The Szemerédi–Petruska conjecture for a few small values

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Received: 29 September 2020 / Revised: 24 February 2021 / Accepted: 14 March 2021 /

Published online: 7 May 2021

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Abstract

Let H be a 3-uniform hypergraph of order n with clique number $\omega(H) = k$. Assume that the union of the k -cliques of H equals its vertex set, the intersection of all maximum cliques of H is empty, but the intersection of all but one k -clique is non-empty. For fixed $m = n - k$, Szemerédi and Petruska conjectured the sharp bound $n \leq \binom{m+2}{2}$. In this note the conjecture is verified for $m = 2, 3$ and 4.

Keywords 3-Uniform hypergraph · Szemerédi–Petruska conjecture · τ -Critical hypergraph · Intersecting set pair systems · τ -Critical graph

Mathematics Subject Classification 05D05 · 05D15 · 05C65

1 Introduction

Let H be a 3-uniform hypergraph of order n with clique number $\omega(H) = k$. Assume that the union of the k -cliques of H equals its vertex set, the intersection of its maximum cliques is empty, but the intersection of all but one maximum clique is non-empty. For fixed $m = n - k$ Szemerédi and Petruska [8] conjectured the tight bound $n \leq \binom{m+2}{2}$.

Here is the construction that shows this bound is sharp for infinitely many $k \geq 3$. For $m \geq 0$ let $k - 1 = \binom{m+1}{2}$, set $n = m + k$, and let X, Y be disjoint sets such that $|X| = k - 1, |Y| = m + 1$. Let $X = \{x_1, \dots, x_{k-1}\}$ and let $\{p_1, \dots, p_{k-1}\}$ be the set of all 2-element sets of Y . For every $i = 1, \dots, k - 1$ set $N_i = (X \setminus \{x_i\}) \cup p_i$. All triples in some $N_i, i = 1, \dots, k - 1$, define a 3-uniform hypergraph H on the n -element

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vertex set $X \cup Y$. It is straightforward to check that $\{N_1, \dots, N_{k-1}\}$ is the family of all maximum cliques of H , in particular $\omega(H) = k$, furthermore, $\bigcap_{i=1}^{k-1} N_i = \emptyset$. Notice also that any $k - 2$ maximum cliques have a common vertex in X ; therefore, H realizes an extremal hypergraph and shows the bound is sharp.

Indeed it had been conjectured this example is the unique extremal example for $m \geq 4$.

In Sect. 2 we prove the Szemerédi–Petruska conjecture for $m = 2, 3$, and 4. The proof, based on τ -critical graphs, makes it possible to characterize all extremal hypergraphs of order $n = \binom{m+2}{2}$ for $m \leq 4$. It is worth noting that there are two-two non-isomorphic extremal hypergraphs for $m = 2$ and 3; meanwhile, the extreme is unique for $m = 4$ and isomorphic to the Szemerédi–Petruska construction described above.

The *transversal number* of a hypergraph (or a graph), $\tau(H)$, is defined as the minimum number of vertices in a *transversal set* containing a vertex from every edge of H . A hypergraph (or graph) is τ -critical if it has no isolated vertex and the removal of every edge decreases its transversal number.

The Szemerédi–Petruska conjecture is equivalent with the statement that $\binom{m+2}{2}$ is the maximum order of a 3-uniform τ -critical hypergraph with transversal number m , see [4] or [9, Problem 18 (a)]. For the maximum order, Tuza¹ obtained the best known bound $\frac{3}{4}m^2 + m + 1$ using the machinery of τ -critical hypergraphs.

An alternative approach to solve the conjecture is proposed by Jobson et al. [6], combining a decomposition process introduced by Szemerédi and Petruska [8] with the skew version of Bollobás's theorem [1], and using tools from linear algebra.

