatorname{div} \tau^* h]\wedge \alpha)$. Then $\operatorname{div} h \cdot \mu \sim 0$ if $\mu \sim 0$ so that the intersection has meaning for $\mu \in \mathcal{B}(Y)$. If h is holomorphic, i.e., $\operatorname{div} h$ is effective, then, in a local frame for L,

$$\operatorname{div} h \cdot \mu = dd^c(\log |h|^2_\circ\, \mu) = \lim_{\epsilon\to 0}\big(dd^c \log(|h|^2_\circ + \epsilon)\wedge \mu\big), \tag{2.18}$$

where $|h|_\circ$ is the norm of the holomorphic function obtained from any fixed local frame for L so that $dd^c \log |h|_\circ$ is well-defined. It follows that $\operatorname{div} h \cdot \mu$ is effective if both $\operatorname{div} h$ and μ are effective. In light of (2.18) it is natural to write $\operatorname{div} h \cdot \mu$ as $[\operatorname{div} h] \wedge \mu$.

Proposition 2.12 (The Poincaré–Lelong formula on a generalized cycle) *Let h be a nontrivial meromorphic section of a Hermitian line bundle $L \to X$. Assume that* $\operatorname{div} h$ *intersects μ properly. Then*

$$dd^c(\log |h|^2 \mu) = [\operatorname{div} h] \wedge \mu - \widehat{c}_1(L) \wedge \mu.$$

Remark 2.13 If $\operatorname{div} h$ does not intersect μ properly we define $[\operatorname{div} h] \wedge \mu = \sum_j [\operatorname{div} h] \wedge \mu_j'$, where μ_j' are the irreducible components of μ that $\operatorname{div} h$ intersects properly, see [4, Section 9].

2.9 Mappings into cohomology groups

In this subsection we assume that X is projective, in particular compact, cf. [4, Section 10]. Let $\widehat{H}^{k,k}(X)$ be the equivalence classes of d-closed (k,k)-currents μ on X of order zero such that $\mu \sim 0$ if there is a current γ of order zero such that $\mu = d\gamma$. If X is smooth there is a natural isomorphism $\widehat{H}^{n-k,n-k}(X) \to H^{n-k,n-k}(X, \mathbb{C})$; the surjectivity is clear and the injectivity follows since a closed current of order zero locally has a potential of order zero. If $i : X \hookrightarrow M$ is an embedding into a smooth manifold M of dimension N, then there is a natural mapping $i_* : \widehat{H}^{n-k,n-k}(X) \to H^{N-k,N-k}(M, \mathbb{C})$ induced by the push-forward of currents.

There are natural cycle class mappings

$$A_X : \mathcal{A}_k(X) \to \widehat{H}^{n-k,n-k}(X), \quad k = 0, 1, \ldots, \tag{2.19}$$

and, [4, Equation (10.8)],

$$A_X(c(E) \cap \mu) = c(E) \wedge A_X \mu,$$

in $\widehat{H}(X)$, where the right-hand side is represented by the wedge product of a smooth form and a current. There are natural mappings

$$B_X : \mathcal{B}_k(X) \to \widehat{H}^{n-k,n-k}(X), \quad k = 0, 1, \ldots, \tag{2.20}$$

and clearly $B_X(c(E) \wedge \mu) = c(E) \wedge B_X \mu$.

Example 2.14 Assume that h is a meromorphic section of a Hermitian line bundle $L \to X$ such that $\operatorname{div} h$ intersects $\mu \in \mathcal{GZ}_k(X)$ properly. It follows from Proposition 2.12 that $[\operatorname{div} h] \wedge \mu$ and $\widehat{c}_1(L) \wedge \mu$ coincide in $\widehat{H}^{n-k+1,n-k+1}(X)$.

Let us recall [4, Proposition 1.6], that the images of $\mathcal{A}_k(X)$ and $\mathcal{B}_k(X)$ in $\widehat{H}^{n-k,n-k}(X)$ coincide. We have the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_k(X) \arrow[r, hook] \arrow[d] & \mathcal{B}_k(X) \arrow[d, "B_X"] \\ \mathcal{A}_k(X) \arrow[r, "A_X"] & \widehat{H}^{n-k,n-k}(X). \end{tikzcd}$$

Example 2.15 It follows from the dimension principle that $\mathcal{A}_n(X) = \mathcal{Z}_n(X) = \mathcal{B}_n(X)$. If X has the irreducible components $X_1, X_2, \ldots$, then the image in $\widehat{H}^{0,0}(X)$ of the cycle $a_1X_1 + a_2X_2 + \cdots$ on X is the d-closed $(0,0)$-current $a_1\mathbf{1}_{X_1} + a_2\mathbf{1}_{X_2} + \cdots$. It follows that the mappings into $\widehat{H}^{0,0}(X)$ are injective.

More generally, we have [4, Proposition 1.7].

Proposition 2.16 *Assume that $\mathcal{J} \to X$ defines a regular embedding $Z_{\mathcal{J}} \hookrightarrow X$ of codimension κ and let μ be a cycle. The images in $\widehat{H}^{*,*}(Z)$ of the Gysin and the $\mathcal{B}$-Gysin mappings of μ,* (2.16) *and* (2.17)*, coincide.*

3 Local intersection numbers

Let Y be a smooth manifold, let $\mu_1, \ldots, \mu_r$ be generalized cycles on Y of pure dimensions and let $d = \dim \mu_1 + \cdots + \dim \mu_r$. Following the ideas of Tworzewski [17] we define the *local intersection numbers* at x, cf. Lemma 2.1 and Sect. 2.6,

$$\epsilon_\ell(\mu_1, \ldots, \mu_r, x) := e_{d-\ell}(\mathcal{J}_\Delta, \mu_1 \times \cdots \times \mu_r, i(x)), \quad \ell = 0, 1, \ldots, d,$$

where $i\colon Y \hookrightarrow Y^r := Y \times \cdots \times Y$ is the parametrization $x \mapsto (x, \ldots, x)$ of the diagonal Δ in Y^r and $\mathcal{J}_\Delta \to Y^r$ is the ideal sheaf that defines Δ. Notice that if $E \to Y \times \cdots \times Y$ is a Hermitian vector bundle and σ is a section of E that generates $\mathcal{J}_\Delta$, then $M^\sigma \wedge (\mu_1 \times \cdots \times \mu_r)$ is a global generalized cycle such that

$$\epsilon_\ell(\mu_1, \ldots, \mu_r, x) = \mathrm{mult}_{i(x)} M^\sigma_{d-\ell} \wedge (\mu_1 \times \cdots \times \mu_r) \tag{3.1}$$

for $\ell \leqslant d$. More invariantly we have, cf. Definition 2.4,

$$\epsilon_\ell(\mu_1, \ldots, \mu_r, x) = \mathrm{mult}_{i(x)} S_{d-\ell}(\mathcal{J}_\Delta, \mu_1 \times \cdots \times \mu_r). \tag{3.2}$$

Given a point x, (3.1) holds as soon as σ defines $\mathcal{J}_\Delta$ in a neighborhood of the point $i(x)$ so we can assume that σ is a section of a trivial bundle. If the μ_j are cycles, therefore these numbers coincide with the local intersection numbers (1.1) introduced by Tworzewski in [17], cf. Sect. 2.6 and [6, Section 10].

Remark 3.1 Tworzewski, [17], proved that there is a unique global cycle μ such that the sum of its multiplicities, of its components of various dimensions, at each point $x \in V$

coincides with the sum of the local intersection numbers at x. Since this definition is local, it cannot carry global information. For instance, the self-intersection, in this sense, of any smooth curve Z in $\mathbb{P}^2$ is just the curve itself, and therefore the Bézout formula, cf. (1.7), is not satisfied unless Z is a line.

4 The $\mathcal{B}$-Stückrad–Vogel class in $\mathbb{P}^M$

Let $\mathbb{P}^M$ be the projectivization of $\mathbb{C}^{M+1}_{x_0,\dots,x_M}$. Let $\eta = (\eta_1, \dots, \eta_m)$ be a tuple of linear forms on $\mathbb{C}^{M+1}$ in general position. As usual we identify the η_j with sections of the line bundle $L = \mathcal{O}(1) \to \mathbb{P}^M$ and η with a section of $E := \bigoplus_1^m L$. Similarly to Sect. 2.8 we let $|\eta|_\circ$ be the norm of the holomorphic tuple obtained from any fixed local frame for L so that $dd^c \log |\eta|_\circ$ is well-defined. Let Z be the plane of codimension m that η defines and let $\mathcal{J} \to \mathbb{P}^M$ be the associated radical ideal sheaf.

Let μ be a fixed generalized cycle in $\mathbb{P}^M$ of pure dimension d. For a generic choice of $a = (a_1, \dots, a_d) \in (\mathbb{P}^{m-1})^d$, the successive intersections[6] by divisors, cf. Sect. 2.8, in

$$v_k^{a\cdot\eta} \wedge \mu := \mathbf{1}_Z[\mathrm{div}(a_k \cdot \eta)] \wedge \mathbf{1}_{X\setminus Z}[\mathrm{div}(a_{k-1} \cdot \eta)] \wedge \cdots \wedge \mathbf{1}_{X\setminus Z}[\mathrm{div}(a_1 \cdot \eta)] \wedge \mathbf{1}_{X\setminus Z}\mu \tag{4.1}$$

for $k = 0, \dots, d$ are proper, and

$$v^{a\cdot\eta} \wedge \mu = \sum_{k=0}^{d} v_k^{a\cdot\eta} \wedge \mu \tag{4.2}$$

is the resulting *Stückrad–Vogel* (*SV*) *cycle*, cf. [4, Section 9].

Proposition 4.1 *If we take the mean value of (4.2) over $(\mathbb{P}^{m-1})^d$, with respect to normalized Haar measure, then we get the generalized cycle*

$$M^{L,\eta} \wedge \mu := \mathbf{1}_Z \mu + \mathbf{1}_Z dd^c \log |\eta|_\circ^2 \wedge \mu + \cdots + \mathbf{1}_Z (dd^c \log |\eta|_\circ^2)^d \wedge \mu.$$

Proof With the convention in Remark 2.13 we can write

$$v_k^{a\cdot\eta} \wedge \mu = \mathbf{1}_Z[\mathrm{div}(a_k \cdot \eta)] \wedge [\mathrm{div}(a_{k-1} \cdot \eta)] \wedge \cdots \wedge [\mathrm{div}(a_1 \cdot \eta)] \wedge \mu.$$

Now the proposition follows from [4, Proposition 9.3]. □

By [4, Proposition 9.5], the class of $M^{L,\eta} \wedge \mu$ in $\mathcal{B}(\mathbb{P}^M)$ only depends on $\mathcal{J}$, L, and μ and not on the choice of generators η.

Definition 4.2 For $\mu \in \mathcal{B}(\mathbb{P}^M)$, we let $V(\mathcal{J}, L, \mu)$, the $\mathcal{B}$-*SV-class* of L and $\mathcal{J}$ on μ, be the class of $M^{L,\eta} \wedge \mu$ in $\mathcal{B}(\mathbb{P}^M)$.

[6] We let $\mathbf{1}_Z$ as well as $[\mathrm{div}(a_j \cdot \eta)]$ act on the whole current on its right, i.e., $\mathbf{1}_Z \gamma \wedge \mu := \mathbf{1}_Z(\gamma \wedge \mu)$ etc.

Notice that $M^{L,\eta}\wedge\mu$ has support in $Z\cap|\mu|$ so that we may identify $V(\mathcal{J}, L, \mu)$ with an element in $\mathcal{B}(Z\cap|\mu|)$, cf. [4, Definition 9.6].

Let $\mathcal{U}\subset\mathbb{P}^M$ be an open set where we have a local frame e for L. For instance, each nontrivial section of L vanishes on a hyperplane H and thus gives rise to a local frame in the open set $\mathbb{P}^M\setminus H$. In $\mathcal{U}$ we have that

$$M^{L,\eta}\wedge\mu = M^{\eta}\wedge\mu \tag{4.3}$$

with the metric on $L|_{\mathcal{U}}$ such that $|e| = 1$, cf. [4, Remark 8.2]. It follows that local statements that hold for $M^{\eta}\wedge\mu$ must hold for $M^{L,\eta}\wedge\mu$ as well. In particular, if η defines the maximal ideal at $x\in\mathbb{P}^M$, then, in view of (2.15),

$$M^{L,\eta}\wedge\mu = \mathrm{mult}_x\mu\cdot[x]. \tag{4.4}$$

By (2.12) and (4.3), in $\mathcal{U}$ we have the regularization

$$M_k^{L,\eta}\wedge\mu = \lim_{\epsilon\to 0}\frac{\epsilon}{(|\eta|_\circ^2+\epsilon)^{k+1}}(dd^c|\eta|_\circ^2)^k\wedge\mu, \quad k = 0, 1, 2, \ldots \tag{4.5}$$

In particular, $M_k^{L,\eta}\wedge\mu$ is effective if μ is; indeed $dd^c|\eta|_\circ^2$ is a positive $(1, 1)$-form.

We have the Fubini–Study norm $|\xi| = \|\xi\|/\|x\|$ on $L = \mathcal{O}(1)$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}_x^{M+1}$.

Proposition 4.3 *With the norm above $M^{L,\eta}\wedge\mu$ is the value at $\lambda = 0$ of the current valued function*

$$\lambda\mapsto\Big(1-|\eta|^{2\lambda}+\sum_{k\geqslant 1}\frac{\bar\partial|\eta|^{2\lambda}\wedge\partial|\eta|^2}{2\pi i|\eta|^2}\wedge\big(dd^c\log|\eta|_\circ^2\big)^{k-1}\Big)\wedge\mu, \tag{4.6}$$

a priori defined when $\mathrm{Re}\,\lambda\gg 0$.

