Semi-fast convergent sequences and k -sums of central Cantor sets

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Abstract

We study k -sums of central Cantor sets and present a full characterization of the case of positive measure for such k -sums. This generalizes the celebrated Sannami's counterexample for the Palis hypothesis. Moreover, we find a characterization of all semi-fast convergent sequences generating central Cantor sets.

Keywords Central Cantor set · Achievement set · Fast convergent sequence · Semi-fast convergent series · Lebesgue measure

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Suppose that $(x_n) = (x_1, x_2, \dots)$ is an absolutely summable sequence. The set

$$E(x_n) := \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0, 1\}^{\mathbb{N}} \right\}$$

of all subsums of the series $\sum_{n=1}^{\infty} x_n$ is called the *achievement set* of (x_n) , see [10]. Since $E(x_n)$ is a translate of $E(|x_n|)$, it suffices to study only summable sequences

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of nonincreasing terms. It is easy to see that for $(x_n) = (\frac{2}{3}, \frac{2}{3^2}, \dots)$ the set $E(x_n)$ is equal to the classic Cantor ternary set $\mathbf{C}$. By a Cantor set we mean any subset of reals homeomorphic to $\mathbf{C}$.

Denote by r_n the number $\sum_{k=n+1}^{\infty} x_k$ and recall that sequences satisfying $x_n > r_n$ for any $n \in \mathbb{N}$ are called *fast convergent*, and sequences with $x_n \leq r_n$ are called *slowly convergent*. Probably Takeya was the first who observed that for any fast convergent sequence (x_n) the set $E(x_n)$ is a Cantor set. He also showed that in this case the Lebesgue measure μ of $E(x_n)$ is equal to $\lim_n 2^n r_n$ [11]. Moreover, Takeya proved that the achievement set $E(x_n)$ of a monotonic sequence is an interval if and only if (x_n) is slowly convergent.

It turned out that achievement sets may take a form different from Cantor sets or finite unions of closed intervals. Recall that a set is called a *Cantorval* if it is homeomorphic to the set $\mathbf{T} = \mathbf{C} \cup \bigcup S_{2n-1}$, where S_n denotes the union of 2^{n-1} open middle thirds which are removed from $[0, 1]$ at the n -th step in the standard construction of the Cantor ternary set $\mathbf{C}$. It can be shown based on [3, Theorem 21.17] that a Cantorval (more precisely, an $\mathcal{M}$ -Cantorval, see [13]) is a nonempty compact subset of the real line that is the closure of its interior, and both endpoints of any of its components with nonempty interior are accumulation points of one-point components. For other characterizations of Cantorvals see for example [13].

In [8] (with a small gap) and in [16] (correctly) it is proven that

Theorem 1 *For any absolutely summable sequence (x_n) with infinitely many nonzero terms, $E(x_n)$ is one of the following sets:*

- A finite union of closed intervals.
- A Cantor set.
- A Cantorval.

It is easy to see that for any summable sequence (x_n) of positive terms, the set $E(x_n)$ is symmetric about the point $r_0/2$, and hence it is a translate of $-E(x_n)$ and $E(x_n) + E(x_n)$ is a translate of $E(x_n) - E(x_n)$. More information on achievement sets can be found in the surveys [3, 14, 15].

By a *gap* in a set E we mean an interval (a, b) such that $a, b \in E$ and $(a, b) \cap E = \emptyset$. The following two lemmas can be found in [3]. The first is obvious.

Lemma 2 (First Gap Lemma) *Let $E = E(x_n)$. If $x_k > r_k$ then (r_k, x_k) is a gap in E .*

The next observation is extracted from the proof of the crucial [16, Lemma 4] (for details see [3]).

Lemma 3 (Second Gap Lemma) *Let (a, b) be a gap in $E = E(x_n)$ and let p be defined by the formula $p := \max \{n : x_n \geq b - a\}$. Then:*

- $b \in F_p$, where $F_p = \{\sum_{i=1}^p \varepsilon_i x_i : (\varepsilon_i) \in \{0, 1\}^p\}$,
- $a \in F_p + r_p$.

For our purpose we need the following strengthening of the Second Gap Lemma, proven in [5] (see also [6]).

Lemma 4 (Third Gap Lemma) *Suppose that (a, b) is a gap in $E = E(x_n)$ such that for any gap (a_1, b_1) with $b_1 < a$ we have $b - a > b_1 - a_1$ (in other words (a, b) is the longest gap from the left). Then $b = x_k$ for some $k \in \mathbb{N}$ and $a = r_k$.*

Knowledge of the Cantor ternary set $\mathbf{C}$ and its construction belongs to the rudimentary education of every mathematician. In this paper we focus on some generalization of the classic notion. A *central Cantor set* is constructed in the following way. At the initial stage we remove from a closed interval $K_0 = [0, r_0]$ a first order gap—the middle open interval (y_1, x_1) (with $y_1 < x_1$). The remaining set, which is the union of two closed intervals with the same length, we denote by K_1 . From each of these two intervals we delete a second order gap—the middle open interval with the same length — obtaining the set K_2 . Denote the left of the two removed intervals by (y_2, x_2) . Proceeding inductively we remove from each of the 2^n closed components of the set K_n the n -th order gaps of length $x_n - y_n$ ($y_n < x_n$). The central Cantor set C is given by the intersection of K_n .