It turns out that the Szemerédi–Petruska conjecture has applications in extremal problems concerning convex sets in the plane, see Jobson et al. [5] and [7]. The validity of the Szemerédi–Petruska conjecture for small values carries relevant information pertaining to those combinatorial geometry problems.

2 Proof of the conjecture for $m = 2, 3$, and 4

In the proof presented here τ -critical graphs play a key role. In particular, we use the tight bounds (1) and (2) shown below due to Erdős and Gallai [2] and Gyárfás and Lehel [3], respectively.

Lemma 2.1 ([2,3]) *If $G = (V, E)$ is a τ -critical graph with $\tau(G) = t$, then*

$$|V| \leq 2t, \quad (1)$$

$$|V| + |E| \leq \binom{t+2}{2}. \quad (2)$$

Proposition 2.2 *Let $m = 2, 3$, or 4, and $n > m$. If H is a 3-uniform hypergraph of order n with clique number $\omega(H) = n - m = k \geq 3$, and the intersection of the k -cliques of H is empty, then $n \leq \binom{m+2}{2}$.*

¹ Personal communication.

Proof Let $\mathcal{N} = \{N_1, \dots, N_\ell\}$ be a collection of k -cliques of $H = (V, E)$ such that $\bigcap_{i=1}^\ell N_i = \emptyset$, but $\bigcap_{j \neq i} N_j \neq \emptyset$ for all $i = 1, \dots, \ell$. Notice that if $\bigcup N_i \neq V$, say $v \in V \setminus \bigcup N_i$, then we may consider $H - v$ thereby reducing both n and m by one. Hence we may assume $\bigcup_{i=1}^\ell N_i = V$ and $|V| = n$. Furthermore, we assume that $\ell \geq 3$, since H needs to have at least two cliques, and the claim is obvious for $\ell = 2$, even if H consists of two disjoint k -cliques.

Szemerédi and Petruska [8, Lemma 4] observed that each $N_i, i = 1, \dots, \ell$, contains a *private pair*: $p_i \subset V, |p_i| = 2$, such that $p_i \subset N_j$ if and only if $i = j$. Let $G = (V, E)$ be the graph of the private pairs $p_i = \{a_i, b_i\}, 1 \leq i \leq \ell$, as its edges.

Set $M_i = V \setminus N_i$, and notice that $|M_i| = n - k = m$. By the definition of private pairs, $(p_i, M_i), 1 \leq i \leq \ell$, form an *intersecting* $(2, m)$ -system: $p_i \cap M_j = \emptyset$ if and only if $i = j$. Furthermore, $\bigcup p_i \subseteq \bigcup M_i = V$ (since $\bigcap N_i = \emptyset$).

For an edge $p_j = \{a_j, b_j\}$ of G , define the *truncated subgraph* $G \setminus p_j$ on vertex set $V \setminus \{a_j, b_j\}$ by including $p_h \setminus \{a_j, b_j\}$ as an edge or a loop, for all $h \neq j$. Since $M_j \cap p_h = M_j \cap (p_h \setminus p_j) \neq \emptyset$ for all $h \neq j$, we have

$$\tau(G \setminus p_j) \leq |M_j| = m, \quad (3)$$

for every $1 \leq j \leq \ell$. For each edge $p_j \in G$ we define the *weight*

$$w(p_j) = m - \tau(G \setminus p_j).$$

Let $G - p_j$ denote the graph obtained by the removal of edge p_j from G (and nothing else). Obviously we have

$$\tau(G) - 1 \leq \tau(G - p_j) \leq \tau(G \setminus p_j) \leq m. \quad (4)$$

The conditions $\bigcap N_i = \emptyset$ and (4) imply $2 \leq \tau(G) \leq m + 1$. Furthermore, $0 \leq w(p_j) = m - \tau(G \setminus p_j) \leq m - \tau(G) + 1$.

Let $V_0(G) = \bigcup_{p \in E} p$, set $Z(G) = V \setminus V_0(G)$ and $w(G) = \sum_{p \in E} w(p)$.