Proof The statement follows directly from Proposition 2.6 in a set where we have a local frame for L if we replace each occurrence of $|\eta|$ in (4.6) by $|\eta|_\circ$. However one can verify, cf. [3, Proof of Lemma 2.1], that the value at $\lambda = 0$ is independent of the choice of norm on L, and thus the proposition follows. □

Notice that the Fubini–Study form $\widehat{\omega} = dd^c\log|x|_\circ^2 = dd^c\log\|x\|^2$ represents the first Chern class $\omega = c_1(L)$. We have van Gastel's formulas for generalized cycles [4, Theorem 9.7],

$$M^{L,\eta}\wedge\mu = \sum_{j\geqslant 0}\Big(\frac{1}{1-\widehat{\omega}}\Big)^j\wedge M_j^{\eta}\wedge\mu \tag{4.7}$$

and

$$M^{\eta}\wedge\mu = \sum_{j\geqslant 0}\Big(\frac{1}{1+\widehat{\omega}}\Big)^j\wedge M_j^{L,\eta}\wedge\mu. \tag{4.8}$$

From [4, Proposition 9.12] we get, cf. (1.3),

Proposition 4.4 *Assume that $\mu \in \mathcal{GZ}_d(X)$. We have the mass formula*

$$\deg \mu = \deg M_0^{L,\eta} \wedge \mu + \cdots + \deg M_d^{L,\eta} \wedge \mu + \deg\big(\mathbf{1}_{X\setminus Z}(dd^c \log|\eta|_\circ^2)^d \wedge \mu\big). \quad (4.9)$$

If $m \leqslant d$, then the last term in (4.9) vanishes since $(dd^c \log|\eta|_\circ^2)^m = 0$ outside Z.

For future reference we also point out the following invariance result. Assume that $i\colon \mathbb{P}^M \to \mathbb{P}^{M'}$ is a linear embedding of $\mathbb{P}^M$ in $\mathbb{P}^{M'}$. Let $p\colon \mathbb{P}^{M'} \dashrightarrow \mathbb{P}^M$ be a projective (generically defined) projection, i.e., induced by an affine projection $\mathbb{C}^{M'+1} \to \mathbb{C}^{M+1}$, so that $p \circ i$ is the identity on $\mathbb{P}^M$. Then $p^*\eta_j$ are well-defined linear forms on $\mathbb{P}^{M'}$. Let η' be some additional linear forms on $\mathbb{P}^{M'}$ that vanish on $i(\mathbb{P}^M)$.

Proposition 4.5 *If $\mu \in \mathcal{GZ}(\mathbb{P}^M)$, then*

$$M^{L,(p^*\eta,\eta')} \wedge i_*\mu = i_*(M^{L,\eta} \wedge \mu).$$

Proof Since $\eta' = 0$ on the Zariski support of $i_*\mu$,

$$M^{L,(p^*\eta,\eta')} \wedge i_*\mu = M^{L,(p^*\eta,0)} \wedge i_*\mu.$$

Now the proposition follows from (2.1) and Proposition 4.3, or (4.5), since $\eta = i^*p^*\eta$. □

5 $\mathcal{B}$-Intersection products on manifolds

Assume that $\mu_1, \ldots, \mu_r$ are cycles on a complex manifold Y of dimension n as in the introduction. It is well-known that if they intersect properly, then, see, e.g., [9, Chapter 12], one can define the wedge product $[\mu_1] \wedge \cdots \wedge [\mu_r]$ by means of appropriate regularizations, see, e.g., [10, Chapter III.3], and this current coincides with (the Lelong current of) the proper intersection cycle $\mu_1 \cdot_Y \cdots \cdot_Y \mu_r$, see, e.g., [9, p. 212]. It is easy to see that the cycle $\mu = \mu_1 \times \cdots \times \mu_r$ and the diagonal Δ in $Y^r = Y \times \cdots \times Y$ intersect properly, and one can prove that if we identify Δ and Y, then the proper intersection $\Delta \cdot_{Y^r} \mu$ coincides with $\mu_1 \cdot_Y \cdots \cdot_Y \mu_r$. If the μ_j do not intersect properly the basic idea is to consider the intersection of Δ and $\mu_1 \times \cdots \times \mu_r$, cf. Sect. 3. The advantage then is that one of the factors is a regular embedding.

We now recall the classical nonproper intersection product. If $\iota\colon Z_{\mathcal{J}} \to Y$ is a regular embedding of codimension κ and $\mu \in \mathcal{A}_k(Y)$, then we have, cf. (2.16), the product

$$Z_{\mathcal{J}} \diamond_Y \mu = \iota^! \mu,$$

see, e.g., [12, Chapter 6.1] for background and motivation. Let

$$i\colon Y \hookrightarrow Y^r, \quad x \mapsto (x, \ldots, x), \quad (5.1)$$

be the diagonal Δ; notice that this is a regular embedding. Given arbitrary cycles $\mu_1, \ldots, \mu_r$, we define the intersection product

$$\mu_1 \cdot_Y \cdots \cdot_Y \mu_r := i^!(\mu_1 \times \cdots \times \mu_r),$$

see, e.g., [12, Chapter 8.1]. After identification of Y and Δ we have $\mu_1 \cdot_Y \cdots \cdot_Y \mu_r = \Delta \diamond_{Y^r} (\mu_1 \times \cdots \times \mu_r)$. In case $\mu_1 = Z_{\mathcal{J}}$ is a regular embedding and μ_2 is an arbitrary cycle, then $\mu_1 \cdot_Y \mu_2 = \mu_1 \diamond_Y \mu_2$, see [12, Corollary 8.1.1].

We will define analogues for $\mathcal{B}(Y)$, cf. Definition 2.4, Lemma 2.1, and (2.11).

Definition 5.1 Assume that $\iota\colon Z_{\mathcal{J}} \to Y$ is a regular embedding. For $\mu \in \mathcal{B}(Y)$ we define, cf. (2.17), the product

$$Z_{\mathcal{J}} \diamond_{\mathcal{B}(Y)} \mu = \iota^! \mu.$$

Notice that if $Z_{\mathcal{J}}$ has codimension κ and $\mu \in \mathcal{B}_k(Y)$, then $Z_{\mathcal{J}} \diamond_{\mathcal{B}(Y)} \mu \in \mathcal{B}_{k-\kappa}(Z)$; recall that Z is the zero set of $\mathcal{J}$. Moreover, the Zariski support of $Z_{\mathcal{J}} \diamond_{\mathcal{B}(Y)} \mu$ is contained in $Z \cap |\mu|$ and so we can identify $Z_{\mathcal{J}} \diamond_{\mathcal{B}(Y)} \mu$ with an element in $\mathcal{B}_{k-\kappa}(Z \cap |\mu|)$.

Remark 5.2 If $\mathcal{J}$ is the radical ideal of a submanifold or a reduced locally complete intersection $i\colon Z \hookrightarrow Y$ of codimension κ and μ is a k-cycle in Y intersecting Z properly, then $i_*(Z \diamond_{\mathcal{B}(Y)} \mu)$ is the proper intersection $[Z] \wedge \mu$. In fact, in view of Definition 2.4 and Proposition 2.10,

$$\begin{aligned} S(\mathcal{J}, \mu) &= i_* S(i^*\mathcal{J}, \mu) = i_*\big(s(N_{i^*\mathcal{J}}\mu) \wedge [Z_{i^*\mathcal{J}}]\big) \\ &= s(N_{\mathcal{J}}Y) \wedge i_*[Z_{i^*\mathcal{J}}] = s(N_{\mathcal{J}}Y) \wedge [Z] \wedge \mu. \end{aligned}$$

Thus, by (2.17),

$$\begin{aligned} i_*(Z \diamond_{\mathcal{B}(Y)} \mu) &= i_* i^! \mu = \big(c(N_{\mathcal{J}}Y) \wedge S(\mathcal{J}, \mu)\big)_{k-\kappa} \\ &= \big(c(N_{\mathcal{J}}Y) \wedge s(N_{\mathcal{J}}Y)\big)_0 \wedge [Z] \wedge \mu = [Z] \wedge \mu. \end{aligned}$$

Definition 5.3 If $\mu_1, \ldots, \mu_r$ are elements in $\mathcal{B}(Y)$, we define

$$\mu_1 \cdot_{\mathcal{B}(Y)} \cdots \cdot_{\mathcal{B}(Y)} \mu_r := i^!(\mu_1 \times \cdots \times \mu_r).$$

As above, notice that after identification of Y and Δ we have $\mu_1 \cdot_{\mathcal{B}(Y)} \cdots \cdot_{\mathcal{B}(Y)} \mu_r = \Delta \diamond_{\mathcal{B}(Y^r)} \mu_1 \times \cdots \times \mu_r$.

Remark 5.4 Let $p\colon Y^r \to Y$ be the projection on one of the factors. Then $p \circ i = \mathrm{id}$, hence $p_* i_* = \mathrm{id}$ and thus $\mu_1 \cdot_{\mathcal{B}(Y)} \cdots \cdot_{\mathcal{B}(Y)} \mu_r = p_*(\Delta \diamond_{\mathcal{B}(Y^r)} \mu_1 \times \cdots \times \mu_r)$.

Assume that μ_1 is a regular embedding. Contrary to the classical intersection product case it is *not* true in general that $\mu_1 \diamond_{\mathcal{B}(Y)} \mu_2$ and $\mu_1 \cdot_{\mathcal{B}(Y)} \mu_2$ coincide. Example 8.14 below shows that the $\mathcal{B}$-self-intersection of the cusp $\mu = \{x_1^3 - x_0 x_2^2 = 0\} \subset \mathbb{P}^2$ is different from $\mu \diamond_{\mathcal{B}(Y)} \mu$. This example also shows that the $\mathcal{B}$-analogue of the classical self-intersection formula does not hold in general. However, it is true for smooth cycles.

Proposition 5.5 (Self-intersection formula) *Let $V \hookrightarrow Y$ be a smooth subvariety of Y of codimension m. Then*

$$V \cdot_{\mathcal{B}(Y)} V = c_m(N_V Y) \wedge [V]. \tag{5.2}$$

Proof Notice that the diagonal Δ_Y is smooth in $Y \times Y$ and that $N_{\Delta_Y}(Y \times Y) = T\Delta_Y$. If $j \colon V \times V \to Y \times Y$ is the product embedding, then $j^*\mathcal{J}_{\Delta_Y} = \mathcal{J}_{\Delta_V}$. Therefore $i_*(V \cdot_{\mathcal{B}(Y)} V) = \Delta_Y \diamond_{\mathcal{B}(Y \times Y)} V \times V$ is the component of dimension $n - 2m$ of

$$\begin{aligned} c(N_{\Delta_Y}(Y \times Y)) \wedge S(j^*\mathcal{J}_{\Delta_Y}, V \times V) &= c(T\Delta_Y) \wedge S(\mathcal{J}_{\Delta_V}, V \times V) \\ &= c(T\Delta_Y) \wedge s(T\Delta_V) \wedge [\Delta_V], \end{aligned}$$

where the last equality follows from Proposition 2.10 and, since V is smooth, that $N_{\Delta_V}(V \times V) = T\Delta_V$. Via the natural isomorphisms $Y \simeq \Delta_Y$ and $V \simeq \Delta_V$ thus $V \cdot_{\mathcal{B}(Y)} V$ is the component of dimension $n - 2m$ of

$$\begin{aligned} c(TY)|_V \wedge s(TV) \wedge [V] &= c(TY)|_V \wedge \frac{1}{c(TV)} \wedge [V] \\ &= c(TY/TV)|_V \wedge [V] = c(N_V Y) \wedge [V], \end{aligned}$$

cf. (2.4). Thus we get (5.2). □

Example 5.6 Let E be the exceptional divisor of the blow-up $Y = \mathrm{Bl}_a\mathbb{P}^2 \to \mathbb{P}^2$ at a point $a \in \mathbb{P}^2$. Let $L_E \to Y$ be the line bundle with a section that defines E. It follows from (5.2) that $E \cdot_{\mathcal{B}(Y)} E = c_1(L_E) \wedge [E]$. Since $-c_1(L_E)$ is positive $E \cdot_{\mathcal{B}(Y)} E$ is negative, which is expected in view of the classical self-intersection of E.

We have always coincidence of the various intersection products on cohomology level; recall the mappings (2.19) and (2.20).

Proposition 5.7 *Assume that $\mu_1, \ldots, \mu_r$ are cycles in Y and let $V = |\mu_1| \cap \cdots \cap |\mu_r|$. Then*

$$A_V(\mu_1 \cdot_Y \cdots \cdot_Y \mu_r) = B_V(\mu_1 \cdot_{\mathcal{B}(Y)} \cdots \cdot_{\mathcal{B}(Y)} \mu_r) \tag{5.3}$$

in $\widehat{H}(V)$. Moreover, if $r = 2$ and μ_1 is a regular embedding, then

$$B_V(\mu_1 \diamond_{\mathcal{B}(Y)} \mu_2) = B_V(\mu_1 \cdot_{\mathcal{B}(Y)} \mu_2). \tag{5.4}$$

Proof The equality (5.3) follows directly from the definitions and Proposition 2.16. Since the two possible definitions of $\mu_1 \cdot_Y \mu_2$ coincide when μ_1 is a regular embedding, (5.4) follows by another application of Proposition 2.16. □

Proposition 5.8 (i) *If $\mu_1, \ldots, \mu_r$ are cycles in Y that intersect properly, then*

$$\mu_1 \cdot_{\mathcal{B}(Y)} \cdots \cdot_{\mathcal{B}(Y)} \mu_r = \mu_1 \cdot_Y \cdots \cdot_Y \mu_r. \tag{5.5}$$

(ii) *If h is a holomorphic section of $L \to Y$ such that* $\operatorname{div} h$ *intersects $\mu \in \mathcal{B}(Y)$ properly, then*

$$\operatorname{div} h \diamond_{\mathcal{B}(Y)} \mu = \operatorname{div} h \cdot \mu = \operatorname{div} h \cdot_{\mathcal{B}(Y)} \mu. \tag{5.6}$$

Proof Assume that the μ_j have dimensions d_j, respectively. The assumption about proper intersection means that the set-theoretic intersection $V = |\mu_1| \cap \cdots \cap |\mu_r|$ has the expected dimension $k := d_1 + \cdots + d_r - (r-1)n$ and that $\mu_1 \cdot_Y \cdots \cdot_Y \mu_r$ and $\mu_1 \cdot_{\mathcal{B}(Y)} \cdots \cdot_{\mathcal{B}(Y)} \mu_r$ are elements in $\mathcal{A}_k(V)$ and $\mathcal{B}_k(V)$, respectively. Now (5.5) follows from (5.3) and Example 2.15.

Let us now consider part (ii). We may assume that $\mu = \tau_* \alpha$, where $\tau \colon W \to Y$ is proper holomorphic and α is a product of components of Chern or Segre forms, cf. (2.6). The assumption of proper intersection implies that h is not identically zero on $|\mu| = \tau(W)$ so that $M_0^h \wedge \mu = \mathbf{1}_{h=0}\, \mu = \tau_* \mathbf{1}_{\tau^* h = 0}\, \alpha = 0$. Let ι be the regular embedding given by the ideal sheaf $\mathcal{J}_h$ generated by h. We have $N_{\mathcal{J}_h} Y = L|_{h=0}$, cf. Sect. 2.7. Thus

$$\begin{aligned} \operatorname{div} h \diamond_{\mathcal{B}(Y)} \mu = \iota^! \mu &= (c(L) \wedge S(\mathcal{J}_h, \mu))_{\dim \mu - 1} = (c(L) \wedge M^h \wedge \mu)_{\dim \mu - 1} \\ &= c_0(L) \wedge M_1^h \wedge \mu = M_1^h \wedge \mu = \operatorname{div} h \cdot \mu; \end{aligned}$$

for the last equality, cf. [4, Equation (8.4)].