Set $y_0 = r_0$ and $\lambda_n = \frac{y_n}{y_{n-1}}$ for $n \geq 1$. Clearly, any λ_n is less than one half. It is also easy to see that the sequence (λ_n) defines C up to multiplication by a number. We denote it by $C_{(\lambda_n)}$ and, for simplicity of the notation, we often suppose that $K_0 = [0, 1]$.

On the other hand, if $C = E(x_n)$, then $y_n = r_n = \sum_{k=n+1}^{\infty} x_k$ and $r_0 = \sum_{n=1}^{\infty} x_n$. Consequently, $\lambda_n = \frac{r_n}{r_{n-1}}$, $r_n = \lambda_1 \cdots \lambda_n r_0$, and $x_n = \lambda_1 \cdots \lambda_{n-1} (1 - \lambda_n) r_0$. Note that $x_n > r_n$ for any n . It means that there is a natural duality between central Cantor sets and fast convergent sequences. In particular, for any central Cantor set C with $K_0 = [0, 1]$ there is exactly one fast convergent sequence (x_n) such that $C = E(x_n)$. If for some $q < \frac{1}{2}$ and for all n we have $\lambda_n = q$, then we obtain the central Cantor set C_q which is an achievement set of the sequence (q^n) . The Cantor ternary set $\mathbf{C}$ is equal to $2C_{1/3} = E(\frac{2}{3^n})$.

Suppose that C is a central Cantor set with $K_0 = [0, 1]$. There may exist a sequence (b_n) such that $E(b_n) = C$ and (b_n) is not fast convergent. As an example we will use so-called multigeometric sequences. We call a sequence *multigeometric* if it is of the form

$$(k_1, \dots, k_m, k_1 q, \dots, k_m q, k_1 q^2, \dots, k_m q^2, k_1 q^3, \dots)$$

for some positive numbers $k_1, \dots, k_m$ and $q \in (0, 1)$. We will denote such a sequence by $(k_1, \dots, k_m; q)$.

Example 5 Fix a number $q < \frac{1}{4}$ and consider the sequence $(a_n) = (2, 1; q)$. We will use this sequence several times later in the paper. It is easy to check that (a_n) is fast convergent, so $C = E(2, 1; q)$ is a central Cantor set. Moreover, $E(2, 1; q) = E(1, 1, 1; q)$. Clearly, $(1, 1, 1; q)$ is not fast convergent.

It is worth noting that if a central Cantor set C is generated by a sequence (a_n) with $a_n > 2a_{n-1}$ for all $n > 1$, then the sequence (a_n) is the only one which generates C (see [6, Theorem 4.2]). The proof of the following proposition is straightforward.

Proposition 6 Let $C_{(\lambda_n)}$ be a central Cantor set generated by a fast convergent sequence (x_n) . Then

$$(\lambda_n) = (\lambda_1, \dots, \lambda_m, \lambda_1, \dots, \lambda_m, \dots)$$

if and only if (x_n) is multigeometric:

$$(x_n) = (k_1, \dots, k_m; q).$$

Let C be a central Cantor set. Denote by $S_k(C)$ the algebraic sum $C + \dots + C$ of k copies of C . Note that if $C = E(x_n)$, then

$$S_k(C) = E(x_1, \dots, x_1, x_2, \dots, x_2, \dots)$$

where each x_i repeats k times. The algebraic sums and differences of Cantor sets, in particular of sets of the form $S_k(C)$, were studied by many authors (see for example [1, 7, 9, 12, 17–19]).

Example 7 The central Cantor set $C = E(2, 1; q)$ with $q < \frac{1}{4}$ is the sum of three copies of $E(1; q) = E(q^n)$. Despite the known fact that every central Cantor set is the algebraic sum of two central Cantor sets [18, Theorem 1], it is easy to observe that no central Cantor set is the sum of two copies of some central Cantor set.

The next theorem follows from the aforementioned Kakeya's result on slowly convergent sequences (see [1]).

Theorem 8 Suppose that $C = E(x_n)$ is a central Cantor set. Then $S_k(C)$ is an interval if and only if $x_n \leq kr_n$ for all $n \in \mathbb{N}$.

Since

$$x_n \leq kr_n \iff \lambda_1 \cdots \lambda_{n-1}(1 - \lambda_n) \leq k(\lambda_1 \cdots \lambda_n) \iff (1 - \lambda_n) \leq k\lambda_n,$$

we obtain

Corollary 9 Suppose that $C_{(\lambda_n)}$ is a central Cantor set. Then $S_k(C)$ is an interval if and only if $\lambda_n \geq \frac{1}{k+1}$ for all $n \in \mathbb{N}$.