Lemma 2.3 $|V| = |V_0(G)| + |Z(G)| \leq |V_0(G)| + w(G)$.

Proof Set $V_0 = V_0(G)$, and $Z_j = M_j \setminus V_0$. Observe that (3) implies $|M_j \cap V_0| \geq \tau(G \setminus p_j)$, hence $|Z_j| = (|M_j| - |M_j \cap V_0|) \leq m - \tau(G \setminus p_j) = w(p_j)$. In words, M_j has at most $w(p_j)$ vertices not in V_0 . Thus $|Z(G)| \leq \sum_{j=1}^\ell |Z_j| \leq \sum_{j=1}^\ell w(p_j) = w(G)$, and the claim follows. $\blacksquare$

Now we break the proof of Proposition 2.2 into three cases.

Case 1: $\tau(G) = 2$. Let $\{u, v\}$ be a transversal set of G , set $V_0 = V_0(G)$, and let $X = \{a \in V_0 \setminus \{v\} : ua \in E\}$ and $Y = \{a \in V_0 \setminus \{u\} : va \in E\}$. Set $x = |X|$, $y = |Y|$, and $z = |X \cap Y|$.

Case 1.1: $Y \subseteq X$. In this case $|V_0| = |X \cup \{u, v\}| = x + 2$. If $q = \{u, v\} \in E$, then $\tau(G) = 2$ implies $\tau(G \setminus q) = |X|$, hence $w(q) = m - \tau(G \setminus q) = m - x$; if $q \notin E$,

define $w(q) = 0$. We also have

$$\begin{aligned}\tau(G \setminus ua) &= |X \setminus \{a\}| = x - 1, \quad w(ua) = m - x + 1 \text{ if } a \in X \setminus Y; \\ \tau(G \setminus ua) &= |(X \setminus \{a\}) \cup \{v\}| = x, \quad w(ua) = m - x \text{ if } a \in Y; \\ \tau(G \setminus va) &= |(Y \setminus \{a\}) \cup \{u\}| = y, \quad w(va) = m - y \text{ if } a \in Y.\end{aligned}$$

Thus the bound in Lemma 2.3 becomes

$$\begin{aligned}|V| &\leq |V_0| + w(G) = (x + 2) + w(q) + \sum_{p \neq q} w(p) \\ &\leq (x + 2) + (m - x) + (x - y)(m - x + 1) + y(m - x) + y(m - y) \\ &= m + 2 + x(m - x + 1) + y(m - y - 1) \\ &\leq m + 2 + \left\lfloor \frac{m+1}{2} \right\rfloor \cdot \left\lceil \frac{m+1}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor \cdot \left\lceil \frac{m-1}{2} \right\rceil \leq \binom{m+2}{2}.\end{aligned}$$

Case 1.2: $X \setminus Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$. In this case $|V_0| = x + y - z + 2$, and if $q = uv \in E$, then $\tau(G \setminus q) = x + y - z$, $w(q) = m - x - y + z$; furthermore, $\tau(G \setminus ua) = x$, $w(ua) = m - x$ if $a \in X$, and $\tau(G \setminus va) = y$, $w(va) = m - y$ if $a \in Y$. Thus by applying Lemma 2.3 we obtain the bound

$$\begin{aligned}|V| &\leq (x + y - z + 2) + (m - x - y + z) + x(m - x) + y(m - y) \\ &\leq m + 2 + 2 \left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lceil \frac{m}{2} \right\rceil \leq \binom{m+2}{2}.\end{aligned}$$

Therefore, if $\tau(G) = 2$ then $n \leq \binom{m+2}{2}$ follows for every $m \geq 2$.