We now consider the last equality in (5.6). Consider the commutative diagram

$$\begin{array}{ccc} Y \times W & \xrightarrow{\mathrm{id} \times \tau} & Y \times Y \\ \pi \big\downarrow & & \big\downarrow p \\ W & \xrightarrow{\quad\tau\quad} & Y, \end{array} \tag{5.7}$$

where p is the projection on the first factor. By definition, cf. Remark 5.4, $\operatorname{div} h \cdot_{\mathcal{B}(Y)} \mu$ is p_* of

$$\begin{aligned} \Delta \diamond_{\mathcal{B}(Y \times Y)} (\operatorname{div} h \times \mu) &= \big(c(N_\Delta(Y \times Y)) \wedge S(\mathcal{J}_\Delta, \operatorname{div} h \times \mu)\big)_{\dim \mu - 1} \\ &= \sum_{\ell=0}^{n} c_{n-\ell}(N_\Delta(Y \times Y)) \wedge S_\ell(\mathcal{J}_\Delta, \operatorname{div} h \times \mu). \end{aligned} \tag{5.8}$$

Recall that $S_\ell(\mathcal{J}_\Delta, \operatorname{div} h \times \mu) = M_\ell^\sigma \wedge (\operatorname{div} h \times \mu)$ if σ is a section that defines $\Delta \subset Y \times Y$. Now $\operatorname{div} h \times \mu = (\mathrm{id} \times \tau)_* (\operatorname{div} h \times \alpha)$ so if $g = (\mathrm{id} \times \tau)^* \sigma$ we have, cf. (2.14) and (2.13),

$$\begin{aligned} M_\ell^\sigma \wedge (\operatorname{div} h \times \mu) &= (\mathrm{id} \times \tau)_* M_\ell^g \wedge (\operatorname{div} h \times \alpha) \\ &= (\mathrm{id} \times \tau)_* \big((1 \times \alpha) \wedge M_\ell^g \wedge (\operatorname{div} h \times 1)\big) \\ &= (\mathrm{id} \times \tau)_* \big((1 \times \alpha) \wedge M_\ell^g \wedge [\operatorname{div} (h \otimes 1)]\big). \end{aligned}$$

Notice that g defines the graph G of τ in $Y \times W$. Since $\operatorname{div} h$ and μ intersect properly, $\tau^* h$ is generically nonvanishing on W and so $h \otimes 1$ is generically nonvanishing on G. Thus, G and $\operatorname{div}(h \otimes 1)$ intersect properly. The Zariski support of $M_\ell^g \wedge [\operatorname{div}(h \otimes 1)]$ is $G \cap \{h \otimes 1 = 0\}$, which thus has dimension $\dim W - 1$. Since $M_\ell^g \wedge [\operatorname{div}(h \otimes 1)]$ has dimension $\dim W + n - \ell - 1$ it follows from the dimension principle that $M_\ell^g \wedge [\operatorname{div}(h \otimes 1)] = 0$ for $\ell < n$. Thus, $S_\ell(\mathcal{J}_\Delta, \operatorname{div} h \times \mu) = 0$ for $\ell < n$ and from (5.8) we get

$$\begin{aligned} \Delta \diamond_{\mathcal{B}(Y \times Y)} (\operatorname{div} h \times \mu) &= S_n(\mathcal{J}_\Delta, \operatorname{div} h \times \mu) \\ &= (\mathrm{id} \times \tau)_* \big((1 \times \alpha) \wedge M_n^g \wedge [\operatorname{div}(h \otimes 1)] \big). \end{aligned} \tag{5.9}$$

To compute $M_n^g \wedge [\operatorname{div}(h \otimes 1)]$, notice that g defines a regular embedding in $Y \times W$ of codimension n and that, since $\dim(G \cap \{h \otimes 1 = 0\}) = \dim W - 1$, the restriction of g to $\operatorname{div}(h \otimes 1)$ defines a regular embedding in $\operatorname{div}(h \otimes 1)$ of codimension n. Thus, by [4, Corollary 7.5],

$$\begin{aligned} M_n^g \wedge [\operatorname{div}(h \otimes 1)] &= \big(M^g \wedge [\operatorname{div}(h \otimes 1)] \big)_{\dim W - 1} \\ &= \big(S(\mathcal{J}_g, Y \times W) \wedge [G] \wedge [\operatorname{div}(h \otimes 1)] \big)_{\dim W - 1} \\ &= S_0(\mathcal{J}_g, Y \times W) \wedge [G] \wedge [\operatorname{div}(h \otimes 1)] \\ &= [G] \wedge [\operatorname{div}(h \otimes 1)], \end{aligned} \tag{5.10}$$

where $\mathcal{J}_g$ is the ideal sheaf generated by g. Since (5.7) is commutative, (5.9) and (5.10) give

$$\begin{aligned} p_* \big(\Delta \diamond_{\mathcal{B}(Y \times Y)} (\operatorname{div} h \times \mu) \big) &= \tau_* \pi_* \big((1 \times \alpha) \wedge [\operatorname{div}(h \otimes 1)] \wedge [G] \big) \\ &= dd^c \tau_* \pi_* \big((1 \times \alpha) \wedge (\log |h|_\circ^2 \otimes 1) \wedge [G] \big), \end{aligned} \tag{5.11}$$

cf. (2.18). Since $\pi_*((1 \times \alpha) \wedge (\log |h|_\circ^2 \otimes 1) \wedge [G]) = \log |\tau^* h|_\circ^2 \alpha$, by (5.11) we get, cf. (2.1) and (2.18),

$$p_* \big(\Delta \diamond_{\mathcal{B}(Y \times Y)} (\operatorname{div} h \times \mu) \big) = dd^c \tau_* (\log |\tau^* h|_\circ^2 \alpha) = dd^c (\log |h|_\circ^2 \mu) = \operatorname{div} h \cdot \mu,$$

which finishes the proof. □

6 The •-product on $\mathbb{P}^n$

In this section we define the product (1.4) of generalized cycles on $\mathbb{P}^n$ and prove Theorem 1.1. The first step is to define the join of two generalized cycles. For simplicity we first assume that $r = 2$. The mapping

$$\mathbb{P}^{2n+1}_{x,y} \overset{\mathfrak{p}}{\dashrightarrow} \mathbb{P}^n_x \times \mathbb{P}^n_y, \quad [x, y] \mapsto ([x], [y]) \tag{6.1}$$

is well-defined outside the union of the two disjoint n-dimensional planes $x = 0$ and $y = 0$, and it has surjective differential. If $\mu_1, \mu_2 \in \mathcal{GZ}(\mathbb{P}^n)$, therefore $\mathfrak{p}^*(\mu_1 \times \mu_2)$ is a well-defined current outside the indeterminacy set of $\mathfrak{p}$. We will see that $\mathfrak{p}^*(\mu_1 \times \mu_2)$ extends in a natural way to a generalized cycle $\mu_1 \times_J \mu_2$ on $\mathbb{P}^{2n+1}_{x,y}$.

Let $\pi : \mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y} \to \mathbb{P}^{2n+1}_{x,y}$ be the blow-up of $\mathbb{P}^{2n+1}_{x,y}$ along $\{x = 0\}$ and $\{y = 0\}$. Then we have

$$\begin{tikzcd} \mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y} \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ \mathbb{P}^{2n+1}_{x,y} \arrow[r, dashed, "\mathfrak{p}"'] & \mathbb{P}^n_x \times \mathbb{P}^n_y, \end{tikzcd} \tag{6.2}$$

where $p := \mathfrak{p} \circ \pi : \mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y} \to \mathbb{P}^n_x \times \mathbb{P}^n_y$ has surjective differential and hence is smooth, i.e., maps smooth forms onto smooth forms.

Lemma 6.1 (i) *If $\mu \in \mathcal{GZ}(\mathbb{P}^n \times \mathbb{P}^n)$, then $p^*\mu \in \mathcal{GZ}(Y)$.*
(ii) *$\pi_* p^*\mu$ is in $\mathcal{GZ}(\mathbb{P}^{2n+1})$ and coincides with $\mathfrak{p}^*\mu$ where it is defined.*
(iii) *If $\mu = 0$ in $\mathcal{B}(\mathbb{P}^n \times \mathbb{P}^n)$, then $\pi_* p^*\mu = 0$ in $\mathcal{B}(\mathbb{P}^{2n+1})$.*

Proof Note that (ii) is a direct consequence of (i).

Let $X = \mathbb{P}^n \times \mathbb{P}^n$ and $X' = \mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y}$. We may assume that $\mu = \tau_*\alpha$, where $\tau : W \to X$ is proper and α is a product of components of Chern forms. Consider the fibre square

$$\begin{tikzcd} W' \arrow[r, "\rho"] \arrow[d, "\widetilde{\pi}"'] & X' \arrow[d, "p"] \\ W \arrow[r, "\tau"] & X. \end{tikzcd} \tag{6.3}$$

Since p is smooth it follows that the fibre product $W' = W \times_X Y$ is smooth, cf. (6.5) below. The pullback $\widetilde{\pi}^*\alpha$ is a product of Chern forms on W' and thus $\rho_*\widetilde{\pi}^*\alpha$ is a generalized cycle on X'. We claim that

$$\rho_*\widetilde{\pi}^*\gamma = p^*\tau_*\gamma \tag{6.4}$$

for any smooth form γ. Taking (6.4) for granted we conclude that $p^*\mu = p^*\tau_*\alpha$ is a generalized cycle, which proves (i). It is enough to prove (6.4) for all smooth forms γ with small support. Notice that locally in X, say in a small open set $\mathcal{U}$, $X'|_{\mathcal{U}}$ is biholomorphic to $\mathcal{U} \times \mathbb{P}^1$. Let us assume that $\tau_*\gamma$ has support in an open set $\mathcal{U} \subset X$, where $X' = \mathcal{U} \times \mathbb{P}^1_t$. Letting $\widetilde{W} = \tau^{-1}(\mathcal{U})$, by the definition of fiber product,

$$\begin{aligned} \widetilde{W} \times_{\mathcal{U}} \left(\mathcal{U} \times \mathbb{P}^1_t\right) &= \{(w, x, t);\ \tau(w) = p(x, t) = x\} \\ &= \{(w, \tau(w), t);\ w \in \widetilde{W}\} \simeq \widetilde{W} \times \mathbb{P}^1 \end{aligned} \tag{6.5}$$

and $\rho(w, t) = (\tau(w), t)$. Now (6.4) is obvious.

To see (iii), note that if $\mu = \tau_*(\beta\wedge\alpha)$, where β is a component of a B-form, then it follows from (6.4) that $\pi_* p^*\mu = \pi_*\rho_*(\widetilde{\pi}^*\beta\wedge\widetilde{\pi}^*\alpha)$ and hence 0 in $\mathcal{B}(\mathbb{P}^{2n+1})$ since $\widetilde{\pi}^*\beta$ is a component of a B-form. □

If $\mu_1, \mu_2 \in \mathcal{GZ}(\mathbb{P}^n)$, then $\mu_1\times\mu_2 \in \mathcal{GZ}(\mathbb{P}^n\times\mathbb{P}^n)$ by Lemma 2.1, and by virtue of Lemma 6.1 we can make the following definition.

Definition 6.2 For $\mu_1, \mu_2 \in \mathcal{GZ}(\mathbb{P}^n)$ we define the join product $\mu_1\times_J\mu_2$ by

$$\mu_1\times_J\mu_2 := \pi_* p^*(\mu_1\times\mu_2).$$

It follows from the same lemmas that $\mu_1\times_J\mu_2 \in \mathcal{GZ}(\mathbb{P}^n)$ and, moreover, that $\mu_1\times_J\mu_2$ is 0 in $\mathcal{B}(\mathbb{P}^{2n+1})$ if μ_1 or μ_2 is 0 in $\mathcal{B}(\mathbb{P}^n)$. Hence, $\mu_1\times_J\mu_2$ is well-defined for $\mu_j \in \mathcal{B}(\mathbb{P}^n)$.

Example 6.3 (*Relation to the classical join*) Assume that $X_1, X_2 \subset \mathbb{P}^n$ are (irreducible) analytic sets. Let $\widetilde{p}\colon \mathbb{C}^{n+1}\setminus\{0\}\times\mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{P}^n\times\mathbb{P}^n$ and $\widetilde{\pi}\colon \mathbb{C}^{2n+2}\setminus\{0\} \to \mathbb{P}^{2n+1}$ be the natural maps. Notice that $\widetilde{X} = \widetilde{p}^{-1}(X_1\times X_2)$ is homogeneous in $\mathbb{C}^{2n+2}$ and $\widetilde{\pi}(\widetilde{X})$ is the classical join of X_1 and X_2. We claim that

$$X_1\times_J X_2 = \widetilde{\pi}(\widetilde{X}). \tag{6.6}$$

Since $\mathfrak{p}\circ\widetilde{\pi} = \widetilde{p}$ on the common set of definition it follows that (6.6) holds outside the union $V \subset \mathbb{P}^{2n+1}$ of planes where $\mathfrak{p}$ is not defined. To prove (6.6) it is thus enough to show that $\mathbf{1}_V\pi_* p^*(X_1\times X_2)$ vanishes. In view of (2.2), $\mathbf{1}_V\pi_* p^*(X_1\times X_2) = 0$ if $\mathbf{1}_{\pi^{-1}V}\, p^*(X_1\times X_2) = 0$, which may be checked locally in $\mathrm{Bl}\,\mathbb{P}^{2n+1}$. We may therefore consider a subset $\mathcal{U}\times\mathbb{P}^1_t$ of $\mathrm{Bl}\,\mathbb{P}^{2n+1}$, where $\mathcal{U} \subset \mathbb{P}^n\times\mathbb{P}^n$ is open, cf. the proof of Lemma 6.1. Note that, in $\mathcal{U}\times\mathbb{P}^1_t$, $\pi^{-1}V$ is of the form $H := \mathcal{U}\times\{t_0\}$ and that $p^*(X_1\times X_2) = X_1\times X_2\times\mathbb{P}^1_t$. Thus, by the dimension principle, $\mathbf{1}_{\pi^{-1}V}\, p^*(X_1\times X_2) = \mathbf{1}_H(X_1\times X_2\times\mathbb{P}^1) = 0$.