To describe k -sums of central Cantor sets that are Cantor sets we will use the notion of semi-fast convergent series.

Definition 10 A monotonic sequence (a_n) with positive terms converging to 0 (or a series $\sum a_n$) is called *semi-fast convergent* if it satisfies the condition

$$a_n > \sum_{k: a_k < a_n} a_k \quad \text{for all } n \in \mathbb{N}.$$

Of course, a strictly decreasing sequence (a_n) is semi-fast convergent if and only if it is fast convergent. For any convergent series $\sum a_n$ we denote by $F(a_n)$ the set of all sums of finite subseries of $\sum a_n$,

$$F = F(a_n) := \left\{ \sum_{n \in A} a_n : A \subset \mathbb{N}, |A| < \infty \right\},$$

where $|A|$ denotes the cardinality of A . We assume the convention $\sum_{n \in \emptyset} a_n = 0$ and we set $F_0^* = F_0 = \{0\}$. Further, as in Lemma 3, we define for a positive integer k ,

$$F_k := \left\{ \sum_{n \in A} a_n : A \subset \{1, \dots, k\} \right\}$$

and

$$F_k^* := F_k \setminus F_{k-1}.$$

In particular, any F_k is a union of disjoint sets F_i^* for $i = 0, \dots, k$ (we write $F_k = \bigsqcup_{i=0}^k F_i^*$). Further, we have $F = \bigcup_k F_k = \bigsqcup_k F_k^*$.

Let $\sum a_n$ be a semi-fast convergent series. There exist two uniquely determined sequences, (α_k) of positive numbers decreasing to zero and (N_k) of positive integers, such that

$$a_i = \alpha_k \quad \text{for} \quad \sum_{j=0}^{k-1} N_j < i \leq \sum_{j=0}^k N_j,$$

where we put $N_0 := 0$. The numbers α_k are the values of the terms of the series $\sum a_n$ and N_k is the multiplicity of the value α_k in the series. Thus, we can identify

$$\sum a_n = \sum (\alpha_k, N_k) \quad \text{or} \quad (a_n) = (\alpha_k, N_k)$$

and the sum of the series is $\sum a_n = \sum (\alpha_k, N_k) = \sum_{k=1}^{\infty} \alpha_k N_k$.

Proposition 11 *If (a_n) is fast convergent, then $|F_k| = 2^k$ for all k .*

Proof Let I_1 and I_2 be two distinct subsets of $\{1, 2, \dots, k\}$. Set $m := \min I_1 \triangle I_2$ (the symmetric difference). Without loss of generality we may assume that $m \in I_1$. Then

$$\begin{aligned} \sum_{i \in I_1} a_i &= \sum_{\substack{i \in I_1 \\ i < m}} a_i + a_m + \sum_{\substack{i \in I_1 \\ i > m}} a_i \geq \sum_{\substack{i \in I_2 \\ i < m}} a_i + a_m \\ &> \sum_{\substack{i \in I_2 \\ i < m}} a_i + \sum_{i=m+1}^{\infty} a_i \geq \sum_{\substack{i \in I_2 \\ i < m}} a_i + \sum_{\substack{i \in I_2 \\ i > m}} a_i = \sum_{i \in I_2} a_i. \end{aligned}$$

Hence, any two distinct subsets of $\{1, 2, \dots, k\}$ yield distinct values of subsums, and thus the cardinality of the family of F_k is equal to the cardinality of the family of all subsets of $\{1, 2, \dots, k\}$. $\square$

Remark 12 Let us observe that the converse of Proposition 11 is false. In order to show this let us consider a sequence (b_i) of linearly independent elements of $\mathbb{R}[\mathbb{Q}]$. Let $a_1 \in (\frac{1}{2}, 1)$ be a rational multiple of b_1 . Choosing $a_{i+1} \in (\frac{a_i}{2}, \frac{1}{2i})$ to be a rational multiple

of b_{i+1} , we define a convergent series $\sum a_i$ (dominated by $\sum \frac{1}{2^{i-1}}$) of decreasing positive terms. The inequalities $a_{i+1} > \frac{a_i}{2}$ for all i imply that $r_n > a_n$ for all n . On the other hand, $|F_k(\sum a_n)| = 2^k$ as the numbers $a_1, \dots, a_k$ are linearly independent over $\mathbb{Q}$.

Now observe that the cardinality of the sets F_k^* is

$$|F_k^*| = \begin{cases} |F_{k-1}^*| & \text{if } a_k = a_{k-1}, \\ |F_{k-1}| & \text{if } k = 1 \text{ or } a_k < a_{k-1}, \end{cases}$$

since $\sum a_n$ is semi-fast convergent. In particular, if $j-1 < i$ and $a_{j-1} > a_j = a_{j+1} = \dots = a_i$, then $|F_i^*| = |F_{i-1}^*| = \dots = |F_j^*| = |F_{j-1}|$. This observation easily leads to the following lemma.