Case 2: $\tau(G) = m + 1$ or m . Assume $\tau(G) = m + 1$. By (4), $0 \leq w(p_i) = m - \tau(G \setminus p_i) \leq m - \tau(G - p_i)$, for every $1 \leq i \leq \ell$, thus we have $\tau(G - p_i) \leq m$. Therefore, G is a τ -critical graph and the bound (1) in Lemma 2.1 yields $|V| \leq 2(m + 1)$. For $m \geq 2$, $|V| \leq 2(m + 1) \leq \binom{m+2}{2}$ follows.

Assume now that $\tau(G) = m$. Notice that $w(p_i) = 0$ means $M_i \subset V_0(G)$. By successively removing edges from G with $w(p_i) = 0$ we conclude with a subgraph G^* such that $\tau(G^*) = m$ and $0 < w(p) = m - \tau(G^* \setminus p) \leq m - \tau(G^* - p)$, for every $p \in E^*$. Hence G^* is a τ -critical graph. We claim that $|V| \leq |V^*| + |E^*|$, and from the bound (2) in Lemma 2.1 we obtain $|V| \leq |V^*| + |E^*| \leq \binom{m+2}{2}$.

The claim $|V| \leq |V^*| + |E^*|$ follows by repeatedly applying the next general² lemma.

Lemma 2.4 *If $\ell \geq 3$ and $w(p_\ell) = 0$, then $G' = G - p_\ell$ satisfies*

$$|V_0(G)| + w(G) \leq |V_0(G')| + w(G').$$

² The lemma is valid for any value of $\tau(G)$.

Proof For $1 \leq i \leq \ell - 1$, define $w'(p_i) = m - \tau(G' \setminus p_i)$. Since a transversal set in $G \setminus p_i$ is also a transversal set in $G' \setminus p_i$, we have $\tau(G \setminus p_i) \geq \tau(G' \setminus p_i)$. Then we obtain

$$w(G') = \sum_{j=1}^{\ell-1} w'(p_j) = \sum_{j=1}^{\ell-1} (m - \tau(G' \setminus p_j)) \geq \sum_{j=1}^{\ell-1} (m - \tau(G \setminus p_j)) = w(G).$$

If p_ℓ is neither an isolated edge nor a pendant edge, then $V_0(G') = V_0(G)$, therefore $|V_0(G)| + w(G) \leq |V_0(G')| + w(G')$ follows.

If p_ℓ is an isolated edge in G , then $|V_0(G')| = |V_0(G)| - 2$. Because $w(p_\ell) = 0$, we have $\tau(G) = \tau(G \setminus p_\ell) + 1 = m - w(p_\ell) + 1 = m + 1$. Moreover, G is τ -critical, since by (4),

$$\tau(G) - 1 \leq \tau(G - p_j) \leq m = \tau(G) - 1,$$

for every $1 \leq j \leq \ell$. Then $w(p_j) = m - \tau(G \setminus p_j) = 0$ thus $w(G) = 0$. Observe that G' is τ -critical, as well, with $\tau(G') = \tau(G) - 1 = m$, therefore, $w'(p_j) = m - \tau(G' \setminus p_j) = m - \tau(G' - p_j) = 1$. Then by the definition of the edge weights, we have $w(G') = \sum_{j=1}^{\ell-1} 1 = |E'|$. Since $|E'| \geq m \geq 2$, we obtain $|V_0(G)| + w(G) = |V_0(G)| \leq (|V_0(G)| - 2) + |E'| = |V_0(G')| + w(G')$.

If p_ℓ is a pendant edge, then $|V_0(G')| = |V_0(G)| - 1$. For any $p_i \cap p_\ell \neq \emptyset$, $\tau(G' \setminus p_i) = \tau(G \setminus p_i) - 1$, because a minimum transversal in $G \setminus p_i$ must use a vertex just for p_ℓ , and this vertex is not required in a minimum transversal of $G' \setminus p_i$. Hence $w'(p_i) = w(p_i) + 1$, which implies $w(G') \geq w(G) + 1$, and $|V_0(G)| + w(G) = (|V_0(G)| - 1) + (w(G) + 1) \leq |V_0(G')| + w(G')$ follows. ■

Because $2 \leq \tau(G) \leq m + 1$, Steps 1 and 2 imply $n \leq \binom{m+2}{2}$, for $m = 2, 3$.