Example 6.4 Let $\mu_1, \mu_2 \in \mathcal{B}(\mathbb{P}^n)$ and assume that $\Lambda\colon \mathbb{P}^n \to \mathbb{P}^{n'}$ is a linear embedding, i.e., Λ is induced by an injective linear map $\widetilde{\Lambda}\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n'+1}$. Then $\widetilde{\Lambda}\times\widetilde{\Lambda}$ is an injective linear map $\mathbb{C}^{2n+2} \to \mathbb{C}^{2n'+2}$ and we get a linear embedding $\mathbf{\Lambda}\colon \mathbb{P}^{2n+1} \to \mathbb{P}^{2n'+1}$. Let π' and p' be defined in the same way as π and p in (6.2) with n replaced by n'. Similarly to the proof of Lemma 6.1 one shows that $\mathbf{\Lambda}_*\pi_* p^* = \pi'_*(p')^*(\Lambda\times\Lambda)_*$ as operations on currents in $\mathbb{P}^n\times\mathbb{P}^n$. It follows that

$$\mathbf{\Lambda}_*(\mu_1\times_J\mu_2) = \Lambda_*\mu_1\times_J\Lambda_*\mu_2.$$

In a similar way as above we have the mapping

$$\mathbb{P}^{r(n+1)-1}_{x^1,\ldots,x^r} \overset{\mathfrak{p}}{\dashrightarrow} \mathbb{P}^n_{x^1}\times\cdots\times\mathbb{P}^n_{x^r}, \quad \big[x^1,\ldots,x^r\big] \mapsto \big(\big[x^1\big],\ldots,\big[x^r\big]\big). \tag{6.7}$$

Let now $\pi\colon \mathrm{Bl}\,\mathbb{P}^{r(n+1)-1}_{x^1,\ldots,x^r} \to \mathbb{P}^{r(n+1)-1}_{x^1,\ldots,x^r}$ be the blow-up of $\mathbb{P}^{r(n+1)-1}_{x^1,\ldots,x^r}$ along the codimension n-planes $\{x^1 = 0\},\ldots,\{x^r = 0\}$ and set $p := \mathfrak{p}\circ\pi$. We get a diagram analogous to (6.2). As above, given $\mu_1,\ldots,\mu_r$ in $\mathcal{GZ}(\mathbb{P}^n)$ or in $\mathcal{B}(\mathbb{P}^n)$, we define $\mu_1 \times_J \cdots \times_J \mu_r$ in $\mathcal{GZ}(\mathbb{P}^{r(n+1)-1})$ or in $\mathcal{B}(\mathbb{P}^{r(n+1)-1})$, respectively, as $\pi_* p^*(\mu_1 \times \cdots \times \mu_r)$.

Proposition 6.5 *If* $\mu_1, \ldots, \mu_r \in \mathcal{GZ}(\mathbb{P}^n)$, *then*

$$\deg(\mu_1 \times_J \cdots \times_J \mu_r) = \deg \mu_1 \ldots \deg \mu_r.$$

Proof We may assume that the μ_j have pure dimension. There are currents a_j in $\mathbb{P}^n$ such that $dd^c a_j = \mu_j - (\deg \mu_j)\widehat{\omega}^{k_j}$ if $\dim \mu_j = n - k_j$, where $\widehat{\omega}$ is the Fubini–Study form on $\mathbb{P}^n$. It follows that there is a current A on $\mathbb{P}^n_{x^1} \times \cdots \times \mathbb{P}^n_{x^r}$ such that

$$dd^c A = \mu_1 \times \cdots \times \mu_r - (\deg \mu_1 \cdots \deg \mu_r)\,\widehat{\omega}^{k_1} \times \cdots \times \widehat{\omega}^{k_r},$$

cf. Lemma 2.1. Applying $\pi_* p^*$, it is enough to show that $\deg(\widehat{\omega}^{k_1} \times_J \cdots \times_J \widehat{\omega}^{k_r}) = 1$; but this is obvious if we just notice that $\pi_* p^*$ of a hyperplane in $\mathbb{P}^n_{x^1} \times \cdots \times \mathbb{P}^n_{x^r}$ induced by a hyperplane in one of the factors $\mathbb{P}^n_{x^j}$ is a hyperplane in $\mathbb{P}^{r(n+1)-1}_{x^1,\ldots,x^r}$ and replace each $\widehat{\omega}^{k_j}$ by the intersection of k_j generic hyperplanes. $\square$

For the last argument one can also observe that $\log((|x^1|^2 + \cdots + |x^r|^2)/|x^j|^2)$ is a well-defined locally integrable function on $\mathbb{P}^{r(n+1)-1}_{x^1,\ldots,x^r}$ and that

$$dd^c \log\big((|x^1|^2 + \cdots + |x^r|^2)/|x^j|^2\big) = \omega_{x^1,\ldots,x^r} - \pi_* p^* \omega_{x^j}.$$

Let

$$j\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{r(n+1)-1}, \quad [x] \mapsto [x, \ldots, x], \tag{6.8}$$

be the parametrization of the join diagonal Δ_J in $\mathbb{P}^{r(n+1)-1}$ and let $\mathcal{J}_J$ be the associated sheaf. Notice that $\mathcal{J}_J$ is generated by the $(r-1)(n+1)$ linear forms, i.e., sections of $L = \mathcal{O}(1)$,

$$\eta = \big(x_0^2 - x_0^1, x_0^3 - x_0^2, \ldots, x_0^r - x_0^{r-1}, \ldots, x_n^2 - x_n^1, x_n^3 - x_n^2, \ldots, x_n^r - x_n^{r-1}\big). \tag{6.9}$$

Since

$$\operatorname{codim} \Delta_J = r(n+1) - 1 - n = (r-1)(n+1),$$

we see that η is a minimal generating set.

Definition 6.6 Given $\mu_1, \ldots, \mu_r \in \mathcal{B}(\mathbb{P}^n)$, $\mu_1 \bullet \cdots \bullet \mu_r$ is the unique class in $\mathcal{B}(\mathbb{P}^n)$ such that

$$j_*(\mu_1 \bullet \cdots \bullet \mu_r) = V(\mathcal{J}_J, L, \mu_1 \times_J \cdots \times_J \mu_r). \tag{6.10}$$

Since (2.11) is injective, $\mu_1 \bullet \cdots \bullet \mu_r$ is well-defined. It is clear that $\mu_1 \bullet \cdots \bullet \mu_r$ is commutative, multilinear, and that its Zariski support is contained in $|\mu_1| \cap \cdots \cap |\mu_r|$.

If $\mu_1, \dots, \mu_r \in \mathcal{GZ}(\mathbb{P}^n)$ denote representatives of the corresponding classes in $\mathcal{B}(\mathbb{P}^n)$, then the right-hand side of (6.10) is represented by

$$M^{L,\eta} \wedge (\mu_1 \times_J \cdots \times_J \mu_r) \tag{6.11}$$

for any choice of η generating $\mathcal{J}_J$. If the μ_j have pure dimensions, then

$$d := \dim(\mu_1 \times_J \cdots \times_J \mu_r) = \sum_1^r \dim \mu_j + r - 1, \tag{6.12}$$

and thus $j_*(\mu_1 \bullet \cdots \bullet \mu_r)_\ell$ is represented by $M^{L,\eta}_{d-\ell} \wedge (\mu_1 \times_J \cdots \times_J \mu_r)$.

The $\bullet$-product is invariant in the following sense.

Proposition 6.7 *Assume that $\mu_j \in \mathcal{B}(\mathbb{P}^n)$ and let $\Lambda \colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{n'}$ be a linear embedding. Then $\Lambda_*(\mu_1 \bullet \cdots \bullet \mu_r) = \Lambda_*\mu_1 \bullet \cdots \bullet \Lambda_*\mu_r$.*

In particular, if T is a linear automorphism of $\mathbb{P}^n$, then

$$T_*(\mu_1 \bullet \cdots \bullet \mu_r) = T_*\mu_1 \bullet \cdots \bullet T_*\mu_r. \tag{6.13}$$

Proof As in Example 6.4, Λ induces a linear embedding $\mathbf{\Lambda} \colon \mathbb{P}^{r(n+1)-1} \to \mathbb{P}^{r(n'+1)-1}$ and $\mathbf{\Lambda} \circ j = j \circ \Lambda$, where j denotes the join diagonal in both $\mathbb{P}^{r(n+1)-1}$ and $\mathbb{P}^{r(n'+1)-1}$. Therefore, since j_* is injective, to show the proposition it is enough to check that $\mathbf{\Lambda}_* j_*(\mu_1 \bullet \cdots \bullet \mu_r) = j_*(\Lambda_*\mu_1 \bullet \cdots \bullet \Lambda_*\mu_r)$, i.e., that

$$\mathbf{\Lambda}_*\big(V(\mathcal{J}_J, L, \mu_1 \times_J \cdots \times_J \mu_r)\big) = V\big(\mathcal{J}_J, L, \Lambda_*\mu_1 \times_J \cdots \times_J \Lambda_*\mu_r\big). \tag{6.14}$$

In the special case that Λ is a linear automorphism of $\mathbb{P}^n$, (6.14) follows by noticing that $\mathbf{\Lambda}^*$ in this case maps sections of L to sections of L, preserves $\mathcal{J}_J$, and, in view of a simple extension of Example 6.4, that $\Lambda_*\mu_1 \times_J \cdots \times_J \Lambda_*\mu_r = \mathbf{\Lambda}_*(\mu_1 \times_J \cdots \times_J \mu_r)$.

For the general case we may now assume that $\Lambda \colon \mathbb{P}^n_x \to \mathbb{P}^{n'}_{x,y}$ is the map $[x] \mapsto [x:0]$. Then $\mathbf{\Lambda}[x^1 : \dots : x^r] = [x^1 : 0 : \dots : x^r : 0]$. Let η be as in (6.9) and let η' be the tuple of $(r-1)(n'-n)$ linear forms $(y_k^{j+1} - y_k^j)$, $j = 1, \dots, r-1$, $k = 1, \dots, n'-n$. Then η and (η, η') define $\mathcal{J}_J$ in $\mathbb{P}^{r(n+1)-1}$ and $\mathbb{P}^{r(n'+1)-1}$, respectively. By Proposition 4.5 we get

$$M^{L,(\eta,\eta')} \wedge \mathbf{\Lambda}_*(\mu_1 \times_J \cdots \times_J \mu_r) = \mathbf{\Lambda}_*(M^{L,\eta} \wedge \mu_1 \times_J \cdots \times_J \mu_r),$$

which implies (6.14) in view of Example 6.4. $\square$

Proposition 6.8 *If $\mu_1, \dots, \mu_r \in \mathcal{B}(\mathbb{P}^n)$ have pure dimensions, then*

$$\deg(\mu_1 \bullet \cdots \bullet \mu_r) = \prod_1^r \deg \mu_j - \int_{\mathbb{P}^{r(n+1)-1} \setminus \Delta_J} (dd^c \log |\eta|^2_\circ)^d \wedge (\mu_1 \times_J \cdots \times_J \mu_r), \tag{6.15}$$

where d is given by (6.12).

Proof First notice that $j^*\widehat{\omega}_{\mathbb{P}^{r(n+1)-1}} = \widehat{\omega}_{\mathbb{P}^n}$, where j is defined in (6.8) and $\widehat{\omega}_{\mathbb{P}^M}$ denote the Fubini–Study form on $\mathbb{P}^M$. Therefore, for $\mu \in \mathcal{GZ}_k(\mathbb{P}^n)$,

$$\deg j_*\mu = \int_{\mathbb{P}^{r(n+1)-1}} \widehat{\omega}^k_{\mathbb{P}^{r(n+1)-1}} \wedge j_*\mu = \int_{\mathbb{P}^n} \widehat{\omega}^k_{\mathbb{P}^n} \wedge \mu = \deg \mu.$$

In particular, $\deg(\mu_1 \bullet \cdots \bullet \mu_r) = \deg(M^{L,\eta} \wedge (\mu_1 \times_J \cdots \times_J \mu_r))$. Now, by Proposition 4.4,

$$\begin{aligned} &\deg(\mu_1 \bullet \cdots \bullet \mu_r) \\ &= \deg(\mu_1 \times_J \cdots \times_J \mu_r) - \deg\big(\mathbf{1}_{\mathbb{P}^{r(n+1)-1}\setminus\Delta_J}(dd^c \log|\eta|^2_\circ)^d \wedge (\mu_1 \times_J \cdots \times_J \mu_r)\big), \end{aligned}$$

and thus (6.15) follows in view of Proposition 6.5. □

The Bézout formula (1.7) holds if and only if the last term in (6.15) vanishes. This happens if $(r+1)(n-1) \leqslant d$ which is the same as (1.6), cf. the remark after Proposition 4.4, (6.9), and (6.12).

However, as mentioned in the introduction, the condition (1.6) is not necessary for (1.7) to hold. For instance, by Proposition 6.7, the $\bullet$-product is not affected if we perform the multiplication in a larger $\mathbb{P}^{n'}$. Thus, as mentioned already in the introduction, the self-intersection of a k-plane is the k-plane itself, in particular, the self-intersection of a point is the point itself. On the other hand, clearly the product of two distinct points vanishes. In this case the last term in (6.15) carries the "missing mass" in the Bézout formula.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1 The first statements, about multlilinearity, commutativity and the support, are already discussed after Definition 6.6.

Since local intersections numbers (multiplicities) are locally defined we can work in an affinization and use the results from [6, Sections 9 and 10] to prove (1.5). However, we omit the details since it is also a direct consequence of the global Proposition 7.1 below, cf. (3.2) and (7.1).

In the discussion after the proof of Proposition 6.8 is noticed that (1.7) holds if (1.6) is fulfilled. If μ_j are effective, then so is $\mu_1 \times_J \cdots \times_J \mu_r$, and it follows that (6.11), and hence $\mu_1 \bullet \cdots \bullet \mu_r$, are effective, cf. (4.5). Moreover $\deg \mu_j$ are positive and the last term in (6.15) is nonpositive so we get (1.8).