Lemma 13 *If $\sum a_n$ is a semi-fast convergent series, then*

$$|F_{N_1+\dots+N_{k-1}+i}| = (1 + N_1)(1 + N_2) \cdots (1 + N_{k-1})(1 + i)$$

for $1 \leq i \leq N_k$, $k \in \mathbb{N}$.

Given $x \in F$, we define α_x to be the value of a minimal summand in any representation $x = \sum_{n \in A} a_n$. In particular, $\alpha_x := a_{\max A}$ where A is a finite set of indices such that $x = \sum_{n \in A} a_n$. This definition does not depend on the choice of the set A , because $\sum a_n$ is semi-fast convergent. Further, we define

$$\rho_x := \sum_{k: \alpha_x > a_k} a_k.$$

Lemma 14 *If $\sum a_n = \sum(\alpha_k, N_k)$ is semi-fast convergent, then the open interval $(x - \alpha_x + \rho_x, x)$ is a gap of E for every $x \in F(a_n) \setminus \{0\}$.*

Proof Given any $x \in E$, there is a sequence (m_i^x) of nonnegative integers such that $0 \leq m_i^x \leq N_i$ for all i and $x = \sum_i \alpha_i m_i^x$. The sequence (m_i^x) is unique. Indeed, if $x, y \in E$, $x \neq y$, then $m_i^x \neq m_i^y$ for at least one index i . Define

$$i_0 := \min \{i \in \mathbb{N} : m_i^x \neq m_i^y\}.$$

Interchanging the names of the numbers x and y , if necessary, we may assume without loss of generality that $m_{i_0}^x > m_{i_0}^y$. Then

$$\begin{aligned} x &= \sum_{i=1}^{i_0-1} \alpha_i m_i^x + \alpha_{i_0} m_{i_0}^x + \sum_{i>i_0} \alpha_i m_i^x \geq \sum_{i=1}^{i_0-1} \alpha_i m_i^x + \alpha_{i_0} (m_{i_0}^y + 1) \\ &\geq y - \sum_{i>i_0} \alpha_i m_i^y + \alpha_{i_0} \geq y - \sum_{i>i_0} \alpha_i N_i + \alpha_{i_0}. \end{aligned} \quad (1)$$

Since $\sum a_n$ is semi-fast convergent, this implies that $x > y$. Hence for any two distinct elements x, y of E the following equivalence holds: $x > y$ if and only if $m_{i_0}^x > m_{i_0}^y$.

Fix any $x \in F \setminus \{0\}$. Let $k = \max\{n : m_n^x > 0\}$. Then $x = \sum_{i=1}^k \alpha_i m_i^x$. Take any $y \in E, y < x$. If $i_0 = k$, then $\alpha_x = \alpha_k = \alpha_{i_0}$, and hence by (1) we get $y \leq x - \alpha_x + \rho_x$. If $i_0 < k$, then $\alpha_x = \alpha_k < \alpha_{i_0}$. Clearly, $m_{i_0}^y \leq m_{i_0}^x - 1$, and thus,

$$y = \sum_{i=1}^{i_0-1} \alpha_i m_i^x + m_{i_0}^y \alpha_{i_0} + \sum_{i=i_0+1}^{\infty} \alpha_i m_i^y \leq \sum_{i=1}^{i_0} \alpha_i m_i^x - \alpha_{i_0} + \sum_{i=i_0+1}^{\infty} \alpha_i N_i.$$

Since $i_0 < k$ and $m_k^x \geq 1$, this implies that

$$y \leq x - \alpha_k - \alpha_{i_0} + \sum_{i=i_0+1}^{\infty} \alpha_i N_i < x - \alpha_k < x - \alpha_x + \rho_x.$$

As $x - \alpha_x + \rho_x \in E$, we conclude that $(x - \alpha_x + \rho_x, x)$ is a gap of E . $\square$

Lemma 15 Let $\sum a_n = \sum(\alpha_k, N_k)$ be semi-fast convergent. If (a, b) is a gap of E , then $b \in F \setminus \{0\}$ and $a = b - \alpha_b + \rho_b$.