Case 3: $m = 4$, $\tau(G) = 3$. By Lemma 2.4 we may assume that all edges of G have positive weights. Since $\tau(G) = 3$, G contains one of the four τ -critical subgraphs in Fig. 1.

Case 3.1: G contains a K_4 . If G had two more non-isolated vertices, a and b , then $ab \notin E$, since $\tau(G) = 3$. One edge from each of a and b to K_4 would result in an edge of zero weight. Hence $|V_0| = 5$ or 4 , and G can be obtained starting with a K_5 , and successively removing edges incident with a common vertex, say a . The corresponding configurations are depicted in Fig. 2 (unlabeled edges have weight 1).

Fig. 1 τ -critical graphs G with $\tau(G) = 3$

Fig. 2 G has a K_4

The rightmost candidate in Fig. 2 can be rejected as follows. If $n = 16 > \binom{4+2}{2}$, then $Z_i = M_i \setminus V_0$, $1 \leq i \leq 6$, are pairwise disjoint 2-element sets. Let $V_0 = \{b, c, d, e\}$ and $Z_i = \{i, i'\}$. The corresponding $(2, m)$ -system is uniquely determined as follows: $p_1 = bc$, $p_2 = bd$, $p_3 = be$, $p_4 = cd$, $p_5 = ce$, $p_6 = de$, and $M_1 = \{d, e, 1, 1'\}$, $M_2 = \{c, e, 2, 2'\}$, $M_3 = \{c, d, 3, 3'\}$, $M_4 = \{b, e, 4, 4'\}$, $M_5 = \{b, d, 5, 5'\}$, $M_6 = \{b, c, 6, 6'\}$.

We eliminate the ‘fake’ candidate by applying a general observation, the *Triples test*, as follows. If a set $f \subset V$, $|f| = 3$, is such that $f \cap M_i = \emptyset$ for some $1 \leq i \leq \ell$, then $f \subset N_i$ and since N_i is a clique of H , we have $f \in E$.

Let $N = V \setminus \{c, d, e\}$; $|N| = n - 3 = n - (m - 1) = k + 1$. Notice that the four sets $M_i \setminus \{c, d, e\}$, $i = 1, 2, 3, 4$, are pairwise disjoint, thus every 3-element set $f \subset N$ is disjoint from some M_i , $1 \leq i \leq 6$. Hence, by the Triples test, $f \in E$ for every $f \subset N$, $|f| = 3$. Therefore, N induces a clique of order $k + 1$, a contradiction.

Case 3.2: G has a 5-cycle and no K_4 . Since $\tau(G) = 3$, all edges of G are incident with the 5-cycle. Figure 3 lists all configurations with no zero edge weights (unlabeled edges have weights 1). All candidates have at most $\binom{4+2}{2} = 15$ vertices.

Fig. 3 G has a 5-cycle and no K_4

Fig. 4 G has a triangle and no 5-cycle or K_4

Case 3.3: G has a triangle, it has no 5-cycle and no K_4 . The τ -critical subgraph in G is $K_2 + K_3$. To get all candidates at most three edges and one new vertex can be added to $K_2 + K_3$ by keeping the transversal number 3 and creating neither a 5-cycle nor a K_4 . These graphs are listed in Fig. 4; all have at most 15 vertices.

Case 3.4: G has no cycle. The τ -critical subgraph in G is $3K_2$. To get all candidates from this one at most three edges can be added by keeping the transversal number 3 and creating no cycle. These graphs are listed in Fig. 5; all have at most 15 vertices. $\square$

Fig. 5 G has no cycle

Funding Open access funding provided by ELKH Alfréd Rényi Institute of Mathematics.

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