If $\mu_1, \ldots, \mu_r$ are cycles that intersect properly, by the dimension principle only the component of $\mu_1 \bullet \cdots \bullet \mu_r$ of dimension ρ is nonzero, where ρ is as in (1.6), and this is a cycle. In this case the local intersections numbers $\epsilon_\ell(\mu_1, \ldots, \mu_r, x)$ coincide with the multiplicites of the proper intersection cycle $\mu_1 \cdot_{\mathbb{P}^n} \cdots \cdot_{\mathbb{P}^n} \mu_r$, cf. [6, Example 10.2], and thus (1.9) follows. □

We will now look at more explicit representations of the $\bullet$-product. Recall that we have a natural Hermitian metric on $\mathcal{O}(1)$, cf. Sect. 4, and thus, cf. (6.9),

$$|\eta|^2 = \sum_{i=1}^{r-1} \sum_{k=0}^{n} |x_k^{i+1} - x_k^i|^2.$$

From (4.6) we see that if μ_i have pure dimension, then $j_*(\mu_1 \bullet \cdots \bullet \mu_r)_\ell$ is given by the value at $\lambda = 0$ of

$$M_k^{L,\eta,\lambda} \wedge (\mu_1 \times_J \cdots \times_J \mu_r)$$
$$:= \bar\partial |\eta|^{2\lambda} \wedge \frac{\partial |\eta|^2}{2\pi i |\eta|^2} \wedge \big(dd^c \log |\eta|_\circ^2\big)^{k-1} \wedge (\mu_1 \times_J \cdots \times_J \mu_r),$$

where $k = d - \ell = \dim(\mu_1 \times_J \cdots \times_J \mu_r) - \ell$. Notice that

$$k \geqslant \dim(\mu_1 \times_J \cdots \times_J \mu_r) - (\dim \mu_1 + \cdots + \dim \mu_r) = r - 1 \geqslant 1$$

so that term corresponding to $k = 0$ in (4.6) is irrelevant here; indeed $\dim(\mu_1 \bullet \cdots \bullet \mu_r) \leqslant \sum \dim \mu_j$ and so $\ell \leqslant \sum \dim \mu_j$.

In an affinization we can also obtain the $\bullet$-product, cf. (4.5), as a limit of smooth forms times $\mu_1 \times_J \cdots \times_J \mu_r$ by the formula

$$M_k^{L,\eta} \wedge (\mu_1 \times_J \cdots \times_J \mu_r) = \lim_{\epsilon \to 0} \frac{\epsilon (dd^c |\eta|_\circ^2)^k}{(\epsilon + |\eta|_\circ^2)^{k+1}} \wedge (\mu_1 \times_J \cdots \times_J \mu_r).$$

....When computing $M_k^{L,\eta} \wedge (\mu_1 \times_J \cdots \times_J \mu_r)$ it can be convenient to compute the SV-cycle $v^{a \cdot \eta} \wedge (\mu_1 \times_J \cdots \times_J \mu_r)$ for generic hyperplanes $a_0 \cdot \eta, a_1 \cdot \eta, \ldots, a_n \cdot \eta a_j \in \mathbb{P}^n$, and then form the mean value, cf. Sect. 4. See Sect. 8 for examples.

Remark 6.9 Assume that $r = 2$. Given the standard coordinates on $\mathbb{C}^{n+1}$ there is a canonical choice of η defining Δ_J, namely $\eta_j = y_j - x_j$, $j = 0, \ldots, n$, cf. (6.9). Thus, given representatives of μ_j, there are canonical representatives (6.11) of $V(\mathcal{J}_J, L, \mu_1 \times_J \mu_2)$, and since (2.8) is injective we can define the $\bullet$-product on the level of generalized cycles. Indeed, given $\mu_1, \mu_2 \in \mathcal{GZ}(\mathbb{P}^n)$, we define $\mu_1 \bullet \mu_2$ as the unique current in $\mathcal{GZ}(\mathbb{P}^n)$ such that

$$j_*(\mu_1 \bullet \mu_2) = M^{L,\eta} \wedge (\mu_1 \times_J \mu_2).$$

Let T be a linear automorphism of $\mathbb{P}^n$ induced by a unitary mapping $\widetilde{T}$ on $\mathbb{C}^{n+1}$, let $\widetilde{\mathbf{T}} = \widetilde{T} \times \widetilde{T}$, and let $\mathbf{T}$ be the induced linear automorphism of $\mathbb{P}^{2n+1}$; cf. Example 6.4 and the proof of Proposition 6.7. Then, considering η as a tuple of linear forms on $\mathbb{C}^{2n+2}$, $|\widetilde{\mathbf{T}}^* \eta|^2_{\mathbb{C}^{2n+2}} = |\eta|^2_{\mathbb{C}^{2n+2}}$. Moreover, $dd^c \log |\eta|^2_{\mathbb{C}^{2n+2}} = dd^c \log |\eta|_\circ^2$, where we on the right-hand side consider η as a tuple of sections of $L \to \mathbb{P}^{2n+1}$. Hence, $dd^c \log |\mathbf{T}^* \eta|_\circ^2 = dd^c \log |\eta|_\circ^2$, and so

$$M^{L,\mathbf{T}^*\eta} \wedge (\mu_1 \times_J \mu_2) = M^{L,\eta} \wedge (\mu_1 \times_J \mu_2).$$

It follows that $T_*\mu_1 \bullet T_*\mu_2 = T_*(\mu_1 \bullet \mu_2)$ as generalized cycles.

Remark 6.10 Consider (6.7) and the corresponding diagram (6.2). By abuse of notation, let Δ_J denote the preimage under π of the join diagonal, let $\mathcal{J}_J$ denote the sheaf in $\mathrm{Bl}\,\mathbb{P}^{r(n+1)-1}$ corresponding to Δ_J, and let j denote the embedding of $\mathbb{P}^n$ in $\mathrm{Bl}\,\mathbb{P}^{r(n+1)-1}$ as Δ_J induced by (6.8). Since (6.11) has support on Δ_J and $\mathrm{Bl}\,\mathbb{P}^{r(n+1)-1}$ and $\mathbb{P}^{r(n+1)-1}$ coincide in a neighborhood of Δ_J we can alternatively think of (6.11) as a generalized cycle on Y.

Remark 6.11 (*Green currents and the $*$-product*) Recall that a $(p-1, p-1)$-current g is a Green current of a closed subvariety Z of codimension p of a complex manifold Y if $dd^c g + [Z] = \omega$, where ω is a smooth form. If g is locally integrable and smooth outside Z it is called a Green form. The calculus of Green forms, based on the $*$-product, is an important tool in the study of height in arithmetic intersection theory, see, e.g., [8,15]. In the case $p = 1$, if s is a section of a Hermitian line bundle that defines Z, then $g = -\log|s|^2$ is a Green form in virtue of the Poincaré–Lelong formula, cf., e.g., Proposition 2.12. The existence of Green forms of so-called logarithmic type for $p > 1$ is a more delicate matter, see [8]. If g and g' are Green forms of logarithmic type of Z and Z', respectively, then one can form the product

$$g * g' := g \wedge [Z'] + \omega \wedge g'.$$

If Z and Z' intersect properly, then $g * g'$ is a Green current for the proper intersection $Z \cdot_Y Z'$. In general, $g * g'$ is the Green current of a kind of product μ of Z and Z'. However μ it is not comparable to our product $Z \bullet Z$ (assuming $Y = \mathbb{P}^n$) since for degree reasons, μ has pure dimension $\dim Z + \dim Z' - n$. Also compare [4, Remark 5.10].

7 Relation to the $\cdot_{\mathcal{B}(\mathbb{P}^n)}$ product

In this section we prove Theorem 1.2. For simplicity let us restrict from now on to the case $r = 2$; the general case is handled in a similar way.

Consider the mapping

$$i^{!!} \colon \mathcal{B}(\mathbb{P}^n \times \mathbb{P}^n) \to \mathcal{B}(\mathbb{P}^n), \quad i_* i^{!!}\mu = c(N_{\mathcal{J}_\Delta}(\mathbb{P}^n \times \mathbb{P}^n)) \wedge S(\mathcal{J}_\Delta, \mu),$$

where i is given by (5.1). Notice that $\mu_1 \cdot_{\mathcal{B}(\mathbb{P}^n)} \mu_2 = i^!(\mu_1 \times \mu_2)$ is the component of dimension ρ of $i^{!!}(\mu_1 \times \mu_2)$, where ρ is given by (1.6), i.e., $\rho = \dim \mu_1 + \dim \mu_2 - n$.

Next, consider the mapping

$$j^\flat \colon \mathcal{B}(\mathbb{P}^n \times \mathbb{P}^n) \to \mathcal{B}(\mathbb{P}^n), \quad j_* j^\flat \mu = c(N_{\mathcal{J}_J} \mathbb{P}^{2n+1}) \wedge S(\mathcal{J}_J, \pi_* p^* \mu),$$

where we are using the notation from Sect. 6 and where j is given by (6.8).

Proposition 7.1 *The mappings $i^{!!}$ and $j^\flat$ coincide.*

Let η be the section (6.9) of $L = \mathcal{O}(1)$ equipped with the Fubini–Study metric, and let $\widehat{\omega}$ be the first Chern form. Then $\widehat{c}(N_{\mathcal{J}_J}\mathbb{P}^{2n+1}) = (1+\widehat{\omega})^{n+1}$ and thus, by (4.8), $j^\flat$ is represented by

$$(1+\widehat{\omega})^{n+1}\wedge M^\eta\wedge\pi_* p^*\mu = \sum_{j\geqslant 0}(1+\widehat{\omega})^{n+1-j}\wedge M_j^{L,\eta}\wedge\pi_* p^*\mu.$$

Now assume that $\mu = \mu_1\times\mu_2$ and let $d = \dim\pi_* p^*\mu = \dim\mu_1 + \dim\mu_2 + 1$, cf. (6.12). Note that $\rho = d-(n+1)$. It follows that

$$\begin{aligned}\Big(\sum_{j\geqslant 0}(1+\widehat{\omega})^{n+1-j}\wedge M_j^{L,\eta}\wedge\pi_* p^*\mu\Big)_\rho &= \Big(\sum_{\ell\geqslant 0}(1+\widehat{\omega})^{\ell-\rho}\wedge M_{d-\ell}^{L,\eta}\wedge\pi_* p^*\mu\Big)_\rho\\ &= \sum_{\ell\geqslant 0}\widehat{\omega}^{\ell-\rho}\wedge M_{d-\ell}^{L,\eta}\wedge\pi_* p^*\mu.\end{aligned}$$

By Definition 6.6, $j_*(\mu_1\bullet\mu_2)_\ell$ is represented by $M_{d-\ell}^{L,\eta}\wedge\pi_* p^*\mu$ and therefore

$$\big(c(N_{\mathcal{J}_J}\mathbb{P}^{2n+1})\wedge S(\mathcal{J}_J,\pi_* p^*\mu)\big)_\rho = j_*\sum_{\ell\geqslant 0}\omega^{\ell-\rho}\wedge(\mu_1\bullet\mu_2)_\ell \tag{7.1}$$

and thus Theorem 1.2 follows from Proposition 7.1.

Remark 7.2 There are classical mappings $\mathcal{A}(\mathbb{P}^n\times\mathbb{P}^n)\to\mathcal{A}(\mathbb{P}^n)$ analogous to $i^!$ and $j^\flat$. If μ_1 and μ_2 are cycles and $\mu = \mu_1\times\mu_2$, then, see [12, Example 8.4.5], the analogue of Proposition 7.1 holds for the component of dimension ρ, which is the component of main interest also for us. However, the argument given in [12] cannot be transferred to the $\mathcal{B}$-setting.

Proof of Proposition 7.1 Let $\mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y}$ be as in Sect. 6. Since $\mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y}$ coincides with $\mathbb{P}^{2n+1}$ in a neighborhood of Δ_J, the restrictions of $c(N_{\mathcal{J}_J}\mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y})$ and $c(N_{\mathcal{J}_J}\mathbb{P}^{2n+1})$ to Δ_J coincide, and moreover, $\pi_* p^*\mu$ and $p^*\mu$ coincide on Δ_J, cf. Remark 6.10. Therefore $j^\flat$ coincides with the mapping

$$\mathcal{B}(\mathbb{P}^n\times\mathbb{P}^n)\to\mathcal{B}(\mathbb{P}^n),\quad \mu\mapsto c(N_{\mathcal{J}_J}\mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y})\wedge S(\mathcal{J}_J,p^*\mu), \tag{7.2}$$

where we are identifying $\Delta_J\subset\mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y}$ with $\mathbb{P}^n$. Hence it suffices to prove that $i^!$ coincides with (7.2).

Let $M = \mathbb{P}^n$ so that $\Delta = i(M)$ and $\Delta_J = j(M)$ and let $X = \mathbb{P}^n\times\mathbb{P}^n$ and $Y = \mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y}$. Then

$$\begin{array}{ccc} M & \xhookrightarrow{\;j\;} & Y\\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle p}\\ M & \xhookrightarrow{\;i\;} & X\end{array}$$

commutes. Note that $j(M)$ is a divisor in $p^{-1}i(M)$. Let $E \to Y$ and $F \to X$ be Hermitian vector bundles with holomorphic sections ϕ and σ that define $\mathcal{J}_{j(M)}$ and $\mathcal{J}_{i(M)}$, respectively. Fix Hermitian metrics on $N_{j(M)}Y$ and $N_{i(M)}X$ and let $\widehat{c}(N_{j(M)}Y)$ and $\widehat{c}(N_{i(M)}X)$ be the associated Chern forms. Moreover, let $\mu \in \mathcal{GZ}(\mathbb{P}^n \times \mathbb{P}^n)$ denote also a fixed representative of $\mu \in \mathcal{B}(\mathbb{P}^n \times \mathbb{P}^n)$. $\square$

Lemma 7.3 *Let $\mathcal{L} \to p^{-1}i(M)$ be the line bundle associated with the divisor $j(M) \subset p^{-1}i(M)$. Then*

$$N_{j(M)}Y = p^* N_{i(M)}X \oplus \mathcal{L} \quad on \quad j(M), \tag{7.3}$$

and for any Hermitian metric on $\mathcal{L}$,

$$M^\sigma \wedge \mu \sim p_*(\widehat{c}(\mathcal{L}) \wedge M^\phi \wedge p^*\mu) \quad in \quad \mathcal{GZ}(X). \tag{7.4}$$

Taking this lemma for granted we can conclude the proof of Proposition 7.1. We have to prove that if μ_1 and μ_2 are the unique elements in $\mathcal{GZ}(M)$ such that

$$i_*\mu_1 = \widehat{c}(N_{i(M)}X) \wedge M^\sigma \wedge \mu$$

and

$$j_*\mu_2 = \widehat{c}(N_{j(M)}Y) \wedge M^\phi \wedge p^*\mu,$$

then $\mu_1 \sim \mu_2$ in $\mathcal{GZ}(M)$.