Proof Let (a, b) be a gap of E . Define $k := \max\{n \in \mathbb{N} : a_n \geq b - a\}$. Suppose that $b = \sum_{n \in A} a_n$ for some $A \subset \mathbb{N}$ with $A \cap [k+1, +\infty) \neq \emptyset$. Taking any $l \in A \cap [k+1, +\infty)$, we easily see that $b - a_l \in E$ and $b - a_l \in (a, b)$. A contradiction, since (a, b) is a gap of E . Thus $b \in F_k$. Now it follows from Lemma 14 that $(b - \alpha_b + \rho_b, b)$ is a gap of E . Hence, $a = b - \alpha_b + \rho_b$. $\square$

Theorem 16 If $\sum a_n = \sum(\alpha_k, N_k)$ is semi-fast convergent, then $E(a_n)$ is a Cantor set. Moreover, the Lebesgue measure of $E(a_n)$ is given by

$$\mu(E(a_n)) = \sum_{k=1}^{\infty} \alpha_k N_k - \sum_{k=1}^{\infty} \left(\alpha_k - \sum_{j=k+1}^{\infty} \alpha_j N_j \right) \cdot N_k \cdot \prod_{i=1}^{k-1} (1 + N_i).$$

Proof F is a dense subset of $E(a_n)$. Thus, if $E(a_n)$ is dense in some open interval (a, b) , then F is dense in the same interval, which contradicts Lemma 14. Hence $E(a_n)$ is nowhere dense. Since it is also perfect and bounded, it is a Cantor set.

Lemmas 14 and 15 provide us with full knowledge on the gaps of E . In particular, the length $\alpha_x - \rho_x$ of a gap with the right endpoint $x \in F_{N_1+\dots+N_k} \setminus F_{N_1+\dots+N_{k-1}}$ is equal to

$$\alpha_k - \sum_{j=k+1}^{\infty} \alpha_j N_j.$$

On the other hand, by Lemma 13, the number of E -gaps with right endpoints belonging to $F_{N_1+\dots+N_k} \setminus F_{N_1+\dots+N_{k-1}}$ is equal to

$$|F_{N_1+\dots+N_k}| - |F_{N_1+\dots+N_{k-1}}| = N_k \cdot \prod_{i=1}^{k-1} (1 + N_i).$$

Hence

$$\begin{aligned} \mu(E) &= \mu\left([0, \sum a_n]\right) - \mu\left([0, \sum a_n] \setminus E\right) \\ &= \sum a_n - \mu\left(\bigcup_{k=1}^{\infty} \bigcup_{x \in F_k^*} (x - \alpha_x + \rho_x, x)\right) \\ &= \sum_{k=1}^{\infty} \alpha_k N_k - \sum_{k=1}^{\infty} \sum_{x \in F_k^*} (\alpha_x - \rho_x) \\ &= \sum_{k=1}^{\infty} \alpha_k N_k - \sum_{k=1}^{\infty} \sum_{x \in F_{N_1+\dots+N_k} \setminus F_{N_1+\dots+N_{k-1}}} (\alpha_x - \rho_x) \\ &= \sum_{k=1}^{\infty} \alpha_k N_k - \sum_{k=1}^{\infty} \left(\alpha_k - \sum_{j=k+1}^{\infty} \alpha_j N_j \right) |F_{N_1+\dots+N_k} \setminus F_{N_1+\dots+N_{k-1}}| \\ &= \sum_{k=1}^{\infty} \alpha_k N_k - \sum_{k=1}^{\infty} \left(\alpha_k - \sum_{j=k+1}^{\infty} \alpha_j N_j \right) \cdot N_k \cdot \prod_{i=1}^{k-1} (1 + N_i), \end{aligned}$$

and the proof is complete. $\square$

Let us apply Theorem 16 to a k -sum of a central Cantor set $C = E(x_n)$. We get $S_k(C) = E(a_n)$ for

$$(a_n) = (\underbrace{x_1, \dots, x_1}_k, \underbrace{x_2, \dots, x_2}_k, \dots)$$

and $\sum a_n = \sum x_i N_i$, where $N_i = k$ for any $i \in \mathbb{N}$.

Theorem 17 Assume that $C = E(x_n) = C_{(\lambda_n)}$ is a central Cantor set and that k is a positive integer such that $\lambda_n < \frac{1}{k+1}$ for all $n \in \mathbb{N}$. Then $S_k(C)$ is a Cantor set. Moreover, $S_k(C)$ has positive measure if and only if $\prod_{n=1}^{\infty} (k+1)\lambda_n$ is convergent.

Proof The sequence

$$(a_n) = (\underbrace{x_1, \dots, x_1}_k, \underbrace{x_2, \dots, x_2}_k, \dots)$$

is semi-fast convergent if and only if $x_n > kr_n$ for any $n \in \mathbb{N}$. The condition $x_n > kr_n$ is equivalent to the condition $\lambda_n < \frac{1}{k+1}$. Therefore, if $\lambda_n < \frac{1}{k+1}$ for any $n \in \mathbb{N}$, then $E(a_n)$ is a Cantor set.