In view of (7.3) and (2.10) we have

$$\widehat{c}(N_{j(M)}Y) \wedge M^\phi \wedge p^*\mu \sim \widehat{c}(p^*N_{i(M)}X) \wedge \widehat{c}(\mathcal{L}) \wedge M^\phi \wedge p^*\mu$$

in $\mathcal{GZ}(Y)$. Therefore, cf. (2.5),

$$p_*\big(\widehat{c}(N_{j(M)}Y) \wedge M^\phi \wedge p^*\mu\big) \sim \widehat{c}(N_{i(M)}X) \wedge p_*(\widehat{c}(\mathcal{L}) \wedge M^\phi \wedge p^*\mu). \tag{7.5}$$

From (7.4) and (7.5) we get

$$p_*\big(\widehat{c}(N_{j(M)}Y) \wedge M^\phi \wedge p^*\mu\big) \sim \widehat{c}(N_{i(M)}X) \wedge M^\sigma \wedge \mu,$$

which means that $p_*j_*\mu_2 \sim i_*\mu_1$ on X. Since $p_*j_* = i_*$ and (2.11) is injective, we conclude that $\mu_1 \sim \mu_2$ on M. Thus Proposition 7.1 is proved. $\square$

Proof of Lemma 7.3 Let us use the notation $N_\sigma X$ for $N_{i(M)}X$ etc. We first consider (7.3). Notice that, with the notation from [4, Section 7], for any columns of minimal sets of generators s, s' of $\mathcal{J}_{i(M)} = \mathcal{J}_\sigma$ at points on $iM \subset X$ there is an invertible matrix g such that $s' = gs$. A section ξ of the normal bundle $N_\sigma X$ can be defined as a set of holomorphic tuples $\xi(s)$ such that $g\xi(s) = \xi(gs)$ in $i(M)$, i.e, the restriction to $i(M)$ of such matrices are transition matrices for $N_\sigma X$. Let t and t' be holomorphic functions in a neighborhood of a point on $p^{-1}i(M)$ such that both $t|_{p^{-1}i(M)}$ and $t'|_{p^{-1}i(M)}$ generate the sheaf associated with the divisor $j(M)$ in $p^{-1}i(M)$. Then $t' = ht$ for a holomorphic function h, which is nonvanishing on $p^{-1}i(M)$, and $h|_{p^{-1}i(M)}$ is a transition function for $\mathcal{L}$. Moreover, (p^*s, t) and (p^*s', t') are minimal sets of generators for $\mathcal{J}_{j(M)} = \mathcal{J}_\phi$. It follows that for given such minimal sets of generators at a point on $j(M)$ we have

$$\begin{bmatrix} p^*s' \\ t' \end{bmatrix} = \begin{bmatrix} p^*g & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} p^*s \\ t \end{bmatrix}.$$

Thus the restriction to $j(M)$ of the matrices

$$G = \begin{bmatrix} p^*g & 0 \\ 0 & h \end{bmatrix}$$

are transition matrices for $N_{j(M)}Y$; it is then clear that (7.3) holds. For future use let η be the section of $\mathcal{L} \to p^{-1}i(M)$ that defines $j(M)$.

To prove (7.4) we must return to the definition of p^*, so let us assume that $\mu = \tau_*\alpha$ and recall the fiber square (6.3). We may also assume that W is chosen so that $\tau^*\sigma$ is principal and hence $\rho^*\phi$ is a regular embedding of codimension 2 in W'. We claim that

$$N_{\rho^*\phi}W' = \widetilde{\pi}^* N_{\tau^*\sigma} W \oplus \rho^*\mathcal{L} \quad \text{on } \{\rho^*\phi = 0\}. \tag{7.6}$$

In fact, notice that $\widetilde{\pi}^*\tau^*\sigma$ combined with the section $\rho^*\eta$ generate the same sheaf as $\rho^*\phi$. Arguing precisely as above for (7.3) we then get (7.6).

We now claim that

$$[Z_{\tau^*\sigma}] = \widetilde{\pi}_*[Z_{\rho^*\phi}], \tag{7.7}$$

where $Z_{\tau^*\sigma}$ is the fundamental cycle of the ideal sheaf generated by $\tau^*\sigma$ etc. Since it is an equality of currents it is a local statement. By the dimension principle it is then enough to check it in an open set $\mathcal{U} \subset W$ where $Z_{\tau^*\sigma}$ is smooth and $\widetilde{\pi}^{-1}\mathcal{U} \simeq \mathcal{U} \times \mathbb{P}^1_t$ in suitable coordinates (x, t) so that $\widetilde{\pi}$ is $(x, t) \mapsto x$, cf. the proof of Lemma 6.1. Thus, we may assume that the ideal generated by $\tau^*\sigma$ is generated by x_1^ℓ in $\mathcal{U}$. Then $\rho^*\phi$ is generated by (x_1^ℓ, t) and (7.7) is reduced to the equality $\ell[x_1 = 0] = \widetilde{\pi}_*(\ell[x_1 = 0] \times [t = 0])$.

Next we claim that

$$M^{\tau^*\sigma} \sim \widetilde{\pi}_*\big(\widehat{c}(\rho^*\mathcal{L}) \wedge M^{\rho^*\phi}\big) \tag{7.8}$$

on W. In fact, from [4, Proposition 1.5] we have

$$M^{\rho^*\phi} = \widehat{s}(N_{\rho^*\phi}W') \wedge [Z_{\rho^*\phi}].$$

By (7.6), noting that (2.10) holds for Segre forms as well in view of (2.4), we have that

$$\begin{aligned}\widehat{c}(\rho^*\mathcal{L})\wedge M^{\rho^*\phi} &\sim \widehat{c}(\rho^*\mathcal{L})\wedge\widehat{s}(\widetilde{\pi}^* N_{\tau^*\sigma} W)\wedge\widehat{s}(\rho^*\mathcal{L})\wedge[Z_{\rho^*\phi}]\\ &= \widehat{s}(\widetilde{\pi}^* N_{\tau^*\sigma} W)\wedge[Z_{\rho^*\phi}].\end{aligned}$$

By (2.1) and (2.5) for Segre forms, thus

$$\widetilde{\pi}_*\big(\widehat{c}(\rho^*\mathcal{L})\wedge M^{\rho^*\phi}\big) \sim \widehat{s}(N_{\tau^*\sigma} W)\wedge\widetilde{\pi}_*[Z_{\rho^*\phi}].$$

In view of (7.7) and [4, Proposition 1.5], now (7.8) follows.

We can now conclude (7.4). Since α is smooth, from (7.8) we have, cf. (2.1), that

$$M^{\tau^*\sigma}\wedge\alpha \sim \widetilde{\pi}_*\big(\widehat{c}(\rho^*\mathcal{L})\wedge M^{\rho^*\phi}\wedge\widetilde{\pi}^*\alpha\big),$$

and hence, by (2.14) and the commutivity of (6.3),

$$\begin{aligned}M^{\sigma}\wedge\mu &= \tau_*(M^{\tau^*\sigma}\wedge\alpha)\\ &\sim \tau_*\widetilde{\pi}_*\big(\widehat{c}(\rho^*\mathcal{L})\wedge M^{\rho^*\phi}\wedge\widetilde{\pi}^*\alpha\big) = p_*\rho_*\big(\widehat{c}(\rho^*\mathcal{L})\wedge M^{\rho^*\phi}\wedge\widetilde{\pi}^*\alpha\big).\end{aligned}$$

Now, by (2.5) and (2.14),

$$\rho_*\big(\widehat{c}(\rho^*\mathcal{L})\wedge M^{\rho^*\phi}\wedge\widetilde{\pi}^*\alpha\big) = \widehat{c}(\mathcal{L})\wedge M^{\phi}\wedge\rho_*\widetilde{\pi}^*\alpha,$$

so

$$M^{\sigma}\wedge\mu \sim p_*\big(\widehat{c}(\mathcal{L})\wedge M^{\phi}\wedge\rho_*\widetilde{\pi}^*\alpha\big),$$

and since $\rho_*\widetilde{\pi}^*\alpha = p^*\mu$, cf. (6.4), therefore (7.4) follows. □

8 Examples

We shall now present some further results on our products and various examples. We first consider an embedding $i\colon \mathbb{P}^M \to \mathbb{P}^{M+1}$ as a linear hyperplane defined by the linear form ξ. Let $a \in \mathbb{P}^{M+1}$ be a point outside this hyperplane and let $\mathfrak{p}\colon \mathbb{P}^{M+1} \dashrightarrow \mathbb{P}^M$ be the induced projection. If Y is the blow-up of $\mathbb{P}^{M+1}$ at a we have the diagram

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ \mathbb{P}^{M+1} \arrow[r, dashed, "\mathfrak{p}"'] & \mathbb{P}^M. \end{tikzcd}$$

As in Sect. 6 we see that given $\mu \in \mathcal{GZ}(\mathbb{P}^M)$, the current $\mathfrak{p}^*\mu$ has a well-defined extension to an element $\pi_* p^*\mu$ in $\mathcal{GZ}(\mathbb{P}^{M+1})$, cf. Lemma 6.1.

Proposition 8.1 *Let η be a tuple of linear forms on $\mathbb{P}^M$. With the notation above we have*

$$i_*(M^{L,\eta}\wedge\mu) = M^{L,\mathfrak{p}^*\eta}\wedge i_*\mu \tag{8.1}$$

and

$$i_*(M^{L,\eta}\wedge\mu) = M^{L,(\mathfrak{p}^*\eta,\xi)}\wedge\pi_* p^*\mu. \tag{8.2}$$

Proof Since the support of $i_*\mu$ is contained in the hyperplane $i(\mathbb{P}^M)$ and Y and $\mathbb{P}^{M+1}$ coincide in a neighborhood of $i(\mathbb{P}^M)$, the right-hand side of (8.1) is well-defined. Now (8.1) follows from (2.14) and (4.3) since $\mathfrak{p}\circ i = \mathrm{id}$ so that $i^*\mathfrak{p}^*\eta = \eta$.

For the second equality first notice that both sides of (8.2) have support on $i(\mathbb{P}^M)$ and that Y and $\mathbb{P}^{M+1}$ coincide in a neighborhood of $i(\mathbb{P}^M)$. For the rest of this proof let i denote also the inclusion of $\mathbb{P}^M$ in Y. Since η defines a regular embedding, it follows from [4, Example 7.8] that

$$i_*(M^\eta\wedge\mu) = \widehat{c}(\pi^*L)\wedge M^{(p^*\eta,\pi^*\xi)}\wedge p^*\mu$$

if μ is a smooth form; here we use the standard metric on L. It follows in general, by assuming that $\mu = \tau_*\alpha$, $\tau\colon W\to\mathbb{P}^M$, and pulling back to W and W' according to the fibre square

$$\begin{tikzcd} W' \arrow[r, "\tau'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ W \arrow[r, "\tau"] & \mathbb{P}^M, \end{tikzcd}$$

cf. the proofs of Lemmas 6.1 and 7.3 above. Since $\widehat{c}(\pi^*L) = 1+\pi^*\widehat{\omega}$ we get

$$i_*(M_j^\eta\wedge\mu) = M_{j+1}^{p^*\eta,\pi^*\xi}\wedge p^*\mu + \pi^*\widehat{\omega}\wedge M_j^{p^*\eta,\pi^*\xi}\wedge p^*\mu.$$

Thus, in view of (4.7),

$$\begin{aligned} i_*(M^{L,\eta}\wedge\mu) &= \sum_{j\geqslant 0}\left(\frac{1}{1-\pi^*\widehat{\omega}}\right)^j\wedge i_*(M_j^\eta\wedge\mu) \\ &= (1-\pi^*\widehat{\omega})\wedge\sum_{j\geqslant 1}\left(\frac{1}{1-\pi^*\widehat{\omega}}\right)^j\wedge M_j^{p^*\eta,\pi^*\xi}\wedge p^*\mu \\ &\quad + \pi^*\widehat{\omega}\wedge\sum_{j\geqslant 0}\left(\frac{1}{1-\pi^*\widehat{\omega}}\right)^j\wedge M_j^{p^*\eta,\pi^*\xi}\wedge p^*\mu \\ &= M^{\pi^*L,(p^*\eta,\pi^*\xi)}\wedge p^*\mu, \end{aligned}$$

where we for the last equality have used that $M_0^{p^*\eta,\pi^*\xi}\wedge p^*\mu = 0$ so that we may let the sum start from $j = 0$; indeed, $M_0^{p^*\eta,\pi^*\xi}\wedge p^*\mu = 0$ since ξ is generically nonvanishing on the Zariski support of $p^*\mu$. Thus, (8.2) follows by applying π_*. □

We will now deduce a formula for $A\bullet\mu$ when A is a linear subspace.

Proposition 8.2 *Assume that A is a linear subspace of $\mathbb{P}^n$ of dimension m, defined by $n-m$ linear forms $\sigma_1,\dots,\sigma_{n-m}$. If $\mu\in\mathcal{GZ}_d(\mathbb{P}^n)$, then*

$$(A\bullet\mu)_{d-k} = M_k^{L,\sigma}\wedge\mu \tag{8.3}$$

in $\mathcal{B}(\mathbb{P}^n)$.

Proof Let us use the notation from Sect. 6. By (6.13) the $\bullet$-product is not affected by a linear change of coordinates on $\mathbb{C}^{n+1}_x$ and therefore we can assume that $x=(x',x'')$ and $\sigma = x''$. Then we need to prove that

$$\mu\bullet[x''=0] = M^{L,x''}\wedge\mu$$

in $\mathcal{B}(\mathbb{P}^n)$. Recall that $\eta = x-y$. By definition we have, cf. (2.14) and (4.3),

$$\begin{aligned} j_*(\mu\bullet[x''=0]) &= M^{L,\eta}\wedge\pi_*p^*(\mu\times[y''=0]) \\ &= M^{L,(x'-y',x'')}\wedge\pi_*p^*(\mu\times[y''=0]). \end{aligned} \tag{8.4}$$

Recall the diagram (6.2) associated with the mapping (6.1) and, as in the proof of Proposition 7.1, let $Y=\mathrm{Bl}\,\mathbb{P}^{2n+1}_{x,y}$. Consider the mapping $\mathfrak{p}'\colon\mathbb{P}^{n+m+1}_{x,y'} \dashrightarrow \mathbb{P}^n_x\times\mathbb{P}^m_{y'}$, $[x,y]\mapsto([x],[y'])$, and let $\pi'\colon Y'\to\mathbb{P}^{n+m+1}_{x,y'}$ be the blow-up of $\mathbb{P}^{n+m+1}_{x,y'}$ along $\{x=0\}$ and $\{y'=0\}$. Similarly to (6.2) we then have

$$\begin{tikzcd} Y' \arrow[d, "\pi'"'] \arrow[dr, "p'"] & \\ \mathbb{P}^{n+m+1}_{x,y'} \arrow[r, dashed, "\mathfrak{p}'"'] & \mathbb{P}^n_x\times\mathbb{P}^m_{y'}. \end{tikzcd}$$

Let $\iota\colon\mathbb{P}^{n+m+1}\hookrightarrow\mathbb{P}^{2n+1}$, $[x,y']\mapsto[x,y',0]$. Then ι extends to a mapping $\widetilde{\iota}\colon Y'\to Y$. Also, let $\iota'\colon\mathbb{P}^n\times\mathbb{P}^m\hookrightarrow\mathbb{P}^n\times\mathbb{P}^n$, $([x],[y'])\mapsto([x],[y',0])$. Consider the fibre square

$$\begin{tikzcd} Y' \arrow[r, "\widetilde{\iota}"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ \mathbb{P}^n\times\mathbb{P}^m \arrow[r, "\iota'"] & \mathbb{P}^n\times\mathbb{P}^n, \end{tikzcd}$$

cf. (6.3). Notice that $\mu \times [y''=0] = \iota'_*(\mu \times 1)$. By the same arguments as in the proof of Lemma 6.1, we get

$$p^*(\mu \times [y''=0]) = p^*\iota'_*(\mu \times 1) = \widetilde{\iota}_*(p')^*(\mu \times 1), \tag{8.5}$$

cf. (6.4). It is straightforward to check that $\pi \circ \widetilde{\iota} = \iota \circ \pi'$ and so, by applying π_* to (8.5), we get

$$\pi_* p^*(\mu \times [y''=0]) = \pi_* \widetilde{\iota}_*(p')^*(\mu \times 1) = \iota_* \pi'_*(p')^*(\mu \times 1). \tag{8.6}$$

Let $p'' \colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ be projection on the first factor and set $p''' := p'' \circ p'$. Then $\mu \times 1 = (p'')^*\mu$ and $(p')^*(\mu \times 1) = (p''')^*\mu$. Thus, by (8.6),

$$\pi_* p^*(\mu \times [y''=0]) = \iota_* \pi'_*(p''')^*\mu. \tag{8.7}$$

By (8.4), (8.7), and repeated use of (8.1) we get

$$\begin{aligned} j_*(\mu \bullet [x''=0]) &= M^{L,(x'-y',x'')} \wedge \iota_* \pi'_*(p''')^*\mu \\ &= \iota_*\big(M^{L,(x'-y',x'')} \wedge \pi'_*(p''')^*\mu\big). \end{aligned} \tag{8.8}$$

Let $j' \colon \mathbb{P}^n \to \mathbb{P}^{n+m+1}$, $[x] \mapsto [x, x']$, and let $\mathfrak{q} \colon \mathbb{P}^{n+m+1} \dashrightarrow \mathbb{P}^n$, $[x, y'] \mapsto [x]$. Then we have the commutative diagram

$$\begin{tikzcd} Y' \arrow[d, "\pi'"'] \arrow[dr, "p'''"] & \\ \mathbb{P}^{n+m+1}_{x,y'} \arrow[r, dashed, "\mathfrak{q}"'] & \mathbb{P}^n_x. \end{tikzcd}$$

By repeated use of (8.2), with $\xi_j = x'_j - y'_j$, $j = 0, \ldots, m$, we get

$$M^{L,(x'-y',x'')} \wedge \pi'_*(p''')^*\mu = j'_*(M^{L,x''} \wedge \mu)$$

and so, by (8.8),

$$j_*(\mu \bullet [x''=0]) = \iota_* j'_*(M^{L,x''} \wedge \mu).$$

Since j_* is injective, to finish the proof it suffices to check that we may replace $\iota_* j'_*$ by j_* in the right-hand side. Notice that $\nu := M^{L,x''} \wedge \mu$ is a generalized cycle with support $\{x''=0\}$ so that $\nu = i_*\nu'$ for some $\nu' \in \mathcal{GZ}(\{x''=0\})$, where $i \colon \{x''=0\} \hookrightarrow \mathbb{P}^n$ is the inclusion. Since $\iota \circ j' \circ i = j \circ i$ we obtain

$$\iota_* j'_* \nu = \iota_* j'_* i_* \nu' = j_* i_* \nu' = j_* \nu. \qquad \square$$

Proposition 8.3 *Assume that* $\mu \in \mathcal{B}(\mathbb{P}^n)$. *Then*

$$\mathbf{1}_{\mathbb{P}^n} \bullet \mu = \mu. \tag{8.9}$$

If a is a point, then

$$a \bullet \mu = \mathrm{mult}_a \mu \cdot [a]. \tag{8.10}$$

Proof From Proposition 8.2 we have that $\mathbf{1}_{\mathbb{P}^n} \bullet \mu = M^{L,0} \wedge \mu = \mu$ and so (8.9) follows. To see (8.10) let ξ be linear forms that define a. By (8.3) and (4.4) we have $a \bullet \mu = M^{L,\xi} \wedge \mu = \mathrm{mult}_a \mu \cdot [a]$. □

Let η be a fixed choice of a tuple of linear forms defining the join diagonal Δ_J in $\mathbb{P}^{r(n+1)-1}$. Then, using the notation of Sect. 6, we can define a $\bullet$-product of $\mu_1, \ldots, \mu_r \in \mathcal{GZ}(\mathbb{P}^n)$ by

$$j_*(\mu_1 \bullet \cdots \bullet \mu_r) := M^{L,\eta} \wedge \pi_* p^*(\mu_1 \times \cdots \times \mu_r), \tag{8.11}$$

cf. Definition 6.6. With this definition, for $\mu \in \mathcal{GZ}(\mathbb{P}^n)$, (8.9) and (8.10) hold in $\mathcal{GZ}(\mathbb{P}^n)$.

Proposition 8.4 *Let* η *be a fixed choice as above. Assume that* $\mu_0, \mu_1, \ldots, \mu_r \in \mathcal{GZ}(\mathbb{P}^n)$ *and that* $\mu_0 = \gamma \wedge \mu_1$ *in an open set* $\mathcal{U} \subset \mathbb{P}^n$, *and* γ *is a smooth and closed form. Then*

$$\mu_0 \bullet \mu_2 \bullet \cdots \bullet \mu_r = \gamma \wedge (\mu_1 \bullet \cdots \bullet \mu_r) \tag{8.12}$$

in $\mathcal{U}$.

Combined with (8.9) we see that

$$\gamma \bullet \mu = \gamma \wedge \mu \tag{8.13}$$

in $\mathcal{U}$ if $\gamma \in \mathcal{GZ}(\mathbb{P}^n)$ is a smooth form there.

Proof In view of (2.14) and (4.3) we have

$$j_*(\mu_0 \bullet \mu_2 \bullet \cdots \bullet \mu_r) = \pi_* M^{L,\pi^*\eta} \wedge p^*(\mu_0 \times \mu_2 \times \cdots \times \mu_r). \tag{8.14}$$

Now

$$\mu_0 \times \mu_2 \times \cdots \times \mu_r = (\gamma \times \mathbf{1} \times \cdots \times \mathbf{1}) \wedge (\mu_1 \times \cdots \times \mu_r)$$

in $\mathcal{U} \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n$. Since $\gamma \times \mathbf{1} \times \cdots \times \mathbf{1}$ is a smooth and closed form, it follows from (2.13) that the right-hand side of (8.14) equals

$$\pi_*\big(p^*(\gamma \times \mathbf{1} \times \cdots \times \mathbf{1}) \wedge M^{L,\pi^*\eta} \wedge p^*(\mu_1 \times \cdots \times \mu_r)\big) \tag{8.15}$$

in $\pi(p^{-1}(\mathfrak{U}\times\mathbb{P}^n\times\cdots\times\mathbb{P}^n))$. In a neighborhood of $\Delta_J=\{\eta=0\}$, $\mathfrak{p}$ is defined and so

$$p^*(\gamma\times\mathbf{1}\times\cdots\times\mathbf{1})=\pi^*\mathfrak{p}^*(\gamma\times\mathbf{1}\times\cdots\times\mathbf{1})$$

in a neighborhood of $\{\pi^*\eta=0\}$ in $p^{-1}(\mathfrak{U}\times\mathbb{P}^n\times\cdots\times\mathbb{P}^n)$. Thus, (8.15) equals

$$\mathfrak{p}^*(\gamma\times\mathbf{1}\times\cdots\times\mathbf{1})\wedge M^{L,\eta}\wedge\pi_*p^*(\mu_1\times\cdots\times\mu_r) \tag{8.16}$$

on $j(\mathfrak{U})$. Since $j^*\mathfrak{p}^*(\gamma\times\mathbf{1}\times\cdots\times\mathbf{1})=\gamma$ in $\mathfrak{U}$, by (8.11) we see that (8.16) equals

$$j_*(\gamma\wedge\mu_1\bullet\cdots\bullet\mu_r)$$

on $j(\mathfrak{U})$. Using that j_* is injective on currents we get (8.12). □

Example 8.5 Let $\widehat{\omega}$ be the Fubini–Study metric form on $\mathbb{P}^n$. Then $\widehat{\omega}$ is a generalized cycle of degree 1 and with multiplicity 0 at each point. Given any choice of η as above, it follows from Proposition 8.4 that $\widehat{\omega}\bullet\widehat{\omega}=\widehat{\omega}\wedge\widehat{\omega}$ and, more generally, $\widehat{\omega}\bullet\cdots\bullet\widehat{\omega}=:\widehat{\omega}^{k\bullet}=\widehat{\omega}^k$.

Example 8.6 Let $a=[1,0,\ldots,0]\in\mathbb{P}^n$ and let $\theta=dd^c\log(|x_1|^2+\cdots+|x_n|^2)$ in $\mathbb{P}^n_{x_0,\ldots,x_n}$. For each k, θ^k is a well-defined positive closed current, see, e.g., [10, Chapter III]. It is an irreducible generalized cycle of dimension $n-k$ and degree 1, with $\mathrm{mult}_a\theta^k=1$ and $\mathrm{mult}_x\theta^k=0$ for $x\neq a$; for $k<n$, θ^k has Zariski-support equal to $\mathbb{P}^n$ whereas $\theta^n=[a]$, see [4, Example 6.3] and cf. Example 2.9. One can think of θ^k as an $(n-k)$-plane through a moving around a. We claim that

$$\theta\bullet\cdots\bullet\theta=:\theta^{k\bullet}=\theta^k,\quad k\leqslant n. \tag{8.17}$$

In fact, notice that both sides coincide outside a in virtue of Proposition 8.4. Thus they can only differ on a generalized cycle with Zariski support at a, that is, $m[a]$ for some integer m. Since the degree of θ is 1, also the degree of $\theta^{k\bullet}$ must be 1 by the Bézout formula (1.7); indeed note that ρ in (1.6) in this case equals $n-k\geqslant 0$. Since the degree of the right-hand side is 1 it follows that $m=0$ and hence (8.17) holds.

Example 8.7 Let $n=2$, let a and θ be as in the previous example, and let ℓ be a line through a. Then

$$\theta\bullet[\ell]=[a]. \tag{8.18}$$

In fact, in view of (8.13), outside a, $\theta\bullet[\ell]=\theta\wedge[\ell]$, which vanishes since the pullback of θ to ℓ vanishes. By the same argument as in Example 8.6, using Bézout's formula (1.7), we get (8.18).

Example 8.8 Let $\mu_1,\ldots,\mu_r$, $r\geqslant 2$, be different lines through $a\in\mathbb{P}^n$. We claim that $\mu_1\bullet\cdots\bullet\mu_r=[a]$. In fact, since the set-theoretic intersection is a, the product must be $m[a]$ for some integer m. Since the μ_j are effective it follows from (1.8) that m is 1 or

0. By (1.5) it is enough to determine the local intersection number $\epsilon_0(\mu_1, \dots, \mu_r, a)$, and thus we can assume that the μ_j are lines through $a = 0$ in $\mathbb{C}^n$. In view of (3.1) and (4.3) this equals the multiplicity of $M_r^{L,\eta} \wedge (\mu_1 \times \cdots \times \mu_r)$, where η is a tuple of linear forms defining the diagonal in $(\mathbb{C}^n)^r = \mathbb{C}^n \times \cdots \times \mathbb{C}^n$. This, in turn, can be computed by intersecting $\mu_1 \times \cdots \times \mu_r$ by r generic hyperplanes $\operatorname{div}(\alpha \cdot \eta)$, see Sect. 4. Doing this, we get [0] with multiplicity 1, which proves the claim.

Example 8.9 Let G be the graph in $\mathbb{C}^6_{x,y} = \mathbb{C}^6_{x_1,x_2,x_3,y_1,y_2,y_3}$ of the function

$$\mathbb{C}^3_x \to \mathbb{C}^3_y, \quad (x_1, x_2, x_3) \mapsto (x_1x_3, x_2x_3, x_3^2),$$

and let Z be the closure in $\mathbb{P}^6_{x_0,x,y}$. Clearly Z is irreducible of dimension 3. We want to compute $A \bullet Z$, where $A = \{y = 0\}$. By (8.3),

$$(A \bullet Z)_{3-k} = M_k^{L,y} \wedge [Z].$$

In view of Sect. 4 we can compute the right-hand side by successively intersecting $[Z]$ by hyperplanes $\operatorname{div} h_j$, where $h_1 = \alpha \cdot y$, $h_2 = \beta \cdot y$, and $h_3 = \gamma \cdot y$ for generic $\alpha, \beta, \gamma \in \mathbb{P}^2$, and then taking averages.

The map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^6$, $[t_0, t_1, t_2, t_3] \mapsto [t_0^2, t_0t_1, t_0t_2, t_0t_3, t_3t_1, t_3t_2, t_3^2]$, lifts to an injective holomorphic map from the blow-up $Y = \mathrm{Bl}_{t_0=t_3=0}\mathbb{P}^3$ to $\mathbb{P}^6$ with image Z. Then Z can be parametrized by two copies of $\mathbb{P}^2 \times \mathbb{C}$,

$$\mathbb{P}^2 \times \mathbb{C} \ni ([s, t_1, t_2], \sigma) \mapsto [s, t_1, t_2, s\sigma, \sigma t_1, \sigma t_2, \sigma^2 s] \in Z,$$
$$\mathbb{P}^2 \times \mathbb{C} \ni ([u, t_1, t_2], v) \mapsto [uv^2, vt_1, vt_2, uv, t_1, t_2, u] \in Z,$$

identified by $s = uv$, $s\sigma = u$. Let Z_1 and Z_2 be the image of the first and second map, respectively. Since $Z_2 \cap A = \varnothing$, the SV-cycle we are to compute is contained in Z_1.