In this case, the series $\sum a_n = \sum x_i N_i$, where $N_i = k$ for any $i \in \mathbb{N}$, satisfies the assumptions of Theorem 16 and $S_k(C) = E(a_n)$, so

$$\begin{aligned} \mu(S_k(C)) &= kr_0 - \sum_{i=1}^{\infty} (x_i - kr_i) \cdot k \cdot (1+k)^{i-1} \\ &= k \left(r_0 - \sum_{i=1}^{\infty} (r_{i-1} - (k+1)r_i)(1+k)^{i-1} \right) \\ &= k \left(r_0 - \sum_{i=1}^{\infty} (r_{i-1}(1+k)^{i-1} - r_i(1+k)^i) \right) \\ &= k \lim_{n \rightarrow \infty} r_n(1+k)^n \end{aligned}$$

and, consequently,

$$\begin{aligned} \mu(S_k(C)) > 0 &\iff \lim_{n \rightarrow \infty} r_n(k+1)^n > 0 \\ &\iff \lim_{n \rightarrow \infty} (k+1)^n \lambda_1 \cdots \lambda_n > 0 \\ &\iff \prod_{n=1}^{\infty} (k+1)\lambda_n \text{ is convergent} \\ &\iff \sum (1 - (k+1)\lambda_n) \text{ is convergent.} \end{aligned}$$

□

The above results were shown for $k = 2$ in the paper [19] and for constant sequence $\lambda_n = \lambda$ in [1]. To the best of our knowledge, results for $k > 2$ and nonconstant sequences (λ_n) are new.

Remark 18 If k is even, then $S_k(C)$ is not a central Cantor set. This follows from the fact that the point $\frac{kr_0}{2}$ belongs to

$$S_k(C) = E(\underbrace{x_1, \dots, x_1}_k, \underbrace{x_2, \dots, x_2}_k, \dots).$$

Theorem 19 Let $(\alpha_i, N_i)_{i \in \mathbb{N}}$ be a semi-fast convergent sequence. Then $E(\alpha_i, N_i)$ is a central Cantor set if and only if

$$\text{for any } i \text{ the number } N_i \text{ is equal to } 2^{m_i} - 1 \text{ for some } m_i \in \mathbb{N}. \quad (2)$$

Proof First assume that (2) holds. Consider the sequence

$$(2^{m_1-1}\alpha_1, 2^{m_1-2}\alpha_1, \dots, 2\alpha_1, \alpha_1, 2^{m_2-1}\alpha_2, \dots, 2\alpha_2, \alpha_2, 2^{m_3-1}\alpha_3, \dots). \quad (3)$$

It is not difficult to see that semi-fast convergence of (α_i, N_i) is equivalent to fast convergence of (3). Moreover, the achievement sets of these two sequences are identical. Hence, $E(\alpha_i, N_i)$ is a central Cantor set.

The reverse implication is less evident. Before we give a proof of it, we introduce the following, very handy symbol: given a nonnegative integer k we denote

$$\rho_k := \sum_{i>k} \alpha_i N_i.$$

Assume that $E = E(\alpha_i, N_i)$ is a central Cantor set. There is a unique fast convergent sequence $(y_l)_{l \in \mathbb{N}}$ such that $E = E(y_l)$. The set of all possible lengths of E -gaps is equal to $\{\alpha_k - \rho_k : k \in \mathbb{N}\}$ by Lemma 15. Since $\alpha_k \rightarrow 0$, there is an increasing sequence (k_i) of positive integers such that all gaps of E lying to the left of $(\rho_{k_i}, \alpha_{k_i})$ are shorter than $\alpha_{k_i} - \rho_{k_i}$.

Fix any $i \in \mathbb{N}$ and denote $k := k_i$. By the Third Gap Lemma the gap (ρ_k, α_k) , as the longest from the left, is of the form (r_n, y_n) for some $n \in \mathbb{N}$ where $r_n := \sum_{l>n} y_l$, that is, $y_n = \alpha_k$ and $r_n = \rho_k$. Define

$$B := \max \{ p \in \mathbb{N} : ((i-1)\alpha_k + \rho_k, i\alpha_k) \text{ is an } E\text{-gap for all } i = 1, \dots, p \}.$$

We will refer to the gaps listed in the definition of B as good gaps.

If $B = 1$, then $N_k = 1$ and hence N_k is of the required form.

We will show that if $B > 1$, then it is of the form $B = 2^{m+1} - 1$ for some $m \in \mathbb{N}$. Indeed, since $B \geq 2$, $2\alpha_k \in E$ and $(\alpha_k + \rho_k, 2\alpha_k)$ is an E -gap by the definition of B . Since $\alpha_k + \rho_k = y_n + r_n < \max E$, we must have $n > 1$. The First Gap Lemma implies then that $y_n + r_n$ is the left endpoint of the E -gap (r_{n-1}, y_{n-1}) and hence $y_{n-1} = 2\alpha_k$.