Expressed in the $([s, t_1, t_2], \sigma)$-coordinates, $A = \{\sigma t_1 = \sigma t_2 = \sigma^2 s = 0\} = \{\sigma = 0\}$ and so, clearly, $v_0^h \wedge [Z] = 0$, cf. (4.1). Moreover, $\operatorname{div} h_1$ is given by

$$\sigma(\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 s\sigma) = 0.$$

Hence $\operatorname{div} h_1$ has two irreducible components; the component $\sigma = 0$ is contained in A and thus contributes to $v_1^h \wedge [Z]$ whereas the component $\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 s\sigma = 0$ is not contained in A. Intersecting the latter component by $\operatorname{div} h_2$ gives

$$\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 s\sigma = \sigma(\beta_1 t_1 + \beta_2 t_2 + \beta_3 s\sigma) = 0.$$

Again we get two irreducible components. The component $\{\sigma = \alpha_1 t_1 + \alpha_2 t_2 = 0\}$ is contained in A and contributes to $v_2^h \wedge [Z]$ while the component $\{\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 s\sigma = \beta_1 t_1 + \beta_2 t_2 + \beta_3 s\sigma = 0\}$ is not contained in A. Intersecting the latter one by $\operatorname{div} h_3$ gives

$$\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 s\sigma = \beta_1 t_1 + \beta_2 t_2 + \beta_3 s\sigma = \sigma(\gamma_1 t_1 + \gamma_2 t_2 + \gamma_3 s\sigma) = 0.$$

The case $\sigma \neq 0$ forces $t_1 = t_2 = s = 0$, which is impossible. The other case gives 2 times the point $\{\sigma = t_1 = t_2 = 0\}$ as contribution to $v_3^h \wedge [Z]$.

We thus get the SV-cycle

$$v^h \wedge [Z] = P + L_\alpha + 2a, \tag{8.19}$$

where $P = \{x_3 = y = 0\}$, $L_\alpha = \{x_3 = y = \alpha_1 x_1 + \alpha_2 x_2 = 0\}$, and $a = [1, 0, 0, 0, 0, 0, 0]$ expressed in the original x_0, x, y-coordinates. Taking the average of (8.19) over $(\alpha, \beta, \gamma) \in (\mathbb{P}^2)^3$ we get

$$A \bullet Z = M^{L,y} \wedge [Z] = P + \mu + 2[a],$$

where μ is the generalized cycle $[x_3 = y = 0] \wedge dd^c \log(|x_1|^2 + |x_2|^2)$ obtained as the average of L_α.

Note that the degree of $A \bullet Z$ is 4 since each term has degree 1 except for the double point $2[a]$. Thus, in view of (1.7), $\deg Z = 4$; indeed ρ in (1.6) is 0 in this case. Moreover, by (1.5), the local intersection numbers at a are $\epsilon_0(A, Z, a) = \mathrm{mult}_a 2[a] = 2$, $\epsilon_1(A, Z, a) = \mathrm{mult}_a \mu = 1$, and $\epsilon_2(A, Z, a) = \mathrm{mult}_a P = 1$. Here we have used that μ has multiplicity 1 at a since it is a mean value of lines through a in the 4-plane $\{x_3 = y = 0\}$, cf. Example 8.6.

We now give an example that shows that the $\bullet$-product is not associative.

Example 8.10 Consider the hypersurface $Z = \{x_2 x_1^m - x_3^2 x_0^{m-1} = 0\}$ in $\mathbb{P}^3$, let $H_2 = \{x_2 = 0\}$ and $H_3 = \{x_3 = 0\}$. Since H_2 and Z intersect properly,

$$H_2 \bullet Z = H_2 \cdot_{\mathbb{P}^3} Z = 2\{x_2 = x_3 = 0\} + (m-1)\{x_0 = x_2 = 0\}$$

cf. (1.9). Let $A = \{x_2 = x_3 = 0\}$. It follows from Proposition 6.7 and (8.9) that $\{x_3 = 0\} \bullet A = A$; this can also be verified by a symmetry argument and the Bézout formula (1.7). Moreover, $\{x_3 = 0\}$ and $\{x_0 = x_2 = 0\}$ intersect properly and the intersection is $b = [0, 1, 0, 0]$. Thus

$$H_3 \bullet (H_2 \bullet Z) = 2A + (m-1)[b]. \tag{8.20}$$

Next note that $H_3 \bullet H_2 = A$. It is showed in [6, Example 11.5] that the local intersection number for A and Z in dimension 0 is m at $a = [1, 0, 0, 0]$, and 1 in dimension 1 at all points $x \in A$. It follows that A and $m[a]$ are components of $A \bullet Z$. Moreover, since A and Z are effective, by Theorem 1.1, $A \bullet Z$ is effective and of degree at most $\deg A \cdot \deg Z = m + 1$. Hence

$$(H_3 \bullet H_2) \bullet Z = A \bullet Z = A + m[a]. \tag{8.21}$$

It follows that neither $\cdot_{\mathcal{B}(\mathbb{P}^N)}$ is associative in $\mathcal{B}(\mathbb{P}^n)$. In fact, it follows from (8.20), (8.21), and Theorem 1.2, that

$$H_3 \cdot_{\mathcal{B}(\mathbb{P}^n)} (H_2 \cdot_{\mathcal{B}(\mathbb{P}^n)} Z) = 2\omega \wedge A + (m-1)[b],$$

whereas

$$(H_3 \cdot_{\mathcal{B}(\mathbb{P}^n)} H_2) \cdot_{\mathcal{B}(\mathbb{P}^n)} Z = \omega \wedge A + m[a]$$

and these right-hand sides are not equal in $\mathcal{B}(\mathbb{P}^n)$.

Example 8.11 Let γ be a smooth curve in $\mathbb{P}^2$ of degree d. It is well-known, see, e.g., [6], that local intersection numbers are biholomorphic invariants. Therefore, since the $\bullet$-self-intersection of a line is the line itself, cf. the discussion after Proposition 6.8, it follows from (1.5) that at each $x \in \gamma$, $\mathrm{mult}_x(\gamma \bullet \gamma)_1 = 1$ and $\mathrm{mult}_x(\gamma \bullet \gamma)_0 = 0$. Thus, since $|\gamma \bullet \gamma| \subset \gamma$, in view of the dimension principle, $\gamma \bullet \gamma = \gamma + \mu$ where μ has dimension 0 and Zariski support equal to γ. By the Bézout formula (1.7) the degree of μ must be $d^2 - d$. We can think of μ as $d^2 - d$ points moving around on γ.

Example 8.12 We want to compute the $\bullet$-self-intersection of a curve Z in $\mathbb{P}^2$. Assume that $Z = \{F = 0\}$ where F is a section of $\mathcal{O}(d)$ with differential generically nonvanishing on Z. Let $\eta_j = y_j - x_j$, $j = 0, 1, 2$, on $\mathbb{P}^5_{x,y} = \mathbb{P}^2_x \times_J \mathbb{P}^2_y$. Then η defines the join diagonal Δ_J. Following Sect. 4 we can compute $M^{L,\eta} \wedge (Z \times_J Z)$ by successively intersecting $Z \times_J Z$ by hyperplanes $\mathrm{div}\, h_j$, where $h_j = \eta \cdot \alpha^j$ for generic $\alpha^j \in \mathbb{P}^2$, and then averaging over $\alpha = (\alpha^1, \alpha^2, \alpha^2) \in (\mathbb{P}^2)^3$. Note that we can write

$$F(y) - F(x) = \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2$$

for suitable homogeneous forms A_j, and thus

$$Z \times_J Z = \{F(x) = 0, F(y) = 0\} = \{F(x) = 0,\ \eta_0 A_0 + \eta_1 A_1 + \eta_2 A_2 = 0\},$$

cf. Example 6.3. It turns out that

$$\begin{aligned}
&[\mathrm{div}\, h_2] \wedge [\mathrm{div}\, h_1] \wedge (Z \times_J Z) \\
&\quad = \left\{F(x) = 0, \eta_2(\beta_0 A_0 + \beta_1 A_1 + \beta_2 A_2) = 0,\ \eta_1 = \gamma_1 \eta_2, \eta_0 = \gamma_0 \eta_2\right\}
\end{aligned}$$

for some $\beta, \gamma \in \mathbb{P}^2$. The second equation gives rise to two components. The component corresponding to $\eta_2 = 0$ is contained in Δ_J and equals

$$\{F(x) = 0, \eta = 0\} = \{F = 0\} \cap \Delta_J = v_2^h \wedge (Z \times_J Z) = j_* Z,$$

where j is the parametrization (6.8) of Δ_J. Next, since $A_j = F_j := \partial F / \partial x_j$ on Δ_J we get that

$$\begin{aligned}
&v_3^h \wedge (Z \times_J Z) \\
&\quad = [\mathrm{div}\, h_3] \wedge \left\{F(x) = 0,\ \sum_{j=0}^{2} \beta_j A_j = 0,\ \eta_1 = \gamma_1 \eta_2,\ \eta_0 = \gamma_0 \eta_2\right\} \qquad (8.22) \\
&\quad = \left\{F(x) = 0,\ \sum_{j=0}^{2} \beta_j F_j = 0,\ \eta = 0\right\}.
\end{aligned}$$

The curve defined by $\beta_0 F_0 + \beta_1 F_1 + \beta_2 F_2$ is a so-called *polar curve* to Z; it is clear that it passes through all singular points $a_1 \dots, a_r$ of Z, since the gradient must vanish there. More precisely, in view of the Bézout formula (1.7), for generic β,

$$v_3^h \wedge (Z \times_J Z) = m_1[a_1] + \cdots + m_r[a_r] + r_\beta,$$

where m_j are the multiplicities of a_j and r_β are $d^2 - d - (m_1 + \cdots + m_r)$ points on Z depending on β, cf. Example 8.11. Thus, taking averages over $\alpha \in (\mathbb{P}^2)^3$, we get that

$$Z \bullet Z = Z + m_1[a_1] + \cdots m_r[a_r] + \mu, \tag{8.23}$$

where $j_*\mu$ is the average of the r_β. In particular, μ has dimension 0, Zariski-support equal to Z, and degree $d^2 - d - (m_1 + \cdots + m_r)$. Moreover, in view of Example 8.11, μ has multiplicity 0 at each point.

Let us now consider a simple cusp.

Example 8.13 Let us consider the situation of the previous example and let $F = x_1^3 - x_0 x_2^2$ so that $Z \subset \mathbb{P}^2$ is a cusp with a singularity only at the point $a = [1, 0, 0]$. Now

$$v_3^h \wedge (Z \times_J Z) = \left\{x_1^3 - x_0 x_2^2 = 0,\ \beta_0 x_2^2 + \beta_1 x_1^2 + \beta_2 x_0 x_2 = 0,\ \eta = 0\right\}$$

for some $\beta \in \mathbb{P}^2$, see (8.22). For generic choices of $\alpha \in (\mathbb{P}^2)^3$, $\beta_2 \neq 0$ and we can identify this with the set of points

$$\varrho_\beta = \left\{x_1^3 - x_0 x_2^2 = 0,\ \beta_0 x_2^2 + \beta_1 x_1^2 + x_0 x_2 = 0\right\} \subset \mathbb{P}^2.$$

To compute the order of the zero at a, we can use affine coordinates and thus let $x_0 = 1$. Then $\varrho_\beta = \left\{x_1^3 - x_2^2 = 0,\ \beta_0 x_2^2 + \beta_1 x_1^2 + x_2 = 0\right\}$. If we choose new coordinates $z_1 = x_1$, $z_2 = x_2 + \beta_0 x_2^2 + \beta_1 x_1^2$, then $x_2 = z_2 + \mathcal{O}(z^2)$, and thus ϱ_β is defined by the equations

$$z_1^3 - (z_2 + \mathcal{O}(z^2))^2 = 0, \quad z_2 = 0.$$

Hence the zero at $a = (0, 0)$ has order 3. In fact, for a complete intersection, as here, the order of the zero coincides with the degree of the associated mapping. From (8.23) we conclude that

$$Z \bullet Z = Z + 3[a] + \mu, \tag{8.24}$$

where μ has dimension 0, Zariski-support equal to Z, multiplicity 0 at each point, and degree 3.

Example 8.14 Let $Z \subset \mathbb{P}^2$ be the cusp as in the previous example. In view of Theorem 1.2 and (8.24) we get

$$Z \cdot_{\mathcal{B}(\mathbb{P}^2)} Z = \omega \wedge [Z] + 3[a] + \mu. \tag{8.25}$$

Since Z is a regular embedding in $\mathbb{P}^2$ we can also form the product $Z \diamond_{\mathcal{B}(\mathbb{P}^2)} Z$. Let $\mathcal{J} \to \mathbb{P}^2$ be the sheaf defining Z. If $i\colon Z \hookrightarrow \mathbb{P}^2$, then $i^*\mathcal{J}_Z = 0$ so that $S(\mathcal{J}_Z, Z) = S(0, Z) = [Z]$, cf. Sect. 2.5. Moreover, $N_Z\mathbb{P}^2 = \mathcal{O}(3)|_Z$, so that $c_1(N_Z\mathbb{P}^2) = 3\omega$. Thus

$$Z \diamond_{\mathcal{B}(\mathbb{P}^2)} Z = (c(N_Z\mathbb{P}^2) \wedge S(\mathcal{J}_Z, Z))_0 = 3\omega \wedge [Z], \tag{8.26}$$

cf. Definition 5.1. Notice that (8.25) and (8.26) do not coincide in $\mathcal{B}(\mathbb{P}^2)$. For instance, the first one has multiplicity 3 at a, whereas the second one has multiplicity 0 at a.

However, in view of Proposition 5.7 their images in $\widehat{H}^{2,2}(Z)$ coincide. Clearly the image of $Z \diamond_{\mathcal{B}(\mathbb{P}^2)} Z$ is represented by the restriction to Z of the form 3ω. It is easy to see that $3a$ is cohomologous with ω on Z as

$$3[a] - \omega \wedge [Z] = \mathrm{mult}_a Z \cdot [a] - \omega \wedge [Z] = dd^c(\log(|z_1|^2/|z|^2)[Z]).$$

It is somewhat less obvious that μ is cohomologous with ω on Z.

Example 8.14 also shows that the self-intersection formula, Proposition 5.5, does not generalize to nonsmooth Z.

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