If $B \geq 4$, then $(3\alpha_k + \rho_k, 4\alpha_k)$ is a good gap with the left endpoint $y_{n-1} + y_n + r_n$ and thus $y_{n-2} = 4\alpha_k$. Continuing in this fashion, we conclude that

$$\text{if } B \geq 2^p \text{ for some } p \in \mathbb{N}, \text{ then } n > p \text{ and } y_{nj} = 2^j \alpha_k \text{ for } j = 0, 1, \dots, p. \quad (4)$$

Next, let $m := \max \{s \in \mathbb{N} : 2^s \leq B\}$. Then $y_n = \alpha_k, y_{n-1} = 2\alpha_k, \dots, y_{n-m} = 2^m \alpha_k$. There are 2^m good gaps in $E \cap [0, y_{n-m}]$ and, since

$$E \cap [y_{n-m}, y_{n-m} + r_{n-m}] = y_{n-m} + (E \cap [0, r_{n-m}]),$$

there are $2^m - 1$ good gaps in $E \cap (y_{n-m}, y_{n-m} + r_{n-m})$. Thus

$$B \geq 2^m + (2^m - 1) = 2^{m+1} - 1,$$

which yields $B = 2^{m+1} - 1$.

Clearly, $N_k \leq B$. If $N_k = B$, then N_k is of the required form. Suppose that $N_k < B$. Then $(\alpha_k N_k + \rho_k, \alpha_k (N_k + 1))$ is a good gap. Hence $k > 1$ and $\alpha_{k-1} = \alpha_k (N_k + 1)$. If $N_{k-1}(N_k + 1) < B$, then analogous reasoning leads to the conclusion that $\alpha_{k-2} =$

$[N_{k-1}(N_k + 1) + N_k + 1]\alpha_k$. Continuing in this fashion we notice that there exists $t \in \mathbb{N}$ such that

$$N_k\alpha_k + N_{k-1}\alpha_{k-1} + \cdots + N_{k-t}\alpha_{k-t} = B\alpha_k, \quad (5)$$

where

$$\alpha_{k-p} = \prod_{j=0}^{p-1} (N_{k-j} + 1)\alpha_k \quad \text{for } p = 1, \dots, t.$$

Combining the above equalities with (5) we get

$$(N_k + 1)\alpha_k + \sum_{p=1}^t N_{k-p} \prod_{j=0}^{p-1} (N_{k-j} + 1)\alpha_k = 2^{m+1}\alpha_k.$$

Since $N_k + 1$ is a divisor of the left-hand side, it must be a divisor of 2^{m+1} , hence $N_k = 2^z - 1$ for some $z = 1, \dots, m$.

We have shown that if $k_i \in \mathbb{N}$ is such that $(\rho_{k_i}, \alpha_{k_i}) = (r_n, y_n)$ for some n , then $k_i \in 2^{\mathbb{N}} - 1$. Now, if $k_i \geq 2$, then $N_{k_i}\alpha_{k_i} = y_p + r_p$ for some $p < n$ by (4). Since $N_{k_i}\alpha_{k_i} < \max E$, we must have that $p > 1$. The open interval $(N_{k_i}\alpha_{k_i} + \rho_{k_i}, \alpha_{k_{i-1}})$ is an E -gap and the gap is also of the form (r_{p-1}, y_{p-1}) . Thus, $k_{i-1} = k_i - 1$ and hence $\alpha_{k_{i-1}} = y_{p-1}$ and $\rho_{k_{i-1}} = r_{p-1}$. Therefore $N_{k_{i-1}} \in 2^{\mathbb{N}} - 1$. Repeating the argument, we conclude that $n_k \in 2^{\mathbb{N}} - 1$ for all $k \leq k_i$.

Since such a k_i can be chosen arbitrarily large, the proof is completed. $\square$

Corollary 20 Let $C = C_{(\lambda_n)}$ be a central Cantor set and let k be a positive integer such that $\lambda_n < \frac{1}{k+1}$ for all n . Then $S_k(C)$ is a central Cantor set if and only if $k \in 2^{\mathbb{N}} - 1$.

Example 21 From Theorems 8 and 17 we know that a k -sum of central Cantor sets can form an interval or a Cantor set (of positive measure or of measure zero). To show that $S_k(C)$ can be a Cantor set we will use some properties of achievement sets generated by multigeometric sequences. Denote by $C(q)$ the set $E(3, 2; q)$. The sequence $(3, 2; \frac{1}{4})$, called the Guthrie–Nymann sequence (see [8]), is probably the simplest example of a sequence whose achievement set is a Cantor set. It is also known that for $q < \frac{1}{4}$ the set $C(q)$ is a Cantor set (see for example [4] and [8]). For $q < \frac{1}{6}$ the sequences $(3, 2; q)$ are fast convergent, so such $C(q)$'s are central Cantors. Since

$$C(q) + C(q) = E(3, 3, 2, 2; q),$$

let us look closely at the multigeometric sequence $(3, 3, 2, 2; q)$. Recall that the achievement sets of multigeometric sequences are special cases of self-similar sets of the form

$$K(\Sigma; q) := \left\{ \sum_{n=0}^{\infty} a_n q^n : (a_n)_{n=0}^{\infty} \in \Sigma^{\omega} \right\},$$

where $\Sigma \subset \mathbb{R}$ is a compact set of real numbers and $q \in (0, 1)$ (see for example [2]). The set $K(\Sigma; q)$ is self-similar in the sense that $K(\Sigma; q) = \Sigma + q \cdot K(\Sigma; q)$. Moreover, the set $K(\Sigma; q)$ can be found as the unique compact solution $K \subset \mathbb{R}$ of the equation $K = \Sigma + qK$. This follows from the Banach Fixed Point Theorem applied to the contraction mapping $K \mapsto \Sigma + qK$ on the hyperspace $H(\mathbb{R})$ of compact sets of real numbers.

It is easily seen that for a multigeometric sequence $x_q = (k_1, \dots, k_m; q)$ the achievement set $E(x_q)$ coincides with the self-similar set $K(\Sigma; q)$ for the set

$$\Sigma := \left\{ \sum_{n=1}^m k_n \varepsilon_n : (\varepsilon_n)_{n=1}^m \in \{0, 1\}^m \right\}$$

of all possible sums of the numbers $k_1, \dots, k_m$. If $(x_n) = (3, 3, 2, 2; q)$, then $\Sigma = \{0, 2, 3, 4, 5, 6, 7, 8, 10\}$. Applying [2, Theorem 1.3] to this set, we obtain that the set $K(\Sigma; q) = C(q) + C(q)$ is a Cantorval for $q = \frac{1}{9}$ and $q \in [\frac{1}{7}, \frac{1}{6})$. Moreover, $C(q) + C(q)$ is an interval for any $q \geq \frac{1}{6}$ (a deeper analysis of the structure of $E(3, 3, 2, 2; \frac{1}{9})$ was recently presented in [6, Example 7.1, Figure 1]).

It is worth noting that we do not know what is the “type” of $K(\Sigma; q)$ for $q \in (\frac{1}{9}, \frac{1}{7})$. We only know that for almost all q ’s from this interval the set $K(\Sigma; q)$ has positive measure (see [20] and [2, Theorem 1.3 (d)]). Interestingly, we can find a sequence (q_n) of algebraic numbers such that $\lim_n q_n = \frac{1}{9}$, $q_n \in (\frac{1}{9}, \frac{1}{7})$ and $K(\Sigma; q_n)$ are Cantor sets with measure zero for all $n \in \mathbb{N}$ (cf. [2, Theorem 1.3 (f)]).

By Proposition 6 we know that $E(3, 2; q)$ is equal to $C_{(\lambda_n)}$, where

$$(\lambda_n) = (\lambda_1, \lambda_2, \lambda_1, \lambda_2, \dots)$$

for

$$\lambda_1 = \frac{r_1}{r_0} = \frac{2+3q}{5} \quad \text{and} \quad \lambda_2 = \frac{r_2}{r_1} = \frac{5q}{2+3q}.$$

In particular, for $q = \frac{1}{9}$ we obtain

$$\lambda_1 = \frac{7}{15} > \frac{1}{3} \quad \text{and} \quad \lambda_2 = \frac{5}{21} < \frac{1}{3}.$$

This example shows that the assumption that $\lambda_n < \frac{1}{k+1}$ cannot be removed from Theorem 17. The same assumption cannot be removed from Corollary 20 either. To see that, consider the sum of three copies of $C(q)$. We get

$$\begin{aligned} C(q) + C(q) + C(q) &= E(3, 3, 3, 2, 2, 2; q), \\ \Sigma &= \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15\}, \end{aligned}$$

and, as we already know,

$$\lambda_1 = \frac{2+3q}{5} > \frac{1}{4} \quad \text{and} \quad \lambda_2 = \frac{5q}{2+3q}.$$

Note that, by [2, Theorem 1.3], in the range $q \in (0, \frac{1}{6})$ the set $S_3(C(q))$ can be of any admissible topological type, depending on the value of q . For $q > \frac{2}{17}$ the set $S_3(C(q))$ is an interval and for $q = \frac{1}{14}$ it is a Cantorval. Finally, for $q < \frac{1}{16}$ we have that $\lambda_1 > \frac{1}{4}$, $\lambda_2 < \frac{1}{4}$ and $S_3(C(q))$ is a Cantor set but not a central Cantor set.

Indeed, suppose that $S_3(C(q))$ is a central Cantor set for some $q < \frac{1}{16}$. Then there is a unique fast convergent sequence (y_n) such that

$$S_3(C(q)) = E(y_n).$$

The central gap of this set has to be of the form (r_1, y_1) . Hence $S_3(C(q)) \cap [0, r_1]$ is a central Cantor, too. However, since $q < \frac{1}{16}$, the open interval $(\frac{15q}{1-q}, 2)$ is the single $(S_3(C(q)) \cap [0, r_1])$ -gap longer than 1. Since this gap is not central in $[0, r_1]$, $S_3(C(q)) \cap [0, r_1]$ is not a central Cantor set. A contradiction. (In fact $S_3(C(q))$ is a Cantor set that is not a central Cantor set for all $q < \frac{1}{14}$, but the proof is a bit more complicated.)

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